1. Introduction

The theory of Rosenthal compacta, that is, of compact subsets of the first Baire class on a Polish space X, was initiated with the pioneering work of H. P. Rosenthal [Ro2]. Significant contributions of many researchers coming from divergent areas have revealed the deep structural properties of this class. Our aim is to study some aspects of separable Rosenthal compacta, as well as to present some of their applications.

The present work consists of three parts. In the first one we determine the prototypes of separable Rosenthal compacta and we provide a classification theorem. The second part concerns an extension of a theorem of S. Todorčević included in his profound study of Rosenthal compacta [To1]. The last one is devoted to applications.

Our results concerning the first part are mainly included in Theorems 2 and 3 below. Roughly speaking, we assert that there exist seven separable Rosenthal compacta such that every $\mathcal{K}$ in that class contains one of them in a very canonical way. We start with the following.

DEFINITION 1.

(a) Let I be a countable set and X, Y be Polish spaces. Let $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ be two pointwise bounded families of real-valued functions on X and Y respectively, indexed by the set I. We say that $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are *equivalent* if the natural map $f_i \mapsto g_i$ extends to a topological homeomorphism between $\overline{\{f_i\}}^p_{i\in I}$ and $\overline{\{g_i\}}^p_{i\in I}$.

(b) Let X be a Polish space and $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ be relatively compact in $\mathcal{B}_1(X)$. We say that $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ is *minimal* if for every dyadic subtree $S = (s_t)_{t\in 2^{<\mathbb{N}}}$ of the Cantor tree $2^{<\mathbb{N}}$, the families $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ and $\{f_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ are equivalent.

In connection with the above notions, the following is proved.

THEOREM 2.

(a) *Up to equivalence, there are exactly seven minimal families.*

(b) *For every family $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ relatively compact in $\mathcal{B}_1(X)$, with X Polish, there exists a regular dyadic subtree $S = (s_t)_{t\in 2^{<\mathbb{N}}}$ of $2^{<\mathbb{N}}$ such that $\{f_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to one of the seven minimal families.*

For any of the seven minimal families the corresponding pointwise closure is a separable Rosenthal compact containing the family as a discrete set. We denote them as follows:

$$A(2^{<\mathbb{N}}),\ 2^{\leqslant\mathbb{N}},\ \hat{S}_+(2^{\mathbb{N}}),\ \hat{S}_-(2^{\mathbb{N}}),\ \hat{A}(2^{\mathbb{N}}),\ \hat{D}(2^{\mathbb{N}}),\ \hat{D}(S(2^{\mathbb{N}})).$$

The precise description of the families and the corresponding compacta is given in §4.3. The first two in the above list are metrizable spaces. The next two are hereditarily separable, non-metrizable and mutually homeomorphic (thus, the above-defined notion of equivalence of families is stronger than saying that the corresponding closures are homeomorphic). The space $\hat{S}_{+}(2^{\mathbb{N}})$, and so the space $\hat{S}_{-}(2^{\mathbb{N}})$ as well, can be realized as a closed subspace of the split interval $S(I)$. Following [E], we shall denote by $A(2^{\mathbb{N}})$ the one-point compactification of the Cantor set $2^{\mathbb{N}}$. The space $\hat{A}(2^{\mathbb{N}})$ is the standard separable extension of $A(2^{\mathbb{N}})$ (see [Po2], [Ma]). This is the only non-first countable space from the above list. The space $\hat{D}(2^{\mathbb{N}})$ is the separable extension of the Aleksandrov duplicate of the Cantor set $D(2^{\mathbb{N}})$, as described in [To1]. Finally, the space $\hat{D}(S(2^{\mathbb{N}}))$ can be realized as a closed subspace of the Helly space. Its accumulation points form the closure of the standard uncountable discrete subset of the Helly space.

Theorem 2 is essentially a success of the infinite-dimensional Ramsey theory for trees and perfect sets. There is a long history of the interaction between Ramsey theory and Rosenthal compacta, which can be traced back to J. Farahat's classical proof [F] of H. P. Rosenthal's ℓ_1 Theorem [Ro1] and its tree extension due to J. Stern [Ste]. This interaction was further expanded by S. Todorčević in [To1] with the use of the parameterized Ramsey theory for perfect sets.

The new Ramsey-theoretic ingredient in the proof of Theorem 2 is a result concerning partitions of two classes of antichains of the Cantor tree, which we call *increasing* and *decreasing*. We will briefly comment on the proof of Theorem 2 and the critical role of this result. One starts with a family $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ relatively compact in $\mathcal{B}_1(X)$. A first topological reduction shows that in order to understand the closure of $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ in $\mathbb{R}^X$ it is enough to determine all subsets of the Cantor tree for which the corresponding subsequence of $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ is pointwise convergent. A second reduction shows that it is enough to determine only a cofinal subset of convergent subsequences. One is then led to analyze which classes of subsets of the Cantor tree are Ramsey and cofinal. First, we observe that every infinite subset of $2^{<\mathbb{N}}$ contains either an infinite chain or an infinite antichain. It is well-known, and goes back to Stern, that chains are Ramsey. On the other hand, the set of all antichains is not. However, the classes of increasing and decreasing antichains are Ramsey and, moreover, they are cofinal in the set of all antichains. Using the above properties of chains and of increasing and decreasing antichains we are able to have a satisfactory control over the convergent subsequences of $\{f_t\}_{t\in 2^{<\mathbb{N}}}$. Finally, repeated applications of F. Galvin's theorem on partitions of doubletons of perfect sets of reals permit us to fully canonicalize the topological behavior of $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ yielding the proof of Theorem 2.

A direct consequence of Theorem 2(b) is that for every separable Rosenthal compact and for every countable dense subset $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ of it, there exists a regular dyadic subtree $S=(s_t)_{t\in 2^{<\mathbb{N}}}$ such that the pointwise closure of $\{f_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is homeomorphic to one of the above-described compacta. In general, for a given countable dense subset $\{f_n\}_n$ of a separable Rosenthal compact $\mathcal{K}$, we say that one of the minimal families *canonically embeds* into $\mathcal{K}$ with respect to $\{f_n\}_n$ if there exists an increasing injection $\phi: 2^{<\mathbb{N}}\to\mathbb{N}$ such that the family $\{f_{\phi(t)}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to it. The next theorem is a supplement to Theorem 2, showing that the minimal families can be chosen to characterize certain topological properties of $\mathcal{K}$.

THEOREM 3. *Let $\mathcal{K}$ be a separable Rosenthal compact and $\{f_n\}_n$ a countable dense subset of $\mathcal{K}$.*

(a) *If $\mathcal{K}$ consists of bounded functions in $\mathcal{B}_1(X)$, is metrizable and non-separable in the supremum norm, then $2^{\leq\mathbb{N}}$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}_n$ such that its image is norm non-separable.*
(b) *If $\mathcal{K}$ is non-metrizable and hereditarily separable, then either $\hat{S}_+(2^{\mathbb{N}})$ or $\hat{S}_-(2^{\mathbb{N}})$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}_n$.*
(c) *If $\mathcal{K}$ is not hereditarily separable and first countable, then either $\hat{D}(2^{\mathbb{N}})$ or $\hat{D}(S(2^{\mathbb{N}}))$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}_n$.*
(d) *If $\mathcal{K}$ is not first countable, then $\hat{A}(2^{\mathbb{N}})$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}_n$.*

In particular, if $\mathcal{K}$ is non-metrizable, then one of the non-metrizable prototypes canonically embeds into $\mathcal{K}$ with respect to any dense subset of $\mathcal{K}$.

Part (a) is an extension of Ch. Stegall's classical result [St], which led to the characterization of the Radon–Nikodym property in dual Banach spaces. We mention that Todorčević [To1] has shown that in case (b) above the split interval $S(I)$ embeds into $\mathcal{K}$. It is an immediate consequence of the above theorem that every non-hereditarily separable $\mathcal{K}$ contains an uncountable discrete subspace of the size of the continuum, a result due to R. Pol [Po1]. The proofs of parts (a), (b) and (c) use variants of Stegall's fundamental construction, similar in spirit to the work of G. Godefroy and M. Talagrand [GT]. Part (d) is a consequence of a more general structural result concerning non-G_δ points which we are about to describe. To this end, we start with the following.

DEFINITION 4. Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X and $\mathcal{C}$ a closed subspace of $\mathcal{K}$. We say that $\mathcal{C}$ is an *analytic subspace* if there exist a countable dense subset $\{f_n\}_n$ of $\mathcal{K}$ and an analytic subset A of $[\mathbb{N}]$ such that:

(1) For every $L \in A$ the accumulation points of the set $\{f_n : n \in L\}$ in $\mathbb{R}^X$ form a subset of $\mathcal{C}$.
(2) For every $g \in \mathcal{C}$ which is an accumulation point of $\mathcal{K}$ there exists $L \in A$ with $g \in \overline{\{f_n\}}^p_{n\in L}$.

Observe that every separable Rosenthal compact $\mathcal{K}$ is an analytic subspace of itself with respect to any countable dense set. Let us point out that while the class of analytic subspaces is strictly wider than the class of separable ones, it shares all the structural properties of the separable Rosenthal compacta. This will become clear in what follows.

A natural question raised by the above definition is whether the concept of an analytic subspace depends on the choice of the countable dense subset of $\mathcal{K}$. We believe that it is independent. This is supported by the fact that it is indeed so for analytic subspaces of separable Rosenthal compacta in $\mathcal{B}_1(X)$ with X compact metrizable.

To state our results concerning analytic subspaces, we also need the following.

DEFINITION 5. Let $\mathcal{K}$ be a separable Rosenthal compact, $\{f_n\}_n$ a countable dense subset of $\mathcal{K}$ and $\mathcal{C}$ a closed subspace of $\mathcal{K}$. We say that one of the prototypes $\mathcal{K}_i$ $(1 \le i \le 7)$ *canonically embeds* into $\mathcal{K}$ with respect to $\{f_n\}_n$ and $\mathcal{C}$ if there exists a subfamily $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ of $\{f_n\}_n$ which is equivalent to the canonical dense family of $\mathcal{K}_i$ and such that all accumulation points of $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ are in $\mathcal{C}$.

The following theorem describes the structure of non-first countable analytic subspaces.

THEOREM 6. *Let $\mathcal{K}$ be a separable Rosenthal compact, $\mathcal{C}$ an analytic subspace of $\mathcal{K}$ and $\{f_n\}_n$ a countable dense subset of $\mathcal{K}$ witnessing the analyticity of $\mathcal{C}$. Let also $f \in \mathcal{C}$ be a non-G_δ point of $\mathcal{C}$. Then $\hat{A}(2^{\mathbb{N}})$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}_n$ and $\mathcal{C}$ and is such that f is the unique non-G_δ point of its image.*

Theorem 6 is the last step of a series of results initiated by a fruitful problem concerning the character of points in separable Rosenthal compacta, posed by R. Pol [Po1]. The first decisive step towards the solution of this problem was made by A. Krawczyk [Kr]. He proved that a point $f \in \mathcal{K}$ is non-G_δ if and only if the set

$$\mathcal{L}_{\mathbf{f},f} = \{L \in [\mathbb{N}] : (f_n)_{n\in L} \text{ is pointwise convergent to } f\}$$

is co-analytic non-Borel. His analysis revealed a fundamental construction, which we call the *Krawczyk tree* (K-tree) with respect to the given point f and any countable dense subset $\mathbf{f} = \{f_n\}_n$ of $\mathcal{K}$. He actually showed that there exists a subfamily $\{f_t\}_{t\in \mathbb{N}^{<\mathbb{N}}}$ of $\{f_n\}_n$ such that:

(P1) For every $\sigma \in \mathbb{N}^{\mathbb{N}}$, $f \notin \overline{\{f_{\sigma|n}\}_n}^p$.
(P2) If $A \subseteq \mathbb{N}^{<\mathbb{N}}$ is such that $f \notin \overline{\{f_t\}_{t\in A}}^p$, then for $n \in \mathbb{N}$ there exist $t_0, \dots, t_k \in \mathbb{N}^n$ such that A is almost included in the set of successors of the t_i's.

Using K-trees, the second named author has shown ([Do]) that the set

$$\mathcal{L}_{\mathbf{f}} = \{L \in [\mathbb{N}] : (f_n)_{n\in L} \text{ is pointwise convergent}\}$$

is complete co-analytic if there exists a non-G_δ point $f \in \mathcal{K}$. Let us also point out that the deep effective version of G. Debs' theorem [De] implies that for any separable Rosenthal compact the set $\mathcal{L}_{\mathbf{f}}$ contains a Borel cofinal subset.

There is strong evidence, like Debs' theorem mentioned above, that separable Rosenthal compacta are definable objects, hence, they are naturally connected to descriptive set theory (see also [ADK1], [B], [Do]). One of the first results illustrating this connection was proved in the late 70's by G. Godefroy [Go], namely that a separable compact $\mathcal{K}$ is Rosenthal if and only if $C(\mathcal{K})$ is an analytic subset of $\mathbb{R}^D$ for every countable dense subset D of $\mathcal{K}$. In this connection, R. Pol has conjectured that a separable Rosenthal compact $\mathcal{K}$ embeds into $\mathcal{B}_1(2^{\mathbb{N}})$ if and only if $C(\mathcal{K})$ is a Borel subset of $\mathbb{R}^D$ (see [Ma] and [Po2]). It is worth mentioning that for a separable $\mathcal{K}$ in $\mathcal{B}_1(2^{\mathbb{N}})$, for every countable dense subset $\{f_n\}_n$ of $\mathcal{K}$ and every $f \in \mathcal{K}$, there exists a Borel cofinal subset of the corresponding set $\mathcal{L}_{\mathbf{f},f}$, a property not shared by all separable Rosenthal compacta.

The final step in the solution of Pol's problem was made by S. Todorčević [To1]. He proved that if f is a non-G_δ point of $\mathcal{K}$, then the space $A(2^{\mathbb{N}})$ is homeomorphic to a closed subset of $\mathcal{K}$ with f as the unique limit point. His remarkable proof involves metamathematical arguments like forcing and absoluteness.

Let us proceed to a discussion of the proof of Theorem 6. The first decisive step is the following theorem, concerning the existence of K-trees.

THEOREM 7. *Let $\mathcal{K}$, $\mathcal{C}$, $\{f_n\}_n$ and $f \in \mathcal{C}$ be as in Theorem 6. Then there exists a K-tree $\{f_t\}_{t\in\mathbb{N}^{<\mathbb{N}}}$ with respect to the point f and the dense sequence $\{f_n\}_n$ such that for every $\sigma \in \mathbb{N}^{\mathbb{N}}$ all accumulation points of the set $\{f_{\sigma|n} : n \in \mathbb{N}\}$ are in $\mathcal{C}$.*

The proof of the above result is a rather direct extension of the results of A. Krawczyk from [Kr] and is based on the key property of bi-sequentiality, established for separable Rosenthal compacta by R. Pol [Po3]. We will briefly comment on some further properties of the K-tree $\{f_t\}_{t\in\mathbb{N}^{<\mathbb{N}}}$ obtained by Theorem 7. To this end, let us call an antichain $\{t_n\}_n$ of $\mathbb{N}^{<\mathbb{N}}$ a *fan* if there exist $s \in \mathbb{N}^{<\mathbb{N}}$ and a strictly increasing sequence $(m_n)_n$ in $\mathbb{N}$ such that $s^\frown m_n \sqsubseteq t_n$ for every $n \in \mathbb{N}$. Let us also say that an antichain $\{t_n\}_n$ *converges* to $\sigma \in \mathbb{N}^{\mathbb{N}}$ if for every $k \in \mathbb{N}$ the set $\{t_n\}_n$ is almost contained in the set of successors of $\sigma|k$. Property (P2) of K-trees implies that for every fan $\{t_n\}_n$ of $\mathbb{N}^{<\mathbb{N}}$ the sequence $(f_{t_n})_n$ must be pointwise convergent to f. This fact combined with the bi-sequentiality of separable Rosenthal compacta yields the following:

(P3) For every $\sigma \in \mathbb{N}^{\mathbb{N}}$ there exists an antichain $\{t_n\}_n$ of $\mathbb{N}^{<\mathbb{N}}$ which converges to σ and is such that the sequence $(f_{t_n})_n$ is pointwise convergent to f.

In the second crucial step, we use the infinite-dimensional extension of Hindman's theorem, due to K. Milliken [Mil1], to pass to an infinitely splitting subtree T of $\mathbb{N}^{<\mathbb{N}}$ such that for every $\sigma \in [T]$ the corresponding antichain $\{t_n\}_n$, described in property (P3), is found in a canonical way. We should point out that, although Milliken's theorem is a result concerning partitions of block sequences, it can also be considered as a partition theorem for a certain class of infinitely splitting subtrees of $\mathbb{N}^{<\mathbb{N}}$. This fact was first realized by W. Henson, in his alternative proof of Stern's theorem (see [Od]), and it is used in the proof of Theorem 6 in a similar spirit. The proof is then completed by choosing an appropriate dyadic subtree S of T and applying the canonicalization method (Theorem 2) to the family $\{f_s\}_{s\in S}$.

The following consequence of Theorem 6 describes the universal property of $\hat{A}(2^{\mathbb{N}})$ among all fundamental prototypes.

COROLLARY 8. *Let $\mathcal{K}$ be a non-metrizable separable Rosenthal compact and $D = \{f_n\}_n$ a countable dense subset of $\mathcal{K}$. Then the space $\hat{A}(2^{\mathbb{N}})$ canonically embeds into $\mathcal{K} - \mathcal{K}$ with respect to $D - D$ and with the constant function 0 as the unique non-G_δ point.*

We notice that the above corollary remains valid within the class of analytic subspaces.

The embedding of $\hat{A}(2^{\mathbb{N}})$ in an analytic subspace $\mathcal{C}$ of a separable Rosenthal compact $\mathcal{K}$ yields unconditional families of elements of $\mathcal{C}$ as follows.

THEOREM 9. *Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X consisting of bounded functions. Let also $\mathcal{C}$ be an analytic subspace of $\mathcal{K}$ having the constant function* 0

as a non-G_δ point. Then there exists a family $\{f_\sigma : \sigma \in 2^{\mathbb{N}}\}$ in $\mathcal{C}$ which is 1-unconditional in the supremum norm, pointwise discrete and having 0 as the unique accumulation point.

Theorem 9 follows from Theorem 6 and the "perfect unconditionality theorem" of [ADK2].

A second application concerns representable Banach spaces, a class introduced in [GT] and closely related to separable Rosenthal compacta.

THEOREM 10. *Let X be a non-separable representable Banach space. Then X^* contains an unconditional family of size $|X^*|$.*

We also introduce the concept of spreading and level unconditional tree bases. This notion is implicitly contained in [ADK2] where their existence was established in every separable Banach space not containing ℓ_1 and with non-separable dual. We present some extensions of this result in the framework of separable Rosenthal compacta.

We proceed to discuss how this work is organized. In §2, we set up our notations concerning trees and we present the Ramsey-theoretic preliminaries needed in the rest of the paper. In the next section we define and study the classes of increasing and decreasing antichains. The main result in §3 is Theorem 10, which establishes the Ramsey properties of these classes. Section 4 is exclusively devoted to the proof of Theorem 2. It consists of four subsections. In the first one, we prove a theorem (Theorem 16 in the main text) which is the first step towards the proof of Theorem 2. Theorem 16 is a consequence of the Ramsey and structural properties of chains and of increasing and decreasing antichains. In §4.2, we introduce the notion of equivalence of families of functions and we provide a criterion for establishing it. As already mentioned, in §4.3 we describe the seven minimal families. The proof of Theorem 2 is completed in §4.4.

In §5.1, we introduce the class of analytic subspaces of separable Rosenthal compacta and we present some of their properties, while in §5.2 we study separable Rosenthal compacta in $\mathcal{B}_1(2^{\mathbb{N}})$. In §6, we prove parts (a), (b) and (c) of Theorem 3. Actually, Theorem 3 is proved for the wider class of analytic subspaces and within the context of Definition 5. The precise statement is as follows.

THEOREM 11. *Let $\mathcal{K}$ be a separable Rosenthal compact, $\mathcal{C}$ an analytic subspace of $\mathcal{K}$ and $\{f_n\}_n$ a countable dense subset of $\mathcal{K}$ witnessing the analyticity of $\mathcal{C}$.*

(a) *If $\mathcal{C}$ is metrizable in the pointwise topology, consists of bounded functions and is non-separable in the supremum norm of $\mathcal{B}_1(X)$, then $2^{\leq\mathbb{N}}$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}_n$ and $\mathcal{C}$, so that its image is norm non-separable.*
(b) *If $\mathcal{C}$ is hereditarily separable and non-metrizable, then either $\hat{S}_+(2^{\mathbb{N}})$ or $\hat{S}_-(2^{\mathbb{N}})$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}_n$ and $\mathcal{C}$.*
(c) *If $\mathcal{C}$ is not hereditarily separable and first countable, then either $\hat{D}(2^{\mathbb{N}})$ or $\hat{D}(S(2^{\mathbb{N}}))$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}_n$ and $\mathcal{C}$.*

Section 7 is devoted to the study of non-first countable analytic subspaces. In §7.1 we prove Theorem 7, while §7.2 is devoted to the proof of Theorem 6. The final section is devoted to applications and in particular to the proofs of Theorem 9 and Theorem 10.

We thank Stevo Todorčević for his valuable remarks and comments.

2. Ramsey properties of perfect sets and of subtrees of the Cantor tree

The aim of this section is to present the Ramsey-theoretic preliminaries needed in the rest of the paper, as well as to set up our notation concerning trees.

Ramsey theory for trees was initiated with the fundamental Halpern–Läuchli partition theorem [HL]. The original proof used metamathematical arguments. The proof avoiding metamathematics was given in [AFK]. Partition theorems related to the ones presented in this section can be found in the work of K. Milliken [Mil2], A. Blass [Bl] and A. Louveau, S. Shelah and B. Veličković [LSV].

2.1. Notations. We let $\mathbb{N} = \{0, 1, \ldots\}$. We denote by $[\mathbb{N}]$ the set of all infinite subsets of $\mathbb{N}$, while for every $L \in [\mathbb{N}]$ we let $[L]$ denote the set of all infinite subsets of L. If $k \geq 1$ and $L \in [\mathbb{N}]$, then $[L]^k$ stands for the set of all finite subsets of L of cardinality k.

A. We denote by $2^{<\mathbb{N}}$ the set of all finite sequences of 0's and 1's (the empty sequence is included). We view $2^{<\mathbb{N}}$ as a tree equipped with the (strict) partial order $\sqsubset$ of extension. If $t \in 2^{<\mathbb{N}}$, then the *length* $|t|$ of t is defined to be the cardinality of the set $\{s : s \sqsubset t\}$. If $s, t \in 2^{<\mathbb{N}}$, then we denote by $s^\frown t$ their concatenation. Two nodes s, t are said to be *comparable* if either $s \sqsubseteq t$ or $t \sqsubseteq s$; otherwise are said to be *incomparable*. A subset of $2^{<\mathbb{N}}$ consisting of pairwise comparable nodes is said to be a *chain*, while a subset of $2^{<\mathbb{N}}$ consisting of pairwise incomparable nodes is said to be an *antichain*. For every $x \in 2^{\mathbb{N}}$ and every $n \geq 1$ we set $x|n = (x(0), \ldots, x(n-1)) \in 2^{<\mathbb{N}}$ and $x|0 = (\emptyset)$. For $x, y \in (2^{<\mathbb{N}} \cup 2^{\mathbb{N}})$ with $x \neq y$ we denote by $x \wedge y$ the $\sqsubset$-maximal node t of $2^{<\mathbb{N}}$ with $t \sqsubseteq x$ and $t \sqsubseteq y$. Moreover, we write $x \prec y$ if $w^\frown 0 \sqsubseteq x$ and $w^\frown 1 \sqsubseteq y$, where $w = x \wedge y$. The ordering $\prec$ restricted to $2^{\mathbb{N}}$ is the usual lexicographical ordering of the Cantor set.

B. We view every subset of $2^{<\mathbb{N}}$ as a *subtree* with the induced partial ordering. A subtree T of $2^{<\mathbb{N}}$ is said to be *pruned* if for every $t \in T$ there exists $s \in T$ with $t \sqsubset s$. It is said to be *downwards closed* if for every $t \in T$ and every $s \sqsubset t$ we have that $s \in T$. For a subtree T of $2^{<\mathbb{N}}$ (not necessarily downwards closed) we set $\hat{T} = \{s : \exists t \in T \text{ with } s \sqsubseteq t\}$. If T is downwards closed, then the *body* $[T]$ of T is the set $\{x \in 2^{\mathbb{N}} : x|n \in T \ \forall n\}$.

C. Let T be a subtree of $2^{<\mathbb{N}}$ (not necessarily downwards closed). For every $t \in T$ we denote by $|t|_T$ the cardinality of the set $\{s \in T : s \sqsubset t\}$ and for every $n \in \mathbb{N}$ we set $T(n) = \{t \in T : |t|_T = n\}$. Moreover, for every $t_1, t_2 \in T$ we let $t_1 \wedge_T t_2$ denote the $\sqsubset$-maximal node w of T such that $w \sqsubseteq t_1$ and $w \sqsubseteq t_2$. Observe that $t_1 \wedge_T t_2 \sqsubseteq t_1 \wedge t_2$. Given two subtrees S and T of $2^{<\mathbb{N}}$, we say that S is a *regular* subtree of T if $S \subseteq T$ and for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $S(n) \subseteq T(m)$. For a regular subtree T of $2^{<\mathbb{N}}$, the *level set* L_T of T is the set $\{l_n : T(n) \subseteq 2^{l_n}\} \subseteq \mathbb{N}$. Notice that for every $x \in [\hat{T}]$ and every $m \in \mathbb{N}$ we have $x|m \in T$ if and only if $m \in L_T$. Hence, the chains of T are naturally identified with the product $[\hat{T}] \times [L_T]$. A pruned subtree T of $2^{<\mathbb{N}}$ is said to be *skew* if for every $n \in \mathbb{N}$ there exists at most one splitting node of T in $T(n)$ with exactly two immediate successors in T; it is said to be *dyadic* if every $t \in T$ has exactly two immediate successors in T. We remark that a subtree T of the Cantor tree is regular dyadic if there exists a (necessarily unique) bijection $i_T : 2^{<\mathbb{N}} \to T$ such that:

(1) For all $t_1, t_2 \in 2^{<\mathbb{N}}$ we have $|t_1| = |t_2|$ if and only if $|i_T(t_1)|_T = |i_T(t_2)|_T$.
(2) For all $t_1, t_2 \in 2^{<\mathbb{N}}$ we have $t_1 \sqsubset t_2$ (respectively $t_1 \prec t_2$) if and only if $i_T(t_1) \sqsubset i_T(t_2)$ (respectively $i_T(t_1) \prec i_T(t_2)$).

When we write $T = (s_t)_{t\in 2^{<\mathbb{N}}}$, where T is a regular dyadic subtree of $2^{<\mathbb{N}}$, we mean that $s_t = i_T(t)$ for all $t \in 2^{<\mathbb{N}}$. Finally, we notice the following. If T is a regular dyadic subtree of $2^{<\mathbb{N}}$ and R is a regular dyadic subtree of T, then R is a regular dyadic subtree of $2^{<\mathbb{N}}$ as well.

2.2. Partitions of trees. We begin by recalling the following notion from [Ka].

Definition 1. Let T be a skew subtree of $2^{<\mathbb{N}}$. We define $f_T : \mathbb{N} \to \{1,2\}^{<\mathbb{N}}$ as follows. For every $n \in \mathbb{N}$, let $T(n) = \{s_0 \prec \cdots \prec s_{m-1}\}$ be the $\prec$-increasing enumeration of $T(n)$. We set $f_T(n) = (e_0, \ldots, e_{m-1}) \in \{1,2\}^m$, where for every $i \in \{0, \ldots, m-1\}$, e_i is the cardinality of the set of immediate successors of s_i in T. The function f_T will be called the *code* of the tree T. If $f : \mathbb{N} \to \{1,2\}^{<\mathbb{N}}$ is a function such that there exists a skew tree T with $f = f_T$, then f will be called a *skew tree code*.

For instance, if $f_T(n) = (1)$ for all $n \in \mathbb{N}$, then the tree T is a chain. Also, if $f_T(0) = (2)$ and $f_T(n) = (1,1)$ for all $n \geq 1$, then T consists of two chains. Moreover, observe that if T and S are two skew subtrees of $2^{<\mathbb{N}}$ with $f_T = f_S$, then T and S are isomorphic with respect to both $\prec$ and $\sqsubset$. If f is a skew tree code and T is a regular dyadic subtree of $2^{<\mathbb{N}}$, then we denote by $[T]_f$ the set of all regular skew subtrees of T of code f. It is easy to see that the set $[T]_f$ is a Polish subspace of 2^T. Also observe that if R is a regular dyadic tree of T, then $[R]_f = [T]_f \cap 2^R$. We will need the following theorem, which is a consequence of Theorem 46 in [Ka].

Theorem 2. *Let T be a regular dyadic subtree of $2^{<\mathbb{N}}$, f a skew tree code and A an analytic subset of $[T]_f$. Then there exists a regular dyadic subtree R of T such that either $[R]_f \subseteq A$ or $[R]_f \cap A = \emptyset$.*

For a regular dyadic subtree T of $2^{<\mathbb{N}}$, denote by $[T]_{\rm chains}$ the set of all infinite chains of T. Theorem 2 includes the following result due to J. Stern [Ste], A. W. Miller, S. Todorčević [Mi] and J. Pawlikowski [Pa].

Theorem 3. *Let T be a regular dyadic subtree of $2^{<\mathbb{N}}$ and A be an analytic subset of $[T]_{\rm chains}$. Then there exists a regular dyadic subtree R of T such that either $[R]_{\rm chains} \subseteq A$ or $[R]_{\rm chains} \cap A = \emptyset$.*

Theorem 2 will essentially be applied to the following classes of skew subtrees.

Definition 4. Let T be a regular dyadic subtree of $2^{<\mathbb{N}}$. A subtree S of T will be called *increasing* (respectively *decreasing*) if:

(a) S is uniquely rooted, regular, skew and pruned.
(b) For every $n \in \mathbb{N}$, there exists a splitting node of S in $S(n)$ which is the $\prec$-maximum (respectively $\prec$-minimum) node of $S(n)$ and has two immediate successors in S.

The class of increasing (respectively decreasing) subtrees of T will be denoted by $[T]_{\mathrm{Incr}}$ (respectively $[T]_{\mathrm{Decr}}$).

It is easy to see that every increasing (respectively decreasing) subtree is of fixed code. Thus Theorem 2 can be applied to give the following.

COROLLARY 5. *Let T be a regular dyadic subtree of $2^{<\mathbb{N}}$ and A be an analytic subset of $[T]_{\mathrm{Incr}}$. Then there exists a regular dyadic subtree R of T such that either $[R]_{\mathrm{Incr}} \subseteq A$ or $[R]_{\mathrm{Incr}} \cap A = \emptyset$. Similarly for the case of $[T]_{\mathrm{Decr}}$.*

The above corollary may be considered as a parameterized version of the Louveau–Shelah–Veličković theorem [LSV].

2.3. Partitions of perfect sets. For every subset X of $2^{\mathbb{N}}$, we denote by $[X]^2$ the set of all doubletons of X. We identify $[X]^2$ with the set of all $(\sigma,\tau) \in X^2$ with $\sigma \prec \tau$. We will need the following partition theorem due to F. Galvin (see [Ke, Theorem 19.7]).

THEOREM 6. *Let P be a perfect subset of $2^{\mathbb{N}}$. If A is a subset of $[P]^2$ with the Baire property, then there exists a perfect subset Q of P such that either $[Q]^2 \subseteq A$ or $[Q]^2 \cap A = \emptyset$.*

3. Increasing and decreasing antichains of a regular dyadic tree

In this section we define the increasing and decreasing antichains and we establish their fundamental Ramsey properties.

As we have already seen in §2, the class of infinite chains of the Cantor tree is Ramsey. On the other hand, an analogue of Theorem 3 for infinite antichains is not valid. For instance, color an antichain $(t_n)_n$ of $2^{<\mathbb{N}}$ red if $t_0 \prec t_1$; otherwise color it blue. It is easy to see that this is an open partition, yet there is no dyadic subtree of $2^{<\mathbb{N}}$ all of whose antichains are monochromatic. So, in order to have a Ramsey result for antichains, it is necessary to restrict our attention to those which are monotone with respect to $\prec$. Still, however, this is not enough. Indeed, consider the set of all $\prec$-increasing antichains and color such an antichain $(t_n)_n$ red if $|t_0| \le |t_1 \wedge t_2|$; otherwise color it blue. Again we see that this is an open partition which is not Ramsey.

The following definition incorporates all the restrictions indicated by the above discussion and which are, as we shall see, essentially the only obstacles to a Ramsey result for antichains.

DEFINITION 7. Let T be a regular dyadic subtree of the Cantor tree $2^{<\mathbb{N}}$. An infinite antichain $(t_n)_n$ of T will be called *increasing* if:

(1) For all $n, m \in \mathbb{N}$ with $n < m$, $|t_n|_T < |t_m|_T$.
(2) For all $n, m, l \in \mathbb{N}$ with $n < m < l$, $|t_n|_T \le |t_m \wedge_T t_l|_T$.
(3I) For all $n, m \in \mathbb{N}$ with $n < m$, $t_n \prec t_m$.

The set of all increasing antichains of T will be denoted by $\mathrm{Incr}(T)$. Similarly, an infinite antichain $(t_n)_n$ of T will be called *decreasing* if (1) and (2) above are satisfied and (3I) is replaced by the following:

(3D) For all $n, m \in \mathbb{N}$ with $n < m$, $t_m \prec t_n$.

The set of all decreasing antichains of T will be denoted by $\mathrm{Decr}(T)$.

The classes of increasing and decreasing antichains of T have the following crucial stability properties.

LEMMA 8. *Let T be a regular dyadic subtree of $2^{<\mathbb{N}}$. Then*:

(1) (Hereditariness) *Let $(t_n)_n \in \mathrm{Incr}(T)$ and $L = \{l_0 < l_1 < \cdots\}$ be an infinite subset of $\mathbb{N}$. Then $(t_{l_n})_n \in \mathrm{Incr}(T)$. Similarly, if $(t_n)_n \in \mathrm{Decr}(T)$, then $(t_{l_n})_n \in \mathrm{Decr}(T)$.*
(2) (Cofinality) *Let $(t_n)_n$ be an infinite antichain of T. Then there exists $L = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]$ such that either $(t_{l_n})_n \in \mathrm{Incr}(T)$ or $(t_{l_n})_n \in \mathrm{Decr}(T)$.*
(3) (Coherence) *We have $\mathrm{Incr}(T) = \mathrm{Incr}(2^{<\mathbb{N}}) \cap 2^T$ and similarly for the decreasing antichains.*

Proof. (1) Straightforward.

(2) The point is that all three properties in the definition of increasing and decreasing antichains are cofinal in the set of all antichains of T. Indeed, let $(t_n)_n$ be an infinite antichain of T. Clearly, there exists $N \in [\mathbb{N}]$ such that the sequence $(|t_n|_T)_{n\in N}$ is strictly increasing. Moreover, by Ramsey's theorem, there exists $M \in [N]$ such that the sequence $(t_n)_{n\in M}$ is either $\prec$-increasing or $\prec$-decreasing. Finally, to see that condition (2) in Definition 7 is cofinal, let

$$A = \{(n, m, l) \in [M]^3 : |t_n|_T \le |t_m \wedge_T t_l|_T\}.$$

By Ramsey's theorem again, there exists $L \in [M]$ such that either $[L]^3 \subseteq A$ or $[L]^3 \cap A = \emptyset$. We claim that $[L]^3 \subseteq A$, which clearly completes the proof. Assume not, i.e. $[L]^3 \cap A = \emptyset$. Let $n = \min L$ and $L' = L \setminus \{n\} \in [L]$. Let also $k = |t_n|_T$. Then for every $(m, l) \in [L']^2$ we have $|t_m \wedge_T t_l|_T < k$. The set $\{t \in T : |t|_T < k\}$ is finite. Hence, by another application of Ramsey's theorem, there exist $s \in T$ with $|s|_T < k$ and $L'' \in [L']$ such that for every $(m, l) \in [L'']^2$ we have $s = t_m \wedge_T t_l$. But this is clearly impossible as the tree T is dyadic.

(3) First we observe the following. As the tree T is regular, for every $t, s \in T$ we have $|t|_T < |s|_T$ (respectively $|t|_T = |s|_T$) if and only if $|t| < |s|$ (respectively $|t| = |s|$).

Now, let $(t_n)_n \in \mathrm{Incr}(T)$. In order to show that $(t_n)_n \in \mathrm{Incr}(2^{<\mathbb{N}}) \cap 2^T$ it is enough to prove that for every $n < m < l$ we have $|t_n| \le |t_m \wedge t_l|$. By the above remarks, $|t_n| \le |t_m \wedge_T t_l|$. As $t_m \wedge_T t_l \sqsubseteq t_m \wedge t_l$, we are done.

Conversely, assume that $(t_n)_n \in \mathrm{Incr}(2^{<\mathbb{N}}) \cap 2^T$. Again it is enough to check that condition (2) in Definition 7 is satisfied. So let $n < m < l$. There exist $s_m, s_l \in T$ with $|s_m|_T = |s_l|_T = |t_n|_T$, $s_m \sqsubseteq t_m$ and $s_l \sqsubseteq t_l$. We claim that $s_m = s_l$. Indeed, if not, then $|t_m \wedge t_l| = |s_m \wedge s_l| < |t_n|$, contradicting the fact that the antichain $(t_n)_n$ is increasing in $2^{<\mathbb{N}}$. It follows that $t_m \wedge_T t_l \sqsupseteq s_m$, and so $|s_m|_T = |t_n|_T \le |t_m \wedge_T t_l|_T$, as desired. The proof for the decreasing antichains is identical. ■

A corollary of property (3) of Lemma 8 is the following.

COROLLARY 9. *Let T be a regular dyadic subtree of $2^{<\mathbb{N}}$ and R a regular dyadic subtree of T. Then $\mathrm{Incr}(R) = \mathrm{Incr}(T) \cap 2^R$ and $\mathrm{Decr}(R) = \mathrm{Decr}(T) \cap 2^R$.*

We notice that for every regular dyadic subtree T of the Cantor tree $2^{<\mathbb{N}}$ the sets $\mathrm{Incr}(T)$ and $\mathrm{Decr}(T)$ are Polish subspaces of 2^T. The main result of this section is the following.

THEOREM 10. *Let T be a regular dyadic subtree of $2^{<\mathbb{N}}$ and A be an analytic subset of* $\mathrm{Incr}(T)$ (*respectively of* $\mathrm{Decr}(T)$). *Then there exists a regular dyadic subtree R of T such that either* $\mathrm{Incr}(R) \subseteq A$ *or* $\mathrm{Incr}(R) \cap A = \emptyset$ (*respectively, either* $\mathrm{Decr}(R) \subseteq A$ *or* $\mathrm{Decr}(R) \cap A = \emptyset$).

We notice that, after a first draft of the present paper was finished, S. Todorčević informed us that he is also aware of the above result with a proof based on K. Milliken's theorem for strong subtrees ([To2]).

The proof of Theorem 10 is based on Corollary 5. The method is to reduce the coloring of $\mathrm{Incr}(T)$ (respectively of $\mathrm{Decr}(T)$) in Theorem 10 to a coloring of the class $[T]_{\mathrm{Incr}}$ (respectively $[T]_{\mathrm{Decr}}$) of increasing (respectively decreasing) regular subtrees of T (see Definition 4). To this end, we need the following easy fact concerning the classes $[T]_{\mathrm{Incr}}$ and $[T]_{\mathrm{Decr}}$.

FACT 11. *Let T be a regular dyadic subtree of $2^{<\mathbb{N}}$. If $S \in [T]_{\mathrm{Incr}}$ or $S \in [T]_{\mathrm{Decr}}$, then $|S(n)| = n+1$ for every $n \in \mathbb{N}$.*

As we have indicated, the crucial fact in the present setting is that there is a canonical correspondence between $[T]_{\mathrm{Incr}}$ and $\mathrm{Incr}(T)$ (and similarly for the decreasing antichains), which we are about to describe. For every $S \in [2^{<\mathbb{N}}]_{\mathrm{Incr}}$ or $S \in [2^{<\mathbb{N}}]_{\mathrm{Decr}}$ and every $n \in \mathbb{N}$, let $\{s_0^n \prec \dots \prec s_n^n\}$ be the $\prec$-increasing enumeration of $S(n)$. Define $\Phi : [2^{<\mathbb{N}}]_{\mathrm{Incr}} \to \mathrm{Incr}(2^{<\mathbb{N}})$ by

$$\Phi(S) = (s_n^{n+1})_n.$$

It is easy to see that Φ is a well-defined continuous map. Moreover, define $\Psi : [2^{<\mathbb{N}}]_{\mathrm{Decr}} \to \mathrm{Decr}(2^{<\mathbb{N}})$ by $\Psi(S) = (s_1^{n+1})_n$. Again it is easy to see that Ψ is well-defined and continuous.

LEMMA 12. *Let T be a regular dyadic subtree of $2^{<\mathbb{N}}$. Then $\Phi([T]_{\mathrm{Incr}}) = \mathrm{Incr}(T)$ and $\Psi([T]_{\mathrm{Decr}}) = \mathrm{Decr}(T)$.*

Proof. We shall give the proof only for the case of increasing subtrees. The proof of the other case is similar. First, we notice that for every $S \in [T]_{\mathrm{Incr}}$ we have $\Phi(S) \in \mathrm{Incr}(2^{<\mathbb{N}}) \cap 2^T$, and so by Lemma 8(3) we find that $\Phi([T]_{\mathrm{Incr}}) \subseteq \mathrm{Incr}(T)$. Conversely, let $(t_n)_n \in \mathrm{Incr}(T)$.

CLAIM 1. *For every $n < m < l$ we have $t_n \wedge_T t_m = t_n \wedge_T t_l$.*

Proof of Claim. Let $n < m < l$. By condition (2) in Definition 7, there exists $s \in T$ with $|s|_T = |t_n|_T$ and such that $s \sqsubseteq t_m \wedge_T t_l$. Moreover, observe that $t_n \prec s$, as $t_n \prec t_m$. It follows that $t_n \wedge_T t_m = t_n \wedge_T s = t_n \wedge_T t_l$, as claimed. ♦

For every $n \in \mathbb{N}$, we set $c_n = t_n \wedge_T t_{n+1}$.

CLAIM 2. *For every $n < m$ we have $c_n \sqsubset c_m$. That is, the sequence $(c_n)_n$ is an infinite chain of T.*

Proof of Claim. Let $n < m$. By Claim 1, c_n and c_m are compatible, since $c_n = t_n \wedge_T t_m$ and, by definition, $c_m = t_m \wedge_T t_{m+1}$. Now notice that $|c_n|_T < |t_n|_T \leq |t_m \wedge_T t_{m+1}|_T = |c_m|_T$. ♦

For every $n \geq 1$, let c'_n be the unique node of T such that $c'_n \sqsubseteq c_n$ and $|c'_n|_T = |t_{n-1}|_T$. We define recursively $S \in [T]_{\mathrm{Incr}}$ as follows. We set $S(0) = \{c_0\}$ and $S(1) = \{t_0, c'_1\}$. Assume that $S(n) = \{s^n_0 \prec \cdots \prec s^n_n\}$ has been defined so that $s^n_{n-1} = t_{n-1}$ and $s^n_n = c'_n$. For every $0 \leq i \leq n-1$, we chose nodes s^{n+1}_i such that $s^n_i \sqsubset s^{n+1}_i$ and $|s^{n+1}_i|_T = |t_n|_T$. We set $S(n+1) = \{s^{n+1}_0 \prec \cdots \prec s^{n+1}_{n-1} \prec t_n \prec c'_{n+1}\}$. It is easy to check that $S \in [T]_{\mathrm{Incr}}$ and $\Phi(S) = (t_n)_n$. The proof is complete. ■

Proof of Theorem 10. Let A be an analytic subset of $\mathrm{Incr}(T)$. By Lemma 12, the set $B = \Phi^{-1}(A) \cap [T]_{\mathrm{Incr}}$ is an analytic subset of $[T]_{\mathrm{Incr}}$. By Corollary 5, there exists a regular dyadic subtree R of T such that either $[R]_{\mathrm{Incr}} \subseteq B$ or $[R]_{\mathrm{Incr}} \cap B = \emptyset$. By Lemma 12, the first case implies that $\mathrm{Incr}(R) = \Phi([R]_{\mathrm{Incr}}) \subseteq \Phi(B) \subseteq A$, while the second that $\mathrm{Incr}(R) \cap A = \Phi([R]_{\mathrm{Incr}}) \cap A = \emptyset$. The proof for the case of decreasing antichains is similar. ■

4. Canonicalizing sequential compactness of trees of functions

The present section consists of four subsections. In the first one, using the Ramsey properties of chains and of increasing and decreasing antichains, we prove a strengthening of a result of J. Stern [Ste]. In the second one, we introduce the notion of equivalence of families of functions and we provide a criterion for establishing it. In the third subsection, we define the seven minimal families. The last subsection is devoted to the proof of the main result of the section, concerning the canonical embedding of one of the minimal families in any separable Rosenthal compact.

4.1. Sequential compactness of trees of functions. We start with the following definition.

DEFINITION 13. Let L be an infinite subset of $2^{<\mathbb{N}}$ and $\sigma \in 2^{\mathbb{N}}$. We say that L *converges* to σ if for every $k \in \mathbb{N}$ the set L is almost included in $\{t \in 2^{<\mathbb{N}} : \sigma|k \sqsubseteq t\}$. The element σ will be called the *limit* of the set L. We write $L \to \sigma$ to denote that L converges to σ.

It is clear that the limit of a subset L of $2^{<\mathbb{N}}$ is unique, if it exists.

FACT 14. *Let $(t_n)_n$ be an increasing (respectively decreasing) antichain of $2^{<\mathbb{N}}$. Then $(t_n)_n$ converges to σ, where σ is the unique element of $2^{\mathbb{N}}$ determined by the chain $(c_n)_n$ with $c_n = t_n \wedge t_{n+1}$ (see the proof of Lemma 12).*

We will also need the following notations.

NOTATION. For every $L \subseteq 2^{<\mathbb{N}}$ infinite and every $\sigma \in 2^{\mathbb{N}}$ we write $L \prec^* \sigma$ if L is almost included in $\{t : t \prec \sigma\}$. We write $L \preceq^* \sigma$ if L is almost included in $\{t : t \prec \sigma\} \cup \{\sigma|n : n \in \mathbb{N}\}$. The notations $\sigma \prec^* L$ and $\sigma \preceq^* L$ have the obvious meaning. We also write $L \subseteq^* \sigma$ if for all but finitely many $t \in L$ we have $t \sqsubset \sigma$, while by $L \perp \sigma$ we mean that $L \cap \{\sigma|n : n \in \mathbb{N}\}$ is finite.

The following fact is essentially a consequence of Lemma 8(2).

FACT 15. *If L is an infinite subset of $2^{<\mathbb{N}}$ and $\sigma \in 2^{\mathbb{N}}$ is such that $L \to \sigma$ and $L \prec^* \sigma$ (respectively $\sigma \prec^* L$), then every infinite subset of L contains an increasing (respectively decreasing) antichain converging to σ.*

The aim of this subsection is to prove the following result.

THEOREM 16. *Let X be a Polish space and $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ be a family relatively compact in $\mathcal{B}_1(X)$. Then there exist a regular dyadic subtree T of $2^{<\mathbb{N}}$ and a family $\{g^0_\sigma, g^+_\sigma, g^-_\sigma : \sigma \in P\}$, where $P = [\hat{T}]$, such that for every $\sigma \in P$ the following are satisfied:*

(1) *The sequence $(f_{\sigma|n})_{n\in L_T}$ converges pointwise to g^0_σ (recall that L_T stands for the level set of T).*
(2) *For every sequence $(\sigma_n)_n$ in P converging to σ such that $\sigma_n \prec \sigma$ for all $n \in \mathbb{N}$, the sequence $(g^{\varepsilon_n}_{\sigma_n})_n$ converges pointwise to g^+_σ for any choice of $\varepsilon_n \in \{0,+,-\}$. If such a sequence $(\sigma_n)_n$ does not exist, then $g^+_\sigma = g^0_\sigma$.*
(3) *For every sequence $(\sigma_n)_n$ in P converging to σ such that $\sigma \prec \sigma_n$ for all $n \in \mathbb{N}$, the sequence $(g^{\varepsilon_n}_{\sigma_n})_n$ converges pointwise to g^-_σ for any choice of $\varepsilon_n \in \{0,+,-\}$. If such a sequence $(\sigma_n)_n$ does not exist, then $g^-_\sigma = g^0_\sigma$.*
(4) *For every infinite subset L of T converging to σ with $L \prec^* \sigma$, the sequence $(f_t)_{t\in L}$ converges pointwise to g^+_σ.*
(5) *For every infinite subset L of T converging to σ with $\sigma \prec^* L$, the sequence $(f_t)_{t\in L}$ converges pointwise to g^-_σ.*

Moreover, the functions $0,+,- : P \times X \to \mathbb{R}$ defined by

$$0(\sigma,x) = g^0_\sigma(x), \qquad +(\sigma,x) = g^+_\sigma(x), \qquad -(\sigma,x) = g^-_\sigma(x)$$

are all Borel.

Before we proceed to the proof of Theorem 16 we notice the following fact (the proof of which is left to the reader).

FACT 17.

(1) *Let $A_1 = (t^1_n)_n$ and $A_2 = (t^2_n)_n$ be two increasing (respectively decreasing) antichains of $2^{<\mathbb{N}}$ converging to the same $\sigma \in 2^{\mathbb{N}}$. Then there exists an increasing (respectively decreasing) antichain $(t_n)_n$ of $2^{<\mathbb{N}}$ converging to σ such that $t_{2n} \in A_1$ and $t_{2n+1} \in A_2$ for every $n \in \mathbb{N}$.*
(2) *Let $(\sigma_n)_n$ be a sequence in $2^{\mathbb{N}}$ converging to $\sigma \in 2^{\mathbb{N}}$. For every $n \in \mathbb{N}$, let $N_n = (t^n_k)_k$ be a sequence in $2^{<\mathbb{N}}$ converging to σ_n. If $\sigma_n \prec \sigma$ (respectively $\sigma_n \succ \sigma$) for all n, then there exist an increasing (respectively decreasing) antichain $(t_m)_m$ and $L = \{n_m : m \in \mathbb{N}\}$ such that $(t_m)_m$ converges to σ and $t_m \in N_{n_m}$ for every $m \in \mathbb{N}$.*

Proof of Theorem 16. Our hypotheses imply that for every sequence $(g_n)_n$ belonging to the closure of $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ in $\mathbb{R}^X$, there exists a subsequence of $(g_n)_n$ which is pointwise convergent. Consider the subset Π_1 of $[2^{<\mathbb{N}}]_{\text{chains}}$ defined by

$$\Pi_1 = \{c \in [2^{<\mathbb{N}}]_{\text{chains}} : \text{the sequence } (f_t)_{t\in c} \text{ is pointwise convergent}\}.$$

Then Π_1 is a co-analytic subset of $[2^{<\mathbb{N}}]_{\mathrm{chains}}$ (see [Ste]). Applying Theorem 3 and invoking our hypotheses, we get a regular dyadic subtree T_1 of $2^{<\mathbb{N}}$ such that $[T_1]_{\mathrm{chains}} \subseteq \Pi_1$. Now consider the subset Π_2 of $\mathrm{Incr}(T_1)$, defined by

$$\Pi_2 = \{(t_n)_n \in \mathrm{Incr}(T_1) : \text{the sequence } (f_{t_n})_n \text{ is pointwise convergent}\}.$$

Again Π_2 is co-analytic (this can be checked with similar arguments to those in [Ste]). Applying Theorem 10, we get a regular dyadic subtree T_2 of T_1 such that $\mathrm{Incr}(T_2) \subseteq \Pi_2$. Finally, applying Theorem 10 for the decreasing antichains of T_2 and the color

$$\Pi_3 = \{(t_n)_n \in \mathrm{Decr}(T_2) : \text{the sequence } (f_{t_n})_n \text{ is pointwise convergent}\},$$

we obtain a regular dyadic subtree T of T_2 such that, with $P = [\hat{T}]$, the following are satisfied:

(i) For every increasing antichain $(t_n)_n$ of T, the sequence $(f_{t_n})_n$ is pointwise convergent.
(ii) For every decreasing antichain $(t_n)_n$ of T, the sequence $(f_{t_n})_n$ is pointwise convergent.
(iii) For each $\sigma \in P$, the sequence $(f_{\sigma|n})_{n\in L_T}$ is pointwise convergent to a function g_σ^0.

We notice that, by Fact 17(1), if $(t_n^1)_n$ and $(t_n^2)_n$ are two increasing (respectively decreasing) antichains of T converging to the same σ, then $(f_{t_n^1})_n$ and $(f_{t_n^2})_n$ are both pointwise convergent to the same function. For every $\sigma \in P$, we define g_σ^+ as follows. If there exists an increasing antichain $(t_n)_n$ of T converging to σ, then we set g_σ^+ to be the pointwise limit of $(f_{t_n})_n$ (by the above remarks g_σ^+ is independent of the choice of $(t_n)_n$). Otherwise we set $g_\sigma^+ = g_\sigma^0$. Similarly we define g_σ^- to be the pointwise limit of $(f_{t_n})_n$, with $(t_n)_n$ a decreasing antichain of T converging to σ, if such an antichain exists. Otherwise we set $g_\sigma^- = g_\sigma^0$. By Fact 15 and the above discussion, properties (i) and (ii) can be strengthened as follows:

(iv) For every $\sigma \in P$ and every infinite $L \subseteq T$ converging to σ with $L \prec^* \sigma$, the sequence $(f_t)_{t\in L}$ is pointwise convergent to g_σ^+.
(v) For every $\sigma \in P$ and every infinite $L \subseteq T$ converging to σ with $\sigma \prec^* L$, the sequence $(f_t)_{t\in L}$ is pointwise convergent to g_σ^-.

We claim that the tree T and the family $\{g_\sigma^0, g_\sigma^+, g_\sigma^- : \sigma \in P\}$ are as desired. First we verify properties (1)–(5). Clearly, we only have to check (2) and (3). We will prove only property (2) (the argument is symmetric). We argue by contradiction. So, assume that there exist a sequence $(\sigma_n)_n$ in P, $\sigma \in P$ and $\varepsilon_n \in \{0,+,-\}$ such that $\sigma_n \prec \sigma$, $(\sigma_n)_n$ converges to σ while $(g_{\sigma_n}^{\varepsilon_n})_n$ does not converge pointwise to g_σ^+. Hence there exist $L \in [\mathbb{N}]$ and an open neighborhood V of g_σ^+ in $\mathbb{R}^X$ such that $g_{\sigma_n}^{\varepsilon_n} \notin \overline{V}$ for all $n \in L$. By definition, for every $n \in L$ we may select a sequence $(t_k^n)_k$ in T such that for every $n \in L$ the following hold:

(a) The sequence $N_n = (t_k^n)_k$ converges to σ_n.
(b) The sequence $(f_{t_k^n})_k$ converges pointwise to $g_{\sigma_n}^{\varepsilon_n}$.
(c) For all $k \in \mathbb{N}$, we have $f_{t_k^n} \notin \overline{V}$.
(d) The sequence $(\sigma_n)_{n\in L}$ converges to σ and $\sigma_n \prec \sigma$.

By Fact 17(2), there exist a diagonal increasing antichain $(t_m)_m$ converging to σ. By (c) above, $(f_{t_m})_m$ is not pointwise convergent to g^+_σ. This leads to a contradiction by the definition of g^+_σ.

Now we will check the Borelness of the maps $0, +$ and $-$. Let $L_T = \{l_0 < l_1 < \cdots\}$ be the increasing enumeration of the level set L_T of T. For every $n \in \mathbb{N}$ define $h_n : P \times X \to \mathbb{R}$ by $h_n(\sigma, x) = f_{\sigma|l_n}(x)$. Clearly, h_n is Borel. As for all $(\sigma, x) \in P \times X$ we have

$$0(\sigma, x) = g^0_\sigma(x) = \lim_{n\in\mathbb{N}} h_n(\sigma, x),$$

the Borelness of 0 is clear. We will only check the Borelness of the function $+$ (the argument for the map $-$ is symmetric). For every $n \in \mathbb{N}$ and every $\sigma \in P$, let $l_n(\sigma)$ be the lexicographic minimum of the closed set $\{\tau \in P : \sigma|l_n \sqsubset \tau\}$. The function $P \ni \sigma \mapsto l_n(\sigma) \in P$ is clearly continuous. Invoking the definition of g^+_σ and property (2) in the statement of the theorem we see that for all $(\sigma, x) \in P \times X$,

$$+(\sigma, x) = g^+_\sigma(x) = \lim_{n\in\mathbb{N}} g^0_{l_n(\sigma)}(x) = \lim_{n\in\mathbb{N}} 0(l_n(\sigma), x).$$

Thus $+$ is Borel as well, and the proof is complete. ∎

Remark. We would like to point out that in order to apply the Ramsey theory for trees in the present setting one has to know that all the colors are sufficiently definable. This is also the reason why the Borelness of the functions $0, +$ and $-$ is emphasized in Theorem 16. As a matter of fact, we will need the full strength of the Ramsey theory for trees and perfect sets, in the sense that in certain situations the color will belong to the σ-algebra generated by the analytic sets. It should be noted that this is in contrast with the classical Silver theorem [Si], for which most applications involve Borel partitions.

4.2. Equivalence of families of functions. Let us give the following definition.

Definition 18. Let I be a countable set and X, Y be Polish spaces. Let also $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ be two pointwise bounded families of real-valued functions on X and Y respectively, indexed by the set I. We say that $\{f_i\}_{i\in I}$ is *equivalent* to $\{g_i\}_{i\in I}$ if the map $f_i \mapsto g_i$ extends to a topological homeomorphism between $\overline{\{f_i\}}^p_{i\in I}$ and $\overline{\{g_i\}}^p_{i\in I}$.

The equivalence of the families $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ is stronger than saying that $\overline{\{f_i\}}^p_{i\in I}$ is homeomorphic to $\overline{\{g_i\}}^p_{i\in I}$ (such an example will be given in the next subsection). The crucial point in Definition 18 is that the equivalence of $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ gives a natural homeomorphism between their closures.

The following lemma provides an efficient criterion for checking the equivalence of families of Borel functions. We mention that in its proof we will often make use of the Bourgain–Fremlin–Talagrand theorem [BFT] without making an explicit reference. From the context it will be clear that this is what we use.

Lemma 19. *Let I be a countable set and X, Y be Polish spaces. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two separable Rosenthal compacta on X and Y respectively. Let $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ be two dense families of $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. Assume that for every $i \in I$ the functions f_i and g_i are isolated in $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. Then the following are equivalent*:

(1) *The families* $\{f_i\}_{i\in I}$ *and* $\{g_i\}_{i\in I}$ *are equivalent.*
(2) *For every* $L\subseteq I$ *infinite, the sequence* $(f_i)_{i\in L}$ *converges pointwise if and only if the sequence* $(g_i)_{i\in L}$ *does.*

Proof. The direction (1)⇒(2) is obvious. What remains is to prove the converse. So assume that (2) holds. Let $M\subseteq I$ be infinite. We set $\mathcal{K}_1^M=\overline{\{f_i\}}^p_{i\in M}$ and $\mathcal{K}_2^M=\overline{\{g_i\}}^p_{i\in M}$. Notice that both $\mathcal{K}_1^M$ and $\mathcal{K}_2^M$ are separable Rosenthal compacta. Our assumptions imply that the isolated points of $\mathcal{K}_1^M$ form precisely the set $\{f_i : i\in M\}$ and similarly for $\mathcal{K}_2^M$. Define $\Phi_M:\mathcal{K}_1^M\to\mathcal{K}_2^M$ as follows. First, for every $i\in M$ set $\Phi_M(f_i)=g_i$. If $h\in\mathcal{K}_1^M$ with $h\notin\{f_i : i\in M\}$, then there exists $L\subseteq M$ infinite such that h is the pointwise limit of the sequence $(f_i)_{i\in L}$. Define $\Phi_M(h)$ to be the pointwise limit of the sequence $(g_i)_{i\in L}$ (by our assumptions this limit exists). To simplify notation, let $\Phi=\Phi_I$.

Claim. *Let* $M\subseteq I$ *infinite. Then*:

(1) *The map* Φ_M *is well-defined, 1-1 and onto.*
(2) *We have* $\Phi|_{\mathcal{K}_1^M}=\Phi_M$.

Proof of Claim. (1) Fix $M\subseteq I$ infinite. To see that Φ_M is well-defined, notice that for every $h\in\mathcal{K}_1^M$ with $h\notin\{f_i : i\in M\}$ and every $L_1,L_2\subseteq M$ infinite with $h=\lim_{i\in L_1}f_i=\lim_{i\in L_2}f_i$ we have $\lim_{i\in L_1}g_i=\lim_{i\in L_2}g_i$. For if not, the sequence $(f_i)_{i\in L_1\cup L_2}$ would converge pointwise, while the sequence $(g_i)_{i\in L_1\cup L_2}$ does not, contradicting our assumptions.

We observe the following consequence of our assumptions and the definition of the map Φ_M. For every $h\in\mathcal{K}_1^M$, the point h is isolated in $\mathcal{K}_1^M$ if and only if $\Phi_M(h)$ is isolated in $\mathcal{K}_2^M$. Using this we will show that Φ_M is 1-1. Indeed, let $h_1,h_2\in\mathcal{K}_1^M$ with $\Phi_M(h_1)=\Phi_M(h_2)$. Then either $\Phi_M(h_1)$ is isolated in $\mathcal{K}_2^M$ or not. In the first case, there exists an $i_0\in M$ with $\Phi_M(h_1)=g_{i_0}=\Phi_M(h_2)$. Thus, $h_1=f_{i_0}=h_2$. So, assume that $\Phi_M(h_1)$ is not isolated in $\mathcal{K}_2^m$. Hence, neither is $\Phi_M(h_2)$. It follows that both h_1 and h_2 are non-isolated points of $\mathcal{K}_1^M$. Pick $L_1,L_2\subseteq M$ infinite with $h_1=\lim_{i\in L_1}f_i$ and $h_2=\lim_{i\in L_2}f_i$. As the sequence $(g_i)_{i\in L_1\cup L_2}$ is pointwise convergent to $\Phi_M(h_1)=\Phi_M(h_2)$, our assumptions yield

$$h_1=\lim_{i\in L_1}f_i=\lim_{i\in L_1\cup L_2}f_i=\lim_{i\in L_2}f_i=h_2,$$

which proves that Φ_M is 1-1. Finally, to see that Φ_M is onto, let $w\in\mathcal{K}_2^M$ with $w\notin\{g_i : i\in M\}$. Let $L\subseteq M$ be infinite with $w=\lim_{i\in L}g_i$. By our assumptions, the sequence $(f_i)_{i\in L}$ converges pointwise to an $h\in\mathcal{K}_1^M$ and clearly $\Phi_M(h)=w$.

(2) Use similar arguments to those in (1). ♦

By the above Claim, it is enough to show that the map Φ is continuous. Notice that it is enough to show that if $(h_n)_n$ is a sequence in $\mathcal{K}_1$ that converges pointwise to an $h\in\mathcal{K}_1$, then the sequence $(\Phi(h_n))_n$ converges to $\Phi(h)$. Assume the contrary. Hence, there exist a sequence $(h_n)_n$ in $\mathcal{K}_1$, $h\in\mathcal{K}_1$ and $w\in\mathcal{K}_2$ such that $h=\lim_n h_n$, $w=\lim_n\Phi(h_n)$ and $w\neq\Phi(h)$. As the map Φ is onto, there exists $z\in\mathcal{K}_1$ such that $z\neq h$ and $\Phi(z)=w$. Pick $x\in X$ and $\varepsilon>0$ such that $|h(x)-z(x)|>\varepsilon$. As the sequence $(h_n)_n$ converges pointwise to h we may assume that for all $n \in \mathbb{N}$ we have $|h_n(x) - z(x)| > \varepsilon$. Let

$$M = \{i \in I : |f_i(x) - z(x)| \geq \varepsilon/2\}.$$

Observe the following:

(O1) For all $n \in \mathbb{N}$, $h_n \in \mathcal{K}_1^M$.
(O2) $z \notin \mathcal{K}_1^M$.

By part (2) of the above claim and (O1), we see that $\Phi(h_n) = \Phi_M(h_n) \in \mathcal{K}_2^M$ for all $n \in \mathbb{N}$ and so $w \in \mathcal{K}_2^M$. As Φ_M is onto, there exists $h' \in \mathcal{K}_1^M$ such that $\Phi_M(h') = w$. Hence by (O2) and invoking the claim once more, we have $z \neq h'$ while $\Phi_M(h') = \Phi(h') = \Phi(z)$, contradicting that Φ is 1-1. ∎

4.3. Seven families of functions. The aim of this subsection is to describe seven families

$$\{d_t^i : t \in 2^{<\mathbb{N}}\} \quad (1 \leq i \leq 7)$$

of functions indexed by the Cantor tree. For every $i \in \{1, \ldots, 7\}$, the closure of the family $\{d_t^i : t \in 2^{<\mathbb{N}}\}$ in the pointwise topology is a separable Rosenthal compact $\mathcal{K}_i$. Each one of them is *minimal*, namely, for every dyadic (not necessarily regular) subtree $S = (s_t)_{t \in 2^{<\mathbb{N}}}$ of $2^{<\mathbb{N}}$ and every $i \in \{1, \ldots, 7\}$ the families $\{d_t^i\}_{t \in 2^{<\mathbb{N}}}$ and $\{d_{s_t}^i\}_{t \in 2^{<\mathbb{N}}}$ are equivalent in the sense of Definition 18. Although the families are mutually non-equivalent, the corresponding compacta might be homeomorphic. In all cases, the family $\{d_t^i : t \in 2^{<\mathbb{N}}\}$ will be discrete in its closure. For any of the corresponding compacta $\mathcal{K}_i$ $(1 \leq i \leq 7)$, we shall denote by $\mathcal{L}(\mathcal{K}_i)$ the set of all infinite subsets L of $2^{<\mathbb{N}}$ for which the sequence $(d_t^i)_{t \in L}$ is pointwise convergent. We will name the corresponding compacta (all of them are homeomorphic to closed subspaces of well-known compacta—see [AU], [E]) and we will refer to the families of functions as the canonical dense sequences of them. We will use the following notations.

If $\sigma \in 2^{\mathbb{N}}$, then δ_σ is the Dirac function at σ. We denote by x_σ^+ the characteristic function of the set $\{\tau \in 2^{\mathbb{N}} : \sigma \preceq \tau\}$, and by x_σ^- the characteristic function of the set $\{\tau \in 2^{\mathbb{N}} : \sigma \prec \tau\}$. Notice that if $t \in 2^{<\mathbb{N}}$, then $t^\frown 0^\infty \in 2^{\mathbb{N}}$, and so the function $x^+_{t^\frown 0^\infty}$ is well-defined. It is useful at this point to isolate the following property of the functions x_σ^+ and x_σ^- which will justify the notation g_σ^+ and g_σ^- in Theorem 16. If $(\sigma_n)_n$ is a sequence in $2^{\mathbb{N}}$ converging to σ with $\sigma_n \prec \sigma$ (respectively $\sigma \prec \sigma_n$) for all $n \in \mathbb{N}$, then the sequence $(x_{\sigma_n}^{\varepsilon_n})_n$ converges pointwise to x_σ^+ (respectively to x_σ^-) for any choice of $\varepsilon_n \in \{+, -\}$.

By identifying the Cantor set with a subset of the unit interval, we will identify every $\sigma \in 2^{\mathbb{N}}$ with the real-valued function on $2^{\mathbb{N}}$ which is everywhere equal to σ. Notice that for every $t \in 2^{<\mathbb{N}}$, we have $t^\frown 0^\infty \in 2^{\mathbb{N}}$, and so the function $t^\frown 0^\infty$ is well-defined. For every $t \in 2^{<\mathbb{N}}$, v_t stands for the characteristic function of the clopen set $V_t = \{\sigma \in 2^{\mathbb{N}} : t \sqsubset \sigma\}$. By 0 we denote the constantly zero function on $2^{\mathbb{N}}$. We will also need to deal with functions on $2^{\mathbb{N}} \oplus 2^{\mathbb{N}}$. In this case when we write, for instance, $(\delta_\sigma, x_\sigma^+)$ we mean that this function is δ_σ on the first copy of $2^{\mathbb{N}}$ and x_σ^+ on the second copy.

We also fix a regular dyadic subtree $R = (s_t)_{t \in 2^{<\mathbb{N}}}$ of $2^{<\mathbb{N}}$ with the following property:

(Q) For any $s, s' \in R$, we have $s^\frown 0^\infty \neq s'^\frown 0^\infty$ and $s^\frown 1^\infty \neq s'^\frown 1^\infty$. Hence, the set $[\hat{R}]$ does not contain the eventually constant sequences.

In what follows, we denote by P the perfect set $[\hat{R}]$. We denote by P^+ the subset of P consisting of all σ's for which there exists an increasing antichain $(s_n)_n$ in R converging to σ in the sense of Definition 13. Moreover, we denote by P^- the subset of P consisting of all σ's for which there exists a decreasing antichain $(s_n)_n$ in R converging to σ.

4.3.1. *The Aleksandrov compactification of the Cantor tree,* $A(2^{<\mathbb{N}})$. It is the pointwise closure of the family

$$\left\{\frac{1}{|t|+1}\,v_t : t \in 2^{<\mathbb{N}}\right\}.$$

Clearly, the space $A(2^{<\mathbb{N}})$ is countable compact, as the whole family accumulates to 0. Setting $d_t^1 = \frac{1}{|t|+1}v_t$ for all $t \in 2^{<\mathbb{N}}$, we see that the family $\{d_t^1 : t \in 2^{<\mathbb{N}}\}$ is a dense discrete subset of $A(2^{<\mathbb{N}})$. In this case the description of $\mathcal{L}(A(2^{<\mathbb{N}}))$ is trivial as

$$L \in \mathcal{L}(A(2^{<\mathbb{N}})) \;\Leftrightarrow\; L \subseteq 2^{<\mathbb{N}}.$$

4.3.2. *The space* $2^{\leq\mathbb{N}}$. It is the pointwise closure of the family

$$\{s^\frown 0^\infty : s \in R\}.$$

The accumulation points of $2^{\leq\mathbb{N}}$ form the set $\{\sigma : \sigma \in P\}$, which is clearly homeomorphic to $2^{\mathbb{N}}$. Thus, the space $2^{\leq\mathbb{N}}$ is uncountable compact metrizable. Setting $d_t^2 = s_t{}^\frown 0^\infty$ for all $t \in 2^{<\mathbb{N}}$ and invoking property (Q) above, we see that the family $\{d_t^2 : t \in 2^{<\mathbb{N}}\}$ is a dense discrete subset of $2^{\leq\mathbb{N}}$. The description of $\mathcal{L}(2^{\leq\mathbb{N}})$ is given by

$$L \in \mathcal{L}(2^{\leq\mathbb{N}}) \;\Leftrightarrow\; \exists\sigma \in 2^{\mathbb{N}} \text{ with } L \to \sigma.$$

4.3.3. *The extended split Cantor set* $\hat{S}_+(2^{\mathbb{N}})$. It is the pointwise closure of the family

$$\{x^+_{s^\frown 0^\infty} : s \in R\}.$$

Notice that $\hat{S}_+(2^{\mathbb{N}})$ can be realized as a closed subspace of the split interval $S(I)$. Thus, it is hereditarily separable. For every $\sigma \in P$, the function x^+_σ belongs to $\hat{S}_+(2^{\mathbb{N}})$. However, for an element $\sigma \in P$, the function x^-_σ belongs to $\hat{S}_+(2^{\mathbb{N}})$ if and only if there exists a decreasing antichain $(s_n)_n$ of R converging to σ. Finally, observe that the family $\{x^+_{s^\frown 0^\infty} : s \in R\}$ is a discrete subset of $\hat{S}_+(2^{\mathbb{N}})$ (this is essentially a consequence of property (Q) above). Hence, the accumulation points of $\hat{S}_+(2^{\mathbb{N}})$ form the set

$$\{x^+_\sigma : \sigma \in P\} \cup \{x^-_\sigma : \sigma \in P^-\}.$$

Setting $d_t^3 = x^+_{s_t{}^\frown 0^\infty}$ for all $t \in 2^{<\mathbb{N}}$, we see that the family $\{d_t^3 : t \in 2^{<\mathbb{N}}\}$ is a dense discrete subset of $\hat{S}_+(2^{\mathbb{N}})$. Moreover, we have the following description of $\mathcal{L}(\hat{S}_+(2^{\mathbb{N}}))$:

$$L \in \mathcal{L}(\hat{S}_+(2^{\mathbb{N}})) \;\Leftrightarrow\; \exists\sigma \in 2^{\mathbb{N}} \text{ with } L \to \sigma \text{ and (either } L \preceq^* \sigma \text{ or } \sigma \prec^* L).$$

4.3.4. *The mirror image* $\hat{S}_-(2^{\mathbb{N}})$ *of the extended split Cantor set.* The space $\hat{S}_+(2^{\mathbb{N}})$ has a natural mirror image $\hat{S}_-(2^{\mathbb{N}})$, which is the pointwise closure of the set

$$\{x^-_{s^\frown 1^\infty} : s \in R\}.$$

The spaces $\hat{S}_+(2^{\mathbb{N}})$ and $\hat{S}_-(2^{\mathbb{N}})$ are homeomorphic. To see this, for every $t \in 2^{<\mathbb{N}}$ let $\bar{t} \in 2^{<\mathbb{N}}$ be the finite sequence obtained by exchanging 0 with 1 and 1 with 0 in the finite sequence t. Define $\phi : R \to R$ by $\phi(s_t) = s_{\bar{t}}$ for all $t \in 2^{<\mathbb{N}}$. Then it is easy to see that the map

$$\hat{S}_+(2^{\mathbb{N}}) \ni x^+_{s_t^\frown 0^\infty} \mapsto x^-_{\phi(s_t)^\frown 1^\infty} \in \hat{S}_-(2^{\mathbb{N}})$$

extends to a topological homeomorphism between $\hat{S}_+(2^{\mathbb{N}})$ and $\hat{S}_-(2^{\mathbb{N}})$. However, the canonical dense sequences in them are *not* equivalent. Notice that for every $\sigma \in P$ the function x^-_σ belongs to $\hat{S}_-(2^{\mathbb{N}})$, while the function x^+_σ belongs to $\hat{S}_-(2^{\mathbb{N}})$ if and only if there exists an increasing antichain $(s_n)_n$ of R converging to σ. It follows that the accumulation points of $\hat{S}_-(2^{\mathbb{N}})$ form the set

$$\{x^-_\sigma : \sigma \in P\} \cup \{x^+_\sigma : \sigma \in P^+\}.$$

As before, setting $d^4_t = x^-_{s_t^\frown 1^\infty}$ for all $t \in 2^{<\mathbb{N}}$, we see that the family $\{d^4_t : t \in 2^{<\mathbb{N}}\}$ is a dense discrete subset of $\mathcal{L}(\hat{S}_-(2^{\mathbb{N}}))$ and moreover

$$L \in \mathcal{L}(\hat{S}_-(2^{\mathbb{N}})) \iff \exists \sigma \in 2^{\mathbb{N}} \text{ with } L \to \sigma \text{ and (either } L \prec^* \sigma \text{ or } \sigma \preceq^* L).$$

4.3.5. *The extended Aleksandrov compactification of the Cantor set, $\hat{A}(2^{\mathbb{N}})$.* The space $\hat{A}(2^{\mathbb{N}})$ is the pointwise closure of the family

$$\{v_t : t \in 2^{<\mathbb{N}}\}.$$

For every $\sigma \in 2^{\mathbb{N}}$ the function δ_σ belongs to $\hat{A}(2^{\mathbb{N}})$, the family $\{\delta_\sigma : \sigma \in 2^{\mathbb{N}}\}$ is discrete and accumulates to 0. The function 0 is the only non-G_δ point of $\hat{A}(2^{\mathbb{N}})$ and this is witnessed in the most extreme way. The accumulation points of $\hat{A}(2^{\mathbb{N}})$ form the set

$$\{\delta_\sigma : \sigma \in 2^{\mathbb{N}}\} \cup \{0\}.$$

Setting $d^5_t = v_t$ for all $t \in 2^{<\mathbb{N}}$, we see that the family $\{d^5_t : t \in 2^{<\mathbb{N}}\}$ is a dense discrete subset of $\hat{A}(2^{\mathbb{N}})$ and

$$L \in \mathcal{L}(\hat{A}(2^{\mathbb{N}})) \iff (\exists \sigma \in 2^{\mathbb{N}} \text{ with } L \subseteq^* \sigma) \text{ or } (\forall \sigma \in 2^{\mathbb{N}}\ L \perp \sigma).$$

4.3.6. *The extended duplicate of the Cantor set, $\hat{D}(2^{\mathbb{N}})$.* The space $\hat{D}(2^{\mathbb{N}})$ is the pointwise closure of the family

$$\{(v_t, t^\frown 0^\infty) : t \in 2^{<\mathbb{N}}\}.$$

This is the separable extension of the space $D(2^{\mathbb{N}})$, as described in [To1]. The accumulation points of $\hat{D}(2^{\mathbb{N}})$ form the set

$$\{(\delta_\sigma, \sigma) : \sigma \in 2^{\mathbb{N}}\} \cup \{(0, \sigma) : \sigma \in 2^{\mathbb{N}}\},$$

which is homeomorphic to the Aleksandrov duplicate of the Cantor set. Todorčević was the first to realize that this classical construction can be represented as a compact subset of the first Baire class. The space $\hat{D}(2^{\mathbb{N}})$ is not only first countable but also pre-metric of degree at most two, in the sense of [To1]. As in the previous cases, setting $d^6_t = (v_t, t^\frown 0^\infty)$ for every $t \in 2^{<\mathbb{N}}$, we see that the family $\{d^6_t : t \in 2^{<\mathbb{N}}\}$ is a dense discrete subset of $\hat{D}(2^{\mathbb{N}})$ and

$$L \in \mathcal{L}(\hat{D}(2^{\mathbb{N}})) \iff \exists \sigma \in 2^{\mathbb{N}} \text{ with } L \to \sigma \text{ and (either } L \subseteq^* \sigma \text{ or } L \perp \sigma).$$

4.3.7. *The extended duplicate of the split Cantor set, $\hat{D}(S(2^{\mathbb{N}}))$.* It is the pointwise closure of the family

$$\{(v_s, x^+_{s^\frown 0^\infty}) : s \in R\}.$$

The space $\hat{D}(S(2^{\mathbb{N}}))$ is homeomorphic to a subspace of the Helly space $\mathcal{H}$. To see this, let $\{(a_t, b_t) : t \in 2^{<\mathbb{N}}\}$ be a family in $[0,1]^2$ such that

(i) $a_t = a_{t^\frown 0} < b_{t^\frown 0} < a_{t^\frown 1} < b_{t^\frown 1} = b_t$,
(ii) $b_t - a_t \le 1/3^{|t|}$

for every $t \in 2^{<\mathbb{N}}$. Define $h_t : [0,1] \to [0,1]$ by

$$h_t(x) = \begin{cases} 1, & b_t < x, \\ 1/2, & a_t \le x \le b_t, \\ 0, & x < a_t. \end{cases}$$

It is easy to see that the map

$$\hat{D}(S(2^{\mathbb{N}})) \ni (v_{s_t}, x^+_{s_t^\frown 0^\infty}) \mapsto h_t \in \mathcal{H}$$

extends to a homeomorphic embedding. It follows that the space $\hat{D}(S(2^{\mathbb{N}}))$ is first countable. We notice, however, that it is not pre-metric of degree at most two.

As in all previous cases, we will describe the accumulation points of $\hat{D}(S(2^{\mathbb{N}}))$. First we observe that if $(s_n)_n$ is a chain in R converging to $\sigma \in P$, then the sequence $((v_{s_n}, x^+_{s_n^\frown 0^\infty}))_n$ is pointwise convergent to $(\delta_\sigma, x^+_\sigma)$. If $(s_n)_n$ is an increasing antichain of R converging to σ, then the sequence $((v_{s_n}, x^+_{s_n^\frown 0^\infty}))_n$ is pointwise convergent to $(0, x^+_\sigma)$, while if it is decreasing, then it is pointwise convergent to $(0, x^-_\sigma)$. Thus, the accumulation points of $\hat{D}(S(2^{\mathbb{N}}))$ form the set

$$\{(\delta_\sigma, x^+_\sigma) : \sigma \in P\} \cup \{(0, x^+_\sigma) : \sigma \in P^+\} \cup \{(0, x^-_\sigma) : \sigma \in P^-\}.$$

Finally, setting $d^7_t = (v_{s_t}, x^+_{s_t^\frown 0^\infty})$ for all $t \in 2^{<\mathbb{N}}$, we see that the family $\{d^7_t : t \in 2^{<\mathbb{N}}\}$ is a dense discrete subset of $\hat{D}(S(2^{\mathbb{N}}))$. The description of $\mathcal{L}(\hat{D}(S(2^{\mathbb{N}})))$ is given by

$$L \in \mathcal{L}(\hat{D}(S(2^{\mathbb{N}}))) \Leftrightarrow \exists \sigma \in 2^{\mathbb{N}} \text{ with } L \to \sigma \text{ and } (L \prec^* \sigma \text{ or } L \subseteq^* \sigma \text{ or } \sigma \prec^* L).$$

We close this subsection by noticing the following minimality property of the above-described families.

PROPOSITION 20. *Let $\{d^i_t : t \in 2^{<\mathbb{N}}\}$ with $i \in \{1, \dots, 7\}$ be one of the seven families of functions and let $S = (s_t)_{t \in 2^{<\mathbb{N}}}$ be a dyadic (not necessarily regular) subtree of $2^{<\mathbb{N}}$. Then the family $\{d^i_t : t \in 2^{<\mathbb{N}}\}$ and the corresponding family $\{d^i_{s_t} : t \in 2^{<\mathbb{N}}\}$ determined by the tree S are equivalent.*

We also observe that any two of the seven families are *not* equivalent. Moreover, except for the case of $\hat{S}_+(2^{\mathbb{N}})$ and $\hat{S}_-(2^{\mathbb{N}})$, the corresponding compacta are not mutually homeomorphic either.

4.4. Canonicalization. The main result of this section is the following.

THEOREM 21. *Let $\{f_t\}_{t \in 2^{<\mathbb{N}}}$ be a family of real-valued functions on a Polish space X which is relatively compact in $\mathcal{B}_1(X)$. Let also $\{d^i_t\}_{t \in 2^{<\mathbb{N}}}$ $(1 \le i \le 7)$ be the families described in the previous subsection. Then there exist a regular dyadic subtree $S = (s_t)_{t \in 2^{<\mathbb{N}}}$ of $2^{<\mathbb{N}}$ and $i_0 \in \{1, \dots, 7\}$ such that $\{f_{s_t}\}_{t \in 2^{<\mathbb{N}}}$ is equivalent to $\{d^{i_0}_t\}_{t \in 2^{<\mathbb{N}}}$.*

Proof. The family $\{f_t\}_{t \in 2^{<\mathbb{N}}}$ satisfies all hypotheses of Theorem 16. Thus, there exist a regular dyadic subtree T of $2^{<\mathbb{N}}$ and a family of functions $\{g^0_\sigma, g^+_\sigma, g^-_\sigma : \sigma \in P\}$, with

$P=[\hat{T}]$, as described in Theorem 16. Let also $0,+$ and $-$ be the corresponding Borel functions. We recall that for every subset X of $2^{\mathbb{N}}$ we identify the set $[X]^2$ of doubletons of X with the set of all $(\sigma,\tau)\in X^2$ with $\sigma\prec\tau$. For every $\varepsilon\in\{0,+,-\}$ let

$$A_{\varepsilon,\varepsilon}=\{(\sigma_1,\sigma_2)\in[P]^2: g^{\varepsilon}_{\sigma_1}\neq g^{\varepsilon}_{\sigma_2}\}.$$

Then $A_{\varepsilon,\varepsilon}$ is an analytic subset of $[P]^2$. To see this, notice that

$$(\sigma_1,\sigma_2)\in A_{\varepsilon,\varepsilon} \;\Leftrightarrow\; \exists x\in X \text{ with } g^{\varepsilon}_{\sigma_1}(x)\neq g^{\varepsilon}_{\sigma_2}(x) \;\Leftrightarrow\; \exists x\in X \text{ with } \varepsilon(\sigma_1,x)\neq\varepsilon(\sigma_2,x).$$

Invoking the Borelness of the functions $0,+,-$ we see that $A_{\varepsilon,\varepsilon}$ is analytic, as desired. Notice that for every $Q\subseteq P$ perfect and every $\varepsilon\in\{0,+,-\}$, the set $A_{\varepsilon,\varepsilon}\cap[Q]^2$ is analytic in $[Q]^2$. Thus, applying Theorem 6 successively three times, we get a perfect subset Q_0 of P such that for all $\varepsilon\in\{0,+,-\}$ we have

$$\text{either } [Q_0]^2\subseteq A_{\varepsilon,\varepsilon}, \text{ or } A_{\varepsilon,\varepsilon}\cap[Q_0]^2=\emptyset.$$

Case 1: $A_{0,0}\cap[Q_0]^2=\emptyset$. In this case, we have $g^0_{\sigma_1}=g^0_{\sigma_2}$ for all $(\sigma_1,\sigma_2)\in[Q_0]^2$. Thus, there exists a function g such that $g^0_\sigma=g$ for all $\sigma\in Q_0$. By properties (2) and (3) in Theorem 16 and the homogeneity of Q_0, we see that $g^+_\sigma=g^-_\sigma=g^0_\sigma=g$ for all $\sigma\in Q_0$. Pick a regular dyadic subtree $S=(s_t)_{t\in 2^{<\mathbb{N}}}$ of T such that $[\hat{S}]\subseteq Q_0$ and $f_s\neq g$ for all $s\in S$. Invoking properties (1), (4) and (5) of Theorem 16 as well as Lemma 8(2), we see that for every infinite subset A of S, the sequence $(f_t)_{t\in A}$ accumulates to g. It follows that $\overline{\{f_s\}}^p_{s\in S}=\{f_s\}_{s\in S}\cup\{g\}$, and so $\{f_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of $A(2^{<\mathbb{N}})$.

Case 2: $[Q_0]^2\subseteq A_{0,0}$. Then for every $(\sigma_1,\sigma_2)\in[Q_0]^2$ we have $g^0_{\sigma_1}\neq g^0_{\sigma_2}$. By passing to a further perfect subset of Q_0 if necessary, we may also assume that

(P1) $g^0_\sigma\neq f_t$ for every $\sigma\in Q_0$ and every $t\in T$.

Case 2.1: Either $A_{+,+}\cap[Q_0]^2=\emptyset$ or $A_{-,-}\cap[Q_0]^2=\emptyset$. Assume first that $A_{+,+}\cap[Q_0]^2=\emptyset$. In this case, there exists a function g such that $g^+_\sigma=g$ for all $\sigma\in Q_0$. By property (3) in Theorem 16 and the homogeneity of Q_0, we must also have $g^-_\sigma=g$ for all $\sigma\in Q_0$. This means that $A_{-,-}\cap[Q_0]^2=\emptyset$. Thus, by symmetry, this case is equivalent to saying that $A_{+,+}\cap[Q_0]^2=\emptyset$ and $A_{-,-}\cap[Q_0]^2=\emptyset$. It follows that there exists a function g such that $g^+_\sigma=g^-_\sigma=g$ for all $\sigma\in Q_0$. By passing to a further perfect subset of Q_0 if necessary, we may also assume that $g^0_\sigma\neq g$ for all $\sigma\in Q_0$. We select a regular dyadic subtree $S=(s_t)_{t\in 2^{<\mathbb{N}}}$ of T such that $[\hat{S}]\subseteq Q_0$ and $f_s\neq g$ for all $s\in S$. This property combined with (P1) implies that for every $s\in S$ the function f_s is isolated in $\overline{\{f_s\}}^p_{s\in S}$.

We claim that $\{f_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of $\hat{A}(2^{\mathbb{N}})$. We will give a detailed argument which will serve as a prototype for the other cases as well. First, we notice that, by Lemma 19 and the description of $\mathcal{L}(\hat{A}(2^{\mathbb{N}}))$, it is enough to show that for a subset A of S, the sequence $(f_s)_{s\in A}$ converges pointwise if and only if either A is almost included in a chain, or A does not contain an infinite chain. For the "if" part, we observe that if A is almost contained in a chain, then by property (1) of Theorem 16, the sequence $(f_s)_{s\in A}$ is pointwise convergent. Assume that A does not contain an infinite chain. Since $g^+_\sigma=g^-_\sigma=g$ for all $\sigma\in Q_0$, we see that for every increasing and every decreasing antichain $(s_n)_n$ of S, the sequence $(f_{s_n})_n$ converges pointwise to g. Thus,

$(f_s)_{s\in A}$ is pointwise convergent to g. For the "only if" part we argue by contradiction. If there exist $\sigma_1 \neq \sigma_2$ contained in $[\hat{S}]$ with $A \cap \{\sigma_1|n : n \in \mathbb{N}\}$ and $A \cap \{\sigma_2|n : n \in \mathbb{N}\}$ infinite, then the fact that $g^0_{\sigma_1} \neq g^0_{\sigma_2}$ implies that the sequence $(f_s)_{s\in A}$ is not pointwise convergent. Finally, if A contains an infinite chain and an infinite antichain, then the fact that $g^0_\sigma \neq g$ for all $\sigma \in [\hat{S}]$ implies that $(f_s)_{s\in A}$ is not pointwise convergent eighter.

CASE 2.2: $[Q_0]^2 \subseteq A_{+,+}$ and $[Q_0]^2 \subseteq A_{-,-}$. In this case we have

(P2) $g^\varepsilon_{\sigma_1} \neq g^\varepsilon_{\sigma_2}$ for all $(\sigma_1,\sigma_2) \in [Q_0]^2$ and $\varepsilon \in \{0,+,-\}$.

Moreover, by passing to a further perfect subset of Q_0 if necessary, we may strengthen (P1) to

(P3) $g^\varepsilon_\sigma \neq f_t$ for all $\sigma \in Q_0$, $\varepsilon \in \{0,+,-\}$ and $t \in T$.

Observe that (P3) implies the following. For every regular dyadic subtree S of T with $[\hat{S}] \subseteq Q_0$ and every $s \in S$, the function f_s is isolated in the closure of $\{f_s\}_{s\in S}$ in $\mathbb{R}^X$. Thus, as in Case 2.1, in what follows, Lemma 19 will be applicable.

For every $\varepsilon_1, \varepsilon_2 \in \{0,+,-\}$ with $\varepsilon_1 \neq \varepsilon_2$ let

$$A_{\varepsilon_1,\varepsilon_2} = \{(\sigma_1,\sigma_2) \in [Q_0]^2 : g^{\varepsilon_1}_{\sigma_1} \neq g^{\varepsilon_2}_{\sigma_2}\}.$$

Then $A_{\varepsilon_1,\varepsilon_2}$ is an analytic subset of $[Q_0]^2$. Applying Theorem 6 successively six times, we find $Q_1 \subseteq Q_0$ perfect such that for all $\varepsilon_1, \varepsilon_2 \in \{0,+,-\}$ with $\varepsilon_1 \neq \varepsilon_2$ we have

$$\text{either } [Q_1]^2 \subseteq A_{\varepsilon_1,\varepsilon_2} \text{ or } A_{\varepsilon_1,\varepsilon_2} \cap [Q_1]^2 = \emptyset.$$

We claim that for each pair $\varepsilon_1, \varepsilon_2$ the first alternative must occur. Assume on the contrary that there exist $\varepsilon_1, \varepsilon_2$ with $\varepsilon_1 \neq \varepsilon_2$ and such that $A_{\varepsilon_1,\varepsilon_2} \cap [Q_1]^2 = \emptyset$. Let τ be the lexicographical minimum of Q_1. Then for every $\sigma, \sigma' \in Q_1$ with $\tau \prec \sigma \prec \sigma'$ we have $g^{\varepsilon_2}_\sigma = g^{\varepsilon_1}_\tau = g^{\varepsilon_2}_{\sigma'}$, which contradicts (P2). Summing up, by passing to Q_1, we have strengthened (P2) to

(P4) $g^{\varepsilon_1}_{\sigma_1} \neq g^{\varepsilon_2}_{\sigma_2}$ for all $(\sigma_1,\sigma_2) \in [Q_1]^2$ and $\varepsilon_1, \varepsilon_2 \in \{0,+,-\}$.

For every $\varepsilon \in \{+,-\}$, define $B_{0,\varepsilon} \subseteq Q_1$ by

$$B_{0,\varepsilon} = \{\sigma \in Q_1 : g^0_\sigma \neq g^\varepsilon_\sigma\}.$$

It is easy to see that $B_{0,\varepsilon}$ is an analytic subset of Q_1. Thus, by the classical perfect set theorem, we find $Q_2 \subseteq Q_1$ perfect such that for every $\varepsilon \in \{+,-\}$ we have

$$\text{either } Q_2 \subseteq B_{0,\varepsilon} \text{ or } B_{0,\varepsilon} \cap Q_2 = \emptyset.$$

CASE 2.2.a: $B_{0,+} \cap Q_2 = \emptyset$ and $B_{0,-} \cap Q_2 = \emptyset$. In this case, for every $\sigma \in Q_2$ there exists a function g_σ such that $g_\sigma = g^0_\sigma = g^+_\sigma = g^-_\sigma$. Moreover, $g_{\sigma_1} \neq g_{\sigma_2}$ for all $\sigma_1 \neq \sigma_2$ in Q_2, as $Q_2 \subseteq Q_1$. Invoking properties (2) and (3) in Theorem 16, we see that the set $\{g_\sigma : \sigma \in Q_2\}$ is homeomorphic to Q_2. We select a regular dyadic subtree $S = (s_t)_{t\in 2^{<\mathbb{N}}}$ of T such that $[\hat{S}] \subseteq Q_2 \subseteq Q_0$. It follows that $\overline{\{f_s\}}^p_{s\in S} = \{f_s\}_{s\in S} \cup \{g_\sigma : \sigma \in [\hat{S}]\}$, and so the family $\{f_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of $2^{\leq\mathbb{N}}$.

CASE 2.2.b: $B_{0,+} \cap Q_2 = \emptyset$ and $Q_2 \subseteq B_{0,-}$. This means that $g^0_\sigma = g^+_\sigma$ and $g^0_\sigma \neq g^-_\sigma$ for all $\sigma \in Q_2$. Let $S = (s_t)_{t\in 2^{<\mathbb{N}}}$ be a regular dyadic subtree of T such that $[\hat{S}] \subseteq Q_2 \subseteq Q_0$. Invoking (P3) and the remarks following it, the description of $\mathcal{L}(\hat{S}_+(2^{\mathbb{N}}))$ and Lemma 19, arguing precisely as in Case 2.1, we see that $\{f_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of $\hat{S}_+(2^{\mathbb{N}})$.

Case 2.2.c: $Q_2 \subseteq B_{0,+}$ and $B_{0,-}\cap Q_2 = \emptyset$. This means that $g^0_\sigma = g^-_\sigma$ and $g^0_\sigma \neq g^+_\sigma$ for all $\sigma\in Q_2$. As in the previous case, let $S=(s_t)_{t\in 2^{<\mathbb{N}}}$ be a regular dyadic subtree of T such that $[\hat{S}]\subseteq Q_2\subseteq Q_0$. In this case $\{f_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of the mirror image $\hat{S}_-(2^{\mathbb{N}})$ of the extended split Cantor set (the argument is as in Case 2.1).

Case 2.2.d: $Q_2 \subseteq B_{0,+}$ and $Q_2\subseteq B_{0,-}$. In this case we have

(P5) $g^0_\sigma \neq g^+_\sigma$ and $g^0_\sigma\neq g^-_\sigma$ for all $\sigma\in Q_2$.

Let

$$B_{+,-}=\{\sigma\in Q_2 : g^+_\sigma \neq g^-_\sigma\}$$

Again, $B_{+,-}$ is an analytic subset of Q_2. Thus there exists $Q_3\subseteq Q_2$ perfect such that either $Q_3\subseteq B_{+,-}$, or $Q_3\cap B_{+,-}=\emptyset$.

Case 2.2.d.I: $Q_3\cap B_{+,-}=\emptyset$. This means that for every $\sigma\in Q_3$ there exists a function g_σ such that $g_\sigma = g^+_\sigma = g^-_\sigma$ and $g_\sigma\neq g^0_\sigma$. Moreover, by property (P4) above, we have $g_{\sigma_1}\neq g_{\sigma_2}$ and $g^0_{\sigma_1}\neq g^0_{\sigma_2}$ for all $(\sigma_1,\sigma_2)\in[Q_3]^2$, as $Q_3\subseteq Q_2\subseteq Q_1$. Let $S=(s_t)_{t\in 2^{<\mathbb{N}}}$ be a regular dyadic subtree of T such that $[\hat{S}]\subseteq Q_3\subseteq Q_0$. In this case $\{f_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of $\hat{D}(2^{\mathbb{N}})$. The verification is similar to the previous cases.

Case 2.2.d.II: $Q_3\subseteq B_{+,-}$. This means that $g^+_\sigma\neq g^-_\sigma$ for all $\sigma\in Q_3$. Combining this with (P4) and (P5), we see that $g^{\varepsilon_1}_{\sigma_1}\neq g^{\varepsilon_2}_{\sigma_2}$ if either $\varepsilon_1\neq\varepsilon_2$ or $\sigma_1\neq\sigma_2$. As before, let $S=(s_t)_{t\in 2^{<\mathbb{N}}}$ be a regular dyadic subtree of T such that $[\hat{S}]\subseteq Q_3\subseteq Q_0$. Then $\{f_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of $\hat{D}(S(2^{\mathbb{N}}))$.

The above cases exhaust all possibilities, so the proof is complete. ∎

By Theorem 21 and Proposition 20 we get the following corollary.

Corollary 22. *Let X be a Polish space and $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ be family of functions relatively compact in $\mathcal{B}_1(X)$. Then for every regular dyadic subtree T of $2^{<\mathbb{N}}$ there exist a regular dyadic subtree S of T and $i_0\in\{1,\dots,7\}$ such that for every regular dyadic subtree $R=(r_t)_{t\in 2^{<\mathbb{N}}}$ of S, the family $\{f_{r_t}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to $\{d^{i_0}_t\}_{t\in 2^{<\mathbb{N}}}$.*

5. Analytic subspaces of separable Rosenthal compacta

In this section we introduce a class of subspaces of separable Rosenthal compacta and we present some of their basic properties.

5.1. Definitions and basic properties. Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X. For every subset $\mathcal{F}$ of $\mathcal{K}$ we denote by $\mathrm{Acc}(\mathcal{F})$ the set of accumulation points of $\mathcal{F}$ in $\mathbb{R}^X$. We start with the following definition.

Definition 23. Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X and $\mathcal{C}$ a closed subspace of $\mathcal{K}$. We say that $\mathcal{C}$ is an *analytic subspace* of $\mathcal{K}$ if there exist a countable dense subset $\{f_n\}_n$ of $\mathcal{K}$ and an analytic subset A of $[\mathbb{N}]$ such that the following are satisfied:

(1) For every $L \in A$ we have $\mathrm{Acc}(\{f_n : n \in L\}) \subseteq \mathcal{C}$.
(2) For every $g \in \mathcal{C} \cap \mathrm{Acc}(\mathcal{K})$ there exists $L \in A$ with $g \in \overline{\{f_n\}}^p_{n\in L}$.

Let us make some remarks concerning the above notion. First we notice that the analytic set A witnessing the analyticity of $\mathcal{C}$ can always be assumed to be hereditary. We also observe that an analytic subspace of $\mathcal{K}$ is not necessarily separable. For instance, if $\mathcal{K} = \hat{A}(2^{\mathbb{N}})$ and $\mathcal{C} = A(2^{\mathbb{N}})$, then it is easy to see that $\mathcal{C}$ is an analytic subspace of $\mathcal{K}$. The following proposition gives some examples of analytic subspaces.

Proposition 24. *Let $\mathcal{K}$ be a separable Rosenthal compact. Then*:

(1) *$\mathcal{K}$ is analytic with respect to any countable dense subset $\{f_n\}_n$ of $\mathcal{K}$.*
(2) *Every closed G_δ subspace $\mathcal{C}$ of $\mathcal{K}$ is analytic.*
(3) *Every closed separable subspace $\mathcal{C}$ of $\mathcal{K}$ is analytic.*

Proof. (1) Take $A = [\mathbb{N}]$.

(2) Let $(U_k)_k$ be a sequence of open subsets of $\mathcal{K}$ with $\overline{U}_{k+1} \subseteq U_k$ for all $k \in \mathbb{N}$ and such that $\mathcal{C} = \bigcap_k U_k$. Let also $\{f_n\}_n$ be a countable dense subset of $\mathcal{K}$. For every $k \in \mathbb{N}$, let $M_k = \{n \in \mathbb{N} : f_n \in U_k\}$. Notice that the sequence $(M_k)_k$ is decreasing. Let $A \subseteq [\mathbb{N}]$ be defined by

$$L \in A \Leftrightarrow \forall k \in \mathbb{N}\ (L \subseteq^* M_k).$$

Clearly, A is Borel. It is easy to see that A satisfies condition (1) of Definition 23 for $\mathcal{C}$. To see that condition (2) is also satisfied, let $g \in \mathcal{C} \cap \mathrm{Acc}(\mathcal{K})$. By the Bourgain–Fremlin–Talagrand theorem [BFT], there exists an infinite subset L on $\mathbb{N}$ such that g is the pointwise limit of the sequence $(f_n)_{n\in L}$. As $g \in U_k$ for all $k \in \mathbb{N}$, we see that $L \subseteq^* M_k$ for all k. Hence the set A witnesses the analyticity of $\mathcal{C}$.

(3) Let D_1 be a countable dense subset of $\mathcal{K}$ and D_2 a countable dense subset of $\mathcal{C}$. Let $\{f_n\}_n$ be an enumeration of the set $D_1 \cup D_2$ and set $L = \{n \in \mathbb{N} : f_n \in D_2\}$. Let also $M = \{k \in L : f_k \in \mathrm{Acc}(\mathcal{K})\}$ and for every $k \in M$ select $L_k \in [\mathbb{N}]$ such that f_k is the pointwise limit of the sequence $(f_n)_{n\in L_k}$. Define $A = [L] \cup \bigcup_{k\in M} [L_k]$. The countable dense subset $\{f_n\}_n$ of $\mathcal{K}$ and the set A prove the analyticity of $\mathcal{C}$. ∎

To proceed with our discussion on the properties of analytic subspaces we need some pieces of notation. Let $\mathcal{K}$ be a separable Rosenthal compact and $\mathbf{f} = \{f_n\}_n$ a countable dense subset of $\mathcal{K}$. We set

$$\mathcal{L}_{\mathbf{f}} = \{L \in [\mathbb{N}] : (f_n)_{n\in L} \text{ is pointwise convergent}\}.$$

Moreover, for every accumulation point f of $\mathcal{K}$ we let

$$\mathcal{L}_{\mathbf{f},f} = \{L \in [\mathbb{N}] : (f_n)_{n\in L} \text{ is pointwise convergent to } f\}.$$

We notice that both $\mathcal{L}_{\mathbf{f}}$ and $\mathcal{L}_{\mathbf{f},f}$ are co-analytic. The first result relating the topological behavior of a point f in $\mathcal{K}$ to the descriptive set-theoretic properties of the set $\mathcal{L}_{\mathbf{f},f}$ is the result of A. Krawczyk from [Kr] asserting that a point $f \in \mathcal{K}$ is G_δ if and only if the set $\mathcal{L}_{\mathbf{f},f}$ is Borel. Another important structural property is the following consequence of the effective version of the Bourgain–Fremlin–Talagrand theorem, proved by G. Debs in [De].

THEOREM 25. *Let $\mathcal{K}$ be a separable Rosenthal compact. Then for every countable dense subset* $\mathbf{f} = \{f_n\}_n$ *of* $\mathcal{K}$, *there exists a Borel, hereditary and cofinal subset* C *of* $\mathcal{L}_{\mathbf{f}}$.

We refer the reader to [Do] for an explanation of how Debs' theorem yields the above result.

Let $\mathcal{K}$ and $\mathbf{f} = \{f_n\}_n$ be as above. For every $A \subseteq \mathcal{L}_{\mathbf{f}}$ we set

$$\mathcal{K}_{A,\mathbf{f}} = \{g \in \mathcal{K} : \exists L \in A \text{ with } g = \lim_{n\in L} f_n\}.$$

We have the following characterization of analytic subspaces which is essentially a consequence of Theorem 25.

PROPOSITION 26. *Let $\mathcal{K}$ be a separable Rosenthal compact and $\mathcal{C}$ a closed subspace of $\mathcal{K}$. Then $\mathcal{C}$ is analytic if and only if there exist a countable dense subset* $\mathbf{f} = \{f_n\}_n$ *of $\mathcal{K}$ and a hereditary and analytic subset A' of $\mathcal{L}_{\mathbf{f}}$ such that* $\mathcal{K}_{A',\mathbf{f}} = \mathcal{C} \cap \mathrm{Acc}(\mathcal{K})$.

Proof. The direction ($\Leftarrow$) is immediate. Conversely, assume that $\mathcal{C}$ is analytic and let $\mathbf{f} = \{f_n\}_n$ and $A \subseteq [\mathbb{N}]$ witness that. As we have already remarked, we may assume that A is hereditary. By Theorem 25, there exists a Borel, hereditary and cofinal subset C of $\mathcal{L}_{\mathbf{f}}$. We set $A' = A \cap C$. We claim that A' is as desired. Clearly, A' is a hereditary and analytic subset of $\mathcal{L}_{\mathbf{f}}$. Also observe that, by condition (1) of Definition 23, for every $L \in A'$ the sequence $(f_n)_{n\in L}$ must be pointwise convergent to a function $g \in \mathcal{C}$. Hence $\mathcal{K}_{A',\mathbf{f}} \subseteq \mathcal{C} \cap \mathrm{Acc}(\mathcal{K})$. Conversely, let $g \in \mathcal{C} \cap \mathrm{Acc}(\mathcal{K})$. There exists $M \in A$ with $g \in \overline{\{f_n\}}^p_{n\in M}$. By the Bourgain–Fremlin–Talagrand theorem, there exists $N \in [M]$ such that g is the pointwise limit of the sequence $(f_n)_{n\in N}$. Clearly, $N \in \mathcal{L}_{\mathbf{f}}$. As C is cofinal in $\mathcal{L}_{\mathbf{f}}$, there exists $L \in [N]$ with $L \in C$. As A is hereditary, we see that $L \in A \cap C = A'$. ■

5.2. Separable Rosenthal compacta in $\mathcal{B}_1(2^{\mathbb{N}})$. Let $\mathcal{K}$ be separable Rosenthal compact on a Polish space X and $\mathbf{f} = \{f_n\}_n$ a countable dense subset of $\mathcal{K}$. By Theorem 25, there exists a Borel cofinal subset of $\mathcal{L}_{\mathbf{f}}$. The following proposition shows that if X is compact metrizable, then the global property of $\mathcal{L}_{\mathbf{f}}$ (namely that it contains a Borel cofinal set) is also valid locally. We notice that in the argument below we make use of the Arsenin–Kunugui theorem in the spirit similar to [Po2].

PROPOSITION 27. *Let X be a compact metrizable space, $\mathcal{K}$ a separable Rosenthal compact on X and* $\mathbf{f} = \{f_n\}_n$ *a countable dense subset of $\mathcal{K}$. Then for every $f \in \mathcal{K}$ there exists an analytic hereditary subset B of $\mathcal{L}_{\mathbf{f},f}$ which is cofinal in $\mathcal{L}_{\mathbf{f},f}$.*

Proof. We apply Theorem 25 and we get a hereditary, Borel and cofinal subset C of $\mathcal{L}_{\mathbf{f}}$. Consider the function $\Phi : C \times X \to \mathbb{R}$ defined by $\Phi(L, x) = f_L(x)$, where f_L denotes the pointwise limit of the sequence $(f_n)_{n\in L}$. Then Φ is Borel. To see this, for every $n \in \mathbb{N}$ let $\Phi_n : C \times X \to \mathbb{R}$ be defined by $\Phi_n(L, x) = f_{l_n}(x)$, where l_n is the nth element of the increasing enumeration of L. Clearly, Φ_n is Borel. As $\Phi(L, x) = \lim_n \Phi_n(L, x)$ for all $(L, x) \in C \times X$, the Borelness of Φ is shown. For every $m \in \mathbb{N}$ define $P_m \subseteq C \times X$ by

$$(L,x)\in P_m \Leftrightarrow |f_L(x)-f(x)| > \frac{1}{m+1}$$
$$\Leftrightarrow (c,x)\in \Phi^{-1}\Big(\Big(-\infty,-\frac{1}{m+1}\Big)\cup\Big(\frac{1}{m+1},+\infty\Big)\Big).$$

Clearly, P_m is Borel. For every $L\in C$ the function $x\mapsto |f_L(x)-f(x)|$ is Baire-1. Hence, for every $L\in C$ the section $(P_m)_L=\{x\in X:(c,x)\in P_m\}$ of P_m at L is F_σ, and as X is compact metrizable, it is K_σ. By the Arsenin–Kunugui theorem (see [Ke, Theorem 35.46]), the set

$$G_m=\mathrm{proj}_C P_m$$

is Borel. It follows that $G=\bigcup_m G_m$ is a Borel subset of C. Put $D=C\setminus G$. Now observe that for every $L\in C$ we have $L\in\mathcal{L}_{\mathbf{f},f}$ if and only if $L\notin G$. Hence, D is a Borel subset of $\mathcal{L}_{\mathbf{f},f}$, and as C is cofinal, we find that D is cofinal in $\mathcal{L}_{\mathbf{f},f}$. Thus, letting B be the hereditary closure of D, we see that B is as desired. ∎

REMARKS. (1) Notice that Proposition 27 is not valid for an arbitrary separable Rosenthal compact. A counterexample, taken from [Po2] (see also [Ma]), is the following. Let A be an analytic non-Borel subset of $2^{\mathbb{N}}$ and denote by $\mathcal{K}_A$ the separable Rosenthal compact obtained by restricting every function of $\hat{A}(2^{\mathbb{N}})$ to A. Clearly, the function $0|_A$ belongs to $\mathcal{K}_A$ and is a non-G_δ point of $\mathcal{K}_A$. It is easy to check that, in this case, there does not exist a Borel cofinal subset of $\mathcal{L}_{0|_A}$.

(2) We should point out that the hereditary and cofinal subset B of $\mathcal{L}_{\mathbf{f},f}$, obtained by Proposition 27, can be chosen to be Borel. To see this, start with an analytic and cofinal subset A_0 of $\mathcal{L}_{\mathbf{f},f}$. Using Suslin's separation theorem we construct two sequences $(B_n)_n$ and $(C_n)_n$ such that B_n is Borel, C_n is the hereditary closure of B_n and $A_0\subseteq B_n\subseteq C_n\subseteq B_{n+1}\subseteq\mathcal{L}_{\mathbf{f},f}$ for all $n\in\mathbb{N}$. Setting $B=\bigcup_n B_n$, we see that B is as desired.

The arguments in the proof of Proposition 27 can be used to derive certain properties of analytic subspaces of separable Rosenthal compacta. To state them we need one more piece of notation. For a separable Rosenthal compact $\mathcal{K}$ on a Polish space X, $\mathbf{f}=\{f_n\}_n$ a countable dense subset of $\mathcal{K}$ and $\mathcal{C}$ a closed subspace of $\mathcal{K}$ we set

$$\mathcal{L}_{\mathbf{f},\mathcal{C}}=\{L\in[\mathbb{N}]:\exists g\in\mathcal{C}\text{ with } g=\lim_{n\in L} f_n\}.$$

Clearly, $\mathcal{L}_{\mathbf{f},\mathcal{C}}$ is a subset of $\mathcal{L}_{\mathbf{f}}$. Also notice that if $\mathcal{C}=\{f\}$ for some $f\in\mathcal{K}$, then $\mathcal{L}_{\mathbf{f},\mathcal{C}}=\mathcal{L}_{\mathbf{f},f}$.

Part (1) of the following proposition extends Proposition 27 for analytic subspaces. The second part shows that the notion of an analytic subspace of $\mathcal{K}$ is independent of the choice of the dense sequence, for every separable Rosenthal compact $\mathcal{K}$ in $\mathcal{B}_1(2^{\mathbb{N}})$.

PROPOSITION 28. *Let X be a compact metrizable space, $\mathcal{K}$ be a separable Rosenthal compact on X and $\mathcal{C}$ and analytic subspace of $\mathcal{K}$. Let $\mathbf{f}=\{f_n\}_n$ be a countable dense subset of $\mathcal{K}$ and $A\subseteq[\mathbb{N}]$ witnessing the analyticity of $\mathcal{C}$. Then:*

(1) *There exists an analytic cofinal subset A_1 of $\mathcal{L}_{\mathbf{f},\mathcal{C}}$.*
(2) *For every countable dense subset $\mathbf{g}=\{g_n\}_n$ of $\mathcal{K}$ there exists an analytic subset A_2 of $\mathcal{L}_{\mathbf{g}}$ such that $\mathcal{K}_{A_2,\mathbf{g}}=\mathcal{C}\cap\mathrm{Acc}(\mathcal{K})$.*

Proof. (1) By Proposition 26, there exists a hereditary and analytic subset A' of $\mathcal{L}_{\mathbf{f}}$ such that $\mathcal{K}_{A',\mathbf{f}} = \mathcal{C} \cap \mathrm{Acc}(\mathcal{K})$. Applying Theorem 25, we get a Borel, hereditary and cofinal subset C of $\mathcal{L}_{\mathbf{f}}$. As in Proposition 27, for every $L \in C$ we denote by f_L the pointwise limit of the sequence $(f_n)_{n\in L}$. Let $A'' = A' \cap C$. Clearly A'' is analytic and hereditary. Moreover, it is easy to see that $\mathcal{K}_{A'',\mathbf{f}} = \mathcal{C} \cap \mathrm{Acc}(\mathcal{K})$ (i.e. the set A'' codes all functions in $\mathrm{Acc}(\mathcal{K}) \cap \mathcal{C}$). Consider the equivalence relation $\sim$ on C, defined by

$$L \sim M \ \Leftrightarrow\ f_L = f_M \ \Leftrightarrow\ \forall x \in X\ f_L(x) = f_M(x).$$

We claim that $\sim$ is Borel. To see this notice that the map

$$C \times C \times X \ni (L, M, x) \mapsto |f_L(x) - f_M(x)|$$

is Borel (this can be easily checked arguing as in Proposition 27). Moreover, for every $(L, M) \in C \times C$, the map $x \mapsto |f_L(x) - f_M(x)|$ is Baire-1. Observe that

$$\neg(L \sim M) \Leftrightarrow \exists x \in X\ \exists \varepsilon > 0 \text{ with } |f_L(x) - f_M(x)| > \varepsilon.$$

By the fact that X is compact metrizable and by the Arsenin–Kunugui theorem we see that $\sim$ is Borel. We let A_1 be the $\sim$ saturation of A'', i.e.

$$A_1 = \{M \in C : \exists L \in A'' \text{ with } M \sim L\}.$$

As A'' is analytic and $\sim$ is Borel, we see that A_1 is analytic. As C is cofinal, it is easy to check that A_1 is cofinal in $\mathcal{L}_{\mathbf{f},\mathcal{C}}$. Thus, A_1 is as desired.

(2) Let C_1 and C_2 be two hereditary, Borel subsets of $\mathcal{L}_{\mathbf{f}}$ and $\mathcal{L}_{\mathbf{g}}$ cofinal in $\mathcal{L}_{\mathbf{f}}$ and $\mathcal{L}_{\mathbf{g}}$ respectively. By part (1), there exists a hereditary and analytic subset A_1 of $\mathcal{L}_{\mathbf{f}}$ which is cofinal in $\mathcal{L}_{\mathbf{f},\mathcal{C}}$. We set $A'_1 = A_1 \cap C_1$. Consider the subset S of $C_1 \times C_2$ defined by

$$(L, M) \in S \ \Leftrightarrow\ f_L = g_M \ \Leftrightarrow\ \forall x \in X\ f_L(x) = g_M(x),$$

where f_L denotes the pointwise limit of $(f_n)_{n\in L}$ while g_M denotes the pointwise limit of $(g_n)_{n\in M}$. As X is compact metrizable, arguing as in part (1), it is easy to see that S is Borel. We set

$$A_2 = \{M \in C_2 : \exists L \in A'_1 \text{ with } (L, M) \in S\}.$$

Then A_2 is as desired. ∎

We close this subsection with the following proposition which provides further examples of analytic subspaces.

Proposition 29. *Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X. Let also F be a K_σ subset of X. Then the subspace $\mathcal{C}_F = \{f \in \mathcal{K} : f|_F = 0\}$ of $\mathcal{K}$ is analytic with respect to any countable dense subset $\mathbf{f} = \{f_n\}_n$ of $\mathcal{K}$.*

Proof. Let C be a hereditary, Borel and cofinal subset of $\mathcal{L}_{\mathbf{f}}$. Let Z be the subset of $C \times X$ defined by

$$(L, x) \in Z \ \Leftrightarrow\ (x \in F) \text{ and } (\exists \varepsilon > 0 \text{ with } |f_L(x)| > \varepsilon).$$

The set Z is Borel. As F is K_σ, we see that for every $L \in C$ the section $Z_L = \{x \in X : (L, x) \in Z\}$ of Z at L is K_σ. Thus, setting $A = C \setminus \mathrm{proj}_C Z$ and invoking the Arsenin–Kunugui theorem, we see that the set A witnesses the analyticity of $\mathcal{C}_F$ with respect to $\{f_n\}_n$. ∎

Related to the above propositions and the concept of an analytic subspace of $\mathcal{K}$, the following questions are open to us.

PROBLEM 1. Is it true that the concept of an analytic subspace is independent of the choice of the countable dense subset of $\mathcal{K}$? More precisely, if $\mathcal{C}$ is an analytic subspace of a separable Rosenthal compact $\mathcal{K}$ on a Polish space X and $\mathbf{f} = \{f_n\}_n$ is an arbitrary countable dense subset of $\mathcal{K}$, does there exist $A \subseteq \mathcal{L}_{\mathbf{f}}$ analytic with $\mathcal{K}_{A,\mathbf{f}} = \mathcal{C} \cap \mathrm{Acc}(\mathcal{K})$?

PROBLEM 2. Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X and let $B \subseteq X$ be Borel. Is the subspace $\mathcal{C}_B = \{f \in \mathcal{K} : f|_B = 0\}$ analytic?

6. Canonical embeddings in analytic subspaces

This section is devoted to the canonical embedding of the most representative prototype, among the seven minimal families, into a given analytic subspace of a separable Rosenthal compact. The section is divided into two subsections. The first concerns metrizable Rosenthal compacta and the second the non-metrizable ones. We start with the following definitions.

DEFINITION 30. An injection $\phi : 2^{<\mathbb{N}} \to \mathbb{N}$ is said to be *canonical* provided that $\phi(s) < \phi(t)$ if either $|s| < |t|$, or $|s| = |t|$ and $s \prec t$. We denote by ϕ_0 the unique canonical bijection between $2^{<\mathbb{N}}$ and $\mathbb{N}$.

DEFINITION 31. Let $\mathcal{K}$ be a separable Rosenthal compact, $\{f_n\}_n$ a countable dense subset of $\mathcal{K}$, and $\mathcal{C}$ a closed subspace of $\mathcal{K}$. Let also $\{d^i_t\}_{t\in 2^{<\mathbb{N}}}$ $(1 \leq i \leq 7)$ be the canonical families described in §4.3 and let $\mathcal{K}_i$ $(1 \leq i \leq 7)$ be the corresponding separable Rosenthal compacta. For every $i \in \{1, \dots, 7\}$, we say that $\mathcal{K}_i$ *canonically embeds into* $\mathcal{K}$ *with respect to* $\{f_n\}_n$ *and* $\mathcal{C}$ if there exists a canonical injection $\phi : 2^{<\mathbb{N}} \to \mathbb{N}$ such that the families $\{d^i_t\}_{t\in 2^{<\mathbb{N}}}$ and $\{f_{\phi(t)}\}_{t\in 2^{<\mathbb{N}}}$ are equivalent, that is, if the map

$$\mathcal{K}_i \ni d^i_t \mapsto f_{\phi(t)} \in \mathcal{K}$$

extends to a homeomorphism between $\mathcal{K}_i$ and $\overline{\{f_{\phi(t)}\}}^p_{t\in 2^{<\mathbb{N}}}$, and moreover,

$$\mathrm{Acc}(\{f_{\phi(t)} : t \in 2^{<\mathbb{N}}\}) \subseteq \mathcal{C}.$$

If $\mathcal{C} = \mathcal{K}$, then we simply say that $\mathcal{K}_i$ *canonically embeds into* $\mathcal{K}$ *with respect to* $\{f_n\}_n$.

6.1. Metrizable Rosenthal compacta. This subsection is devoted to the proof of the following theorem.

THEOREM 32. *Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X consisting of bounded functions. Let also $\{f_n\}_n$ be a countable dense subset of $\mathcal{K}$. Assume that $\mathcal{K}$ is metrizable in the pointwise topology and non-separable in the supremum norm of $\mathcal{B}_1(X)$. Then there exists a canonical embedding of $2^{\leq\mathbb{N}}$ into $\mathcal{K}$ with respect to $\{f_n\}_n$ whose accumulation points are ε-separated in the supremum norm for some $\varepsilon > 0$. In particular, its image is non-separable in the supremum norm.*

Proof. Fix a compatible metric ϱ for the pointwise topology of $\mathcal{K}$. Our assumptions on $\mathcal{K}$ imply that there exist $\varepsilon > 0$ and a family $\Gamma = \{f_\xi : \xi < \omega_1\} \subseteq \mathcal{K}$ such that Γ is ε-separated in the supremum norm and each f_ξ is a condensation point of the family Γ in the pointwise topology.

By recursion on the length of finite sequences in $2^{<\mathbb{N}}$ we shall construct:

(C1) a family $(B_t)_{t\in 2^{<\mathbb{N}}}$ of open subsets of $\mathcal{K}$,
(C2) a family $(x_t)_{t\in 2^{<\mathbb{N}}}$ in X,
(C3) two families $(r_t)_{t\in 2^{<\mathbb{N}}}$, $(q_t)_{t\in 2^{<\mathbb{N}}}$ of reals,
(C4) a canonical injection $\phi : 2^{<\mathbb{N}} \to \mathbb{N}$

such that for every $t \in 2^{<\mathbb{N}}$ the following are satisfied:

(P1) $\overline{B}_{t^\frown 0} \cap \overline{B}_{t^\frown 1} = \emptyset$, $\overline{B}_{t^\frown 0} \cup \overline{B}_{t^\frown 1} \subseteq B_t$ and $\varrho\text{-diam}(B_t) \le 1/(|t|+1)$.
(P2) $|B_t \cap \Gamma| = \aleph_1$.
(P3) $r_t < q_t$ and $q_t - r_t > \varepsilon$.
(P4) For every $f \in \overline{B}_{t^\frown 0}$, $f(x_t) < r_t$, while for every $f \in \overline{B}_{t^\frown 1}$, $f(x_t) > q_t$.
(P5) $f_{\phi(t)} \in B_t$.

We set $B_{(\emptyset)} = \mathcal{K}$ and $\phi((\emptyset)) = 0$. We choose $f, g \in \Gamma$ and we pick $x \in X$ and $r, q \in \mathbb{R}$ such that $f(x) < r < q < g(x)$ and $q - r > \varepsilon$. We set $x_{(\emptyset)} = x$, $r_{(\emptyset)} = r$ and $q_{(\emptyset)} = q$. We select open subsets $B_{(0)}, B_{(1)}$ of $\mathcal{K}$ such that $f \in B_{(0)} \subseteq \{h \in \mathbb{R}^X : h(x_{(\emptyset)}) < r_{(\emptyset)}\}$, $g \in B_{(1)} \subseteq \{h \in \mathbb{R}^X : h(x_{(\emptyset)}) > q_{(\emptyset)}\}$, $\varrho\text{-diam}(B_{(0)}) < 1/2$ and $\varrho\text{-diam}(B_{(1)}) < 1/2$. Observe that $x_{(\emptyset)}$, $r_{(\emptyset)}$, $q_{(\emptyset)}, B_{(0)}$ and $B_{(1)}$ have properties (P1)–(P4) above. Notice also that $B_{(0)}, B_{(1)}$ are uncountable, hence they intersect the dense set $\{f_n\}_n$ at an infinite set. So, we may select $\phi((\emptyset)) < \phi((0)) < \phi((1))$ satisfying (P5). The general inductive step proceeds in a similar manner assuming that:

(a) for each $t \in 2^{<\mathbb{N}}$ with $|t| < n-1$, x_t, r_t and q_t have been chosen,
(b) for each $t \in 2^{<\mathbb{N}}$ with $|t| < n$, B_t and $\phi(t)$ have been chosen

such that (P1)–(P5) are satisfied. This completes the recursive construction.

Notice that for every $\sigma \in 2^{\mathbb{N}}$ we have $\bigcap_n B_{\sigma|n} = \{f_\sigma\}$. The map $2^{\mathbb{N}} \ni \sigma \mapsto f_\sigma \in \mathcal{K}$ is a homeomorphic embedding. Moreover, for every $\sigma \in 2^{\mathbb{N}}$, the sequence $(f_{\phi(\sigma|n)})_n$ is pointwise convergent to f_σ. We also observe the following consequence of properties (P3) and (P4). If $\sigma < \tau \in 2^{\mathbb{N}}$, then, setting $t = \sigma \wedge \tau$, we have $f_\sigma(x_t) \le r_t < q_t \le f_\tau(x_t)$ and so $\|f_\sigma - f_\tau\|_\infty > \varepsilon$. As there are at most countably many $\sigma \in 2^{\mathbb{N}}$ with $f_\sigma \in \{f_n\}_n$, by passing to a regular dyadic subtree of $2^{<\mathbb{N}}$ if necessary, we may assume that for every $t \in 2^{<\mathbb{N}}$ the function $f_{\phi(t)}$ is isolated in $\overline{\{f_{\phi(t)}\}}^p_{t\in 2^{<\mathbb{N}}}$. This easily shows that the family $\{f_{\phi(t)}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of $2^{\le\mathbb{N}}$. ∎

6.2. Non-metrizable separable Rosenthal compacta. The main results of this subsection are the following.

THEOREM 33. *Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X and let $\mathcal{C}$ be an analytic subspace of $\mathcal{K}$. Let also $\{f_n\}_n$ be a countable dense subset of $\mathcal{K}$ and $A \subseteq [\mathbb{N}]$ analytic, witnessing the analyticity of $\mathcal{C}$. Assume that $\mathcal{C}$ is not hereditarily separable. Then either $\hat{A}(2^{\mathbb{N}})$, $\hat{D}(2^{\mathbb{N}})$, or $\hat{D}(S(2^{\mathbb{N}}))$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}_n$ and $\mathcal{C}$. In particular, if $\mathcal{K}$ is first countable and not hereditarily separable, then either*

$\hat{D}(2^{\mathbb{N}})$ or $\hat{D}(S(2^{\mathbb{N}}))$ canonically embeds into $\mathcal{K}$ with respect to every countable dense subset $\{f_n\}_n$ of $\mathcal{K}$.

As shown in Corollary 45, if $\mathcal{K}$ is not first countable, then $\hat{A}(2^{\mathbb{N}})$ canonically embeds into $\mathcal{K}$.

THEOREM 34. *Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X and $\{f_n\}_n$ a countable dense subset of $\mathcal{K}$. Assume that $\mathcal{K}$ is hereditarily separable and non-metrizable. Then either $\hat{S}_+(2^{\mathbb{N}})$ or $\hat{S}_-(2^{\mathbb{N}})$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}_n$.*

6.2.1. *Proof of Theorem 33.* The main goal is to prove the following.

PROPOSITION 35. *Let $\mathcal{K}$, $\mathcal{C}$ and $\{f_n\}_n$ be as in Theorem 33. Then there exists a canonical injection $\psi : 2^{<\mathbb{N}} \to \mathbb{N}$ such that, if we set*

$$\mathcal{K}_\sigma = \overline{\{f_{\psi(\sigma|n)}\}_n}^p \setminus \{f_{\psi(\sigma|n)}\}_n$$

for all $\sigma \in 2^{\mathbb{N}}$, then there exists an open subset $V_\sigma \subseteq \mathbb{R}^X$ with $\mathcal{K}_\sigma \subseteq V_\sigma \cap \mathcal{C}$ and such that $\mathcal{K}_\tau \cap V_\sigma = \emptyset$ for every $\tau \in 2^{\mathbb{N}}$ with $\tau \neq \sigma$.

Granted Proposition 35, we complete the proof as follows. Let ψ be the canonical injection obtained by the above proposition and define $f_t = f_{\psi(t)}$ for all $t \in 2^{<\mathbb{N}}$. We apply Theorem 21 to get a regular dyadic subtree $S = (s_t)_{t\in 2^{<\mathbb{N}}}$ of $2^{<\mathbb{N}}$ and $i_0 \in \{1, \dots, 7\}$ such that $\{f_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to $\{d_t^{i_0}\}_{t\in 2^{<\mathbb{N}}}$. By Proposition 35, the closure of $\{f_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ in $\mathbb{R}^X$ contains an uncountable discrete set. Thus $\{f_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of either $\hat{A}(2^{\mathbb{N}})$, $\hat{D}(2^{\mathbb{N}})$, or $\hat{D}(S(2^{\mathbb{N}}))$. Setting $\phi = \psi \circ i_S$ we see that ϕ is a canonical injection imposing an embedding of either $\hat{A}(2^{\mathbb{N}})$, $\hat{D}(2^{\mathbb{N}})$, or $\hat{D}(S(2^{\mathbb{N}}))$ into $\mathcal{K}$ with respect to $\{f_n\}_n$ and $\mathcal{C}$.

We proceed to the proof of Proposition 35. By enlarging the topology on X if necessary (see [Ke]), we may assume that the functions $\{f_n\}_n$ are continuous. We may also assume that the set A is hereditary. It follows by condition (2) of Definition 23 and the Bourgain–Fremlin–Talagrand theorem that for every $g \in \mathcal{C} \cap \mathrm{Acc}(\mathcal{K})$ there exists $L \in A$ such that g is the pointwise limit of the sequence $(f_n)_{n\in L}$. We fix a continuous map $\Phi : \mathbb{N}^{\mathbb{N}} \to [\mathbb{N}]$ with $\Phi(\mathbb{N}^{\mathbb{N}}) = A$.

We will need the following notation. For every $m \in \mathbb{N}$, $y = (x_1, \dots, x_m) \in X^m$, $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$ and $\varepsilon > 0$ we set

$$V(y, \lambda, \varepsilon) = \{g \in \mathbb{R}^X : \lambda_i - \varepsilon < g(x_i) < \lambda_i + \varepsilon \ \forall i = 1, \dots, m\}.$$

We denote by $\overline{V}(y, \lambda, \varepsilon)$ the closure of $V(y, \lambda, \varepsilon)$ in $\mathbb{R}^X$.

Using the fact that $\mathcal{C}$ is not hereditarily separable, by recursion on countable ordinals we get:

(1) $m \in \mathbb{N}$, $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Q}^m$ and positive rationals ε and δ,
(2) a family $\Gamma = \{y_\xi = (x_1^\xi, \dots, x_m^\xi) : \xi < \omega_1\} \subseteq X^m$,
(3) a family $\{f_\xi : \xi < \omega_1\} \subseteq \mathcal{C}$,
(4) a family $\{M_\xi : \xi < \omega_1\} \subseteq [\mathbb{N}]$,
(5) a family $\{b_\xi : \xi < \omega_1\} \subseteq \mathbb{N}^{\mathbb{N}}$

such that for every $\xi < \omega_1$ the following are satisfied:

(i) $f_\xi \in \mathrm{Acc}(\mathcal{K})$.
(ii) $f_\xi \in V(y_\xi,\lambda,\varepsilon)$, while for every $\zeta<\xi$ we have $f_\zeta \notin \overline{V}(y_\xi,\lambda,\varepsilon+\delta)$.
(iii) y_ξ is a condensation point of Γ in X^m.
(iv) $\Phi(b_\xi)=M_\xi$ and f_ξ is the pointwise limit of the sequence $(f_n)_{n\in M_\xi}$.

Now, by induction on the length of the finite sequences in $2^{<\mathbb{N}}$ we shall construct:

(C1) a canonical injection $\psi : 2^{<\mathbb{N}} \to \mathbb{N}$,
(C2) a family $(B_t)_{t\in 2^{<\mathbb{N}}}$ of open balls in X^m, taken with respect to a compatible complete metric ϱ of X^m,
(C3) a family $(\Delta_t)_{t\in 2^{<\mathbb{N}}}$ of uncountable subsets of ω_1.

This will be done so that for every $t\in 2^{<\mathbb{N}}$ the following are satisfied:

(P1) If $t\neq(\emptyset)$, then $f_{\psi(t)} \in V(y,\lambda,\varepsilon)$ for all $y\in B_t$.
(P2) For all $t',t\in 2^{<\mathbb{N}}$ with $|t'|=|t|$ and $t'\neq t$ we have $f_{\psi(t)}\notin \overline{V}(y,\lambda,\varepsilon+\delta)$ for every $y\in B_{t'}$.
(P3) $\overline{B}_{t^\frown 0}\cap \overline{B}_{t^\frown 1}=\emptyset$, $\overline{B}_{t^\frown 0}\cup \overline{B}_{t^\frown 1}\subseteq B_t$ and $\varrho\text{-diam}(B_t)\le 1/(|t|+1)$.
(P4) $\Delta_{t^\frown 0}\cap\Delta_{t^\frown 1}=\emptyset$ and $\Delta_{t^\frown 0}\cup\Delta_{t^\frown 1}\subseteq\Delta_t$.
(P5) $\mathrm{diam}(\{b_\xi:\xi\in\Delta_t\})\le 1/2^{|t|}$.
(P6) $\{y_\xi:\xi\in\Delta_t\}\subseteq B_t$.
(P7) If $t\neq(\emptyset)$, then $\psi(t)\in M_\xi$ for every $\xi\in\Delta_t$.

Assume that the construction has been carried out. We set $y_\sigma=\bigcap_n B_{\sigma|n}$ and $V_\sigma = V(y_\sigma,\lambda,\varepsilon+\delta/2)$ for all $\sigma\in 2^{\mathbb{N}}$. Using (P1) and (P2), it is easy to see that $\mathcal{K}_\sigma\subseteq V_\sigma$ and $\mathcal{K}_\sigma\cap V_\tau=\emptyset$ if $\sigma\neq\tau$. We only need to check that $\mathcal{K}_\sigma\subseteq\mathcal{C}$ for every $\sigma\in 2^{\mathbb{N}}$. So, let $\sigma\in 2^{\mathbb{N}}$ arbitrary. We set $M=\{\psi(\sigma|n):n\ge 1\}\in[\mathbb{N}]$. It is enough to show that $M\in A$. For every $k\ge 1$ we select $\xi_k\in\Delta_{\sigma|k}$. By properties (P4), (P5) and (P7), the sequence $(b_{\xi_k})_{k\ge 1}$ converges to a unique $b\in\mathbb{N}^{\mathbb{N}}$, and moreover $\psi(\sigma|n)\in M_{\xi_k}=\Phi(b_{\xi_k})$ for every $1\le n\le k$. By the continuity of Φ we see that $M_{\xi_k}\to\Phi(b)$, and so $M\subseteq\Phi(b)$. As A is hereditary, we deduce that $M\in A$, as desired.

We proceed to the construction. We set $\psi((\emptyset))=0$, $B_{(\emptyset)}=X^m$ and $\Delta_{(\emptyset)}=\omega_1$. Assume that for some $n\ge 1$ and for all $t\in 2^{<n}$ the values $\psi(t)\in\mathbb{N}$, the open balls B_t and the sets Δ_t have been constructed. Refining if necessary, we may assume that for every $t\in 2^{<n}$ and every $\xi\in\Delta_t$ the point y_ξ is a condensation point of the set $\{y_\zeta:\zeta\in\Delta_t\}$.

Let $\{t_0\prec\dots\prec t_{2^{n-1}-1}\}$ be the $\prec$-increasing enumeration of 2^{n-1}. For every $j\in\{0,\dots,2^n-1\}$ we choose an open ball B_j^{-1} in X^m and an uncountable subset Δ_j^{-1} of ω_1 such that $\varrho\text{-diam}(B_j^{-1})<1/(n+1)$, $\{y_\xi:\xi\in\Delta_j^{-1}\}\subseteq B_j^{-1}$ and $\mathrm{diam}\{b_\xi:\xi\in\Delta_j^{-1}\}\le 1/2^n$. Moreover, the selection is done so that for j even we have $\overline{B_j^{-1}}\cap\overline{B_{j+1}^{-1}}=\emptyset$, $\overline{B_j^{-1}}\cup\overline{B_{j+1}^{-1}}\subseteq B_{t_{j/2}}$ and $\Delta_j^{-1}\cup\Delta_{j+1}^{-1}\subseteq\Delta_{t_{j/2}}$. We set $m_{-1}=\max\{\psi(t):t\in 2^{<n}\}$.

By finite recursion on $k\in\{0,\dots,2^n-1\}$, we will construct a family $\{B_j^k:j=0,\dots,2^n-1\}$ of open balls of X^m, a family $\{\Delta_j^k:j=0,\dots,2^n-1\}$ of uncountable subsets of ω_1 and a positive integer $m_k\in\mathbb{N}$ such that for every $k\in\{0,\dots,2^n-1\}$ the following are satisfied:

(a) For every $j\in\{0,\dots,2^n-1\}$ we have $B_j^{k-1}\supseteq B_j^k$, $\Delta_j^{k-1}\supseteq\Delta_j^k$ and $\{y_\xi:\xi\in\Delta_j^k\}\subseteq B_j^k$. Moreover, y_ξ is a condensation point of $\{y_\zeta:\zeta\in\Delta_j^k\}$ for all j and $\xi\in\Delta_j^k$.

(b) $m_{k-1} < m_k$.
(c) For every $y \in B^k_k$ we have $f_{m_k} \in V(y,\lambda,\varepsilon)$, while for every $j \in \{0,\dots,2^n-1\}$ with $j \neq k$ and every $y \in B^k_j$ we have $f_{m_k} \notin \overline{V}(y,\lambda,\varepsilon+\delta)$.
(d) $m_k \in M_\xi$ for all $\xi \in \Delta^k_k$.

As the first step of this construction is identical to the general one, we may assume that the construction has been carried out for all $k' < k$, where $k \in \{0,\dots,2^n-1\}$. Fix a countable base $\mathcal{B}$ of open balls of X^m. We first observe that for each $\xi \in \Delta^{k-1}_k$ and every $j \in \{0,\dots,2^n-1\}$ there exist:

(e) a basic open ball $B^{k,\xi}_j \subseteq B^{k-1}_j$,
(f) a positive integer $m_\xi \in M_\xi$ with $m_{k-1} < m_\xi$

such that

(g) for every $y \in B^{k,\xi}_k$ we have $f_{m_\xi} \in V(y,\lambda,\varepsilon)$, while for every $j \neq k$ and every $y \in B^{k,\xi}_j$ we have $f_{m_\xi} \notin \overline{V}(y,\lambda,\varepsilon+\delta)$.

To see that such choices are possible, fix $\xi \in \Delta^{k-1}_k$. We can choose $\xi_0,\dots,\xi_{2^n-1}$ distinct countable ordinals such that:

(h) for all $j \in \{0,\dots,2^n-1\}$ we have $\xi_j \in \Delta^{k-1}_j$,
(k) $\xi = \xi_k = \min\{\xi_j : j = 0,\dots,2^n-1\}$.

By (ii) and (k) above, we have $f_{\xi_k} \in V(y_{\xi_k},\lambda,\varepsilon)$ while $f_{\xi_k} \notin \overline{V}(y_{\xi_j},\lambda,\varepsilon+\delta)$ for all $j \neq k$. By (iv), we can choose $m_\xi \in M_\xi$ with $m_{k-1} < m_\xi$ (thus condition (f) above is satisfied) and such that $f_{m_\xi} \in V(y_\xi,\lambda,\varepsilon)$ while $f_{m_\xi} \notin \overline{V}(y_{\xi_j},\lambda,\varepsilon+\delta)$ for all $j \neq k$. As f_{m_ξ} is continuous, for every $j \in \{0,\dots,2^n-1\}$ we can find a basic open ball $B^{k,\xi}_j$ in X^m containing y_{ξ_j} such that conditions (e) and (g) above are satisfied.

By cardinality arguments, there exist $\Delta^k_k \subseteq \Delta^{k-1}_k$ uncountable, $m_k \in \mathbb{N}$ and for every j a ball B^k_j such that $m_\xi = m_k$ and $B^{k,\xi}_j = B^k_j$ for all $\xi \in \Delta^k_k$. Setting $\Delta^k_j = \{\xi : y_\xi \in B^k_j\} \cap \Delta^{k-1}_j$ completes the recursive construction, described by (a)–(d).

Now let $\{t_0 \prec \dots \prec t_{2^n-1}\}$ be the $\prec$-increasing enumeration of 2^n. We set $\psi(t_k) = m_k$, $B_{t_k} = B^{2^n-1}_k$ and $\Delta_{t_k} = \Delta^{2^n-1}_k$ for all $k \in \{0,\dots,2^n-1\}$. It is easy to check that (P1)–(P7) are satisfied. This completes the construction described in (C1), (C2) and (C3). The proof of Proposition 35 being complete, Theorem 33 follows. ■

6.2.2. *Proof of Theorem 34.* As in the proof of Theorem 33, the main goal is to show

PROPOSITION 36. *Let $\{f_n\}_n$ be a family of continuous functions relatively compact in $\mathcal{B}_1(X)$. If the closure $\mathcal{K}$ of $\{f_n\}_n$ in $\mathbb{R}^X$ is non-metrizable, then there exist canonical injections $\psi_1 : 2^{<\mathbb{N}} \to \mathbb{N}$ and $\psi_2 : 2^{<\mathbb{N}} \to \mathbb{N}$ such that, if we set*

$$\mathcal{K}^1_\sigma = \overline{\{f_{\psi_1(\sigma|n)}\}_n}^p \setminus \{f_{\psi_(\sigma|n)}\}_n \quad \textit{and} \quad \mathcal{K}^2_\sigma = \overline{\{f_{\psi_2(\sigma|n)}\}_n}^p \setminus \{f_{\psi_2(\sigma|n)}\}_n$$

for all $\sigma \in 2^{\mathbb{N}}$, then there exists an open subset V_σ of $\mathbb{R}^X$ with $\mathcal{K}^1_\sigma - \mathcal{K}^2_\sigma \subseteq V_\sigma$ and $(\mathcal{K}^1_\tau - \mathcal{K}^2_\tau) \cap V_\sigma = \emptyset$ for every $\tau \in 2^{\mathbb{N}}$ with $\tau \neq \sigma$.

Granted Proposition 36, we complete the proof of Theorem 34 as follows. Let $\{f_n\}_n$ be a countable dense subset of $\mathcal{K}$. As already remarked, we may assume that the functions $\{f_n\}_n$ are actually continuous. We apply Proposition 36 to the family $\{f_n\}_n$ and we get the canonical injections $\psi_1 : 2^{<\mathbb{N}} \to \mathbb{N}$ and $\psi_2 : 2^{<\mathbb{N}} \to \mathbb{N}$ as described above. We define $g_t = f_{\psi_1(t)}$ and $h_t = f_{\psi_2(t)}$ for every $t \in 2^{<\mathbb{N}}$. Applying Corollary 22 successively two times, we get a regular dyadic subtree $S = (s_t)_{t\in 2^{<\mathbb{N}}}$ of $2^{<\mathbb{N}}$ such that the families $\{g_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ and $\{h_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ are canonicalized. The fact that $\mathcal{K}$ is hereditarily separable implies that each of the above families must be equivalent to the canonical dense family of either $A(2^{<\mathbb{N}})$, $2^{\leq\mathbb{N}}$, $\hat{S}_+(2^{\mathbb{N}})$, or $\hat{S}_-(2^{\mathbb{N}})$. By Proposition 36, it cannot be the case that both $\overline{\{g_{s_t}\}}^p_{t\in 2^{<\mathbb{N}}}$ and $\overline{\{h_{s_t}\}}^p_{t\in 2^{<\mathbb{N}}}$ are metrizable. Thus, at least one of them is equivalent to either $\hat{S}_+(2^{\mathbb{N}})$ or $\hat{S}_-(2^{\mathbb{N}})$, which clearly implies Theorem 34.

We proceed to the proof of Proposition 36, which is similar to that of Proposition 35 and relies on the fact that $\mathcal{K}$ is metrizable if and only if there exists $D \subseteq X$ countable such that the map $\mathrm{Acc}(\mathcal{K}) \ni f \mapsto f|_D \in \mathbb{R}^D$ is 1-1. Thus, by our assumptions and by transfinite recursion on countable ordinals, we get:

(1) $p < q \in \mathbb{Q}$,
(2) a set $\Gamma = \{x_\xi : \xi < \omega_1\} \subseteq X$,
(3) two families $\{g_\xi : \xi < \omega_1\}$ and $\{h_\xi : \xi < \omega_1\}$ in $\mathrm{Acc}(\mathcal{K})$

such that for every $\xi < \omega_1$ the following are satisfied:

(i) $g_\xi(x_\zeta) = h_\xi(x_\zeta)$ for all $\zeta < \xi$.
(ii) $g_\xi(x_\xi) < p < q < h_\xi(x_\xi)$.
(iii) x_ξ is a condensation point of Γ.

By recursion on the length of the finite sequences in $2^{<\mathbb{N}}$ we shall construct:

(C1) two canonical injections $\psi_1 : 2^{<\mathbb{N}} \to \mathbb{N}$ and $\psi_2 : 2^{<\mathbb{N}} \to \mathbb{N}$,
(C2) a family $(B_t)_{t\in 2^{<\mathbb{N}}}$ of open balls in X, taken with respect to a compatible complete metric ϱ of X.

This will be done so that for every $t \in 2^{<\mathbb{N}}$ the following are satisfied:

(P1) If $t \neq (\emptyset)$, then $f_{\psi_1(t)}(x) < p < q < f_{\psi_2(t)}(x)$ for every $x \in B_t$.
(P2) For every $t' \in 2^{<\mathbb{N}}$ with $|t'| = |t|$ and $t' \neq t$ and for every $x' \in B_{t'}$ we have $|f_{\psi_1(t)}(x') - f_{\psi_2(t)}(x')| < 1/(|t|+1)$.
(P3) $\overline{B}_{t^\frown 0} \cap \overline{B}_{t^\frown 1} = \emptyset$, $\overline{B}_{t^\frown 0} \cup \overline{B}_{t^\frown 1} \subseteq B_t$ and $\varrho\text{-diam}(B_t) \leq 1/(|t|+1)$.
(P4) $|B_t \cap \Gamma| = \aleph_1$.

Assuming that the construction has been carried out, setting $x_\sigma = \bigcap_n B_{\sigma|n}$ and

$$V_\sigma = \{w \in \mathbb{R}^X : |w(x_\sigma)| > (q-p)/2\}$$

for all $\sigma \in 2^{\mathbb{N}}$, it is easy to check that ψ_1, ψ_2 and $\{V_\sigma : \sigma \in 2^{\mathbb{N}}\}$ satisfy all the requirements of Proposition 36.

We proceed to the construction. We set $\psi_1((\emptyset)) = \psi_2((\emptyset)) = 0$ and $B_{(\emptyset)} = X$. Assume that for some $n \geq 1$ and for all $t \in 2^{<n}$ the values $\psi_1(t), \psi_2(t) \in \mathbb{N}$ and the open balls B_t have been constructed. As in Proposition 35, in order to determine $\psi_1(t), \psi_2(t)$ and B_t for every $t \in 2^n$ we shall follow a finite recursion.

Let $\{t_0 \prec \cdots \prec t_{2^{n-1}-1}\}$ be the $\prec$-increasing enumeration of 2^{n-1}. For every $j \in \{0, \dots, 2^n - 1\}$ we choose an open ball B_j^{-1} in X such that $\varrho\text{-diam}(B_j^{-1}) < 1/(n+1)$ and $|B_j^{-1} \cap \Gamma| = \aleph_1$. Moreover, the selection is done so that for j even we have $\overline{B_j^{-1}} \cap \overline{B_{j+1}^{-1}} = \emptyset$

and $\overline{B_j^{-1}} \cup \overline{B_{j+1}^{-1}} \subseteq B_{t_{j/2}}$. We set $m_{-1} = \max\{\psi_1(t) : t \in 2^{<n}\}$ and $l_{-1} = \max\{\psi_2(t) : t \in 2^{<n}\}$.

By finite recursion on $k \in \{0, \dots, 2^n-1\}$, we construct a family $\{B_j^k : j = 0, \dots, 2^n-1\}$ of open balls of X and $m_k, l_k \in \mathbb{N}$ such that for all $k \in \{0, \dots, 2^n - 1\}$ the following are satisfied:

(a) For every $j \in \{0, \dots, 2^n - 1\}$ we have $B_j^{k-1} \supseteq B_j^k$.
(b) $m_{k-1} < m_k$ and $l_{k-1} < l_k$.
(c) For every $x \in B_k^k$ we have $f_{m_k}(x) < p < q < f_{l_k}(x)$, while for every $j \in \{0, \dots, 2^n - 1\}$ with $j \neq k$ and every $x' \in B_j^k$ we have $|f_{m_k}(x') - f_{l_k}(x')| < 1/(n+1)$.
(d) For every $j \in \{0, \dots, 2^n - 1\}$ we have $|B_j^k \cap \Gamma| = \aleph_1$.

We omit the details of the above construction as it is similar to the one in Proposition 35. We only notice that condition (k) is replaced by

(k′) $\xi_k = \max\{\xi_j : j = 0, \dots, 2^n - 1\}$.

Now let $\{t_0 \prec \dots \prec t_{2^n-1}\}$ be the $\prec$-increasing enumeration of 2^n. We set $\psi_1(t_k) = m_k$, $\psi_2(t_k) = l_k$ and $B_{t_k} = B_k^{2^n-1}$ for every $k \in \{0, \dots, 2^n - 1\}$. It is easy to check that (P1)-(P4) are satisfied. The proof of Proposition 36, and hence of Theorem 34, is complete. ∎

REMARK. We notice that Theorem 32 (respectively Theorem 34) is valid for an analytic and metrizable (respectively hereditarily separable) subspace $\mathcal{C}$ of $\mathcal{K}$. In particular, we have the following.

THEOREM 37. *Let $\mathcal{K}$ be a separable Rosenthal compact and $\mathcal{C}$ an analytic subspace of $\mathcal{K}$. Let $\{f_n\}_n$ be a countable dense subset of $\mathcal{K}$ and $A \subseteq [\mathbb{N}]$ witnessing the analyticity of $\mathcal{C}$. If $\mathcal{C}$ is metrizable in the pointwise topology, consists of bounded functions and is norm non-separable, then $2^{\leq \mathbb{N}}$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}_n$ and $\mathcal{C}$, so that its image is norm non-separable. Furthermore, if $\mathcal{C}$ is hereditarily separable and not metrizable, then either $\hat{S}_+(2^{\mathbb{N}})$ or $\hat{S}_-(2^{\mathbb{N}})$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}_n$ and $\mathcal{C}$.*

The additional information provided by Theorem 37 is that the canonical embedding of the corresponding prototype is found with respect to the dense subset $\{f_n\}_n$ of $\mathcal{K}$ witnessing the analyticity of $\mathcal{C}$, which is not necessarily a subset of $\mathcal{C}$. The proof of Theorem 37 follows the lines of Theorems 32 and 34, using the arguments of the proof of Proposition 35.

7. Non-G_δ points in analytic subspaces

This section is devoted to the study of the structure of non-first countable analytic subspaces. The first subsection is devoted to the presentation of an extension of a result of A. Krawczyk from [Kr]. The proof follows the same lines as in [Kr]. In the second one we show that $\hat{A}(2^{\mathbb{N}})$ canonically embeds into any non-first countable analytic subspace $\mathcal{C}$ of a separable Rosenthal compact $\mathcal{K}$ and with respect to any countable subset D of $\mathcal{K}$ witnessing the analyticity of $\mathcal{C}$.

7.1. Krawczyk trees. We begin by introducing some pieces of notation and by recalling some standard terminology. We denote by Σ the set of all non-empty strictly increasing finite sequences in $\mathbb{N}$. We view Σ as a tree equipped with the (strict) partial order $\sqsubset$ of extension. We view, however, every $t \in \Sigma$ not only as a finite increasing sequence, but also as a finite subset of $\mathbb{N}$. Thus, for every $t, s \in \Sigma$ with $\max s < \min t$ we will frequently denote by $s \cup t$ the concatenation of s and t. We denote by $[\Sigma]$ the branches of Σ, i.e. the set $\{\sigma \in \mathbb{N}^{\mathbb{N}} : \sigma|n \in \Sigma \ \forall n \geq 1\}$. For every $t \in \Sigma$ we denote by Σ_t the set $\{s \in \Sigma : t \sqsubseteq s\}$.

For any $A, B \in [\mathbb{N}]$ we write $A \subseteq^* B$ if the set $A \setminus B$ is finite. If $\mathcal{A} \subseteq [\mathbb{N}]$, then we set $\mathcal{A}^* = \{\mathbb{N} \setminus A : A \in \mathcal{A}\}$. For a pair $\mathcal{A}, \mathcal{B} \subseteq [\mathbb{N}]$ we say that $\mathcal{A}$ is *$\mathcal{B}$-generated* if for every $A \in \mathcal{A}$ there exist $B_0, \ldots, B_k \in \mathcal{B}$ such that $A \subseteq B_0 \cup \cdots \cup B_k$. We say that $\mathcal{A}$ is *countably $\mathcal{B}$-generated* if there exists a sequence $(B_n)_n$ is $\mathcal{B}$ such that $\mathcal{A}$ is $\{B_n : n \in \mathbb{N}\}$-generated. An ideal $\mathcal{I}$ on $\mathbb{N}$ is said to be *bi-sequential* if for every $p \in \beta\mathbb{N}$ with $\mathcal{I} \subseteq p^*$, the family $\mathcal{I}$ is countably p^*-generated. Finally, for every family $\mathcal{F}$ of subsets of $\mathbb{N}$ and every $A \subseteq \mathbb{N}$ we let $\mathcal{F}[A] = \{L \cap A : L \in \mathcal{F}\}$ be the *trace* of $\mathcal{F}$ on A. Observe that if $\mathcal{F}$ is hereditary, then $\mathcal{F}[A] = \mathcal{F} \cap \mathcal{P}(A) = \{L \in \mathcal{F} : L \subseteq A\}$. The following result is essentially Lemma 1 from [Kr].

Lemma 38. *Let $\mathcal{I}$ be a bi-sequential ideal. Let also $\mathcal{F} \subseteq \mathcal{I}$ and $A \in [\mathbb{N}]$. Assume that $\mathcal{F}[A]$ is not countably $\mathcal{I}$-generated. Then there exists a sequence $(A_n)_n$ of pairwise disjoint infinite subsets of A such that, if we set $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$, then $\mathcal{I}[A]$ is $\mathcal{A}$-generated, while $\mathcal{F}[A_n]$ is non-countably $\mathcal{I}$-generated for every $n \in \mathbb{N}$.*

Proof (sketch). It suffices to prove the lemma for $A = \mathbb{N}$. We set

$$\mathcal{J} = \{C \subseteq \mathbb{N} : \mathcal{F}[C] \text{ is countably } \mathcal{I}\text{-generated}\}.$$

Then $\mathcal{J}$ is an ideal, $\mathbb{N} \notin \mathcal{J}$ and $\mathcal{I} \subseteq \mathcal{J}$. We select $p \in \beta\mathbb{N}$ with $\mathcal{J} \subseteq p^*$. By the bi-sequentiality of $\mathcal{I}$, there exists a sequence $(D_n)_n$ in p^* such that $\mathcal{I}$ is $\{D_n : n \in \mathbb{N}\}$-generated. As p^* is an ideal, we may assume that $D_n \cap D_m = \emptyset$ if $n \neq m$. Define $M = \{n \in \mathbb{N} : D_n \in \mathcal{J}\}$. By the fact that $\mathcal{I}$ is $\{D_n : n \in \mathbb{N}\}$-generated and $\mathcal{F} \subseteq \mathcal{I}$, we infer that $\mathcal{F}$ is $\{D_n : n \in \mathbb{N}\}$-generated. This observation and the fact that $(D_n)_n$ are disjoint imply that the set $D = \bigcup_{n \in M} D_n$ belongs to $\mathcal{J}$. Moreover, the set $\mathbb{N} \setminus M$ is infinite (for if not we would get $\mathbb{N} \in p^*$). Let $\{k_0 < k_1 < \cdots\}$ be the increasing enumeration of $\mathbb{N} \setminus M$ and define $A_0 = D \cup D_{k_0}$ and $A_n = D_{k_n}$ for $n \geq 1$. It is easy to see that the sequence $(A_n)_n$ is as desired. ∎

The main result of this subsection is the following theorem, which corresponds to Lemma 2 in [Kr]. We notice that it is one of the basic ingredients in the proof of the embeddability of $\hat{A}(2^{\mathbb{N}})$ in non-first countable separable Rosenthal compacta.

Theorem 39. *Let $\mathcal{I}$ be a bi-sequential ideal and $\mathcal{F} \subseteq \mathcal{I}$ analytic and hereditary. Assume that $\mathcal{F}$ is non-countably $\mathcal{I}$-generated. Then there exists a 1-1 map $\kappa : \Sigma \to \mathbb{N}$ such that, if we set $\mathcal{J}_{\mathcal{F}} = \{\kappa^{-1}(L) : L \in \mathcal{F}\}$ and $\mathcal{J} = \{\kappa^{-1}(M) : M \in \mathcal{I}\}$, then:*

(1) *For every $\sigma \in [\Sigma]$, $\{\sigma|n : n \geq 1\} \in \mathcal{J}_{\mathcal{F}}$.*
(2) (Domination property) *For every $B \in \mathcal{J}$ and every $n \geq 1$ there exist $t_0, \ldots, t_k \in \Sigma$ with $|t_0| = \cdots = |t_k| = n$ and such that $B \subseteq^* \Sigma_{t_0} \cup \cdots \cup \Sigma_{t_k}$.*

It is easy to see that property (2) in Theorem 39 is equivalent to saying that B is contained in a finitely splitting subtree of Σ.

Proof. We fix a continuous map $\phi : \mathbb{N}^{\mathbb{N}} \to [\mathbb{N}]$ with $\phi(\mathbb{N}^{\mathbb{N}}) = \mathcal{F}$. Recursively, we shall construct:

(C1) a family $(A_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of infinite subsets of $\mathbb{N}$,
(C2) a family $(a_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of finite subsets of $\mathbb{N}$,
(C3) a family $(U_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of basic clopen subsets of $\mathbb{N}^{\mathbb{N}}$.

This will be done in such a way that:

(P1) $A_s \subseteq A_t$ if $t \sqsubset s$, and $A_s \cap A_t = \emptyset$ if s and t are incomparable.
(P2) For every $s \in \mathbb{N}^{<\mathbb{N}}$ we have $|a_s| = |s|$ and $\max a_s \in A_s$ for all $s \in \mathbb{N}^{<\mathbb{N}}$ with $s \neq (\emptyset)$. Moreover $a_s \sqsubset a_t$ if and only if $s \sqsubset t$.
(P3) $U_s \subseteq U_t$ if $t \sqsubset s$ and $\mathrm{diam}(U_s) \leq 1/2^{|s|}$. Moreover, $U_s \cap U_t = \emptyset$ if s and t are incomparable.
(P4) For every $s \in \mathbb{N}^{<\mathbb{N}}$, $\phi(U_s)[A_s]$ is non-countably $\mathcal{I}$-generated.
(P5) For every $s \in \mathbb{N}^{<\mathbb{N}}$ and every $\tau \in U_s$ we have $a_s \subseteq \phi(\tau)$.
(P6) For every $s \in \mathbb{N}^{<\mathbb{N}}$, $\mathcal{I}[\bigcup_n A_{s^\frown n}]$ is $\{A_{s^\frown n} : n \in \mathbb{N}\}$-generated.

Assuming that the construction has been carried out we complete the proof as follows. We define $\lambda : \mathbb{N}^{<\mathbb{N}} \setminus \{(\emptyset)\} \to \mathbb{N}$ by $\lambda(s) = \max a_s$. By (P1) and (P2) above, we see that λ is 1-1. Let $\sigma \in \mathbb{N}^{\mathbb{N}}$. We claim that $\{\lambda(\sigma|n) : n \geq 1\} = \bigcup_n a_{\sigma|n} \in \mathcal{F}$. To see this, by (P3), let τ be the unique element of $\bigcap_n U_{\sigma|n}$. Then, by (P5), we have $a_{\sigma|n} \subseteq \phi(\tau)$ for all $n \in \mathbb{N}$. Thus, $\bigcup_n a_{\sigma|n} \subseteq \phi(\tau) \in \mathcal{F}$. As $\mathcal{F}$ is hereditary, our claim is proved. Now, let $B \subseteq \mathbb{N}^{<\mathbb{N}} \setminus \{(\emptyset)\}$ be such that $\{\lambda(t) : t \in B\} \in \mathcal{I}$. We claim that B must be dominated, i.e. for every $n \geq 1$ there exist $s_0, \ldots, s_k \in \mathbb{N}^n$ such that B is almost included in the set of successors of the s_i's in $\mathbb{N}^{<\mathbb{N}}$. If not, then we may find a subset $\{t_n\}_n$ of B, a node s of $\mathbb{N}^{<\mathbb{N}}$ and a subset $\{m_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ such that $s^\frown m_n \sqsubseteq t_n$ for every $n \in \mathbb{N}$. Notice that $\{\lambda(t_n) : n \in \mathbb{N}\} \in \mathcal{I}$, as $\mathcal{I}$ is hereditary. Moreover, by the definition of λ and by properties (P1) and (P2) above, we see that $\lambda(t_n) \in A_{s^\frown m_n}$ for every $n \in \mathbb{N}$. Hence $\{\lambda(t_n) : n \in \mathbb{N}\} \in \mathcal{I}[\bigcup_n A_{s^\frown n}]$. This leads to a contradiction by properties (P1) and (P6) above. We set $\kappa = \lambda|_\Sigma$. Clearly, κ is as desired.

We proceed to the construction. We set $A_{(\emptyset)} = \mathbb{N}$, $a_{(\emptyset)} = \emptyset$ and $U_{(\emptyset)} = \mathbb{N}^{\mathbb{N}}$. Assume that A_s, a_s and U_s have been constructed for some $s \in \mathbb{N}^{<\mathbb{N}}$. We set $\mathcal{F}_s = \phi(U_s)$. By property (P4) above and by Lemma 38, there exists a sequence $(A_n)_n$ of pairwise disjoint infinite subsets of A_s such that $\mathcal{F}_s[A_n]$ is non-countably $\mathcal{I}$-generated for every $n \in \mathbb{N}$, while $\mathcal{I}[A_s]$ is $\{A_n : n \in \mathbb{N}\}$-generated. Recursively, we may select a subset $\{\tau_n : n \in \mathbb{N}\}$ in U_s such that for all $n \in \mathbb{N}$ the following are satisfied:

(i) $\phi(\tau_n) \cap A_n$ is infinite.
(ii) $\phi(V_{\tau_n|k})[A_n]$ is non-countably $\mathcal{I}$-generated for every $k \in \mathbb{N}$, where $V_{\tau_n|k} = \{\sigma \in \mathbb{N}^{\mathbb{N}} : \tau_n|k \sqsubset \sigma\}$.

This can be easily done, as $\phi(U_s)[A_n] = \mathcal{F}_s[A_n]$ is non-countably $\mathcal{I}$-generated for every $n \in \mathbb{N}$. We may select $L = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]$ and a sequence $(k_n)_n$ in $\mathbb{N}$ such that, setting $\sigma_n = \tau_{l_n}$ for all n, the following are satisfied:

(iii) $V_{\sigma_n|k_n} \cap V_{\sigma_m|k_m} = \emptyset$ if $n \neq m$.
(iv) For every $n \in \mathbb{N}$ we have $V_{\sigma_n|k_n} \subseteq U_s$.
(v) For every $n \in \mathbb{N}$ we have $\mathrm{diam}(V_{\sigma_n|k_n}) < 1/2^{|s|+1}$.

Using the continuity of the map ϕ, for every $n \in \mathbb{N}$ we may also select $k'_n, i_n \in \mathbb{N}$ such that:

(vi) $i_n \in \phi(\sigma_n) \cap A_{l_n}$.
(vii) $\max a_s < i_n$ and $k_n < k'_n$.
(viii) For every $\tau \in V_{\sigma_n|k'_n}$ we have $i_n \in \phi(\tau)$.

For every $n \in \mathbb{N}$ we set $a_{s^\frown n} = a_s \cup \{i_n\}$, $A_{s^\frown n} = A_{l_n}$ and $U_{s^\frown n} = V_{\sigma_n|k'_n}$. It is easy to check that conditions (P1)–(P6) are satisfied. ∎

7.2. The embedding of $\hat{A}(2^{\mathbb{N}})$ in analytic subspaces. The main result of this subsection is the following.

THEOREM 40. *Let $\mathcal{K}$ be a separable Rosenthal compact and $\mathcal{C}$ an analytic subspace of $\mathcal{K}$. Let $\{f_n\}_n$ be a countable dense subset of $\mathcal{K}$ and $A \subseteq [\mathbb{N}]$ analytic, witnessing the analyticity of $\mathcal{C}$. Let also $f \in \mathcal{C}$ be a non-G_δ point of $\mathcal{C}$. Then there exists a canonical homeomorphic embedding of $\hat{A}(2^{\mathbb{N}})$ into $\mathcal{K}$ with respect to $\{f_n\}_n$ and $\mathcal{C}$ which sends 0 to f.*

For the proof we need to make some preliminary steps. Let $\mathcal{K}$, $\mathcal{C}$, $\{f_n\}_n$ and $f \in \mathcal{C}$ be as in Theorem 40. We may assume that $f_n \neq f$ for all $n \in \mathbb{N}$. We let

$$\mathcal{I}_f = \{L \in [\mathbb{N}] : f \notin \overline{\{f_n\}}^p_{n\in L}\}.$$

Then $\mathcal{I}_f$ is an analytic ideal on $\mathbb{N}$ (see [Kr]). A fundamental property of $\mathcal{I}_f$ is that it is bi-sequential. This is due to R. Pol (see [Po3]). We notice that the bi-sequentiality of $\mathcal{I}_f$ can also be derived from the non-effective proof of the Bourgain–Fremlin–Talagrand theorem due to G. Debs (see [De], or [AGR]).

Let also $A \subseteq [\mathbb{N}]$ be as in Theorem 40. As we have already pointed out, we may assume that A is hereditary. We set

$$\mathcal{F} = A \cap \mathcal{I}_f.$$

Clearly, $\mathcal{F}$ is an analytic and hereditary subset of $\mathcal{I}_f$. The assumption that f is a non-G_δ point of $\mathcal{C}$ implies (and in fact is equivalent to) that $\mathcal{F}$ is non-countably $\mathcal{I}_f$-generated, i.e. there does not exist a sequence $(M_k)_k$ in $\mathcal{I}_f$ such that for every $L \in \mathcal{F}$ there exists $k \in \mathbb{N}$ with $L \subseteq M_0 \cup \dots \cup M_k$. To see this, assume on the contrary that such a sequence $(M_k)_k$ exists. We set $N_k = M_0 \cup \dots \cup M_k$ for every $k \in \mathbb{N}$. As $\mathcal{I}_f$ is an ideal, we see that $N_k \in \mathcal{I}_f$ for every k. We set $F_k = \overline{\{f_n\}}^p_{n\in N_k} \cup \{f_k\}$. The fact that $N_k \in \mathcal{I}_f$ implies that $f \notin F_k$ for every $k \in \mathbb{N}$. Let $g \in \mathcal{C} \cap \mathrm{Acc}(\mathcal{K})$ with $g \neq f$. By condition (2) of Definition 23, there exists $L \in A$ with $g \in \overline{\{f_n\}}^p_{n\in L}$. Hence, there exists $M \in [L]$ such that g is the pointwise limit of the sequence $(f_n)_{n\in M}$. As A is hereditary, we see that $M \in \mathcal{F}$, and so there exists $k_0 \in \mathbb{N}$ with $M \subseteq N_{k_0}$. This implies that $g \in F_{k_0}$. It follows by the above discussion that $\{f\} = \bigcap_k (\mathcal{C} \setminus F_k)$, that is, the point f is G_δ in $\mathcal{C}$, a contradiction.

Summarizing, we know that $\mathcal{I}_f$ is bi-sequential, $\mathcal{F} \subseteq \mathcal{I}_f$ is analytic, hereditary and not countably $\mathcal{I}_f$-generated. Thus, we may apply Theorem 39 to get the 1-1 map $\kappa : \Sigma \to \mathbb{N}$

as described in the theorem. Setting $f_t = f_{\kappa(t)}$ for every $t \in \Sigma$ and invoking condition (1) of Definition 23, we get the following corollary.

COROLLARY 41. *There exists a family* $\{f_t\}_{t\in\Sigma} \subseteq \{f_n\}_n$ *such that:*

(1) *For every* $\sigma \in [\Sigma]$, *we have* $f \notin \overline{\{f_{\sigma|n}\}_n}^p$ *and* $\mathrm{Acc}(\{f_{\sigma|n} : n \in \mathbb{N}\}) \subseteq \mathcal{C}$.

(2) *For every* $B \subseteq \Sigma$ *such that* $f \notin \overline{\{f_t\}_{t\in B}}^p$ *and every* $n \geq 1$, *there exist* $t_0, \ldots, t_k \in \Sigma$ *with* $|t_0| = \cdots = |t_k| = n$ *and such that* $B \subseteq^* \Sigma_{t_0} \cup \cdots \cup \Sigma_{t_k}$.

We call the family $\{f_t\}_{t\in\Sigma}$ obtained in Corollary 41 the *Krawczyk tree* of f with respect to $\{f_n\}_n$ and $\mathcal{C}$. Let us isolate the following property of $\{f_t\}_{t\in\Sigma}$ that we will need later on.

(P) Let $i \in \Sigma$ and $(b_n)_n$ be a sequence in Σ such that $\max i < \min b_n$ and $\max b_n < \min b_{n+1}$ for every $n \in \mathbb{N}$. Set $s_n = i \cup b_n$ for all $n \in \mathbb{N}$. Then $(f_{s_n})_n$ is pointwise convergent to f. To see this observe that, by property (2) in Corollary 41, every subsequence of the sequence $(f_{s_n})_n$ accumulates to f. Hence $(f_{s_n})_n$ is pointwise convergent to f.

We will also need the following well-known consequence of the bi-sequentiality of $\mathcal{I}_f$. For the sake of completeness we include a proof.

LEMMA 42. *Let* $(A_l)_l$ *be a sequence in* $[\mathbb{N}]$ *such that* $\lim_{n\in A_l} f_n = f$ *for every* $l \in \mathbb{N}$. *Then there exists* $D \in [\mathbb{N}]$ *with* $\lim_{n\in D} f_n = f$, $D \subseteq \bigcup_l A_l$ *and such that* $D \cap A_l \neq \emptyset$ *for infinitely many* $l \in \mathbb{N}$.

Proof. For every $k \in \mathbb{N}$ we set $B_k = \bigcup_{l\geq k} A_l$. Then $(B_k)_k$ is a decreasing sequence of infinite subsets of $\mathbb{N}$. We may select $p \in \beta\mathbb{N}$ such that $\lim_{n\in p} f_n = f$ and $B_k \in p$ for every $k \in \mathbb{N}$. By the bi-sequentiality of $\mathcal{I}_f$, there exists a sequence $(C_m)_m$ of elements of p converging to f. We select a strictly increasing sequence $(l_k)_k$ in $\mathbb{N}$ such that $l_k \in B_k \cap C_0 \cap \cdots \cap C_k$ for all $k \in \mathbb{N}$ and we set $D = \{l_k : k \in \mathbb{N}\}$. It is easy to check that D is as desired. ■

In what follows, we will apply K. Milliken's theorem [Mil1]. To this end, we need to recall some notation. Given $b, b' \in \Sigma$ we write $b < b'$ if $\max b < \min b'$. We denote by $\mathbf{B}$ the subset of $\Sigma^{\mathbb{N}}$ consisting of all sequences $(b_n)_n$ which are increasing, i.e. $b_n < b_{n+1}$ for every $n \in \mathbb{N}$. It is easy to see that $\mathbf{B}$ is a closed subspace of $\Sigma^{\mathbb{N}}$, where Σ is equipped with the discrete topology and $\Sigma^{\mathbb{N}}$ with the product topology. For every $\mathbf{b} = (b_n)_n \in \mathbf{B}$ we set

$$\langle \mathbf{b} \rangle = \Big\{ \bigcup_{n\in F} b_n : F \subseteq \mathbb{N} \text{ finite} \Big\}, \qquad [\mathbf{b}] = \{(c_n)_n \in \mathbf{B} : c_n \in \langle \mathbf{b} \rangle \ \forall n\}.$$

Let us point out that for every block sequence $\mathbf{b}$ the set $\langle \mathbf{b} \rangle$ corresponds to an infinitely branching subtree of Σ, denoted by $\mathcal{T}_{\mathbf{b}}$. We also recall that the chains of $\mathcal{T}_{\mathbf{b}}$ are in one-to-one correspondence with the family $[\mathbf{b}]$ of all block subsequences of $\mathbf{b}$. More precisely, if $(t_n)_n$ is a chain of $\mathcal{T}_{\mathbf{b}}$, then $(t_0, t_1 \setminus t_0, \ldots, t_{n+1} \setminus t_n, \ldots)$ is the block subsequence of $\mathbf{b}$ corresponding to the chain $(t_n)_n$. This observation was used by W. Henson to derive an alternative proof of J. Stern's theorem (see [Od]). If $\beta = (b_0, \ldots, b_k)$ with $b_0 < \cdots < b_k$

and $\mathbf{d} \in \mathbf{B}$, then we set

$$[\beta, \mathbf{d}] = \{(c_n)_n \in \mathbf{B} : c_n = b_n \ \forall n \le k \text{ and } c_n \in \langle \mathbf{d} \rangle \ \forall n > k\}.$$

We will use the following consequence of Milliken's theorem.

THEOREM 43. *For every $\mathbf{b} \in \mathbf{B}$ and every analytic subset A of $\mathbf{B}$ there exists $\mathbf{c} \in [\mathbf{b}]$ such that either $[\mathbf{c}] \subseteq A$ or $[\mathbf{c}] \cap A = \emptyset$.*

For every $\mathbf{b} = (b_n)_n \in \mathbf{B}$ and every $n \in \mathbb{N}$ we set $i_n = \bigcup_{i=0}^n b_i$. We define $C : \mathbf{B} \to \Sigma^{\mathbb{N}}$ and $A : \mathbf{B} \to \Sigma^{\mathbb{N}}$ by

$$C((b_n)_n) = (i_0, \dots, i_n, \dots), \qquad A((b_n)_n) = (i_0 \cup b_2, \dots, i_{3n} \cup b_{3n+2}, \dots).$$

We notice that for every $\mathbf{b} \in \mathbf{B}$ the sequence $C(\mathbf{b})$ is a chain of Σ while $A(\mathbf{b})$ is an antichain of Σ converging, in the sense of Definition 13, to $\sigma = \bigcup_n i_n \in [\Sigma]$. We also notice that the functions C and A are continuous.

LEMMA 44. *Let $\{f_t\}_{t\in\Sigma}$ be a Krawczyk tree of f with respect to $\{f_n\}_n$ and $\mathcal{C}$. Then there exists a block sequence $\mathbf{b} = (b_n)_n$ such that for every $\mathbf{c} \in [\mathbf{b}]$ the sequence $(f_t)_{t\in C(\mathbf{c})}$ is pointwise convergent to a function belonging to $\mathcal{C}$ and different from f, while the sequence $(f_t)_{t\in A(\mathbf{c})}$ is pointwise convergent to f.*

Proof. Let

$$C_1 = \{\mathbf{c} \in \mathbf{B} : \text{the sequence } (f_t)_{t\in C(\mathbf{c})} \text{ is pointwise convergent}\}.$$

It is easy to see that C_1 is a co-analytic subset of $\mathbf{B}$. By Theorem 43 and the sequential compactness of $\mathcal{K}$, we find $\mathbf{d} \in \mathbf{B}$ such that $[\mathbf{d}]$ is a subset of C_1. As already remarked, for every block sequence $\mathbf{c}$ the sequence $C(\mathbf{c})$ is a chain of Σ. Hence, by Corollary 41(1), for every $\mathbf{c} \in [\mathbf{d}]$ the sequence $(f_t)_{t\in C(\mathbf{c})}$ must be pointwise convergent to a function belonging to $\mathcal{C}$ and different from f.

Now let

$$C_2 = \{\mathbf{c} \in [\mathbf{d}] : \text{the sequence } (f_t)_{t\in A(\mathbf{c})} \text{ is pointwise convergent to } f\}.$$

Again by Milliken's theorem, there exists $\mathbf{b} = (b_n)_n \in [\mathbf{d}]$ such that either $[\mathbf{b}] \subseteq C_2$ or $[\mathbf{b}] \cap C_2 = \emptyset$. We claim that $[\mathbf{b}]$ is subset of C_2. It is enough to show that $[\mathbf{b}] \cap C_2 \neq \emptyset$. Recall that for every $l \in \mathbb{N}$ we have set $i_l = b_0 \cup \dots \cup b_l$. Let

$$A_l = \{i_l \cup b_m : m > l+1\} \subseteq \Sigma.$$

As $(b_n)_n$ is a block sequence, by property (P) above, we see that the sequence $(f_t)_{t\in A_l}$ is pointwise convergent to f. By Lemma 42, there exists $D \subseteq \bigcup_l A_l$ such that the sequence $(f_t)_{t\in D}$ is pointwise convergent to f and $D \cap A_l \neq \emptyset$ for infinitely many l. We may select $L = \{l_0 < l_1 < \cdots\}$, $M = \{m_0 < m_1 < \cdots\} \in [\mathbb{N}]$ such that $l_n + 1 < m_n < l_{n+1}$ and $i_{l_n} \cup b_{m_n} \in D$ for all $n \in \mathbb{N}$. Now we define $\mathbf{c} = (c_n)_n \in [\mathbf{b}]$ as follows. We set $c_0 = i_{l_0}$, $c_1 = b_{l_0+1} \cup \dots \cup b_{m_0-1}$ and $c_2 = b_{m_0}$. For every $n \in \mathbb{N}$ with $n \ge 1$ let $I_n = [m_{n-1}+1, l_n]$ and $J_n = [l_n, m_n - 1]$ and set

$$c_{3n} = \bigcup_{i\in I_n} b_i, \qquad c_{3n+1} = \bigcup_{i\in J_n} b_i, \qquad c_{3n+2} = b_{m_n}.$$

It is easy to see that $\mathbf{c} \in [\mathbf{b}]$ and $A(\mathbf{c}) = (i_{l_n} \cup b_{m_n})_n \subseteq D$. Hence, the sequence $(f_t)_{t\in A(\mathbf{c})}$ is pointwise convergent to f. It follows that $[\mathbf{b}] \cap C_2 \neq \emptyset$ and the proof is complete. ∎

Proof of Theorem 40. Let $\mathbf{b} = (b_n)_n$ be the block sequence obtained by Lemma 44. If $\beta = (b_{n_0}, \ldots, b_{n_k})$ with $n_0 < \cdots < n_k$ is a finite subsequence of $\mathbf{b}$, then we let $\bigcup \beta = b_{n_0} \cup \cdots \cup b_{n_k} \in \Sigma$. Recursively, we shall select a family $(\beta_s)_{s\in 2^{<\mathbb{N}}}$ such that the following are satisfied:

- (C1) For every $s \in 2^{<\mathbb{N}}$, β_s is a finite subsequence of $\mathbf{b}$.
- (C2) For every $s, s' \in 2^{<\mathbb{N}}$ we have $s \sqsubset s'$ if and only if $\beta_s \sqsubset \beta_{s'}$.
- (C3) For every $s \in 2^{<\mathbb{N}}$ and every $\mathbf{c} \in [\beta_{s^\frown 0}, \mathbf{b}]$ we have $\bigcup \beta_{s^\frown 1} \in A(\mathbf{c})$.

The construction proceeds as follows. We set $\beta_{(\emptyset)} = (\emptyset)$. For every $M = \{m_0 < m_1 < \cdots\} \in [\mathbb{N}]$, let $\mathbf{b}_M = (b_{m_n})_n$ be the subsequence of $\mathbf{b}$ determined by M. Assume that for some $s \in 2^{<\mathbb{N}}$ the finite sequence β_s has been defined. We select $M = M_s \in [\mathbb{N}]$ such that $\beta_s \sqsubset \mathbf{b}_M$. The set $A(\mathbf{b}_M)$ converges to the unique branch of Σ determined by the infinite chain $C(\mathbf{b}_M)$. So, we may select a finite subsequence $\beta_{s^\frown 1}$ with $\beta_s \sqsubset \beta_{s^\frown 1}$ and such that $\bigcup \beta_{s^\frown 1} \in A(\mathbf{b}_M)$. The function $A : [\mathbf{b}] \to \Sigma^{\mathbb{N}}$ is continuous. Hence, there exists a finite subsequence $\beta_{s^\frown 0}$ of $\mathbf{b}$ with $\beta_{s^\frown 0} \sqsubset \mathbf{b}_M$ and such that condition (C3) above is satisfied. Finally, notice that $\beta_{s^\frown 0}$ and $\beta_{s^\frown 1}$ are incomparable with respect to the partial order $\sqsubset$ of extension.

One can also provide a recursive formula defining a family $(\beta_s)_{s\in 2^{<\mathbb{N}}}$ satisfying conditions (C1)–(C3) above. Indeed, set $\beta_{(\emptyset)} = (\emptyset)$, $\beta_{(0)} = (b_0, b_1, b_2)$ and $\beta_{(1)} = (b_0, b_2)$. Assume that β_s has been defined for some $s \in 2^{<\mathbb{N}}$. Let $n_s = \max\{n : b_n \in \beta_s\}$. If s ends with 0, then we set

$$\beta_{s^\frown 0} = \beta_s^\frown(b_{n_s+1}, b_{n_s+2}, b_{n_s+3}), \qquad \beta_{s^\frown 1} = \beta_s^\frown(b_{n_s+1}, b_{n_s+3}).$$

If s ends with 1, then we set

$$\beta_{s^\frown 0} = \beta_s^\frown(b_{n_s+1}, b_{n_s+2}, b_{n_s+3}, b_{n_s+4}), \qquad \beta_{s^\frown 1} = \beta_s^\frown(b_{n_s+1}, b_{n_s+2}, b_{n_s+4}).$$

It is easy to see that, with the above choices, conditions (C1)–(C3) are satisfied.

Having defined the family $(\beta_s)_{s\in 2^{<\mathbb{N}}}$ for every $s \in 2^{<\mathbb{N}}$ we let

$$t_s = \bigcup \beta_s \in \Sigma, \qquad h_s = f_{t_s}.$$

Clearly, the family $\{h_s\}_{s\in 2^{<\mathbb{N}}}$ is a dyadic subtree of the Krawczyk tree $\{f_t\}_{t\in\Sigma}$ of f with respect to $\{f_n\}_n$ and $\mathcal{C}$. The basic properties of the family $\{h_s\}_{s\in 2^{<\mathbb{N}}}$ are summarized in the following claim.

Claim 1.

1. *For every $\sigma \in 2^{\mathbb{N}}$ the sequence $(h_{\sigma|n})_n$ is pointwise convergent to a function $g_\sigma \in \mathcal{C}$ with $g_\sigma \neq f$.*
2. *For every $P \subseteq 2^{\mathbb{N}}$ perfect the function f belongs to the closure of the family $\{g_\sigma : \sigma \in P\}$.*

Proof of Claim. (1) Let $\sigma \in 2^{\mathbb{N}}$ and put $\mathbf{b}_\sigma = \bigcup_n \beta_{\sigma|n} \in [\mathbf{b}]$. It is easy to see that $(t_{\sigma|n})_n$ is a subsequence of $C(\mathbf{b}_\sigma)$. So the result follows by Lemma 44.

(2) Assume not. Then there exist $P \subseteq 2^{\mathbb{N}}$ perfect and a neighborhood V of f in $\mathbb{R}^X$ such that $g_\sigma \notin \overline{V}$ for all $\sigma \in P$. By part (1), for every $\sigma \in P$ there exists $n_\sigma \in \mathbb{N}$ such that $h_{\sigma|n} \notin V$ for all $n \geq n_\sigma$. For every $n \in \mathbb{N}$ let $P_n = \{\sigma \in P : n_\sigma \leq n\}$. Then each P_n is a closed subset of P and clearly $P = \bigcup_n P_n$. Thus, there exist $n_0 \in \mathbb{N}$ and $Q \subseteq 2^{\mathbb{N}}$ perfect with $Q \subseteq P_{n_0}$. It follows that $h_{\sigma|n} \notin V$ for all $\sigma \in Q$ and $n \geq n_0$. Let τ be the lexicographical minimum of Q. We may select a sequence $(\sigma_k)_k$ in Q such that, setting $s_k = \tau \wedge \sigma_k$ for all $k \in \mathbb{N}$, we have $\sigma_k \to \tau$, $\tau \prec \sigma_k$ and $|s_k| > n_0$. Notice that $s_k^\frown 0 \sqsubset \tau$ while $s_k^\frown 1 \sqsubset \sigma_k$ and $|s_k^\frown 1| > n_0$. Hence, by our assumptions on the set Q and the definition of $\{h_s\}_{s\in 2^{<\mathbb{N}}}$, we get

$$h_{s_k^\frown 1} = f_{t_{s_k^\frown 1}} \notin V \quad \text{for all } k \in \mathbb{N}. \tag{1}$$

We are ready to derive the contradiction. We set $\mathbf{b}_\tau = \bigcup_n \beta_{\tau|n} \in [\mathbf{b}]$. As $\beta_{s_k^\frown 0} \sqsubset \mathbf{b}_\tau$, by property (C3) in the above construction, we see that $t_{s_k^\frown 1} = \bigcup \beta_{s_k^\frown 1} \in A(\mathbf{b}_\tau)$ for all $k \in \mathbb{N}$. By Lemma 44, the sequence $(f_t)_{t\in A(\mathbf{b}_\tau)}$ is pointwise convergent to the function f. It follows that $(f_{t_{s_k^\frown 1}})_k$ is also pointwise convergent to f, which clearly contradicts (1) above. ♦

We apply Theorem 21 to the family $\{h_s\}_{s\in 2^{<\mathbb{N}}}$ and we get a regular dyadic subtree $T = (s_t)_{t\in 2^{<\mathbb{N}}}$ of $2^{<\mathbb{N}}$ such that the family $\{h_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is canonicalized. The main claim is the following.

Claim 2. *The family $\{h_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of $\hat{A}(2^{\mathbb{N}})$.*

Proof of Claim. In order to prove the claim we will isolate a property of the whole family $\{h_s\}_{s\in 2^{<\mathbb{N}}}$ (property (Q) below). Let S be an arbitrary regular dyadic subtree of $2^{<\mathbb{N}}$. Notice that $g_\sigma \in \overline{\{h_s\}}^p_{s\in S}$ for every $\sigma \in [\hat{S}]$. By property (2) in Claim 1, we see that the function f belongs to the pointwise closure of $\{h_s\}_{s\in S}$ in $\mathbb{R}^X$. By the Bourgain–Fremlin–Talagrand theorem there exists $A \subseteq S$ such that the sequence $(h_s)_{s\in A}$ is pointwise convergent to f. By property (1) in Claim 1, we see that A can be chosen to be an antichain converging to some $\sigma \in [\hat{S}]$. As all these facts hold for every regular dyadic subtree S of $2^{<\mathbb{N}}$ we arrive at the following property of the family $\{h_s\}_{s\in 2^{<\mathbb{N}}}$:

(Q) For every regular dyadic subtree S of $2^{<\mathbb{N}}$, there exist two antichains A_1, A_2 of S and $\sigma_1, \sigma_2 \in [\hat{S}]$ with $\sigma_1 \neq \sigma_2$ such that A_1 converges to σ_1, A_2 converges to σ_2 and both sequences $(h_s)_{s\in A_1}$ and $(h_s)_{s\in A_2}$ are pointwise convergent to f.

Now let $T = (s_t)_{t\in 2^{<\mathbb{N}}}$ be the regular dyadic subtree of $2^{<\mathbb{N}}$ such that the family $\{h_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is canonicalized. Invoking property (Q) above and referring to the description of the families $\{d_t^i : t \in 2^{<\mathbb{N}}\}$ $(1 \leq i \leq 7)$, we see that $\{h_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ must be equivalent to either the canonical dense family of $A(2^{<\mathbb{N}})$ or the canonical dense family of $\hat{A}(2^{\mathbb{N}})$. By property (1) in Claim 1, the first case is impossible. It follows that $\{h_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ must be equivalent to the canonical dense family of $\hat{A}(2^{\mathbb{N}})$. ♦

Let $T = (s_t)_{t\in 2^{<\mathbb{N}}}$ and $\{h_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ be as above. Observe that for every $t \in 2^{<\mathbb{N}}$ there exists a unique $n_t \in \mathbb{N}$ with $h_{s_t} = f_{n_t}$. Thus, by passing to a dyadic subtree of T if necessary and invoking the minimality of the canonical dense family of $\hat{A}(2^{\mathbb{N}})$, we deduce that the function $2^{<\mathbb{N}} \ni t \mapsto n_t \in \mathbb{N}$ is a canonical injection and that the map

$$\hat{A}(2^{\mathbb{N}}) \ni v_t \mapsto f_{n_t} \in \mathcal{K}$$

extends to a homeomorphism Φ between $\hat{A}(2^{\mathbb{N}})$ and $\overline{\{f_{n_t}\}}^p_{t\in 2^{<\mathbb{N}}}$. That this homeomorphism sends 0 to f is an immediate consequence of property (Q) in Claim 2 above. Moreover, by Claim 1(1), we see that $\Phi(\delta_\sigma) \in \mathcal{C}$ for every $\sigma \in 2^{\mathbb{N}}$. ■

By Theorem 40 and Proposition 24(1) we get the following corollary.

Corollary 45. *Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X, $\{f_n\}_n$ a countable dense subset of $\mathcal{K}$ and $f \in \mathcal{K}$. If f is a non-G_δ point of $\mathcal{K}$, then there exists a canonical homeomorphic embedding of $\hat{A}(2^{\mathbb{N}})$ into $\mathcal{K}$ with respect to $\{f_n\}_n$ which sends* 0 *to f.*

After a first draft of the present paper was complete, S. Todorčević informed us ([To3]) that he is aware of the above corollary with a proof based on his approach in [To1].

We notice that if $\mathcal{K}$ is a non-metrizable separable Rosenthal compact on a Polish space X, then the constant function 0 is a non-G_δ point of $\mathcal{K}-\mathcal{K}$. Indeed, since $\mathcal{K}$ is non-metrizable, for every $D \subseteq X$ countable there exist $f, g \in \mathcal{K}$ with $f \neq g$ and such that $f|_D = g|_D$. This easily implies that 0 is a non-G_δ point of $\mathcal{K}-\mathcal{K}$. By Corollary 45, we see that there exists a homeomorphic embedding of $\hat{A}(2^{\mathbb{N}})$ into $\mathcal{K}-\mathcal{K}$ with 0 as the unique non-G_δ point of its image. This fact can be lifted to the class of analytic subspaces, as follows.

Corollary 46. *Let $\mathcal{K}$ be a separable Rosenthal compact and $\mathcal{C}$ an analytic subspace of $\mathcal{K}$ which is non-metrizable. Let also $D = \{f_n\}_n$ be a countable dense subset of $\mathcal{K}$ witnessing the analyticity of $\mathcal{C}$. Then there exists a family $\{f_t\}_{t\in 2^{<\mathbb{N}}} \subseteq D - D$, equivalent to the canonical dense family of $\hat{A}(2^{\mathbb{N}})$, with $\mathrm{Acc}(\{f_t : t \in 2^{<\mathbb{N}}\}) \subseteq \mathcal{C}-\mathcal{C}$ and such that the constant function* 0 *is the unique non-G_δ point of $\overline{\{f_t\}}^p_{t\in 2^{<\mathbb{N}}}$.*

Proof. Let $\{g_n\}_n$ be an enumeration of the set $D - D$ which is dense in $\mathcal{K}-\mathcal{K}$. It is easy to see that $\mathcal{C}-\mathcal{C}$ is an analytic subspace of $\mathcal{K}-\mathcal{K}$ with this being witnessed by the sequence $\{g_n\}_n$. Moreover, as $\mathcal{C}$ is non-metrizable, the constant function 0 belongs to $\mathcal{C}-\mathcal{C}$ and it is a non-G_δ point of it. By Theorem 40, the result follows. ■

8. Connections with Banach space theory

This section is devoted to applications, motivated by the results obtained in [ADK2], of the embedding of $\hat{A}(2^{\mathbb{N}})$ in analytic subspaces of separable Rosenthal compacta containing 0 as a non-G_δ point. The first one concerns the existence of unconditional families. The second deals with spreading and level unconditional tree bases.

8.1. Existence of unconditional families. We recall that a family $\{x_i\}_{i\in I}$ in a Banach space X is said to be 1-*unconditional* if for every $F \subseteq G \subseteq I$ and every $(a_i)_{i\in G} \in \mathbb{R}^G$ we have

$$\Big\|\sum_{i\in F} a_i x_i\Big\| \leq \Big\|\sum_{i\in G} a_i x_i\Big\|.$$

We will need the following reformulation of Theorem 4 in [ADK2], to which we also refer the reader for the proof.

THEOREM 47. *Let X be a Polish space and $\{f_\sigma : \sigma \in 2^{\mathbb{N}}\}$ be a bounded family of real-valued functions on X which is pointwise discrete and has the constant function 0 as the unique accumulation point in $\mathbb{R}^X$. Assume that the map $\Phi : 2^{\mathbb{N}} \times X \to \mathbb{R}$ defined by $\Phi(\sigma, x) = f_\sigma(x)$ is Borel. Then there exists a perfect subset P of $2^{\mathbb{N}}$ such that the family $\{f_\sigma : \sigma \in P\}$ is 1-unconditional in the supremum norm.*

In [ADK2] it is shown that if X is a separable Banach space not containing ℓ_1 and with non-separable dual, then X^{**} contains a 1-unconditional family of the size of the continuum. This result can be lifted to the framework of separable Rosenthal compacta, as follows.

THEOREM 48. *Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X. Let also $\mathcal{C}$ be an analytic subspace of $\mathcal{K}$ consisting of bounded functions.*

- (a) *If $\mathcal{C}$ contains the function 0 as a non-G_δ point, then there exists a family $\{f_\sigma : \sigma \in 2^{\mathbb{N}}\}$ in $\mathcal{C}$ which is 1-unconditional in the supremum norm, pointwise discrete and having 0 as the unique accumulation point.*
- (b) *If $\mathcal{C}$ is non-metrizable, then there exists a family $\{f_\sigma - g_\sigma : \sigma \in 2^{\mathbb{N}}\}$, where $f_\sigma, g_\sigma \in \mathcal{C}$ for all $\sigma \in 2^{\mathbb{N}}$, which is 1-unconditional in the supremum norm.*

Proof. (a) Let $D = \{f_n\}_n$ be a countable dense subset of $\mathcal{K}$ witnessing the analyticity of $\mathcal{C}$. As 0 is a non-G_δ point of $\mathcal{C}$, by Theorem 40 there exists a family $\{f_t\}_{t\in 2^{<\mathbb{N}}} \subseteq D$, equivalent to the canonical dense family of $\hat{A}(2^{\mathbb{N}})$, with $\mathrm{Acc}(\{f_t : t \in 2^{<\mathbb{N}}\}) \subseteq \mathcal{C}$ and such that the constant function 0 is the unique non-G_δ point of $\overline{\{f_t\}}^p_{t\in 2^{<\mathbb{N}}}$. For every $\sigma \in 2^{\mathbb{N}}$ let f_σ be the pointwise limit of the sequence $(f_{\sigma|n})_n$. Clearly, the family $\{f_\sigma : \sigma \in 2^{\mathbb{N}}\}$ is pointwise discrete and has 0 as the unique accumulation point. Moreover, it is easy to see that the map $\Phi : 2^{\mathbb{N}} \times X \to \mathbb{R}$ defined by $\Phi(\sigma, x) = f_\sigma(x)$ is Borel. By Theorem 47, the result follows.

(b) This follows by Corollary 46 and Theorem 47. ■

Actually, we can strengthen the properties of the family $\{f_\sigma : \sigma \in 2^{\mathbb{N}}\}$ obtained by part (a) of Theorem 48 as follows.

THEOREM 49. *Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X and $\mathcal{C}$ be an analytic subspace of $\mathcal{K}$ consisting of bounded functions. Assume that $\mathcal{C}$ contains the function 0 as a non-G_δ point. Then there exist a family $\{(g_\sigma, x_\sigma) : \sigma \in 2^{\mathbb{N}}\} \subseteq \mathcal{C} \times X$ and $\varepsilon > 0$ satisfying $|g_\sigma(x_\sigma)| > \varepsilon$, $g_\sigma(x_\tau) = 0$ if $\sigma \neq \tau$ and such that the family $\{g_\sigma : \sigma \in 2^{\mathbb{N}}\}$ is 1-unconditional in the supremum norm and has 0 as the unique accumulating point.*

Proof. Let $\{f_\sigma : \sigma \in 2^{\mathbb{N}}\} \subseteq \mathcal{C}$ be the family obtained by Theorem 48(a). We notice that, by the proof of Theorem 48, the map $\Phi : 2^{\mathbb{N}} \times X \to \mathbb{R}$ defined by $\Phi(\sigma, x) = f_\sigma(x)$ is Borel. Using this and passing to a perfect subset of $2^{\mathbb{N}}$ if necessary, we may find $\varepsilon > 0$ such that $\|f_\sigma\|_\infty > \varepsilon$ for all $\sigma \in 2^{\mathbb{N}}$. Define $N \subseteq 2^{\mathbb{N}} \times X$ by

$$(\sigma, z) \in N \ \Leftrightarrow\ |f_\sigma(z)| > \varepsilon.$$

As the map Φ is Borel, we see that the set N is Borel. Moreover, by the choice of ε, for every $\sigma \in 2^{\mathbb{N}}$ the section $N_\sigma = \{z : (\sigma, z) \in N\}$ of N at σ is non-empty. By the Yankov–von Neumann uniformization theorem (see [Ke, Theorem 18.1]), there exists a map

$$2^{\mathbb{N}} \ni \sigma \mapsto z_\sigma \in X$$

which is measurable with respect to the σ-algebra generated by the analytic sets and such that $(\sigma, z_\sigma) \in N$ for every $\sigma \in 2^{\mathbb{N}}$. Invoking the classical fact that analytic sets have the Baire property, by Theorem 8.38 in [Ke] and passing to a further perfect subset of $2^{\mathbb{N}}$ if necessary, we may assume that the map $\sigma \mapsto z_\sigma$ is actually continuous.

For every $m \in \mathbb{N}$ define $A_m \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ by

$$(\sigma, \tau) \in A_m \ \Leftrightarrow\ |f_\tau(z_\sigma)| > \frac{1}{m+1}.$$

Notice that the set A_m is Borel. Since the family $\{f_\sigma : \sigma \in 2^{\mathbb{N}}\}$ accumulates to 0, we see that for every $\sigma \in 2^{\mathbb{N}}$ the section $(A_m)_\sigma = \{\tau : (\sigma, \tau) \in A_m\}$ of A_m at σ is finite, hence meager in $2^{\mathbb{N}}$. By the Kuratowski–Ulam theorem (see [Ke, Theorem 8.41]), the set A_m is meager in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$. Hence so is

$$A = \bigcup_{m \in \mathbb{N}} A_m.$$

By a result of J. Mycielski (see [Ke, Theorem 19.1]) there exists $P \subseteq 2^{\mathbb{N}}$ perfect such that for every $\sigma, \tau \in P$ with $\sigma \neq \tau$ we have $(\sigma, \tau) \notin A$ and $(\tau, \sigma) \notin A$. This implies that $f_\tau(z_\sigma) = 0$ and $f_\sigma(z_\tau) = 0$. We fix a homeomorphism $h : 2^{\mathbb{N}} \to P$ and we set $g_\sigma = f_{h(\sigma)}$ and $x_\sigma = z_{h(\sigma)}$ for every $\sigma \in 2^{\mathbb{N}}$. Clearly, the family $\{(g_\sigma, x_\sigma) : \sigma \in 2^{\mathbb{N}}\}$ is as desired. ∎

The proof of the corresponding result in [ADK2] is based on Ramsey and Banach space tools, avoiding the embedding of $\hat{A}(2^{\mathbb{N}})$ into $(B_{X^{**}}, w^*)$.

We recall that a Banach space X is said to be *representable* if X is isomorphic to a subspace of $\ell_\infty(\mathbb{N})$ which is analytic in the weak* topology (see [GT], [GL] and [AGR]). We close this subsection with the following.

THEOREM 50. *Let X be a non-separable representable Banach space. Then X^* contains an unconditional family of size $|X^*|$.*

Proof. Identify X with its isomorphic copy in $\ell_\infty(\mathbb{N})$. Then B_X is an analytic subset of (B_{ℓ_∞}, w^*). Let $f : \mathbb{N}^{\mathbb{N}} \to B_X$ be an onto continuous map. Let $\{x_n\}_n$ be a norm dense subset of $\ell_1(\mathbb{N})$. Viewing ℓ_1 as a subspace of ℓ_∞^*, we define $f_n : \mathbb{N}^{\mathbb{N}} \to \mathbb{R}$ by $f_n = x_n \circ f$. Then $\{f_n\}_n$ is a uniformly bounded sequence of continuous real-valued functions on $\mathbb{N}^{\mathbb{N}}$. Notice that $\overline{\{f_n\}_n}^p = \{x^* \circ f : x^* \in B_{X^*}\}$, which can be naturally identified with $\{x^*|_{B_X} : x^* \in B_{X^*}\}$. By the non-effective version of Debs' theorem (see [AGR]) we have the following alternatives.

CASE 1: There exist an increasing sequence $(n_k)_k$, a continuous map $\phi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and real numbers $a < b$ such that for every $\sigma \in 2^{\mathbb{N}}$ and every $k \in \mathbb{N}$, if $\sigma(k) = 0$ then $f_{n_k}(\phi(\sigma)) < a$, while if $\sigma(k) = 1$, then $f_{n_k}(\phi(\sigma)) > b$. In this case, for every $p \in \beta\mathbb{N}$ we set

$$g_p = p\text{-}\lim_k f_{n_k}.$$

Then $g_p = x_p^*|_{B_X}$ for some $x_p^* \in X^*$. We claim that $\{x_p^* : p \in \beta\mathbb{N}\}$ is equivalent to the natural basis of $\ell_1(2^{\mathfrak{c}})$. To see this, observe that $g_p(\phi(\sigma)) \leq a$ if and only if $\{k : \sigma(k) = 0\} \in p$, and $g_p(\phi(\sigma)) \geq b$ if and only if $\{k : \sigma(k) = 1\} \in p$. Setting $A_p = [g_p \leq a]$ and $B_p = [g_p \geq b]$ for all $p \in \beta\mathbb{N}$, we see that the family $(A_p, B_p)_{p\in\beta\mathbb{N}}$ is an independent family of disjoint pairs. By Rosenthal's criterion, the family $\{g_p : p \in \beta\mathbb{N}\}$ is equivalent to $\ell_1(2^{\mathfrak{c}})$. Thus, so is $\{x_p^* : p \in \beta\mathbb{N}\}$.

Case 2: The sequence $\{f_n\}_n$ is relatively compact in $\mathcal{B}_1(\mathbb{N}^{\mathbb{N}})$. In this case, as X is non-separable, $0 \in \overline{\{f_n\}}_n^p$ is a non-G_δ point. Thus, by Theorem 48(a), there exists a 1-unconditional family in X^* of the size of the continuum. ■

It can also be shown that every representable Banach space has a separable quotient (see Theorem 15 in [ADK2]). For further applications of the existence of unconditional families we refer the reader to [ADK2].

8.2. Spreading and level unconditional tree bases. We start with the following definition.

Definition 51. Let X be a Banach space.

(1) A *tree basis* is a bounded family $\{x_t\}_{t\in 2^{<\mathbb{N}}}$ in X which is Schauder basic when enumerated according to the canonical bijection ϕ_0 between $2^{<\mathbb{N}}$ and $\mathbb{N}$.
(2) A tree basis $\{x_t\}_{t\in 2^{<\mathbb{N}}}$ is said to be *spreading* if there exists $(\varepsilon_n)_n \downarrow 0$ such that for any $n, m \in \mathbb{N}$ with $n < m$, any $0 \leq d < 2^n$ and any pair $\{s_i\}_{i=0}^d \subseteq 2^n$ and $\{t_i\}_{i=0}^d \subseteq 2^m$ with $s_i \sqsubset t_i$ for all $i \in \{0, \dots, d\}$, we have $\|T\| \cdot \|T^{-1}\| < 1 + \varepsilon_n$ where $T : \mathrm{span}\{x_{s_i} : i = 0, \dots, d\} \to \mathrm{span}\{x_{t_i} : i = 0, \dots, d\}$ is the natural 1-1 and onto linear operator.
(3) A tree basis $\{x_t\}_{t\in 2^{<\mathbb{N}}}$ is said to be *level unconditional* if there exists $(\varepsilon_n)_n \downarrow 0$ such that for every $n \in \mathbb{N}$, the family $\{x_t : t \in 2^n\}$ is $(1 + \varepsilon_n)$-unconditional.

In [ADK2] the existence of spreading and level unconditional tree bases was established for every separable Banach space X not containing ℓ_1 and with non-separable dual.

This result can be extended in the frame of separable Rosenthal compacta, as follows.

Theorem 52. *Let $\mathcal{K}$ be a uniformly bounded separable Rosenthal compact on a compact metrizable space X and having a countable dense subset D of continuous functions. Let also $(\varepsilon_n)_n$ be a decreasing sequence of positive reals with $\varepsilon_n \to 0$. Assume that the constant function 0 is a non-G_δ point of $\mathcal{K}$. Then there exists a family $\{u_t\}_{t\in 2^{<\mathbb{N}}} \subseteq \mathrm{conv}(D)$ equivalent to the canonical dense family of $\hat{A}(2^{\mathbb{N}})$ such that, if we set $g_\sigma = \lim_n u_{\sigma|n}$ for all $\sigma \in 2^{\mathbb{N}}$, the following are satisfied:*

(1) *The function 0 is the unique non-G_δ point of $\overline{\{u_t\}}_{t\in 2^{<\mathbb{N}}}^p$.*
(2) *The family $\{u_t\}_{t\in 2^{<\mathbb{N}}}$ is a tree basis with respect to the supremum norm.*
(3) *The family $\{g_\sigma : \sigma \in 2^{\mathbb{N}}\}$ is a subset of $\mathcal{K}$ and is 1-unconditional.*
(4) *For every $n \in \mathbb{N}$, if $\{t_0 \prec \dots \prec t_{2^n-1}\}$ is the $\prec$-increasing enumeration of 2^n, then for every $\{\sigma_0, \dots, \sigma_{2^n-1}\} \subseteq 2^{\mathbb{N}}$ with $t_i \sqsubset \sigma_i$ for all $i \in \{0, \dots, 2^n - 1\}$ the sequence $(g_{\sigma_i})_{i=0}^{2^n-1}$ is $(1 + \varepsilon_n)$-equivalent to $(u_{t_i})_{i=0}^{2^n-1}$.*

The proof of the above result is a slight modification of Theorem 17 in [ADK2], to wish we also refer the reader for more information.

We close this subsection with the following result whose proof is based on Stegall's construction [St].

THEOREM 53. *Let X be a Banach space such that X^* is separable and X^{**} is non-separable. Let also $\varepsilon > 0$. Then there exists a family $\{u_t\}_{t\in 2^{<\mathbb{N}}} \subseteq B_X$ such that:*

(i) *The family $\{u_t\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of $2^{\leq\mathbb{N}}$.*
(ii) *For every $\sigma \in 2^{\mathbb{N}}$, if y^{**}_σ is the weak* limit of $(u_{\sigma|n})_n$, then there exists $y^{***}_\sigma \in X^{***}$ with $\|y^{***}_\sigma\| \leq 1+\varepsilon$ and such that $y^{***}_\sigma(y^{**}_\sigma) = 1$ while $y^{***}_\sigma(y^{**}_\tau) = 0$ for all $\tau \neq \sigma$.*
(iii) *For every $n \in \mathbb{N}$, if $\{t_0 \prec \cdots \prec t_{2^n-1}\}$ is the $\prec$-increasing enumeration of 2^n, then for every $\{\sigma_0, \ldots, \sigma_{2^n-1}\} \subseteq 2^{\mathbb{N}}$ with $t_i \sqsubset \sigma_i$ for all $i \in \{0, \ldots, 2^n-1\}$, the sequence $(y^{**}_{\sigma_i})_{i=0}^{2^n-1}$ is $(1+1/n)$-equivalent to $(u_{t_i})_{i=0}^{2^n-1}$.*

Proof. Since X^* is separable, $(B_{X^{**}}, w^*)$ is compact metrizable. Fix a compatible metric ϱ for $(B_{X^{**}}, w^*)$. Using Stegall's construction [St], we get:

(C1) a family $\{x^*_t\}_{t\in 2^{<\mathbb{N}}} \subseteq X^*$,
(C2) a family $\{B_t\}_{t\in 2^{<\mathbb{N}}}$ of open subsets of $(B_{X^{**}}, w^*)$

such that for all $t \in 2^{<\mathbb{N}}$ the following are satisfied:

(P1) $1 < \|x^*_t\| < 1+\varepsilon$.
(P2) $\overline{B}_{t^\frown 0} \cap \overline{B}_{t^\frown 1} = \emptyset$, $\overline{B}_{t^\frown 0} \cup \overline{B}_{t^\frown 1} \subseteq B_t$ and ϱ-diam$(B_t) \leq 1/(|t|+1)$.
(P3) For all $x^{**} \in B_t$, $|x^{**}(x^*_t) - 1| < 1/(|t|+1)$.
(P4) For all $t' \neq t$ with $|t| = |t'|$ and for all $x^{**} \in B_{t'}$, $|x^{**}(x^*_t)| < 1/(|t|+1)$.

By property (P2), for every $\sigma \in 2^{\mathbb{N}}$ we have $\bigcap_n B_{\sigma|n} = \{x^{**}\}$. Moreover, the map $2^{\mathbb{N}} \ni \sigma \mapsto x^{**}_\sigma \in (B_{X^{**}}, w^*)$ is a homeomorphic embedding. By Goldstine's theorem, for every $t \in 2^{<\mathbb{N}}$ we can choose $x_t \in B_t \cap X$. Notice that w^*-$\lim_n x_{\sigma|n} = x^{**}_\sigma$ for all $\sigma \in 2^{\mathbb{N}}$. For every $\sigma \in 2^{\mathbb{N}}$ we choose $x^{***}_\sigma \in \bigcap_n \overline{\{x^*_{\sigma|k} : k \geq n\}}^{w*}$. By (P3) we see that $x^{***}_\sigma(x^{**}_\sigma) = 1$ while, by (P4), $x^{***}_\sigma(x^{**}_\tau) = 0$ for all $\tau \neq \sigma$. Moreover,

$$\sup\{|\lambda_i| : i = 0, \ldots, n\} \leq (1+\varepsilon)\Big\|\sum_{i=0}^{n} \lambda_i x^{**}_{\sigma_i}\Big\|$$

for all $n \in \mathbb{N}$, $\{\sigma_0, \ldots, \sigma_n\} \subseteq 2^{\mathbb{N}}$ and every $(\lambda_i)_{i=0}^n \in \mathbb{R}^{n+1}$. Arguing as in the proof of Theorem 17 in [ADK2], we may construct a family $\{u_t\}_{t\in 2^{<\mathbb{N}}} \subseteq \mathrm{conv}\{x_t : t \in 2^{<\mathbb{N}}\}$ and a regular dyadic subtree $S = (s_t)_{t\in 2^{<\mathbb{N}}}$ of $2^{<\mathbb{N}}$ such that:

(1) For all $\sigma \in 2^{\mathbb{N}}$, the sequence $(u_{\sigma|n})_n$ is weak* convergent to y^{**}_σ, where
$$y^{**}_\sigma = \lim_n x_{s_{\sigma|n}}.$$
(2) For every $n \in \mathbb{N}$, if $\{t_0 \prec \cdots \prec t_{2^n-1}\}$ is the $\prec$-increasing enumeration of 2^n, then for every $\{\sigma_0, \ldots, \sigma_{2^n-1}\} \subseteq 2^{\mathbb{N}}$ with $t_i \sqsubset \sigma_i$ for all $i \in \{0, \ldots, 2^n-1\}$ the sequence $(y^{**}_{\sigma_i})_{i=0}^{2^n-1}$ is $(1+1/n)$-equivalent to $(u_{t_i})_{i=0}^{2^n-1}$.

For all $\sigma \in 2^{\mathbb{N}}$, let $\bar\sigma = \bigcup_n s_{\sigma|n} \in 2^{\mathbb{N}}$. Setting $y^{***}_\sigma = x^{***}_{\bar\sigma}$ for all $\sigma \in 2^{\mathbb{N}}$, we see that properties (ii) and (iii) in the statement of the theorem are satisfied. Finally, by passing to a regular dyadic subtree if necessary, we also find that the family $\{u_t\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of $2^{\leq\mathbb{N}}$, i.e. property (i) is satisfied. ∎

REMARK 4. (1) We do not know if the family $\{u_t\}_{t\in 2^{<\mathbb{N}}}$ obtained in Theorem 53 can be chosen to be Schauder basic or an FDD. It also seems to be unknown whether for every Banach space X with X^* separable and X^{**} non-separable, there exists a subspace Y of X with a Schauder basis such that Y^{**} is non-separable.

(2) The family $\{y^{**}_\sigma : \sigma \in 2^{\mathbb{N}}\}$ obtained in Theorem 53 cannot be chosen to be unconditional, as the examples of non-separable HI spaces show (see [AAT], [AT]). However, all these second dual, non-separable HI spaces have quotients with separable kernel which contain unconditional families of the cardinality of the continuum. The following problem is motivated by the previous observation.

PROBLEM. Let X be a separable Banach space with X^{**} non-separable. Does there exist a quotient Y of X^{**} containing an unconditional family of size $|X^{**}|$?

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