Contents

Abstract

We prove an interpolatory estimate linking the directional Haar projection $P^{(\varepsilon)}$ to the Riesz transform in the context of Bochner–Lebesgue spaces $L^p(\mathbb{R}^n; X)$, $1 < p < \infty$, provided X is a UMD-space. If $\varepsilon_{i_0} = 1$, the result is the inequality

$$\|P^{(\varepsilon)}u\|_{L^p(\mathbb{R}^n;X)} \leq C\|u\|^{1/\mathcal{T}}_{L^p(\mathbb{R}^n;X)}\|R_{i_0}u\|^{1-1/\mathcal{T}}_{L^p(\mathbb{R}^n;X)}, \tag{1}$$

where the constant C depends only on n, p, the UMD-constant of X and the Rademacher type $\mathcal{T}$ of $L^p(\mathbb{R}^n; X)$.

In order to obtain the interpolatory result (1) we analyze stripe operators S_λ, $\lambda \geq 0$, which are used as basic building blocks to dominate the directional Haar projection. The main result on stripe operators is the estimate

$$\|S_\lambda u\|_{L^p(\mathbb{R}^n;X)} \leq C \cdot 2^{-\lambda/\mathcal{C}}\|u\|_{L^p(\mathbb{R}^n;X)}, \tag{2}$$

where the constant C depends only on n, p, the UMD-constant of X and the Rademacher cotype $\mathcal{C}$ of $L^p(\mathbb{R}^n; X)$. The proof of (2) relies on a uniform bound for the shift operators T_m, $0 \leq m < 2^\lambda$, acting on the image of S_λ.

Mainly based upon inequality (1), we prove a vector-valued result on sequential weak lower semicontinuity of integrals of the form

$$u \mapsto \int f(u)\, dx,$$

where $f : X^n \to \mathbb{R}^+$ is separately convex satisfying $f(x) \leq C(1 + \|x\|_{X^n})^p$.

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1. Main results

1.1. A brief history of development. The calculus of variations, in particular the theory of compensated compactness, has long been a source of hard problems in harmonic analysis. One development started with the work of F. Murat and L. Tartar, and especially in [Tar78, Tar79, Tar83, Tar84, Tar90, Tar93] and [Mur78, Mur79, Mur81]. Their approach exploited L^p-boundedness of Fourier multipliers to obtain sequential weak lower semicontinuity of integrals such as

$$(u, v) \mapsto \int f(x, u(x), v(x))\, dx.$$

The crucial hypothesis on the integrand f was the so-called constant rank condition. In [Mül99], S. Müller obtained analogous results for separately convex integrands f for which the constant rank condition is not satisfied. The method introduced by S. Müller [Mül99] consists of time-frequency localization in combination with the modern Calderón–Zygmund theory. The result is the following. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be separately convex satisfying $0 \le f(z) \le C(1 + |z|^2)$, let $U \subset \mathbb{R}^2$ be open and suppose that

$$\begin{aligned} u_j &\rightharpoonup u_\infty, & v_j &\rightharpoonup v_\infty, && \text{in } L^2_{\text{loc}}(U),\\ \partial_2 u_j &\rightharpoonup \partial_2 u_\infty, & \partial_1 v_j &\rightharpoonup \partial_1 v_\infty, && \text{in } H^{-1}_{\text{loc}}(U). \end{aligned}$$

Then for every open $V \subset U$,

$$\int_V f(u_\infty, v_\infty) \le \liminf_{j\to\infty} \int_V f(u_j, v_j)\, dx. \tag{1.1}$$

The basis of the result were interpolatory estimates for the directional Haar projection $P^{(\varepsilon)}$, $\varepsilon \in \{0, 1\}^n \setminus \{0\}$, defined below. Let $u \in L^p(\mathbb{R}^n)$ with $n \ge 2$ and $1 < p < \infty$ be fixed. Then $P^{(\varepsilon)} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is given by

$$P^{(\varepsilon)} u = \sum_{Q\in\mathscr{Q}} \langle u, h_Q^{(\varepsilon)}\rangle\, h_Q^{(\varepsilon)}\, |Q|^{-1},$$

where $h_Q^{(\varepsilon)}$ denote Haar functions, which are briefly discussed in Section 2. The crucial interpolatory estimate in [Mül99] is then

$$\|P^{(\varepsilon)} u\|_{L^2(\mathbb{R}^2)} \le C \|u\|^{1/2}_{L^2(\mathbb{R}^2)} \|R_{i_0} u\|^{1-1/2}_{L^2(\mathbb{R}^2)}, \tag{1.2}$$

where R_{i_0} denotes the Riesz transform in direction $i_0 \in \{1, 2\}$, $0 \ne (\varepsilon_1, \varepsilon_2) = \varepsilon \in \{0, 1\}^2$, and $\varepsilon_{i_0} = 1$. The formal definition of R_{i_0} is supplied in Section 2.

This inequality was later extended by J. Lee, P. F. X. Müller and S. Müller [LMM11] for arbitrary $1 < p < \infty$ and dimension $n \ge 2$ to

$$\|P^{(\varepsilon)} u\|_{L^p(\mathbb{R}^n)} \le C \|u\|^{1/\min(2,p)}_{L^p(\mathbb{R}^n)} \|R_{i_0} u\|^{1-1/\min(2,p)}_{L^p(\mathbb{R}^n)}, \tag{1.3}$$

where $\varepsilon \in \{0,1\}^n \setminus \{0\}$, $\varepsilon_{i_0} = 1$. If we rewrite inequality (1.3) using the notion of type $\mathcal{T}(L^p(\mathbb{R}^n)) = \min(2,p)$, it reads

$$\|P^{(\varepsilon)}u\|_{L^p(\mathbb{R}^n)} \le C\|u\|_{L^p(\mathbb{R}^n)}^{1/\mathcal{T}(L^p(\mathbb{R}^n))}\|R_{i_0}u\|_{L^p(\mathbb{R}^n)}^{1-1/\mathcal{T}(L^p(\mathbb{R}^n))}. \tag{1.4}$$

It is in this form that (1.3) will be given a vector-valued extension; see estimate (1.5).

The proofs of (1.2) and (1.3) are based on two consecutive time-frequency localizations of the operator $P^{(\varepsilon)}$ as well as on Littlewood–Paley and wavelet expansions. The L^p-estimates in [LMM11] were obtained by systematically interpolating between the spaces H^1, L^2 and BMO. In the present paper we obtain vector-valued extensions of (1.4) working directly on $L^p(\mathbb{R}^n;X)$, avoiding interpolation and using martingale methods instead.

1.2. The main results. S. Müller asks in [Mül99] whether it is possible to obtain (1.2) in such a way that the original time-frequency decompositions are replaced by the *canonical martingale decomposition* of T. Figiel (see [Fig90]). This paper provides an affirmative answer to this question. The details of the decomposition are worked out in Section 4. This allows us to extend the interpolatory estimate (1.4) to the Bochner–Lebesgue space $L^p_X(\mathbb{R}^n)$, provided X satisfies the UMD-property.

Let $1 < p < \infty$, and let X be a UMD-space (see [Mau75]) with type $\mathcal{T}(X)$. It is well known that X has non-trivial type $\mathcal{T}(X) > 1$ and cotype $\mathcal{C}(X) < \infty$ (see [Mau75], [MP76] and [Ald79]). Consequently, $L^p_X(\mathbb{R}^n)$ has non-trivial type $\mathcal{T}(L^p_X(\mathbb{R}^n))$ and cotype given by $\min(p,\mathcal{T}(X))$ and $\max(p,\mathcal{C}(X))$, respectively (see [LT91, Section 9.2, p. 247]).

We will now briefly give definitions of the objects immediately involved in the formulation of the main theorems below. Consider the collection of dyadic intervals at scale $j \in \mathbb{Z}$ given by

$$\mathscr{D}_j = \{[2^{-j}k, 2^{-j}(k+1)[\, : k \in \mathbb{Z}\},$$

and the collection of the dyadic intervals

$$\mathscr{D} = \bigcup_{j\in\mathbb{Z}} \mathscr{D}_j.$$

Let h_I denote the L^∞-*normalized Haar function*, that is,

$$h_I = \mathbb{1}_{I_0} - \mathbb{1}_{I_1} \qquad \text{for all } I \in \mathscr{D},$$

where $I_0 \in \mathscr{D}$ denotes the left and $I_1 \in \mathscr{D}$ the right half of I. The *Haar system* $\{h_I : I \in \mathscr{D}\}$ is an unconditional basis for $L^p_X(\mathbb{R})$, $1 < p < \infty$, if X has the UMD-property.

In dimensions $n \ge 2$ one can obtain an unconditional basis for $L^p_X(\mathbb{R}^n)$, $1 < p < \infty$, if X is a UMD-space, as follows. For every $\varepsilon = (\varepsilon_1,\dots,\varepsilon_n) \in \{0,1\}^n$, $\varepsilon \ne 0$, define

$$h_Q^{(\varepsilon)}(t) = \prod_{i=1}^n h_{I_i}^{\varepsilon_i}(t_i),$$

where $t = (t_1,\dots,t_n) \in \mathbb{R}^n$, $Q = I_1 \times \cdots \times I_n$, $|I_1| = \cdots = |I_n|$, $I_i \in \mathscr{D}$, and $h_{I_i}^{\varepsilon_i}$ is the function

$$h_{I_i}^{\varepsilon_i} = \begin{cases} h_{I_i}, & \varepsilon_i = 1,\\ \mathbb{1}_{I_i}, & \varepsilon_i = 0.\end{cases}$$

We denote the collection of all such cubes Q by $\mathscr{Q}$, that is,

$$\mathscr{Q} = \{I_1 \times \cdots \times I_n : I_i \in \mathscr{D},\ 1 \le i \le n,\ |I_1| = \cdots = |I_n|\}.$$

For a dyadic cube $Q \in \mathscr{Q}$, the side length of Q is

$$\operatorname{sidelength}(Q) = |I_1|.$$

Let X be a UMD-space, $n \ge 2$ and $1 < p < \infty$. Then the *directional Haar projection* $P^{(\varepsilon)} : L^p_X(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)$ is given by

$$P^{(\varepsilon)} u = \sum_{Q \in \mathscr{Q}} \langle u, h_Q^{(\varepsilon)} \rangle h_Q^{(\varepsilon)} |Q|^{-1}$$

for all $u \in L^p_X(\mathbb{R}^n)$. For details see (4.1).

The main inequality of this paper reads as follows.

THEOREM 1.1. *Let $1 < p < \infty$, and let X be a Banach space with the UMD-property. Denote by $\mathcal{T}(L^p_X(\mathbb{R}^n)) > 1$ the type of $L^p_X(\mathbb{R}^n)$. Let*

$$\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{0,1\}^n \quad \text{with} \quad \varepsilon_{i_0} = 1,$$

and let R_{i_0} denote the Riesz transform in direction i_0 (see (2.10)). Then for every u in $L^p_X(\mathbb{R}^n)$ we have

$$\|P^{(\varepsilon)} u\|_{L^p_X(\mathbb{R}^n)} \le C \|u\|^{1/\mathcal{T}(L^p_X(\mathbb{R}^n))}_{L^p_X(\mathbb{R}^n)} \|R_{i_0} u\|^{1-1/\mathcal{T}(L^p_X(\mathbb{R}^n))}_{L^p_X(\mathbb{R}^n)}, \tag{1.5}$$

where C depends only on n, p, the UMD-constant of X and the type $\mathcal{T}(L^p_X(\mathbb{R}^n))$.

For the proof of Theorem 1.1 see Subsection 1.4.

The L^p-estimates of Theorem 1.1 are obtained directly from estimates of rearrangement operators avoiding the detour to the endpoint spaces H^1 and BMO. The basic tools for the proof of the above theorem are vector-valued estimates of stripe operators S_λ, developed in Section 3. A careful examination of shift operators acting on dyadic stripes will be crucial. We also point out that the L^2-estimates for the stripe operators are obvious in the scalar case, but form the main obstacle in the vector-valued case.

The vector-valued interpolatory estimate (1.5) allows us to extend the scalar-valued result (see inequality (1.1)) on weak lower semi-continuity to the following vector-valued result.

THEOREM 1.2. *Let E and X be Banach spaces, assume that X has the UMD-property, and let $J : E \to X$ be a compact operator. Let $1 < p < \infty$, and consider the differential operator $\mathcal{A}_0 : L^p(\mathbb{R}^n; X^n) \to W^{-1,p}(\mathbb{R}^n; X^n \times X^n)$ given by*

$$(\mathcal{A}_0(u))_{i,j} = \begin{cases} \partial_i u^{(j)}, & i \ne j, \\ 0, & i = j, \end{cases} \tag{1.6}$$

where $u = (u^{(j)})_{j=1}^n$. Assume the function $f : X^n \to \mathbb{R}$ is separately convex and satisfies

$$0 \le f(x) \le C(1 + \|x\|_{X^n})^p \tag{1.7}$$

for all $x \in X^n$*, where* $C > 0$ *does not depend on* x*. Let the sequence* $\{v_r\} \subset L(\mathbb{R}^n; E^n)$ *be such that*

$$v_r \to v \qquad \text{weakly in } L^p(\mathbb{R}^n; E^n), \tag{1.8}$$

$$\mathcal{A}_0(Jv_r) \qquad \text{is precompact in } W^{-1,p}(\mathbb{R}^n; X^n \times X^n). \tag{1.9}$$

Then

$$\liminf_{r\to\infty} \int_{\mathbb{R}^n} f(Jv_r(x))\varphi(x)\,dx \geq \int_{\mathbb{R}^n} f(Jv(x))\varphi(x)\,dx \tag{1.10}$$

for all $\varphi \in C_0^+(\mathbb{R}^n)$*.*

The proof of Theorem 1.2 may be found in Subsection 1.5.

Remark 1.3. Theorem 1.2 remains valid if we replace the hypothesis that J is compact by J being Dunford–Pettis.

1.3. The main inequality and interpolation. The interpolatory main result, Theorem 1.1, concerns interpolation of operators, linking the identity map, the Riesz transforms and the directional Haar projection. We would now like to give a reformulation of Theorem 1.1 which places it in the context of structure theorems for the so-called K-method of interpolation spaces. To this end, we first introduce the K-functional, cite the relevant structure theorem (Proposition 1.4) and apply it to inequality (1.5).

Define the K*-functional*

$$K(f,t) = \inf\{\|g\|_{E_0} + t\|h\|_{E_1} : f = g + h,\ g \in E_0,\ h \in E_1\}$$

for all $f \in E_0 + E_1$ and $t > 0$. For $0 < \theta < 1$, the interpolation space $(E_0, E_1)_{\theta,1}$ is given by

$$(E_0, E_1)_{\theta,1} = \{f : f \in E_0 + E_1,\ \|f\|_{\theta,1} < \infty\},$$

where

$$\|f\|_{\theta,1} = \int_0^\infty t^{-\theta} K(f,t)\,\frac{dt}{t}.$$

The following proposition interprets interpolatory estimates such as the ones obtained in Theorem 1.1 in terms of continuity of the identity map between interpolation spaces. It is a result of general interpolation theory (see [BS88, Proposition 2.10, Chapter 5]).

Proposition 1.4. *Let* (E_0, E_1) *be a compatible couple and suppose* $0 < \theta < 1$*. Then the estimate*

$$\|f\|_E \leq C\|f\|_{\theta,1} \tag{1.11}$$

holds for some constant C *and all* f *in* $(E_0, E_1)_{\theta,1}$ *if and only if*

$$\|f\|_E \leq C\|f\|_{E_0}^{1-\theta}\|f\|_{E_1}^{\theta}$$

for some constant C *and for all* f *in* $E_0 \cap E_1$*.*

In the following we will specify the spaces E, E_0 and E_1 so that the two equivalent conditions of the above proposition match precisely the assertions of Theorem 1.1.

Application of Proposition 1.4 to Theorem 1.1. Let $0 \neq \varepsilon \in \{0,1\}^n$ with $\varepsilon_{i_0} = 1$ be fixed, and let

$$R_{i_0} : L^p_X(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)$$

denote the Riesz transform defined in Section 2. If we define the Banach spaces

$$\begin{aligned} E &= L^p_X(\mathbb{R}^n)/\ker(P^{(\varepsilon)}), & \|u + \ker(P^{(\varepsilon)})\|_E &= \|P^{(\varepsilon)}u\|_{L^p_X(\mathbb{R}^n)}, \\ E_0 &= L^p_X(\mathbb{R}^n), & \|u\|_{E_0} &= \|u\|_{L^p_X(\mathbb{R}^n)}, \\ E_1 &= L^p_X(\mathbb{R}^n)/\ker(R_{i_0}), & \|u + \ker(R_{i_0})\|_{E_1} &= \|R_{i_0}u\|_{L^p_X(\mathbb{R}^n)}, \end{aligned}$$

then Proposition 1.4 together with Theorem 1.1 yields

$$(E_0, E_1)_{\theta,1} \hookrightarrow E.$$

In other words, there exists a constant $C > 0$ such that

$$\|u\|_E \le C\|u\|_{\theta,1}$$

for all $u \in (E_0, E_1)_{\theta,1}$.

We summarize this brief discussion in

THEOREM 1.5. *Let $1 < p < \infty$, and let X be a Banach space with the UMD-property. Denote by $\mathcal{T}(L^p_X(\mathbb{R}^n))$ the (non-trivial) type of $L^p_X(\mathbb{R}^n)$. Furthermore, let*

$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{0,1\}^n \quad \textit{with} \quad \varepsilon_{i_0} = 1,$$

and define

$$\begin{aligned} E_0 &= L^p_X(\mathbb{R}^n), & \|u\|_{E_0} &= \|u\|_{L^p_X(\mathbb{R}^n)}, \\ E_1 &= L^p_X(\mathbb{R}^n)/\ker(R_{i_0}), & \|u + \ker(R_{i_0})\|_{E_1} &= \|R_{i_0}u\|_{L^p_X(\mathbb{R}^n)}. \end{aligned}$$

Then there exists a constant $C > 0$ such that

$$\|P^{(\varepsilon)}u\|_{L^p_X(\mathbb{R}^n)} \le C\|u\|_{\theta,1} \tag{1.12}$$

for all $u \in L^p_X(\mathbb{R}^n)$, where $\theta = 1 - 1/\mathcal{T}(L^p_X)$.

The connection with general interpolation theory was pointed out by S. Geiss.

1.4. Proof of Theorem 1.1. The subsequent proof of Theorem 1.1 merges the vector-valued results of this paper, particularly Theorems 4.7 and 4.5. Apart from replacing the scalar-valued estimates with our vector-valued analogues, we repeat the scalar-valued proof in [LMM11].

Before we give the proof we shall discuss the objects involved. Recall that

$$P^{(\varepsilon)}u = \sum_{Q \in \mathcal{Q}} \langle u, h_Q^{(\varepsilon)} \rangle h_Q^{(\varepsilon)} |Q|^{-1}$$

for all $u \in L^p_X(\mathbb{R}^n)$. Now choose $b \in C_c^\infty(]0,1[^n)$ such that

$$\int b(x)\,dx = 1 \quad \text{and} \quad \int x_i b(x_1, \dots, x_i, \dots, x_n)\,dx_i = 0$$

for all $1 \le i \le n$. For every integer l define

$$\Delta_l u = u * d_l, \quad \text{where} \quad d_l(x) = 2^{ln} d(2^l x) \quad \text{and} \quad d(x) = 2^n b(2x) - b(x).$$

If $\mathscr{Q}_j \subset \mathscr{Q}$ denotes the collection of all dyadic cubes having measure 2^{-jn}, then

$$P_l^{(\varepsilon)} u = \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathscr{Q}_j} \langle u, \Delta_{j+l}(h_Q^{(\varepsilon)})\rangle h_Q^{(\varepsilon)} |Q|^{-1}.$$

Note that

$$P^{(\varepsilon)} = \sum_{l\in\mathbb{Z}} P_l^{(\varepsilon)},$$

and define

$$P_-^{(\varepsilon)} = \sum_{l<0} P_l^{(\varepsilon)}.$$

For details on the above definitions see Subsection 4.1.

Proof of Theorem 1.1. Within this proof we shall abbreviate $L_X^p(\mathbb{R}^n)$ by L_X^p.

First, define $M \in \mathbb{N}$ by

$$2^{M-1} \le \frac{\|R_{i_0} : L_X^p \to L_X^p\| \, \|u\|_{L_X^p}}{\|R_{i_0} u\|_{L_X^p}} \le 2^M. \tag{1.13}$$

Second, we use decomposition (4.2) and (4.8), that is,

$$P^{(\varepsilon)} = P_-^{(\varepsilon)} + \sum_{l\ge 0} P_l^{(\varepsilon)},$$

and observe that

$$\|P^{(\varepsilon)} u\|_{L_X^p} \le \|P_-^{(\varepsilon)} R_{i_0}^{-1} R_{i_0} u\|_{L_X^p} + \sum_{l=0}^{M} \|P_l^{(\varepsilon)} R_{i_0}^{-1} R_{i_0} u\|_{L_X^p} + \sum_{l=M}^{\infty} \|P_l^{(\varepsilon)} u\|_{L_X^p}.$$

If we apply Theorem 4.7 to the first two sums, and inequality (4.45) in Theorem 4.5 to the latter sum, we get

$$\|P_-^{(\varepsilon)} R_{i_0}^{-1} R_{i_0} u\|_{L_X^p} \lesssim \|R_{i_0} u\|_{L_X^p}, \qquad \|P_l^{(\varepsilon)} R_{i_0}^{-1} R_{i_0} u\|_{L_X^p} \lesssim 2^{l/\mathcal{T}(L_X^p)} \|R_{i_0} u\|_{L_X^p},$$

and

$$\|P_l^{(\varepsilon)} u\|_{L_X^p} \lesssim 2^{-l(1-1/\mathcal{T}(L_X^p))} \|u\|_{L_X^p}.$$

Thus, we can dominate $\|P^{(\varepsilon)} u\|_{L_X^p}$ by a constant multiple of

$$\|R_{i_0} u\|_{L_X^p} + \sum_{l=0}^{M} 2^{l/\mathcal{T}(L_X^p)} \|R_{i_0} u\|_{L_X^p} + \sum_{l=M}^{\infty} 2^{-l(1-1/\mathcal{T}(L_X^p))} \|u\|_{L_X^p}.$$

Evaluating the geometric series yields

$$\|P^{(\varepsilon)} u\|_{L_X^p} \lesssim 2^{M/\mathcal{T}(L_X^p)} \|R_{i_0} u\|_{L_X^p} + 2^{-M(1-1/\mathcal{T}(L_X^p))} \|u\|_{L_X^p},$$

and plugging in M concludes the proof. ∎

1.5. Proof of Theorem 1.2. Apart from using vector-valued analogues dealing with the technicalities, the subsequent proof is similar to the scalar-valued case (see [Mül99] and [LMM11]).

We will divide the proof into four steps. Define the projection $P : L^p(\mathbb{R}^n; X^n) \to L^p(\mathbb{R}^n; X^n)$ by

$$P(v) = (P^{(e_1)} v^{(1)}, \dots, P^{(e_n)} v^{(n)}),$$

where $v = (v^{(j)})_{j=1}^n$, and

$$P^{(\varepsilon)}u = \sum_{Q \in \mathscr{Q}} \langle u, h_Q^{(\varepsilon)} \rangle h_Q^{(\varepsilon)} |Q|^{-1}$$

for all $u \in L_X^p(\mathbb{R}^n; X)$ and $\varepsilon \in \{0,1\}^n \setminus \{0\}$.

In the first step the setting is as follows. The operator $J : E \to X$ is compact, $w_r \to 0$ weakly in $L^p(\mathbb{R}^n; E^n)$, and $\{\mathcal{A}_0(Jw_r)\}_r$ is precompact in the Sobolev space $W^{-1,p}(\mathbb{R}^n; X^n \times X^n)$. It is here that we will see how the interpolatory estimate (1.5) is used to obtain the estimate

$$\lim_{r\to\infty} \|\psi_k \cdot Jw_r - P(\psi_k \cdot Jw_r)\|_{L^p(\mathbb{R}^n;X^n)} \le C\frac{1}{k^\theta}$$

for all positive integers k and some $0 < \theta < 1$. The function ψ is a smooth cut-off function and $\psi_k(x) = \psi(x/k)$, $x \in \mathbb{R}^n$.

In the second stage of the proof we will show that for our separately convex function $f : X^n \to \mathbb{R}$ satisfying the growth condition

$$0 \le f(x) \le C(1 + \|x\|_{X^n})^p, \qquad x \in X^n,$$

Jensen's inequality holds on the image of P, that is,

$$f(\mathbb{E}_M(Pv)) \le \mathbb{E}_M(f(Pv))$$

for all $v \in L^p(\mathbb{R}^n; X^n)$, where

$$\mathbb{E}_M\, u = \sum_{Q \in \mathscr{Q}_M} \left(\frac{1}{|Q|} \int_Q u(x)\, dx \right) \cdot 1_Q$$

for all $u \in L^p(\mathbb{R}^n; X^n)$. Recall that $\mathscr{Q}_M$ is the collection of dyadic cubes having measure 2^{-Mn}.

In the third step we will obtain our desired result, that is, the weak lower semicontinuity

$$\liminf_{r\to\infty} \int_{\mathbb{R}^n} f(Jv_r)\varphi\, dx \ge \int_{\mathbb{R}^n} f(Jv)\varphi\, dx,$$

assuming that v is a finite sum of Haar functions and φ has support in $(0,1)^n$.

The restrictions on v and φ will be lifted in step four.

Proof of Theorem 1.2

Step 1. Within this proof we shall use the abbreviations $W^{-1,p}(F)$ for $W^{-1,p}(\mathbb{R}^n; F)$ and $L^p(F)$ for $L^p(\mathbb{R}^n; F)$, where F is a Banach space.

Choose a smooth cut-off function $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $0 \le \psi(x) \le 1$ for all $x \in \mathbb{R}^n$ and

$$\psi(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| \ge 2. \end{cases}$$

For every positive integer k, we set $\psi_k(x) = \psi(x/k)$ for all $x \in \mathbb{R}^n$. Define the projection $P : L^p(X^n) \to L^p(X^n)$ by

$$P(v) = (P^{(e_1)}v^{(1)}, \ldots, P^{(e_n)}v^{(n)}),$$

where $v = (v^{(j)})_{j=1}^n$. We will show that whenever $w_r \to 0$ weakly in $L^p(E^n)$ and $\{\mathcal{A}_0(Jw_r)\}$ is precompact in $W^{-1,p}(X^n \times X^n)$, then

$$\lim_{r\to\infty} \|\psi_k \cdot Jw_r - P(\psi_k \cdot Jw_r)\|_{L^p(X^n)} \le C\frac{1}{k^\theta} \tag{1.14}$$

for all positive integers k and some $0 < \theta < 1$.

To this end, let w_r, converging weakly to zero in $L^p(E^n)$, be fixed. Then, since J is bounded, $Jw_r \to 0$ weakly in $L^p(X^n)$. Note that since $\{\mathcal{A}_0(Jw_r)\}$ is precompact in $W^{-1,p}(X^n \times X^n)$, the operator $\mathcal{A}_0 : L^p(X^n) \to W^{-1,p}(X^n \times X^n)$ being bounded implies

$$\|\mathcal{A}_0(Jw_r)\|_{W^{-1,p}(X^n\times X^n)} \to 0 \quad \text{as } r \to \infty.$$

This means that

$$\lim_{r\to\infty} \|\partial_i(Jw_r^{(j)})\|_{W^{-1,p}(X)} = 0 \quad \text{for all } i \ne j. \tag{1.15}$$

We will prove (1.14) using the interpolatory main result Theorem 1.1. First, with k fixed, we use Theorem 5.5 and the remark thereafter to obtain

$$R_i(\psi_k \cdot Jw_r^{(j)}) = (R_i T_1^{(k)})(w_r^{(j)}) + T_2\big(\mathcal{F}^{-1}(\langle\xi\rangle^{-1}\xi_i \cdot \mathcal{F}(\psi_k \cdot Jw_r^{(j)}))\big), \qquad i \ne j,$$

where $T_1^{(k)} : L^p(E) \to L^p(X)$ is compact and $T_2 : L^p(X) \to L^p(X)$ is bounded. One can see from the proof of Theorem 5.5 that, in fact, T_2 does not depend on k. From the identity above it follows immediately that

$$\|R_i(\psi_k \cdot Jw_r^{(j)})\|_{L^p(X)} \le \|R_i T_1^{(k)}(w_r^{(j)})\|_{L^p(X)} + C\|\partial_i(\psi_k \cdot Jw_r^{(j)})\|_{W^{-1,p}(X)}. \tag{1.16}$$

Since X has the UMD-property, we may use [McC84, Theorem 1.1] and infer that R_i is bounded, and therefore $R_i T_1^{(k)}$ is compact. Since $w_r^{(j)} \to 0$ weakly in $L^p(E)$, we obtain

$$\lim_{r\to\infty} \|R_i T_1^{(k)}(w_r^{(j)})\|_{L^p(X)} = 0 \quad \text{for all } k \text{ and } i \ne j. \tag{1.17}$$

To estimate the second term we apply Theorem 5.4, and since $\sup_r \|Jw_r^{(j)}\|_{W^{-1,p}(X)} < \infty$, we infer that

$$\|\partial_i(\psi_k \cdot Jw_r^{(j)})\|_{W^{-1,p}(X)} \le C\frac{1}{k} + C\|\partial_i(Jw_r^{(j)})\|_{W^{-1,p}(X)}. \tag{1.18}$$

Combining (1.16) with (1.18), and letting $r \to \infty$, we deduce in view of (1.15) and (1.17) that

$$\lim_{r\to\infty} \|R_i(\psi_k \cdot Jw_r^{(j)})\|_{L^p(X)} \le C\frac{1}{k} \quad \text{for all } k \text{ and } i \ne j. \tag{1.19}$$

Since $u = \sum_{\varepsilon\ne 0} P^{(\varepsilon)}u$ for all $u \in L^p(X)$, we have

$$\psi_k \cdot Jw_r^{(j)} - P^{(e_j)}(\psi_k \cdot Jw_r^{(j)}) = \sum_{0\ne\varepsilon\ne e_j} P^{(\varepsilon)}(\psi_k \cdot Jw_r^{(j)}) \quad \text{for all } k \text{ and } 1 \le j \le n.$$

Hence, we can apply the interpolatory estimate (1.5) of Theorem 1.1 to each component of $\psi_k \cdot Jw_r - P(\psi_k \cdot Jw_r)$ and obtain

$$\|\psi_k \cdot Jw_r - P(\psi_k \cdot Jw_r)\|_{L^p(X^n)} \le C\sum_j \sum_{0\ne\varepsilon\ne e_j} \|\psi_k \cdot Jw_r^{(j)}\|_{L^p(X)}^{1-\theta} \|R_{j^*}(\psi_k \cdot Jw_r^{(j)})\|_{L^p(X)}^{\theta},$$

where $0 < \theta < 1$ and j^* is some index in $\{1, \dots, n\} \setminus \{j\}$. The interpolatory estimate together with (1.19) yields the desired result (1.14), concluding the first step of the proof.

STEP 2. We will prove the following version of Jensen's inequality for separately convex functions f on the range of P:

$$f(\mathbb{E}_M(Pv)) \leq \mathbb{E}_M(f(Pv)) \tag{1.20}$$

for all $v \in L^p(\mathbb{R}^n; X^n)$, where

$$\mathbb{E}_M\, u = \sum_{Q \in \mathscr{Q}_M} \left(\frac{1}{|Q|} \int_Q u(x)\, dx \right) \cdot \mathtt{1}_Q$$

for all $u \in L^p(\mathbb{R}^n; X^n)$. Recall that $\mathscr{Q}_M$ is the collection of dyadic cubes having measure 2^{-Mn}.

First, we will show that

$$f\Big(\int_{[0,1]^n} P(v)\, dx \Big) \leq \int_{[0,1]^n} f(P(v))\, dx. \tag{1.21}$$

Then rescaling and translating (1.21) yields the desired inequality (1.20).

Define the *truncated Haar projections*

$$P_k^{(\varepsilon)} u = \sum_{j=-\infty}^{k} \sum_{Q \in \mathscr{Q}_j} \langle u, h_Q^{(\varepsilon)} \rangle h_Q^{(\varepsilon)} |Q|^{-1}$$

for every $u \in L^p(\mathbb{R}^n; X)$, $k \in \mathbb{Z}$, and furthermore

$$P_k v = (P_k^{(e_1)} v^{(1)}, \dots, P_k^{(e_n)} v^{(n)})$$

for all $v \in L^p(\mathbb{R}^n; X^n)$, $k \in \mathbb{Z}$. Note that $P_k \to P$ pointwise in $L^p(\mathbb{R}^n; X^n)$.

Let $k \geq 0$. Then

$$\begin{aligned} \int_{[0,1]^n} f(P_k(v))\, dx &= \sum_{Q \in \mathscr{Q}_k | [0,1]^n} \int_Q f\big((P_k^{(e_j)}(v^{(j)}))_{j=1}^n \big)\, dx \\ &= \sum_{Q \in \mathscr{Q}_k | [0,1]^n} \int_Q f\big((P_{k-1}^{(e_j)}(v^{(j)}) + c_Q^{(j)} h_Q^{(e_j)})_{j=1}^n \big)\, dx. \end{aligned}$$

Observe that $(P_{k-1}^{(e_j)}(v^{(j)}))|Q = a_Q^{(j)}$ is constant, and $h_Q^{(e_j)}(x) = h_Q^{(e_j)}(x_j)$ for all $x \in Q$ and $1 \leq j \leq n$. Since f is separately convex, we apply Jensen's inequality to each direction e_j, $1 \leq j \leq n$, which yields

$$\begin{aligned} \int_{[0,1]^n} f(P_k(v))\, dx &\geq \sum_{Q \in \mathscr{Q}_k | [0,1]^n} |Q| \cdot f\bigg(\Big(\frac{1}{|I_Q^{(j)}|} \int_{I_Q^{(j)}} (a_Q^{(j)} + c_Q^{(j)} h_Q^{(e_j)}(x_j))\, dx_j \Big)_{j=1}^n \bigg) \\ &= \sum_{Q \in \mathscr{Q}_k | [0,1]^n} |Q| \cdot f\big((P_{k-1}^{(e_j)}(v^{(j)}))_{j=1}^n \big), \end{aligned}$$

where $\prod_{j=1}^n I_Q^{(j)} = Q$. Hence,

$$\int_{[0,1]^n} f(P_k(v))\, dx \geq \int_{[0,1]^n} f(P_{k-1}(v))\, dx$$

for all $k \geq 0$. Since $P_{-1}(v)$ is constant on $[0,1]^n$, we certainly have

$$\int_{[0,1]^n} f(P_{-1}(v))\, dx = f\Big(\int_{[0,1]^n} P_{-1}(v)\, dx \Big),$$

so by induction on $k \geq 0$ we obtain

$$\int_{[0,1]^n} f(P_k(v))\,dx \geq f\Big(\int_{[0,1]^n} P_{-1}(v)\,dx\Big)$$

for all $k \geq 0$. First, we use the Lipschitz estimate for f in the Appendix (see Theorem 5.1) and get

$$\begin{aligned}\Big|\int_{[0,1]^n} f(P(v))\,dx - \int_{[0,1]^n} f(P_k(v))\,dx\Big| &\\ &\leq C\int_{[0,1]^n}\big(1+\|f(Pv)\|_{X^n}+\|f(P_kv)\|_{X^n}\big)^{(p-1)}\|(P-P_k)v\|_{X^n}\,dx\\ &\leq C_v\|(P-P_k)v\|_{L^p_{X^n}(\mathbb{R}^n)}\end{aligned}$$

for all $k \in \mathbb{Z}$. Second, note that $\int_{[0,1]^n} P_{-1}(v)\,dx = \int_{[0,1]^n} P(v)\,dx$, thus, letting $k \to \infty$, the latter two inequalities imply estimate (1.21).

As mentioned above, inequality (1.20) follows by rescaling and translating (1.21).

Step 3. The hypothesis in Theorem 1.2 on the sequence $\{v_r\} \subset L(\mathbb{R}^n; E^n)$ is that

$$\begin{aligned}&v_r \to v && \text{weakly in } L^p(\mathbb{R}^n; E^n),\\ &\mathcal{A}_0(Jv_r) && \text{is precompact in } W^{-1,p}(\mathbb{R}^n; X^n \times X^n).\end{aligned}$$

In this step of the proof we will additionally assume that v is a finite Haar series and $\operatorname{supp}(\varphi) \subset (0,1)^n$.

Let $\mathscr{B} \subset \mathscr{Q}$ be a finite collection of pairwise disjoint dyadic cubes such that

$$v = \sum_{Q\in\mathscr{B}} c_Q\, 1_Q. \tag{1.22}$$

Now define

$$f_Q(x) = f(x + Jc_Q) \quad \text{for all } Q \in \mathscr{Q} \text{ and } x \in \mathbb{R}^n. \tag{1.23}$$

Theorem 5.1 asserts that

$$|f_Q(x) - f_Q(y)| \leq A(n,p,c_Q)(1 + \|x\|_{X^n} + \|y\|_{X^n})^{p-1}\|x-y\|_{X^n} \tag{1.24}$$

for all $x, y \in X^n$. We shall abbreviate $A(n,p,c_Q)$ as A. If we set $w_r = v_r - v$, then since $w_r \to 0$ weakly in $L^p(\mathbb{R}^n; E^n)$ and $\{\mathcal{A}_0(Jw_r)\}_r$ is precompact in $W^{-1,p}(\mathbb{R}^n; X^n \times X^n)$, we know from (1.14) in Step 1 that

$$\lim_{r\to\infty} \|\psi_k \cdot Jw_r - P(\psi_k \cdot Jw_r)\|_{L^p(\mathbb{R}^n;X^n)} \leq C\frac{1}{k^\theta} \tag{1.25}$$

for all positive integers k and some $0 < \theta < 1$. At this point we remind the reader that ψ is a smooth cut-off function taking values in $[0,1]$ given by

$$\psi(x) = \begin{cases} 1, & |x| \leq 1,\\ 0, & |x| \geq 2,\end{cases}$$

and $\psi_k(x) = \psi(x/k)$ for all positive integers k.

Let $Q \in \mathscr{B}$ be an arbitrary dyadic cube and let $k, r \geq 1$ be fixed for now. A glance at (1.22), (1.23) and noting that $\psi_k(x) = 1$ for all $x \in \operatorname{supp}(\varphi)$ shows that

$$\int_Q f(Jv_r)\varphi\,dx = \int_Q f_Q(Jw_r)\varphi\,dx = \int_Q f_Q(\psi_k \cdot Jw_r)\varphi\,dx.$$

Now we introduce the projection P via the identity

$$\int_Q f_Q(\psi_k \cdot Jw_r)\varphi\,dx = \int_Q f_Q(P(\psi_k \cdot Jw_r))\varphi\,dx \\ + \int_Q \big(f_Q(\psi_k \cdot Jw_r) - f_Q(P(\psi_k \cdot Jw_r))\big)\varphi\,dx.$$

In view of the Lipschitz estimate (1.24), the latter term is bounded by

$$A\big\|1 + \|\psi_k \cdot Jw_r\|_{X^n} + \|P(\psi_k \cdot Jw_r)\|_{X^n}\big\|^{p-1}_{L^p(0,1)^n} \|\psi_k \cdot Jw_r - P(\psi_k \cdot Jw_r)\|_{L^p(\mathbb{R}^n;X^n)}.$$

Since $\sup_{r,k} \|\psi_k \cdot Jw_r\|_{L^p(\mathbb{R}^n;X^n)} \leq C$ for some constant C, and P maps $L^p(\mathbb{R}^n; X^n)$ boundedly into itself, we get

$$\int_Q f(Jv_r)\varphi\,dx \geq \int_Q f_Q(P(\psi_k \cdot Jw_r))\varphi\,dx - AC\|\psi_k \cdot Jw_r - P(\psi_k \cdot Jw_r)\|_{L^p(\mathbb{R}^n;X^n)}. \quad (1.26)$$

With M fixed, we introduce the conditional expectation $\mathbb{E}_M$:

$$\int_Q f_Q(P(\psi_k \cdot Jw_r))\varphi\,dx = \int_Q f_Q(P(\psi_k \cdot Jw_r))\,\mathbb{E}_M\,\varphi\,dx \\ + \int_Q f_Q(P(\psi_k \cdot Jw_r))(\varphi - \mathbb{E}_M\,\varphi)\,dx. \quad (1.27)$$

Considering that

$$\int_Q f_Q(P(\psi_k \cdot Jw_r))\,\mathbb{E}_M\,\varphi\,dx = \int_Q \mathbb{E}_M\big(f_Q(P(\psi_k \cdot Jw_r))\big)\,\mathbb{E}_M\,\varphi\,dx$$

and applying Jensen's inequality on the range of P, that is, inequality (1.20), yields

$$\int_Q f_Q(P(\psi_k \cdot Jw_r))\,\mathbb{E}_M\,\varphi\,dx \geq \int_Q f_Q\big(\mathbb{E}_M(P(\psi_k \cdot Jw_r))\big)\,\mathbb{E}_M\,\varphi\,dx.$$

Introducing $f_Q(J0)$ we obtain

$$\int_Q f_Q(P(\psi_k \cdot Jw_r))\,\mathbb{E}_M\,\varphi\,dx \\ \geq \int_Q f_Q(J0)\,\mathbb{E}_M\,\varphi\,dx + \int_Q \big(f_Q\big(\mathbb{E}_M(P(\psi_k \cdot Jw_r))\big) - f_Q(J0)\big)\,\mathbb{E}_M\,\varphi\,dx. \quad (1.28)$$

Using the Lipschitz estimate (1.24) and the boundedness of $\{\psi_k \cdot Jw_r\}_r$ in $L^p(\mathbb{R}^n; X^n)$ as we did above, we can dominate the last term of (1.28) by

$$AC\|\mathbb{E}_M\,P(\psi_k \cdot Jw_r)\|_{L^p((0,1)^n;X^n)}.$$

Combining the latter estimate with (1.26), (1.27), (1.28) and using the estimate

$f_Q(P(\psi_k \cdot Jw_r)) \le A(c_Q)(1 + \|P(\psi_k \cdot Jw_r)\|_{X^n})^p$ in the latter term of (1.27) implies

$$\int_Q f(Jv_r)\varphi\,dx \ge \int_Q f_Q(J0)\,\mathbb{E}_M\,\varphi\,dx - AC\|\mathbb{E}_M\,P(\psi_k \cdot Jw_r)\|_{L^p((0,1)^n;X^n)} - C\|\varphi - \mathbb{E}_M\,\varphi\|_\infty - AC\|\psi_k \cdot Jw_r - P(\psi_k \cdot Jw_r)\|_{L^p(\mathbb{R}^n;X^n)}. \quad (1.29)$$

Now let us consider

$$\mathbb{E}_M\,P(\psi_k \cdot Jw_r) = \sum_{2^{-Mn} < |K| < 2^{Mn}} \big(\langle \psi_k \cdot Jw_r^{(j)}, h_K^{(e_j)}\rangle h_K^{(e_j)}|K|^{-1}\big)_{j=1}^n + \sum_{|K| \ge 2^{Mn}} \big(\langle \psi_k \cdot Jw_r^{(j)}, h_K^{(e_j)}\rangle h_K^{(e_j)}|K|^{-1}\big)_{j=1}^n.$$

First, observe that $\psi_k \cdot w_r \to 0$ weakly in $L^p(\mathbb{R}^n; E^n)$ as $r \to \infty$, hence $\langle \psi_k \cdot w_r, h_K^{(e_j)}\rangle \to 0$ weakly in E^n as $r \to \infty$. The operator $J : E \to X$ is compact, and therefore

$$\|(\langle \psi_k \cdot Jw_r, h_K^{(e_j)}\rangle)_{j=1}^n\|_{X^n} \to 0 \quad \text{for all } K \text{ as } r \to \infty;$$

consequently, with M fixed,

$$\Big\| \sum_{2^{-Mn} < |K| < 2^{Mn}} \big(\langle \psi_k \cdot Jw_r^{(j)}, h_K^{(e_j)}\rangle h_K^{(e_j)}|K|^{-1}\big)_{j=1}^n \Big\|_{L^p((0,1)^n;X^n)} \to 0 \quad \text{as } r \to \infty.$$

The $L^p((0,1)^n; X^n)$ norm of the second term in $\mathbb{E}_M\,P(\psi_k \cdot Jw_r)$ is dominated by

$$\sum_{\substack{|K| \ge 2^{Mn} \\ K \supset [0,1]^n}} \|\psi_k \cdot Jw_r\|_{L^p(\mathbb{R}^n;X^n)}|K|^{-1/p} \le C \cdot 2^{-Mn/p}.$$

We now pass to our last two estimates for $\mathbb{E}_M\,P(\psi_k \cdot Jw_r)$. Plugging them into (1.29) as well as using inequality (1.25) yields

$$\liminf_{r\to\infty} \int_Q f(Jv_r)\varphi\,dx \ge \int_Q f_Q(J0)\,\mathbb{E}_M\,\varphi\,dx - C \cdot 2^{-Mn/p} - C\|\varphi - \mathbb{E}_M\,\varphi\|_{L^\infty(0,1)^n} - C\frac{1}{k^\theta}$$

for all M, k and some $0 < \theta < 1$. Letting $M \to \infty$ and $k \to \infty$, recalling (1.22), (1.23) and noting that $f_Q(J0) = f(Jv(x))$ for all $x \in Q$, we obtain

$$\liminf_{r\to\infty} \int_Q f(Jv_r)\varphi\,dx \ge \int_Q f(Jv)\varphi\,dx$$

for every $Q \in \mathscr{Q}$. Since $\mathscr{B}$ is a finite collection, summation over $Q \in \mathscr{B}$ yields

$$\liminf_{r\to\infty} \int_{\mathscr{B}^*} f(Jv_r)\varphi\,dx \ge \int_{\mathscr{B}^*} f(Jv)\varphi\,dx,$$

where $\mathscr{B}^* = \bigcup_{Q\in\mathscr{B}} Q$. Repeating the above argument with f_Q replaced by f shows that

$$\liminf_{r\to\infty} \int_{(\mathscr{B}^*)^c} f(Jv_r)\varphi\,dx \ge \int_{(\mathscr{B}^*)^c} f(Jv)\varphi\,dx.$$

Note that $w_r(x) = v_r(x)$ for all $x \in (\mathscr{B}^*)^c$. Adding the last two estimates yields

$$\liminf_{r\to\infty} \int_{\mathbb{R}^n} f(Jv_r)\varphi\,dx \ge \int_{\mathbb{R}^n} f(Jv)\varphi\,dx, \quad (1.30)$$

under the additional restrictions of v being a finite Haar series and φ having support in $(0,1)^n$.

STEP 4. First, we will lift the restriction that v is a finite Haar series, then we will dispose of the restriction that $\operatorname{supp}(\varphi) \subset (0,1)^n$.

Consider the auxiliary operators P_k, $k \geq 1$, given by

$$P_k u = \sum_{\varepsilon \neq 0} \sum_{j : |j| \leq k} \sum_{\substack{Q \in \mathscr{Q}_j \\ Q \subset B(0,k)}} \big(\langle u^{(i)}, h_Q^{(\varepsilon)} \rangle h_Q^{(\varepsilon)} |Q|^{-1} \big)_{i=1}^n \quad \text{for all } u = (u^{(1)}, \ldots, u^{(n)}),$$

where $B(0,k) = \{x \in \mathbb{R}^n : |x| \leq k\}$. Then one can see that $P_k \to \mathrm{Id}$ pointwise in $L^p(\mathbb{R}^n; X^n)$. Now let us define $v_r^k = v_r + P_k v - v$ for all r, k, and note that $v_r^k \to P_k v$ weakly in $L^p(\mathbb{R}^n; E^n)$ as $r \to \infty$. Since $P_k v$ is a finite Haar series, we know from Step 3, namely inequality (1.30) applied to v_r^k, that

$$\liminf_{r \to \infty} \int_{\mathbb{R}^n} f(J v_r^k) \varphi \, dx \geq \int_{\mathbb{R}^n} f(P_k J v) \varphi \, dx \tag{1.31}$$

for all $k \geq 1$. In view of the Lipschitz estimate (1.24) and $P_k \to \mathrm{Id}$ pointwise in $L^p(\mathbb{R}^n; X^n)$, we may lift the restriction of v being a finite Haar series, by using techniques similar to those in Step 3. To elaborate on this, fix an arbitrary $k \geq 1$ and observe

$$\begin{aligned} \liminf_{r \to \infty} \int_{\mathbb{R}^n} f(J v_r) \varphi \, dx &= \liminf_{r \to \infty} \int_{\mathbb{R}^n} f(J v_r^k) \varphi \, dx + \int_{\mathbb{R}^n} (f(J v_r) - f(J v_r^k)) \varphi \, dx \\ &\geq \int_{\mathbb{R}^n} f(P_k J v) \, dx - AC \|J v - P_k(J v)\|_{L^p((0,1)^n; X^n)}, \end{aligned}$$

where for the former term we used (1.31), and for the latter term the aforementioned Lipschitz estimate (1.24) as in Step 3. Also, note that by definition $v_r - v_r^k = v - P_k v$. Similarly, we estimate

$$\int_{\mathbb{R}^n} f(P_k J v) \, dx \geq \int_{\mathbb{R}^n} f(J v) \, dx - AC \|J v - P_k(J v)\|_{L^p((0,1)^n; X^n)},$$

so since $J v \in L^p(\mathbb{R}^n; X^n)$, combining the above two estimates and letting $k \to \infty$ we obtain

$$\liminf_{r \to \infty} \int_{\mathbb{R}^n} f(J v_r) \varphi \, dx \geq \int_{\mathbb{R}^n} f(J v) \varphi \, dx, \tag{1.32}$$

with $\operatorname{supp}(\varphi) \subset (0,1)^n$ being the only additional restriction imposed, as of now.

To lift this restriction, let $\varphi \in C_0^+(\mathbb{R}^n)$ be arbitrary and let $\eta_k \in C_0^+(0,1)^n$, $k \geq 1$, be functions such that $0 \leq \eta_k \leq 1$ and $\eta_k \to \mathbb{1}_{(0,1)^n}$ pointwise. Now extend η_k periodically to $\mathbb{R}^n$ and note that

$$\liminf_{r \to \infty} \int_{\mathbb{R}^n} f(J v_r) \varphi \, dx \geq \liminf_{r \to \infty} \int_{\mathbb{R}^n} f(J v_r) \varphi \eta_k \, dx = \sum_{|Q|=1} \liminf_{r \to \infty} \int_{\mathbb{R}^n} f(J v_r) \, \mathbb{1}_Q \varphi \eta_k \, dx$$

for all $k \geq 1$. In the above sum the Q are dyadic cubes. Since $\mathbb{1}_Q \varphi \eta_k \in C_0^+(Q)$, translating the integration domain of inequality (1.32) from $[0,1]^n$ to the dyadic cube Q yields

$$\liminf_{r \to \infty} \int_{\mathbb{R}^n} f(J v_r) \varphi \, dx \geq \int_{\mathbb{R}^n} f(J v) \varphi \eta_k \, dx$$

for all $k \geq 1$. Letting $k \to \infty$ concludes the proof of Theorem 1.2. ∎

2. Preliminaries

This brief section provides notions and tools used frequently in this work. First, we introduce the Haar system supported on dyadic cubes. Then the notions of Banach spaces with the UMD-property and type and cotype of Banach spaces are outlined. We recall Kahane's contraction principle and Bourgain's version of Stein's martingale inequality. Then we turn to the shift operators T_m, $m \in \mathbb{Z}^n$.

The Haar system. For the Haar system supported on cubes we refer the reader to [Cie87]. Consider the collection of dyadic intervals at scale $j \in \mathbb{Z}$ given by

$$\mathscr{D}_j = \{[2^{-j}k, 2^{-j}(k+1)[\, : k \in \mathbb{Z}\},$$

and the collection of the dyadic intervals

$$\mathscr{D} = \bigcup_{j \in \mathbb{Z}} \mathscr{D}_j.$$

Let h_I denote the L^∞-*normalized Haar function*, that is,

$$h_I = \mathbb{1}_{I_0} - \mathbb{1}_{I_1} \qquad \text{for all } I \in \mathscr{D},$$

where $I_0 \in \mathscr{D}$ denotes the left and $I_1 \in \mathscr{D}$ the right half of I. The Haar system $\{h_I : I \in \mathscr{D}\}$ is an unconditional basis for $L^p_X(\mathbb{R})$, $1 < p < \infty$, if X has the UMD-property.

In dimensions $n \geq 2$ one can obtain an unconditional basis for $L^p_X(\mathbb{R}^n)$, $1 < p < \infty$, if X is a UMD-space, as follows. For every $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{0,1\}^n$, $\varepsilon \neq 0$, define

$$h_Q^{(\varepsilon)}(t) = \prod_{i=1}^n h_{I_i}^{\varepsilon_i}(t_i),$$

where $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, $Q = I_1 \times \cdots \times I_n$, $|I_1| = \cdots = |I_n|$, $I_i \in \mathscr{D}$, and $h_{I_i}^{\varepsilon_i}$ is the function

$$h_{I_i}^{\varepsilon_i} = \begin{cases} h_{I_i}, & \varepsilon_i = 1, \\ \mathbb{1}_{I_i}, & \varepsilon_i = 0. \end{cases}$$

We denote the collection of all such cubes Q by $\mathscr{Q}$:

$$\mathscr{Q} = \{I_1 \times \cdots \times I_n : I_i \in \mathscr{D}, \, 1 \leq i \leq n, \, |I_1| = \cdots = |I_n|\}.$$

For a dyadic cube $Q \in \mathscr{Q}$ the side length of Q is

$$\operatorname{sidelength}(Q) = |I_1|.$$

Finally, define the dyadic predecessor map $\pi : \mathscr{Q} \to \mathscr{Q}$, where the *dyadic predecessor* $\pi(Q)$ is the unique cube $M \in \mathscr{Q}$ with $M \supset Q$ and $\operatorname{sidelength}(M) = 2 \operatorname{sidelength}(Q)$. By π^λ, $\lambda \geq 1$, we denote the composition of the function π with itself.

Banach spaces with the UMD-property. By $L^p(\Omega, \mu; X)$ we denote the space of functions with values in X, Bochner-integrable with respect to μ. If $\Omega = \mathbb{R}^n$ and μ is the Lebesgue measure $|\cdot|$ on $\mathbb{R}^n$, we write $L^p_X(\mathbb{R}^n; X) = L^p(\mathbb{R}^n, |\cdot|; X)$, if unambiguous further abbreviated as $L^p_X(\mathbb{R}^n)$ or even as L^p_X.

We say X is a *UMD-space* (i.e. a Banach space with the UMD-property) if for every X-valued martingale difference sequence $\{d_j\}_j \subset L^p(\Omega, \mu; X)$ and every choice of signs $\varepsilon_j \in \{-1, 1\}$ one has

$$\Big\|\sum_j \varepsilon_j\, d_j\Big\|_{L^p(\Omega,\mu;X)} \leq \mathscr{U}_p(X)\Big\|\sum_j d_j\Big\|_{L^p(\Omega,\mu;X)}, \tag{2.1}$$

where $\mathscr{U}_p(X)$ does not depend on ε_j or d_j. The constant $\mathscr{U}_p(X)$ is called the *UMD-constant*. We refer the reader to [Bur81].

Type and cotype. A Banach space X is said to be of *type* $\mathcal{T}$, $1 < \mathcal{T} \leq 2$, respectively of *cotype* $\mathcal{C}$, $2 \leq \mathcal{C} < \infty$, if there are constants $A(\mathcal{T}, X) > 0$ and $B(\mathcal{C}, X) > 0$ such that for every finite set $\{x_j\}_j \subset X$ we have

$$\int_0^1 \Big\|\sum_j r_j(t) x_j\Big\|_X \,dt \leq A(\mathcal{T}, X)\Big(\sum_j \|x_j\|_X^{\mathcal{T}}\Big)^{1/\mathcal{T}}, \tag{2.2}$$

respectively

$$\int_0^1 \Big\|\sum_j r_j(t) x_j\Big\|_X \,dt \geq B(\mathcal{C}, X)\Big(\sum_j \|x_j\|_X^{\mathcal{C}}\Big)^{1/\mathcal{C}}, \tag{2.3}$$

where $\{r_j\}_j$ is an independent sequence of Rademacher functions.

It is well known that if X is a UMD-space, then for every $1 < p < \infty$ the space $L^p_X(\mathbb{R}^n)$ has a type and cotype (see [Mau75], [MP76] and [Ald79]).

Kahane's contraction principle. For every Banach space X, $1 < p < \infty$, finite set $\{x_j\} \subset X$ and bounded sequence $\{c_j\}$ of scalars we have

$$\int_0^1 \Big\|\sum_j r_j(t) c_j x_j\Big\|_X^p \,dt \leq \sup_j |c_j|^p \int_0^1 \Big\|\sum_j r_j(t) x_j\Big\|_X^p \,dt, \tag{2.4}$$

where $\{r_j\}_j$ denotes an independent sequence of Rademacher functions. For details see [Kah85].

Remark 2.1. Let X be a Banach space with the UMD-property, and let $1 < p < \infty$. If $\delta_Q, \varepsilon_Q \in \{0, 1\}^n \setminus \{0\}$ for all $Q \in \mathscr{Q}$, then

$$\Big\|\sum_{Q\in\mathscr{Q}} u_Q h_Q^{(\delta_Q)}\Big\|_{L^p_X(\mathbb{R}^n)} \leq (\mathscr{U}_p(X))^2\Big\|\sum_{Q\in\mathscr{Q}} u_Q h_Q^{(\varepsilon_Q)}\Big\|_{L^p_X(\mathbb{R}^n)} \tag{2.5}$$

for all $u_Q \in X$, where only finitely many u_Q are non-zero. Therefore, we will drop the superscripts of the Haar functions and simply denote by h_Q one of the functions $h_Q^{(\varepsilon)}$, $\varepsilon \neq 0$, where appropriate.

The martingale inequality of Stein–Bourgain's version. Let X be a UMD-space and $1 < p < \infty$. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and let $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_m \subset \mathcal{F}$ denote an increasing sequence of σ-algebras. If $r_1, \ldots, r_m$ denote independent Rademacher functions, then for every choice of $f_1, \dots, f_m \in L^p(\Omega, \mu; X)$ we have

$$\int_0^1 \Big\| \sum_{i=1}^m r_i(t)\, \mathbb{E}(f_i \mid \mathcal{F}_i) \Big\|_{L^p(\Omega,\mu;X)} dt \le C \int_0^1 \Big\| \sum_{i=1}^m r_i(t) f_i \Big\|_{L^p(\Omega,\mu;X)} dt, \tag{2.6}$$

where C depends only on p and X.

A Banach space X having the UMD-property ensures $C < \infty$. The scalar-valued version of (2.6) by E. M. Stein can be found in [Ste70b]. The vector-valued extension is due to J. Bourgain [Bou86]. For details we refer the reader to [Mül05].

The shift operators T_m. For every $m \in \mathbb{Z}^n$ let $\tau_m : \mathscr{Q} \to \mathscr{Q}$ denote the rearrangement given by

$$\tau_m(Q) = Q + m \,\mathrm{sidelength}(Q). \tag{2.7}$$

The map τ_m induces the rearrangement operator T_m as the linear extension of

$$T_m h_Q = h_{\tau_m(Q)}, \qquad Q \in \mathscr{Q}. \tag{2.8}$$

Let X be a UMD-space. Then

$$\|T_m : L^p_X(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)\| \le C \log(2 + |m|)^\alpha, \tag{2.9}$$

where $0 < \alpha(X) < 1$ and $C = C(n, p, \mathscr{U}_p(X), \alpha(X))$; for details we refer the reader to [Fig88] and [Fig90].

The Riesz transform. For all $1 \le i \le n$ we define the Riesz transform R_i formally by

$$R_i f = K_i * f, \tag{2.10}$$

$$K_i(x) = c_n \frac{x_i}{|x|^{n+1}}, \qquad x = (x_1, \dots, x_n) \in \mathbb{R}^n. \tag{2.11}$$

Details may be found in [Ste70a] and [Ste93].

If X is a Banach space with the UMD-property and $1 < p < \infty$, then the operator $R_i : L^p(\mathbb{R}^n; X) \to L^p(\mathbb{R}^n; X)$ is bounded because of [McC84, Theorem 1.1].

Dunford–Pettis operators. Let X and Y be Banach spaces. A bounded linear operator $T : X \to Y$ is a *Dunford–Pettis operator* if T is weak-to-norm sequentially continuous, that is, whenever $\{x_n\}_n \subset X$ converges to x weakly, then Tx_n converges to Tx in norm. Clearly, if an operator is compact, then it is Dunford–Pettis. If X is reflexive, then T is compact if and only if T is Dunford–Pettis. For more information on Dunford–Pettis operators see [AK06].

Supplementary definitions. Denote the standard Fourier multiplier $\langle \cdot \rangle$ by

$$\langle \xi \rangle = (1 + |\xi|^2)^{1/2} \quad \text{for all } \xi \in \mathbb{R}^n. \tag{2.12}$$

The *Haar spectrum* of an operator $T : L^p_X(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)$ is defined by

$$\mathscr{Q} \setminus \big\{ Q \in \mathscr{Q} : \langle Tu, h_Q^{(\varepsilon)} \rangle = 0 \text{ for all } u \in L^p_X(\mathbb{R}^n) \text{ and } \varepsilon \in \{0,1\}^n \setminus \{0\} \big\}. \tag{2.13}$$

Given a collection of sets $\mathscr{C}$, we denote by $\sigma(\mathscr{C})$ the smallest σ-algebra containing $\mathscr{C}$, i.e.,

$$\sigma(\mathscr{C}) = \bigcap \{ \mathscr{A} : \mathscr{A} \text{ is a } \sigma\text{-algebra}, \mathscr{C} \subset \mathscr{A} \}.$$

3. The stripe operator S_λ

Here we introduce and study the stripe operator S_λ (defined in (3.6)), mapping h_Q, $Q \in \mathscr{Q}$, onto the blocks $g_{Q,\lambda}$, each supported on a dyadic stripe (see (3.3), (3.5) and Figures 1 and 2). The vector-valued estimates given by

$$\|S_\lambda u\|_{L^p_X(\mathbb{R}^n)} \le C \cdot 2^{-\lambda/\mathcal{C}(L^p_X(\mathbb{R}^n))} \|u\|_{L^p_X(\mathbb{R}^n)} \tag{3.1}$$

constitute the main technical component of this paper (see Theorem 3.6).

The crucial points in the proof of (3.1) are the cotype inequality and Corollary 3.5, that is, the uniform equivalence

$$\frac{1}{C}\|S_\lambda u\|_{L^p_X(\mathbb{R}^n)} \le \|T_{me_1} S_\lambda u\|_{L^p_X(\mathbb{R}^n)} \le C\|S_\lambda u\|_{L^p_X(\mathbb{R}^n)} \tag{3.2}$$

for all $0 \le m \le 2^\lambda - 1$ and $u \in L^p_X(\mathbb{R}^n)$, where C does not depend on u, λ and m. In other words, the operators T_m, $0 \le m \le 2^\lambda - 1$, act as isomorphisms on the image of S_λ, with norm independent of m and λ. This is in contrast to the well known norm estimates $\|T_m : L^p_X(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)\| \approx \log(2+m)^\alpha$, see (2.9).

3.1. Preparation. Within this section the superscripts (ε) are omitted and we generically denote by h_Q one of the functions $\{h_Q^{(\varepsilon)} : \varepsilon \in \{0,1\}^n \setminus \{0\}\}$. Note that ε may depend on Q (see Remark 2.1).

For every $Q \in \mathscr{Q}$ and $\lambda \ge 0$ define the *dyadic stripe*

$$\mathscr{U}_\lambda(Q) = \Big\{E \in \mathscr{Q} : \pi^\lambda(E) = Q,\ \inf_{x\in E} x_1 = \inf_{q \in Q} q_1\Big\}, \tag{3.3}$$

where x_1 respectively q_1 denotes the orthogonal projection of $x \in \mathbb{R}^n$ respectively $q \in \mathbb{R}^n$ onto the vector $e_1 = (1,0,\dots,0)$. Recall that $\pi^\lambda(E)$ is the unique $Q \in \mathscr{Q}$ such that $|Q| = 2^{\lambda n}|E|$ and $Q \supset E$ (see Section 2). The dyadic stripe $\mathscr{U}_\lambda(Q)$ is illustrated in Figure 1.

Additionally, we set

$$\mathscr{U}_\lambda = \bigcup_{Q\in\mathscr{Q}} \mathscr{U}_\lambda(Q). \tag{3.4}$$

We define the *stripe functions* by

$$g_{Q,\lambda} = \sum_{E \in \mathscr{U}_\lambda(Q)} h_E, \tag{3.5}$$

and the *stripe operator* by

$$S_\lambda u = \sum_{Q\in\mathscr{Q}} \langle u, h_Q\rangle g_{Q,\lambda} |Q|^{-1} \tag{3.6}$$

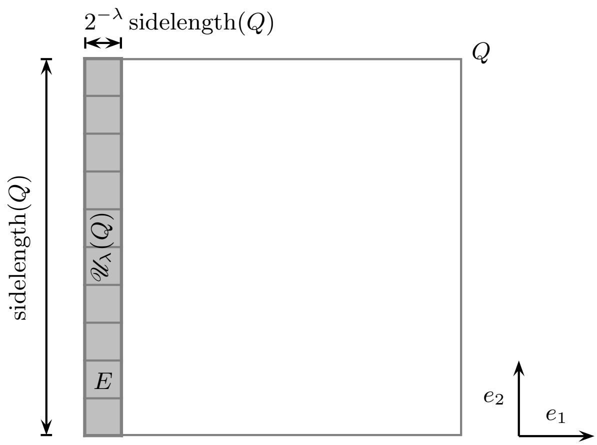

Fig. 1. Dyadic stripe $\mathscr{U}_\lambda(Q)$ in dimension $n = 2$

for all $u \in L^p_X(\mathbb{R}^n)$. The stripe functions are visualized in Figure 2.

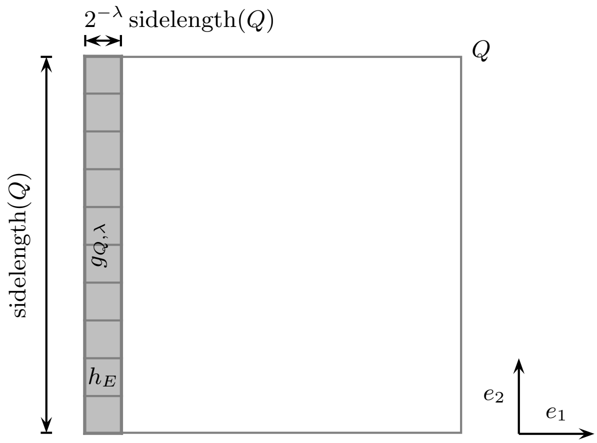

Fig. 2. Stripe functions $g_{Q,\lambda}$ in dimension $n = 2$

Remark 3.1. In (3.5), we used the convention that h_E denotes one of the functions $h_E^{(\varepsilon)}$ for some $\varepsilon \in \{0,1\}^n \setminus \{0\}$, where ε may depend on E. The reason behind this is the following.

For any $E \in \mathscr{Q}$ let $\delta_1(E), \delta_2(E) \in \{0,1\}^n \setminus \{0\}$ define the two functions

$$g_{Q,\lambda}^{(i)} = \sum_{E \in \mathscr{U}_\lambda(Q)} h_E^{(\delta_i(E))}, \qquad i = 1, 2,$$

and the stripe operators

$$S_\lambda^{(i)} u = \sum_{Q \in \mathscr{Q}} \langle u, h_Q \rangle g_{Q,\lambda}^{(i)} |Q|^{-1}, \qquad i = 1, 2.$$

Let us define

$$c_Q = g^{(1)}_{Q,\lambda} g^{(2)}_{Q,\lambda} = \sum_{E \in \mathscr{U}_\lambda(Q)} h_E^{(\delta_1(E))} h_E^{(\delta_2(E))}.$$

Then $c_Q g^{(1)}_{Q,\lambda} = g^{(2)}_{Q,\lambda}$, and c_Q is constant on every subcube of every $E \in \mathscr{U}_\lambda(Q)$. Hence, the UMD-property yields

$$\frac{1}{C} \|S^{(1)}_\lambda u\|_{L^p_X(\mathbb{R}^n)} \le \|S^{(2)}_\lambda u\|_{L^p_X(\mathbb{R}^n)} \le C \|S^{(1)}_\lambda u\|_{L^p_X(\mathbb{R}^n)}$$

for all $u \in L^p_X(\mathbb{R}^n)$, where C does not depend on the choice of $\delta_1(E)$ and $\delta_2(E)$.

This estimate means that stripe operators are, up to a constant, uniformly invariant under multiplication with functions of the form c_Q, and allows us to simply drop the superscripts in the Haar functions h_E defining $g_{Q,\lambda}$.

3.2. Shift operators acting on dyadic stripes. In Lemma 3.2 we will prove a measure estimate regarding one-dimensional dyadic stripes $\mathscr{S}_\lambda$, $\lambda \ge 1$, defined in (3.8), and the action of dyadic shift maps τ_m, $0 \le m \le 2^{\lambda-1}$, given by

$$\tau_m(I) = I + m|I|, \qquad I \in \mathscr{D}.$$

These estimates will then enter Theorem 3.3, where we prove the uniform estimates

$$\frac{1}{C} \|u\|_{L^p_X(\mathbb{R})} \le \|T_m u\|_{L^p_X(\mathbb{R})} \le C \|u\|_{L^p_X(\mathbb{R})}, \tag{3.7}$$

for all u supported on $\mathscr{S}_\lambda$ and $0 \le m \le 2^\lambda - 1$. The constant C does not depend on λ or m. The shift operator T_m is defined in (2.8).

The subsequent Corollary 3.5 states that T_m acts as an isomorphism on the image of S_λ, with norm independent of m and λ.

Before we state Lemma 3.2, we build up some notation. Define $\pi^\lambda : \mathscr{D} \to \mathscr{D}$ for all $I \in \mathscr{D}$ by

$$\pi^\lambda(I) = J,$$

where J is the uniquely determined $J \in \mathscr{D}$ such that $|J| = 2^\lambda\,|I|$ and $J \supset I$. Then define the one-dimensional stripe $\mathscr{S}_\lambda$ by

$$\mathscr{S}_\lambda = \{I \in \mathscr{D} : \inf I = \inf \pi^\lambda(I)\}. \tag{3.8}$$

LEMMA 3.2. *For every $\lambda \ge 1$ let $0 \le m \le 2^{\lambda-1}$, and let*

$$\tau_m(I) = I + m|I|, \qquad I \in \mathscr{D}.$$

Let $\mathscr{B} \subset \mathscr{S}_\lambda$ be such that for all $J, K \in \mathscr{B}$ with $|J| \ne |K|$ either

$$|J| \le \tfrac{1}{4}|K| \quad \text{or} \quad |K| \le \tfrac{1}{4}|J|.$$

Then

$$\Big| I \cap \bigcup_{d=1}^{\lambda-1} \bigcup_{\substack{J \in \mathscr{B} \\ |J| = 2^{-d}|I|}} J \cup \tau_m(J) \Big| \le \tfrac{2}{3}|I|, \qquad \Big| \tau_m(I) \cap \bigcup_{d=1}^{\lambda-1} \bigcup_{\substack{J \in \mathscr{B} \\ |J| = 2^{-d}|I|}} J \cup \tau_m(J) \Big| \le \tfrac{2}{3}|I|,$$

for all $I \in \mathscr{B}$.

Proof. First, we claim that for any $I \in \mathscr{B} \cup \tau_m(\mathscr{B})$, $1 \le d \le \lambda - 1$ and $J, K \in \mathscr{B}$ with $|J| = |K| = 2^{-d}|I|$,

$$\text{whenever } (J \cup \tau_m(J)) \cap I \neq \emptyset \text{ and } (K \cup \tau_m(K)) \cap I \neq \emptyset, \text{ then } J = K. \tag{3.9}$$

Indeed, assume that (3.9) is incorrect. Hence, we can find intervals $I \in \mathscr{B} \cup \tau_m(\mathscr{B})$ and $J, K \in \mathscr{B}$ with $J \neq K$, $|J| = |K| = 2^{-d}\,|I|$ where $1 \le d \le \lambda - 1$, such that

$$(J \cup \tau_m(J)) \cap I \neq \emptyset \quad \text{and} \quad (K \cup \tau_m(K)) \cap I \neq \emptyset.$$

Since $J \neq K$, we see from the definition of $\mathscr{B}$ that

$$\operatorname{dist}(\tau_m(J), \tau_m(K)) = \operatorname{dist}(J, K) \ge (2^\lambda - 1)|J|,$$

and consequently

$$\operatorname{dist}(J \cup \tau_m(J), K \cup \tau_m(K)) \ge (2^\lambda - 1 - m)|J|.$$

We know that I intersects both $J \cup \tau_m(J)$ and $K \cup \tau_m(K)$, so

$$|I| \ge \operatorname{dist}(J \cup \tau_m(J), K \cup \tau_m(K)) + 2|J| \ge (2^\lambda - m + 1)2^{-d}|I| \ge (2^{\lambda-1} + 1)2^{-d}|I| > |I|,$$

which is a contradiction.

Hence, (3.9) holds true, which means that for all $1 \le d \le \lambda - 1$, every interval $I \in \mathscr{B} \cup \tau_m(\mathscr{B})$ intersects at most one element of the set

$$\{J \cup \tau_m(J) \in \mathscr{B} : |J| = 2^{-d}|I|\}.$$

If such a J exists, we denote it by $J_d(I) \in \mathscr{B}$, and set $J_d(I) = \emptyset$ otherwise. Note that for small shift widths m or small J it may happen that $J_d(I) \cup \tau_m(J_d(I)) \subset I$.

Using (3.9) we find that for every $I \in \mathscr{B} \cup \tau_m(\mathscr{B})$,

$$\begin{aligned}\Big| I \cap \bigcup_{d=1}^{\lambda-1} \bigcup_{\substack{J \in \mathscr{B} \\ |J| = 2^{-d}|I|}} J \cup \tau_m(J) \Big| &\le \sum_{d=1}^{\lambda-1} \big| I \cap \big(J_d(I) \cup \tau_m(J_d(I))\big) \big| \\ &\le \sum_{d=1}^{\lambda-1} 2|J_d(I)| \le 2 \sum_{d=1}^{\infty} 2^{-2d}|I| = \tfrac{2}{3}|I|.\end{aligned}$$

The last inequality is true since for $J, K \in \mathscr{B}$, if $|J| \neq |K|$, then either $|J| \le |K|/4$ or $|K| \le |J|/4$. ■

For $m \in \mathbb{Z}$ the shift operator T_m is given by

$$T_m h_I = h_{\tau_m(I)}, \qquad I \in \mathscr{D},$$

where $\tau_m(I) = I + m|I|$, $I \in \mathscr{D}$ (see (2.7) and (2.8)). We will now investigate the action of T_m restricted to functions supported on the dyadic stripe $\mathscr{S}_\lambda$, $\lambda \ge 0$, defined in (3.8). Observe that $\mathscr{S}_\lambda$ is the spectrum of the stripe operator S_λ, when it is restricted to lines in direction $(1, 0, \ldots, 0)$. This will be discussed in more detail in Corollary 3.5. For now we dedicate ourselves to the one-dimensional case.

Theorem 3.3. *Let X be a Banach space with the UMD-property and $1 < p < \infty$. For $\lambda \ge 0$ define the linear subspace Z_λ of $L^p_X(\mathbb{R})$ by*

$$Z_\lambda = \Big\{ \sum_{I \in \mathscr{S}_\lambda} u_I\, h_I |I|^{-1} : u_I \in X \Big\} \cap L^p_X(\mathbb{R}). \tag{3.10}$$

Then there exists a constant $C > 0$ such that for all integers λ and m satisfying $0 \leq m \leq 2^\lambda - 1$ we have

$$\frac{1}{C}\|u\|_{L^p_X(\mathbb{R})} \leq \|T_m u\|_{L^p_X(\mathbb{R})} \leq C\|u\|_{L^p_X(\mathbb{R})} \tag{3.11}$$

for all $u \in Z_\lambda$, where C depends only on p and the UMD-constant of X. In other words, T_m acts as an isomorphism on Z_λ with norm independent of m and λ.

Proof. With $\lambda \geq 0$ fixed, we will first prove

$$\frac{1}{C}\|u\|_{L^p_X(\mathbb{R})} \leq \|T_m u\|_{L^p_X(\mathbb{R})} \leq C\|u\|_{L^p_X(\mathbb{R})} \tag{3.12}$$

for all $0 \leq m \leq 2^{\lambda-1}$ and $u \in Z_\lambda$. Once we have (3.12), it is easy to see by symmetry that also

$$\frac{1}{C}\|T_{2^\lambda-1}u\|_{L^p_X(\mathbb{R})} \leq \|T_m u\|_{L^p_X(\mathbb{R})} \leq C\|T_{2^\lambda-1}u\|_{L^p_X(\mathbb{R})} \tag{3.13}$$

for all $2^{\lambda-1} - 1 \leq m \leq 2^\lambda - 1$ and $u \in Z_\lambda$. Certainly, (3.12) together with (3.13) implies (3.11), since we may join (3.12) and (3.13) at the intersection of the two collections of operators

$$\{T_m : 0 \leq m \leq 2^{\lambda-1}\} \quad \text{and} \quad \{T_m : 2^{\lambda-1} - 1 \leq m \leq 2^\lambda - 1\},$$

that is, at $m = 2^{\lambda-1}$ or at $m = 2^{\lambda-1} - 1$.

We begin the proof of (3.12) by defining the four collections

$$\mathscr{B}^0_{\text{odd}} = \bigcup_{j\in\mathbb{Z}} \bigcup_{\substack{k=0\\ k \text{ odd}}}^{\lambda-1} \mathscr{S}_\lambda \cap \mathscr{D}_{2j\lambda+k}, \qquad \mathscr{B}^0_{\text{even}} = \bigcup_{j\in\mathbb{Z}} \bigcup_{\substack{k=0\\ k \text{ even}}}^{\lambda-1} \mathscr{S}_\lambda \cap \mathscr{D}_{2j\lambda+k},$$

$$\mathscr{B}^1_{\text{odd}} = \bigcup_{j\in\mathbb{Z}} \bigcup_{\substack{k=0\\ k \text{ odd}}}^{\lambda-1} \mathscr{S}_\lambda \cap \mathscr{D}_{(2j+1)\lambda+k}, \qquad \mathscr{B}^1_{\text{even}} = \bigcup_{j\in\mathbb{Z}} \bigcup_{\substack{k=0\\ k \text{ even}}}^{\lambda-1} \mathscr{S}_\lambda \cap \mathscr{D}_{(2j+1)\lambda+k}.$$

For any given $j \in \mathbb{Z}$ we shall call the collections

$$\mathscr{B}^0_{\text{odd}} : \bigcup_{\substack{k=0\\ k \text{ odd}}}^{\lambda-1} \mathscr{S}_\lambda \cap \mathscr{D}_{2j\lambda+k}, \qquad \mathscr{B}^0_{\text{even}} : \bigcup_{\substack{k=0\\ k \text{ even}}}^{\lambda-1} \mathscr{S}_\lambda \cap \mathscr{D}_{2j\lambda+k},$$

$$\mathscr{B}^1_{\text{odd}} : \bigcup_{\substack{k=0\\ k \text{ odd}}}^{\lambda-1} \mathscr{S}_\lambda \cap \mathscr{D}_{(2j+1)\lambda+k}, \qquad \mathscr{B}^1_{\text{even}} : \bigcup_{\substack{k=0\\ k \text{ even}}}^{\lambda-1} \mathscr{S}_\lambda \cap \mathscr{D}_{(2j+1)\lambda+k}$$

λ-*blocks*, each associated to the indicated collection.

Let $\mathscr{B}$ denote one of those four collections. We claim the existence of a filtration $\{\mathcal{F}_j\}_j$ such that for every $j \in \mathbb{Z}$ and $I \in \mathscr{B} \cap \mathscr{D}_j$ there exists an atom $A(I)$ of $\mathcal{F}_j$ satisfying the inequalities

$$|A(I)| \leq 2|I|, \qquad |I \cap A(I)| \geq \tfrac{1}{3}|I|, \qquad |\tau_m(I) \cap A(I)| \geq \tfrac{1}{3}|I|. \tag{3.14}$$

We will now define the atoms within each λ-block $\mathscr{C}$ of $\mathscr{B}$. The resulting atoms are unions of dyadic intervals having length $\min_{I\in\mathscr{C}} |I|$. The construction of the atoms is independent of other λ-blocks of $\mathscr{B}$. Now, for each $I \in \mathscr{B}$ we will define atoms inductively, beginning at the finest level of $\mathscr{C}$. Initially, define

$$A(I) = I \cup \tau_m(I) \tag{3.15}$$

for all $I \in \mathscr{C}$ such that $|I| = \min_{J\in\mathscr{C}} |J|$. Let $I \in \mathscr{C}$, and assume that we already constructed atoms $A(J)$ for all $J \in \mathscr{C}$, $|J| < |I|$. Then we define the atom $A(I)$ by

$$A(I) = (I \cup \tau_m(I)) \setminus \bigcup_{\substack{J\in\mathscr{C}\\ |J|<|I|}} A(J). \tag{3.16}$$

Applying Lemma 3.2 to the atoms $A(I) \subset I \cup \tau_m(I)$ inside the λ-block $\mathscr{C}$, we obtain

$$|I \cap A(I)| = |I| - \Big|I \cap \bigcup_{\substack{J\in\mathscr{C}\\ |J|<|I|}} A(J)\Big| \geq \tfrac{1}{3}|I|,$$

and analogously

$$|\tau_m(I) \cap A(I)| \geq \tfrac{1}{3}|I|,$$

which yields (3.14). Finally, we define the collections

$$\mathscr{A}_j = \{A(I) : I \in \mathscr{B} \cap \mathscr{D}_j\}, \qquad j \in \mathbb{Z}, \tag{3.17}$$

and the filtration

$$\mathcal{F}_j = \sigma\Big(\bigcup_{i\leq j} \mathscr{A}_i\Big), \qquad j \in \mathbb{Z}. \tag{3.18}$$

What is left to show is that every $A \in \mathscr{A}_j$ is an atom for the σ-algebra $\mathcal{F}_j$.

To see this we reason as follows. First, note that any two atoms are either in the same λ-block, or are separated by at least λ levels. If atoms $A(I)$ and $A(I')$ are in the same λ-block, then they do not intersect by construction (see (3.15) and (3.16)). Whenever $A(I)$ and $A(I')$ intersect and $|I'| \leq 2^{-\lambda}|I|$, then since

$$A(I') \subset (I' \cup \tau_m(I')) \subset \pi^\lambda(I'),$$

we have

$$\pi^\lambda(I') \cap A(I) \neq \emptyset.$$

Clearly, $A(I)$ consists of intervals K which are at least as big as $\pi^\lambda(I')$, so $|\pi^\lambda(I')| \leq |K|$, hence

$$A(I') \subset A(I).$$

This means that $\bigcup_j \mathscr{A}_j$ is a nested collection of sets, hence every $A \in \mathscr{A}_j$ is an atom for the σ-algebra $\mathcal{F}_j$.

Now we are prepared to estimate the shift operator T_m. To this end, let $u \in Z_\lambda$ be fixed throughout the rest of the proof. Having (3.14) at hand and knowing that the collection $\mathscr{A}_j$ consists of atoms of $\mathcal{F}_j$, observe that

$$\mathbb{1}_I \leq 18\,\mathbb{E}\big(\mathbb{E}(\mathbb{1}_{\tau_m(I)} \mid \mathcal{F}_j) \bigm| \mathscr{D}_j\big), \qquad I \in \mathscr{B} \cap \mathscr{D}_j, \tag{3.19}$$

and analogously

$$\mathbb{1}_{\tau_m(I)} \leq 18\,\mathbb{E}\big(\mathbb{E}(\mathbb{1}_I \mid \mathcal{F}_j) \bigm| \mathscr{D}_j\big), \qquad I \in \mathscr{B} \cap \mathscr{D}_j. \tag{3.20}$$

The UMD-property and Kahane's contraction principle applied to $|h_I| \le \mathbb{1}_I$ yield

$$\|u\|^p_{L^p_X(\mathbb{R})} \approx \int_0^1 \Big\| \sum_{j\in\mathbb{Z}} r_j(t)(u)_j \Big\|^p_{L^p_X(\mathbb{R})} \, dt,$$

where $(\,\cdot\,)_j$ denotes the restriction of the Haar expansion to intervals in $\mathscr{D}_j$, and the Haar functions h_I, $I \in \mathscr{D}_j$, are replaced by the characteristic functions $\mathbb{1}_I$, $I \in \mathscr{D}_j$. More precisely, if

$$u = \sum_{j\in\mathbb{Z}} \sum_{I\in\mathscr{D}_j} u_I h_I |I|^{-1},$$

then

$$(u)_j = \sum_{I\in\mathscr{D}_j} u_I \mathbb{1}_I |I|^{-1}.$$

Applying Kahane's contraction principle in view of (3.19) yields

$$\|u\|^p_{L^p_X(\mathbb{R})} \lesssim \int_0^1 \Big\| \sum_{j\in\mathbb{Z}} r_j(t)\, \mathbb{E}\big(\mathbb{E}((T_m u)_j \mid \mathcal{F}_j) \bigm| \mathscr{D}_j\big) \Big\|^p_{L^p_X(\mathbb{R})} \, dt.$$

Using Stein's martingale inequality (2.6) with respect to the filtration $\{\mathscr{D}_j\}_j$ gives

$$\|u\|^p_{L^p_X(\mathbb{R})} \lesssim \int_0^1 \Big\| \sum_{j\in\mathbb{Z}} r_j(t)\, \mathbb{E}((T_m u)_j \mid \mathcal{F}_j) \Big\|^p_{L^p_X(\mathbb{R})} \, dt.$$

Now we apply Stein's martingale inequality with respect to the filtration $\{\mathcal{F}_j\}_j$ and get

$$\|u\|^p_{L^p_X(\mathbb{R})} \lesssim \int_0^1 \Big\| \sum_{j\in\mathbb{Z}} r_j(t)(T_m u)_j \Big\|^p_{L^p_X(\mathbb{R})} \, dt.$$

Subsequently, we apply Kahane's contraction principle to $\mathbb{1}_{\tau_m(I)} \le |h_{\tau_m(I)}|$ and make use of the UMD-property to dispose of the Rademacher functions and obtain

$$\|u\|^p_{L^p_X(\mathbb{R})} \lesssim \|T_m u\|^p_{L^p_X(\mathbb{R})}.$$

Repeating this argument with the roles of u and $T_m u$ reversed, and using (3.20) instead of (3.19) we get the converse inequality

$$\|T_m u\|^p_{L^p_X(\mathbb{R})} \lesssim \|u\|^p_{L^p_X(\mathbb{R})}.$$

A fortiori, we proved (3.12), that is,

$$\frac{1}{C}\|u\|_{L^p_X(\mathbb{R})} \le \|T_m u\|_{L^p_X(\mathbb{R})} \le C\|u\|_{L^p_X(\mathbb{R})}$$

for all $\lambda \ge 0$, $0 \le m \le 2^{\lambda-1}$ and $u \in Z_\lambda$, where C depends only on p and the UMD-constant of X.

Observe that due to symmetry we may use the same argument for the operators T_m, $2^{\lambda-1} \le m \le 2^\lambda - 1$, if we reverse the sign of the shift operation and replace u by $T_{2^\lambda-1}u$. Therefore inequality (3.13) holds true as well, i.e.

$$\frac{1}{C}\|T_{2^\lambda-1}u\|_{L^p_X(\mathbb{R})} \le \|T_m u\|_{L^p_X(\mathbb{R})} \le C\|T_{2^\lambda-1}u\|_{L^p_X(\mathbb{R})}$$

for all $2^{\lambda-1} - 1 \le m \le 2^\lambda - 1$ and $u \in Z_\lambda$, where C depends only on p and the UMD-constant of X.

Joining the last two displayed inequalities via $T_{2^{\lambda-1}}$ (or $T_{2^{\lambda-1}-1}$) as indicated above concludes the proof of Theorem 3.3. ∎

REMARK 3.4. The central difficulty of the proof was finding the filtration $\{\mathcal{F}_j\}_j$, given by (3.18), such that each collection $\mathscr{A}_j$, given by (3.17), consists of atoms $A(I)$ of $\mathcal{F}_j$. This was achieved by subtracting the atoms $A(J)$ succeeding $A(I)$ within a λ-block (see (3.15) and (3.16)). The measure estimates in Lemma 3.2 guaranteed inequalities (3.14). As a consequence, we obtained inequalities (3.19) and (3.20), which enabled us to shift h_I to $h_{\tau_m(I)}$ by means of Kahane's contraction principle and Bourgain's version of Stein's martingale inequality.

For a detailed exposition and the development of a method of estimating rearrangement operators that admit a supporting tree, we refer the reader to [KM09] and [MS91]. Given a rearrangement τ such that $|\tau(I)| = |I|$, the existence of a supporting tree is essentially the existence of a filtration having the properties of $\{\mathcal{F}_j\}_j$ listed above, with τ_m replaced by τ.

In order to shift an essential portion of h_I to $h_{\tau_m(I)}$, one can replace Bourgain's version of Stein's martingale inequality by the martingale transforms used in [Fig88, Proposition 2, Step 0]. To this end, we need additional symmetry properties (see (3.21)), which were not needed in the first proof. For our purposes we will refine the above construction of the filtration $\{\mathcal{F}_j\}_j$. The details are given in the proof below.

Alternative proof of Theorem 3.3. We modify the construction of the above collections $\mathscr{B}$ by taking only every fourth level instead of every second level, and denote each of those collections by $\mathscr{C}$. Hence, for all $J, K \in \mathscr{C}$, if $|J| \neq |K|$ we have either

$$|J| \leq \tfrac{1}{16}|K| \quad \text{or} \quad |K| \leq \tfrac{1}{16}|J|.$$

Inspecting the proof of Lemma 3.2 we see that

$$\Big|I \cap \bigcup_{d=1}^{\lambda-1} \bigcup_{\substack{J\in\mathscr{C}\\|J|=2^{-d}|I|}} J \cup \tau_m(J)\Big| \leq \tfrac{2}{15}|I|, \qquad \Big|\tau_m(I) \cap \bigcup_{d=1}^{\lambda-1} \bigcup_{\substack{J\in\mathscr{C}\\|J|=2^{-d}|I|}} J \cup \tau_m(J)\Big| \leq \tfrac{2}{15}|I|.$$

So if we construct the atoms $A(I)$ according to (3.15) and (3.16) (with $\mathscr{B}$ replaced by $\mathscr{C}$), instead of (3.14) we obtain the inequalities

$$|A(I)| \leq 2|I|, \qquad |I \cap A(I)| \geq \tfrac{13}{15}|I|, \qquad |\tau_m(I) \cap A(I)| \geq \tfrac{13}{15}|I|.$$

In what follows we denote the left and right dyadic successors of I by I_0 and I_1, respectively. To be more precise, $I_0, I_1 \in \mathscr{D}$, $|I_0| = |I_1| = |I|/2$, and $\inf I_0 = \inf I$, $\sup I_1 = \sup I$. Consequently, if we define

$$\begin{aligned} B(I) = {} & \big(A(I) \cap (A(I) \cap I_1 - |I|/2)\big) \cup \big(A(I) \cap (A(I) \cap I_0 + |I|/2)\big) \\ & \cup \big(A(I) \cap (A(I) \cap \tau_m(I)_1 - |I|/2)\big) \cup \big(A(I) \cap (A(I) \cap \tau_m(I)_0 + |I|/2)\big) \end{aligned}$$

and furthermore

$$C(I) = \big(B(I) \cap (B(I) - m|I|)\big) \cup \big(B(I) \cap (B(I) + m|I|)\big),$$

we see that

$$|C(I)| \leq 2|I|, \qquad |I \cap C(I)| \geq \tfrac{7}{15}|I|, \qquad |\tau_m(I) \cap C(I)| \geq \tfrac{7}{15}|I|.$$

Since $C(I) \subset A(I)$, the $C(I)$, $I \in \mathscr{C}$, do not intersect inside a λ-block. Retracing our steps, we may replace $A(I)$ by $C(I)$ in the above proof. Observe that additionally we have the following identities at our disposal:

$$C(I) \cap \tau_m(I) = C(I) \cap I + m|I|, \qquad C(I) \cap I_1 = C(I) \cap I_0 + |I|/2; \tag{3.21}$$

they allow us to use the martingale transform in the proof of [Fig88, Proposition 2, Step 0]. To be more precise, if we define

$$d_{I,1} = \tfrac{1}{2}(h_I + h_{\tau_m(I)}) \cdot 1_{C(I)} \quad \text{and} \quad d_{I,2} = \tfrac{1}{2}(h_I - h_{\tau_m(I)}) \cdot 1_{C(I)}, \tag{3.22}$$

then due to (3.21) we see that $\{d_{I,1}, d_{I,2} : I \in \mathscr{C}\}$ is a martingale difference sequence. Furthermore, note that

$$\{h_I \cdot 1_{C(I)} : I \in \mathscr{C}\} \quad \text{and} \quad \{h_{\tau_m(I)} \cdot 1_{C(I)} : I \in \mathscr{C}\}$$

are martingale difference sequences as well. Observe that

$$d_{I,1} + d_{I,2} = h_I \cdot 1_{C(I)} \quad \text{and} \quad d_{I,1} - d_{I,2} = h_{\tau_m(I)} \cdot 1_{C(I)}; \tag{3.23}$$

thus we can swap $h_I \cdot 1_{C(I)}$ with $h_{\tau_m(I)} \cdot 1_{C(I)}$, according to [Fig88, Lemma 2].

Thus we shifted $h_I \cdot 1_{C(I)}$ to $h_{\tau_m(I)} \cdot 1_{C(I)}$ by means of the martingale transformation given by (3.23) instead of applying Bourgain's version of Stein's martingale inequality for this purpose. ∎

The following Corollary 3.5 connects the one-dimensional Theorem 3.3 with the multidimensional stripe operators S_λ. In Figure 3 the action of the shift operators T_m, $0 \le m \le 2^\lambda - 1$, on the image of S_λ is visualized.

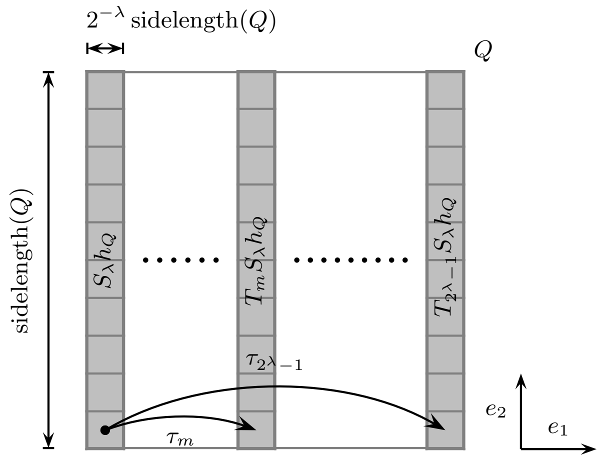

Fig. 3. Shifting the image of a stripe operator S_λ in dimension $n = 2$

Corollary 3.5. *Let X be a UMD-space. Let $1 < p < \infty$, $n \in \mathbb{N}$, and denote by e_1 the unit vector $(1, 0, \dots, 0) \in \mathbb{R}^n$. Then there exists a constant $C > 0$ such that, for all integers λ and m satisfying $0 \le m \le 2^\lambda - 1$ and every $u \in L^p_X(\mathbb{R}^n)$,*

$$\frac{1}{C}\|S_\lambda u\|_{L^p_X(\mathbb{R}^n)} \le \|T_{me_1} S_\lambda u\|_{L^p_X(\mathbb{R}^n)} \le C\|S_\lambda u\|_{L^p_X(\mathbb{R}^n)}, \tag{3.24}$$

where C depends only on n, p and the UMD-constant of X. In other words, T_m acts as an isomorphism on the image of S_λ, having norm estimates independent of m and λ.

Proof. We recall the definitions (3.3) and (3.4), that is,

$$\mathscr{U}_\lambda = \bigcup_{Q\in\mathscr{Q}} \Big\{E \in \mathscr{Q} : \pi^\lambda(E) = Q,\ \inf_{x\in E} x_1 = \inf_{q\in Q} q_1\Big\};$$

by $\cdot_1$ we denoted the projection onto the first coordinate. Observe that due to the definitions (3.5) and (3.6) we have

$$\operatorname{image}(S_\lambda) \subset \Big\{\sum_{Q\in\mathscr{U}_\lambda} u_Q\, h_Q|Q|^{-1} : u_Q \in X\Big\} \cap L^p_X(\mathbb{R}^n).$$

With this in mind we will apply Theorem 3.3 to every line in the direction e_1. Recall that we omitted the superscripts for the Haar functions $h_Q^{(\varepsilon)}$, $\varepsilon \neq 0$, and used the generic notation h_Q instead. Note that Kahane's contraction principle allows us to choose the function $h_Q = h_Q^{(\varepsilon)}$ with $\varepsilon_1 = 1$, at the same time preserving the norm of the operator, up to a constant (see (2.5)). So now we shall assume that each h_Q has zero mean in the first coordinate.

Fix $u \in L^p_X$, define $v = S_\lambda u$, and denote by v_x the function $v(\cdot, x)$ for all $x \in \mathbb{R}^{n-1}$. Due to our assumption above, $v_x \in Z_\lambda$ for almost all $x \in \mathbb{R}^n$. Observe that for all $x \in \mathbb{R}^{n-1}$ and $t \in \mathbb{R}$ we have the identity

$$(T_{me_1} v)(t, x) = (T_m v_x)(t),$$

hence

$$\|T_{me_1} v\|^p_{L^p_X(\mathbb{R}^n)} = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \|(T_m v_x)(t)\|^p_X\, dt\, dx = \int_{\mathbb{R}^{n-1}} \|T_m v_x\|^p_{L^p_X(\mathbb{R})}\, dx.$$

Since $v_x \in Z_\lambda$ for almost every $x \in \mathbb{R}^n$, we may use Theorem 3.3 to get

$$\int_{\mathbb{R}^{n-1}} \|T_m v_x\|^p_{L^p_X(\mathbb{R})}\, dx \approx \int_{\mathbb{R}^{n-1}} \|v_x\|^p_{L^p_X(\mathbb{R})}\, dx = \|v\|^p_{L^p_X(\mathbb{R}^n)}.$$

Substituting $v = S_\lambda u$ finishes the proof. ∎

3.3. Estimates for the stripe operator. Before we formulate and prove the main result on stripe operators S_λ, we will recapitulate the definition of S_λ (see (3.6)). The dyadic stripe $\mathscr{U}_\lambda(Q)$ (for details see (3.3)) was defined to be the collection

$$\Big\{E \in \mathscr{Q} : \pi^\lambda(E) = Q,\ \inf_{x\in E} x_1 = \inf_{q\in Q} q_1\Big\},$$

where $\pi^\lambda(E)$ is the unique $Q \in \mathscr{Q}$ such that $|Q| = 2^{\lambda n}|E|$ and $Q \supset E$. Furthermore, x_1 respectively q_1 denotes the orthogonal projection of $x \in \mathbb{R}^n$ respectively $q \in \mathbb{R}^n$ onto the vector $e_1 = (1, 0, \ldots, 0)$. Then the stripe operator S_λ is given by the linear extension of

$$S_\lambda h_Q = g_{Q,\lambda},$$

and the stripe functions were defined in (3.5) by

$$g_{Q,\lambda} = \sum_{E\in\mathscr{U}_\lambda(Q)} h_E.$$

Having verified Corollary 3.5 we will now present our main theorem on stripe operators.

THEOREM 3.6. *Let X be a UMD-space, $1 < p < \infty$ and $n \in \mathbb{N}$. For $\lambda \geq 0$ let S_λ denote the stripe operator given by*

$$S_\lambda u = \sum_{Q \in \mathscr{Q}} \langle u, h_Q \rangle g_{Q,\lambda} |Q|^{-1}$$

for all $u \in L^p_X(\mathbb{R}^n)$. Recall that h_Q denotes any of the functions $h_Q^{(\varepsilon)}$, $\varepsilon \neq 0$. If $L^p_X(\mathbb{R}^n)$ has cotype $\mathcal{C}(L^p_X(\mathbb{R}^n))$, then there exists a constant $C > 0$ such that for every $u \in L^p_X(\mathbb{R}^n)$ and $\lambda \geq 0$,

$$\|S_\lambda u\|_{L^p_X(\mathbb{R}^n)} \leq C \cdot 2^{-\lambda/\mathcal{C}(L^p_X(\mathbb{R}^n))} \|u\|_{L^p_X(\mathbb{R}^n)}, \tag{3.25}$$

where the constant C depends only on n, p, the UMD-constant of X and the cotype $\mathcal{C}(L^p_X(\mathbb{R}^n))$.

Proof. The UMD-property and Kahane's contraction principle shows that the estimate holds true if we restrict λ to $0 \leq \lambda \leq 1$.

So from now on we may assume that $\lambda \geq 2$. The definition of the dyadic stripe $\mathscr{U}_\lambda$ (see (3.3) and (3.4)) implies that

$$\tau_{ke_1}(\mathscr{U}_\lambda) \cap \tau_{me_1}(\mathscr{U}_\lambda) = \emptyset \tag{3.26}$$

if $0 \leq k < m \leq 2^\lambda - 1$. Furthermore, one has the high frequency cover of $Q \in \mathscr{Q}$ given by

$$\bigcup_{m=0}^{2^\lambda - 1} \tau_{me_1}(\mathscr{U}_\lambda(Q)) = \{E \in \mathscr{Q} : \pi^\lambda(E) = Q\},$$

thus we see that

$$|h_Q| = \Big| \sum_{m=0}^{2^\lambda - 1} T_{me_1} g_{Q,\lambda} \Big| \tag{3.27}$$

by the definition of $g_{Q,\lambda}$ (see Figure 4).

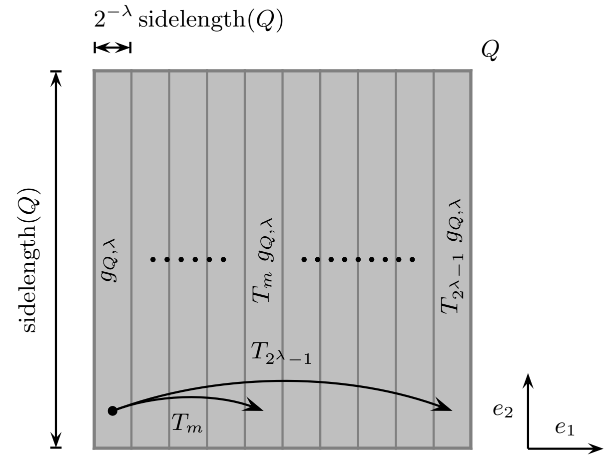

Fig. 4. High frequency cover of the cube Q obtained by shifts of the stripe functions $g_{Q,\lambda}$

Now let $u \in L^p_X(\mathbb{R}^n)$ be fixed. For the rest of the proof we shall write L^p_X for $L^p_X(\mathbb{R}^n)$ and $\mathcal{C}$ for $\mathcal{C}(L^p_X)$. We want to bound $\|u\|_{L^p_X}$ from below by means of the stripe operator S_λ.

First, the UMD-property allows us to introduce the Rademacher means

$$\|u\|_{L^p_X} \approx \int_0^1 \Big\| \sum_j r_j(t) \sum_{Q\in\mathcal{Q}_j} u_Q h_Q |Q|^{-1} \Big\|_{L^p_X} dt.$$

Second, Kahane's contraction principle applied to (3.27) on the right hand side yields

$$\|u\|_{L^p_X} \approx \int_0^1 \Big\| \sum_j r_j(t) \sum_{Q\in\mathcal{Q}_j} u_Q \sum_{m=0}^{2^\lambda-1} T_{me_1} g_{Q,\lambda} |Q|^{-1} \Big\|_{L^p_X} dt. \tag{3.28}$$

Third, if we set

$$d_{(j,m)} = T_{me_1} \sum_{Q\in\mathcal{Q}_j} g_{Q,\lambda} \quad \text{for } j \in \mathbb{Z} \text{ and } 0 \le m \le 2^\lambda - 1,$$

and define the lexicographic ordering relation

$$(j,m) < (j',m') \quad \text{iff} \quad \begin{cases} j < j', \text{ or} \\ j = j' \text{ and } m < m', \end{cases}$$

then $\{d_{(j,m)} : j \in \mathbb{Z},\ 0 \le m \le \lambda\}$ with respect to "$<$" generates a martingale difference sequence. So in view of (3.26) and the UMD-property we may introduce the following new Rademacher means in (3.28):

$$\int_0^1 \Big\| \sum_{m=0}^{2^\lambda-1} r_m(t) T_{me_1} \sum_{Q\in\mathcal{Q}} u_Q g_{Q,\lambda} |Q|^{-1} \Big\|_{L^p_X} dt.$$

Hence, we have

$$\|u\|_{L^p_X} \approx \int_0^1 \Big\| \sum_{m=0}^{2^\lambda-1} r_m(t) T_{me_1} \sum_{Q\in\mathcal{Q}} u_Q g_{Q,\lambda} |Q|^{-1} \Big\|_{L^p_X} dt. \tag{3.29}$$

Fourth, with $g_{Q,\lambda} = S_\lambda h_Q$ in mind, we apply the cotype inequality (2.3) to (3.29) to find that

$$\|u\|_{L^p_X} \gtrsim \Big(\sum_{m=0}^{2^\lambda-1} \|T_{me_1} S_\lambda u\|^{\mathcal{C}}_{L^p_X} \Big)^{1/\mathcal{C}}.$$

Finally, utilizing Corollary 3.5 concludes the proof:

$$\Big(\sum_{m=0}^{2^\lambda-1} \|T_{me_1} S_\lambda u\|^{\mathcal{C}}_{L^p_X} \Big)^{1/\mathcal{C}} \approx \Big(\sum_{m=0}^{2^\lambda-1} \|S_\lambda u\|^{\mathcal{C}}_{L^p_X} \Big)^{1/\mathcal{C}} = 2^{\lambda/\mathcal{C}} \|S_\lambda u\|_{L^p_X}. \ \blacksquare$$

Repeating the proof of Theorem 3.6 without Corollary 3.5, and using Figiel's bound (2.9) on shift operators directly, would lead to the weaker result

$$\|S_\lambda u\|_{L^p_X(\mathbb{R}^n)} \le C\lambda^\alpha 2^{-\lambda/\mathcal{C}(L^p_X(\mathbb{R}^n))} \|u\|_{L^p_X(\mathbb{R}^n)}, \tag{3.30}$$

where the exponent $0 < \alpha < 1$ is the one occurring in (2.9).

3.4. The ring domain operator. We will define the ring domain operator H_λ, which is supported in the vicinity of the set of discontinuities of Haar functions. We will show that H_λ can be written as a finite sum of continuous images of stripe operators S_λ. Thus, estimate (3.6) for the stripe operator carries over to the ring domain operator, that is,

$$\|H_\lambda u\|_{L^p_X(\mathbb{R}^n)} \leq C \cdot 2^{-\lambda/\mathcal{C}(L^p_X(\mathbb{R}^n))} \|u\|_{L^p_X(\mathbb{R}^n)}. \tag{3.31}$$

For every Q denote by $D(Q)$ the set of discontinuities of the Haar function $h_Q^{(1,\dots,1)}$ and define

$$D_\lambda(Q) = \{x \in \mathbb{R}^n : \operatorname{dist}(x, D(Q)) \leq C \cdot 2^{-\lambda} \operatorname{sidelength}(Q)\}$$

for all $\lambda \geq 0$. Note that

$$|D_\lambda(Q)| \leq C \cdot 2^{-\lambda} |Q| \tag{3.32}$$

for all $\lambda \geq 0$ and $Q \in \mathscr{Q}$, where C does not depend on λ or Q. Now we cover the set $D_\lambda(Q)$ using dyadic cubes $E(Q)$ with $\operatorname{sidelength}(E(Q)) = 2^{-\lambda} \operatorname{sidelength}(Q)$, and call the collection of those cubes $\mathscr{V}_\lambda(Q)$. To be more precise,

$$\mathscr{V}_\lambda(Q) = \{E \in \mathscr{Q} : \operatorname{sidelength}(E) = 2^{-\lambda} \operatorname{sidelength}(Q),\ E \cap D_\lambda(Q) \neq \emptyset\}, \tag{3.33}$$

and we define

$$\mathscr{V}_\lambda = \bigcup_{Q \in \mathscr{Q}} \mathscr{V}_\lambda(Q). \tag{3.34}$$

The set covered by $\mathscr{V}_\lambda(Q)$ is illustrated by the shaded region in Figure 5, where the dashed lines represent the set of discontinuities $D(Q)$. The cardinality $\# \mathscr{V}_\lambda(Q)$ does not

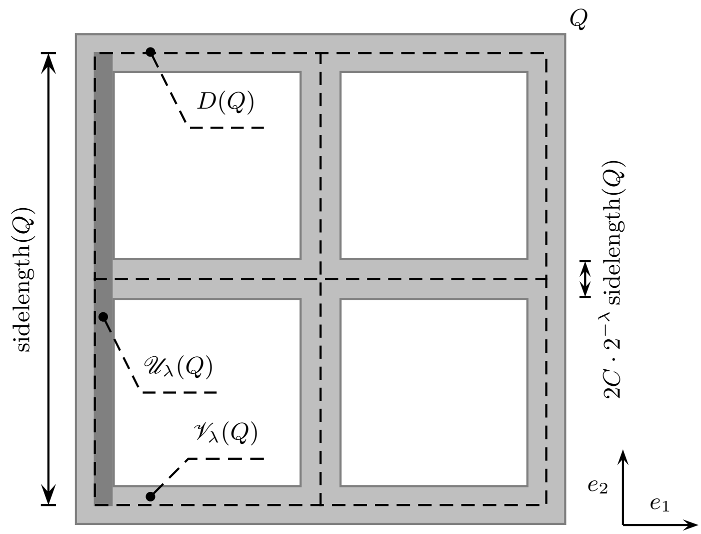

Fig. 5. The dyadic stripe $\mathscr{U}_\lambda(Q)$ embedded in the ring domain $\mathscr{V}_\lambda(Q)$ in dimension $n = 2$. The picture is drawn for $C = 1$.

depend on the choice of Q, so we note that

$$\# \mathscr{V}_\lambda(Q) \approx 2^{\lambda(n-1)}. \tag{3.35}$$

Finally, define the functions $d_{Q,\lambda}$ associated to the ring domain $\mathscr{V}_\lambda(Q)$ by

$$d_{Q,\lambda} = \sum_{E \in \mathscr{V}_\lambda(Q)} h_E, \tag{3.36}$$

and the *ring domain operator* H_λ by

$$H_\lambda u = \sum_{Q \in \mathscr{Q}} \langle u, h_Q \rangle \, d_{Q,\lambda} |Q|^{-1}. \tag{3.37}$$

In the subsequent theorem, H_λ is dominated by the stripe operator S_λ. This is done by covering the ring domain function $d_{Q,\lambda}$ with continuous mappings of the dyadic stripe functions $g_{Q,\lambda}$ (see identity (3.40)).

THEOREM 3.7. *Let X be a UMD-space, $1 < p < \infty$ and $n \in \mathbb{N}$. For $\lambda \geq 0$ we can dominate H_λ by S_λ, that is,*

$$\|H_\lambda u\|_{L^p_X} \leq C \|S_\lambda u\|_{L^p_X} \tag{3.38}$$

for all $u \in L^p_X(\mathbb{R}^n)$, where the constant C depends only on n, p and the UMD-constant of X.

A fortiori, we have the following estimate for H_λ.

COROLLARY 3.8. *Let X be a UMD-space, $1 < p < \infty$ and $n \in \mathbb{N}$. If $L^p_X(\mathbb{R}^n)$ has cotype $\mathcal{C}(L^p_X(\mathbb{R}^n))$, then there exists a constant $C > 0$ such that*

$$\|H_\lambda u\|_{L^p_X(\mathbb{R}^n)} \leq C \cdot 2^{-\lambda/\mathcal{C}(L^p_X(\mathbb{R}^n))} \|u\|_{L^p_X(\mathbb{R}^n)} \tag{3.39}$$

for every $u \in L^p_X(\mathbb{R}^n)$ and $\lambda \geq 0$, where C depends only on n, p, the UMD-constant of X and the cotype $\mathcal{C}(L^p_X(\mathbb{R}^n))$.

Proof. Once we have proved Theorem 3.7, we obtain Corollary 3.8 simply by plugging in estimate (3.25) for the stripe operator. ∎

Proof of Theorem 3.7. Let q denote the lower left corner of Q, that is,

$$q_i = \inf\{x_i : x \in Q\} \quad \text{for all } 1 \leq i \leq n,$$

where x_1 respectively q_1 denotes the orthogonal projection of $x \in \mathbb{R}^n$ respectively $q \in \mathbb{R}^n$ onto the vector $e_1 = (1, 0, \ldots, 0)$. Furthermore, let M_i be the orthogonal transformation swapping e_1 and e_i, that is, the linear extension of

$$M_i e_1 = e_i, \qquad M_i e_i = e_1, \qquad M_i e_j = e_j \quad \text{for all } j \notin \{1, i\},$$

and finally define the stripe functions

$$g_{Q,\lambda,i} = g_{Q,\lambda}(M_i(x - q) + q), \qquad Q \in \mathscr{Q},\ 1 \leq i \leq n,$$

and the stripe operators

$$S_{\lambda,i} h_Q = g_{Q,\lambda,i}, \qquad Q \in \mathscr{Q},\ 1 \leq i \leq n,$$

with respect to the coordinate i. Clearly, the operators $S_{\lambda,i}$, $1 \le i \le n$, have analogous properties to S_λ, in particular they satisfy the estimates

$$\|T_{ke_i} S_{\lambda,i} u\|_{L^p_X(\mathbb{R}^n)} \le C \cdot 2^{-\lambda/\mathcal{C}(L^p_X(\mathbb{R}^n))} \|u\|_{L^p_X(\mathbb{R}^n)}, \qquad 0 \le k \le 2^\lambda - 1,$$

for $u \in L^p_X(\mathbb{R}^n)$ and $\lambda \ge 0$. We can find a constant $C > 0$ and functions $|c_{Q,i,(k_1,\dots,k_n),m}| \le 1$, constant on dyadic cubes of measure $2^{-\lambda n}|Q|$, such that

$$d_{Q,\lambda} = \sum_{i,j=1}^{n} \sum_{|k_j| \le C} \sum_{m \in \{0, 2^{\lambda-1}-1, 2^\lambda - 1\}} T_{k_1 e_1} \circ \cdots \circ T_{k_n e_n} \circ T_{m e_i} c_{Q,i,(k_1,\dots,k_n),m} g_{Q,\lambda,i}. \tag{3.40}$$

The ring domain $\mathscr{V}_\lambda(Q)$ and the dyadic stripe $\mathscr{U}_\lambda(Q)$ are pictured in Figure 5. Plugging the previous identity into (3.37) and using estimate (2.9), we see that $\|H_\lambda u\|_{L^p_X}$ is dominated by a constant multiple of

$$\sum_{i=1}^{n} \sum_{k} \Big\| T_{ke_i} \sum_{Q \in \mathscr{Q}} u_Q c_{Q,i} g_{Q,\lambda,i} |Q|^{-1} \Big\|_{L^p_X},$$

where $u_Q = \langle u, h_Q \rangle$, and the summation over k extends over the set $\{0, 2^{\lambda-1} - 1, 2^\lambda - 1\}$. Also, for the sake of brevity, we dropped the rest of the subscripts for the function $c_{Q,i}$. Because we have the same properties in every coordinate $1 \le i \le n$, we only need to estimate

$$\Big\| T_{ke_1} \sum_{Q \in \mathscr{Q}} u_Q c_{Q,1} g_{Q,\lambda} |Q|^{-1} \Big\|_{L^p_X}$$

for all $k \in \{0, 2^{\lambda-1} - 1, 2^\lambda - 1\}$. Recall that

$$\sum_{Q \in \mathscr{Q}} u_Q c_{Q,1} g_{Q,\lambda} |Q|^{-1} = \sum_{Q \in \mathscr{Q}} \sum_{E \in \mathscr{U}_\lambda(Q)} u_Q c_{Q,1} h_E |Q|^{-1},$$

and observe that the collection

$$\{T_{ke_1} h_E : E \in \mathscr{U}_\lambda(Q),\ Q \in \mathscr{Q}\}$$

forms a martingale difference sequence, separately for every $0 \le k \le 2^\lambda - 1$. Since $|c_{Q,1}| \le 1$, we may estimate

$$\Big\| T_{ke_1} \sum_{Q \in \mathscr{Q}} u_Q c_{Q,1} g_{Q,\lambda} |Q|^{-1} \Big\|_{L^p_X} \lesssim \Big\| T_{ke_1} \sum_{Q \in \mathscr{Q}} u_Q g_{Q,\lambda} |Q|^{-1} \Big\|_{L^p_X}.$$

Since $g_{Q,\lambda} = S_\lambda h_Q$, we can now use estimate (3.24), and collecting all our inequalities yields

$$\|H_\lambda u\|_{L^p_X} \le C \|S_\lambda u\|_{L^p_X},$$

concluding the proof. ■

4. Decomposition of the directional Haar projection $P^{(\varepsilon)}$

Given $1 < p < \infty$ and an integer $n \geq 2$, the *directional Haar projection* $P^{(\varepsilon)} : L^p_X(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)$ is defined by

$$P^{(\varepsilon)}u = \sum_{Q \in \mathscr{Q}} \langle u, h_Q^{(\varepsilon)} \rangle h_Q^{(\varepsilon)} |Q|^{-1} \tag{4.1}$$

for all $u \in L^p_X(\mathbb{R}^n)$.

In order to estimate $P^{(\varepsilon)}$, we decompose it in Subsection 4.1 into a series of mollified operators $\sum_l P_l^{(\varepsilon)}$, following [LMM11]. Subsequently, wavelet expansions are used in [LMM11] to further analyze $P_l^{(\varepsilon)}$.

On the other hand, we decompose $P_l^{(\varepsilon)}$ into a series of stripe operators

$$P_l^{(\varepsilon)} = \sum_{\lambda(l)} c_{\lambda(l)} S_{\lambda(l)},$$

using martingale methods feasible in UMD-spaces. In Subsection 4.2 we use T. Figiel's martingale approach (see [Fig90]) to find a suitable representation for $P_l^{(\varepsilon)}$. In the following Subsection 4.3 we define the main cases for further decomposition of $P_l^{(\varepsilon)}$, which we then dominate by weighted series of ring domain operators H_λ in Subsection 4.4. In Subsection 4.5, we reduce the estimates for $P_l^{(\varepsilon)} R_{i_0}^{-1}$ to inequalities for $P_l^{(\varepsilon)}$.

4.1. Decomposition of $P^{(\varepsilon)}$ into $P_l^{(\varepsilon)}$. We give a brief overview of the Littlewood–Paley decomposition used in [LMM11] and continue with further decompositions in Subsections 4.2 and 4.3, different from the methods in [LMM11].

We utilize a compactly supported, smooth approximation of the identity to obtain a decomposition of $P^{(\varepsilon)}$ into a series of mollified operators $P_l^{(\varepsilon)}$,

$$P^{(\varepsilon)} = \sum_{l \in \mathbb{Z}} P_l^{(\varepsilon)}. \tag{4.2}$$

To this end, we fix $b \in C_c^\infty(]0,1[^n)$ such that

$$\int b(x)\,dx = 1 \quad \text{and} \quad \int x_i b(x_1, \ldots, x_i, \ldots, x_n)\,dx_i = 0 \tag{4.3}$$

for all $1 \leq i \leq n$. For every integer l define

$$\Delta_l u = u * d_l, \quad \text{where} \quad d_l(x) = 2^{ln} d(2^l x) \quad \text{and} \quad d(x) = 2^n\, b(2x) - b(x). \tag{4.4}$$

Then for all $u \in L^p_X(\mathbb{R}^n)$,

$$u = \sum_{l \in \mathbb{Z}} \Delta_l u, \tag{4.5}$$

with the series converging in $L^p_X(\mathbb{R}^n)$. Denoting by $\mathscr{Q}_j \subset \mathscr{Q}$ the collection of all dyadic cubes having measure 2^{-jn}, we set

$$P_l^{(\varepsilon)} u = \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathscr{Q}_j} \langle u, \Delta_{j+l}(h_Q^{(\varepsilon)})\rangle h_Q^{(\varepsilon)} |Q|^{-1}, \tag{4.6}$$

and observe that by (4.5), for all $u \in L^p_X(\mathbb{R}^n)$,

$$P^{(\varepsilon)} u = \sum_{l\in\mathbb{Z}} P_l^{(\varepsilon)} u,$$

where equality holds in the sense of $L^p_X(\mathbb{R}^n)$. Setting $f_{Q,l}^{(\varepsilon)} = \Delta_{j+l} h_Q^{(\varepsilon)}$, if $Q \in \mathscr{Q}_j$, we rewrite (4.6) as

$$P_l^{(\varepsilon)} u = \sum_{Q\in\mathscr{Q}} \langle u, f_{Q,l}^{(\varepsilon)}\rangle h_Q^{(\varepsilon)} |Q|^{-1}. \tag{4.7}$$

In contrast to [LMM11] we will rather estimate the operator

$$P_-^{(\varepsilon)} = \sum_{l<0} P_l^{(\varepsilon)} \tag{4.8}$$

instead of estimating each $P_l^{(\varepsilon)}$, $l < 0$, separately.

4.2. The integral kernels $K_l^{(\varepsilon)}$ and $K_-^{(\varepsilon)}$ of $P_l^{(\varepsilon)}$ and $P_-^{(\varepsilon)}$. In this subsection we identify the integral kernel $K_l^{(\varepsilon)}$ of the operator $P_l^{(\varepsilon)}$. As mentioned in Subsection 1.2, S. Müller asks in [Mül99] whether it is possible to obtain (1.2) in such a way that the original time-frequency decompositions are replaced by the canonical martingale decomposition of T. Figiel (see [Fig90]). This paper provides an affirmative answer to this question. The details of the decomposition are worked out in this subsection.

Note that

$$(P_l^{(\varepsilon)} u)(x) = \int K_l^{(\varepsilon)}(x,y) u(y)\, dy, \tag{4.9}$$

where

$$K_l^{(\varepsilon)}(x,y) = \sum_{Q\in\mathscr{Q}} h_Q^{(\varepsilon)}(x) f_{Q,l}^{(\varepsilon)}(y) |Q|^{-1}. \tag{4.10}$$

Now we expand $K_l^{(\varepsilon)}$ into the series

$$\sum_{\substack{\alpha,\beta\in\{0,1\}^n \\ (\alpha,\beta)\neq 0}} \sum_{\substack{K,M,Q\in\mathscr{Q} \\ |K|=|M|}} \langle h_Q^{(\varepsilon)}, h_K^{(\alpha)}\rangle \langle f_{Q,l}^{(\varepsilon)}, h_M^{(\beta)}\rangle |K|^{-1}|M|^{-1}|Q|^{-1} h_K^{(\alpha)}(x) h_M^{(\beta)}(y). \tag{4.11}$$

We seek a simpler algebraic form of (4.11), and therefore we distinguish the following settings for the parameters α and β, with $(\alpha,\beta) \neq 0$:

(1) $\beta \neq 0$, $\alpha \neq 0$,
(2) $\beta \neq 0$, $\alpha = 0$,
(3) $\beta = 0$.

Note that due to the condition $(\alpha,\beta) \neq 0$ in (4.11), case (3) clearly implies $\alpha \neq 0$.

In case (1), that is, $\beta \neq 0$ and $\alpha \neq 0$, we begin by rewriting the inner sum of (4.11) as

$$\sum_{\substack{K,M,Q\in\mathcal{Q}\\|K|=|M|}} \langle h_Q^{(\varepsilon)}, h_K^{(\alpha)}\rangle\langle f_{Q,l}^{(\varepsilon)}, h_M^{(\beta)}\rangle|K|^{-1}|M|^{-1}|Q|^{-1}h_K^{(\alpha)}(x)h_M^{(\beta)}(y)$$
$$= \sum_{M,Q\in\mathcal{Q}} \langle f_{Q,l}^{(\varepsilon)}, h_M^{(\beta)}\rangle|M|^{-1}|Q|^{-1}h_M^{(\beta)}(y) \sum_{\substack{K\in\mathcal{Q}\\|K|=|M|}} \langle h_Q^{(\varepsilon)}, h_K^{(\alpha)}\rangle|K|^{-1}h_K^{(\alpha)}(x).$$

If we now sum this identity over all $\alpha \neq 0$, we get

$$\sum_{\substack{M,Q\in\mathcal{Q}\\|M|=|Q|}} \langle f_{Q,l}^{(\varepsilon)}, h_M^{(\beta)}\rangle|M|^{-1}|Q|^{-1}h_Q^{(\varepsilon)}(x)h_M^{(\beta)}(y) \tag{4.12}$$

for all $\beta \neq 0$ in case (1).

In case (2), that is, $\beta \neq 0$ and $\alpha = 0$, the inner sum of (4.11) reads

$$\sum_{M,Q\in\mathcal{Q}} \langle f_{Q,l}^{(\varepsilon)}, h_M^{(\beta)}\rangle|M|^{-1}|Q|^{-1}h_M^{(\beta)}(y) \sum_{\substack{K\in\mathcal{Q}\\|K|=|M|}} \langle h_Q^{(\varepsilon)}, \mathbb{1}_K\rangle|K|^{-1}\cdot\mathbb{1}_K(x).$$

Observe that the second sum is the conditional expectation of $h_Q^{(\varepsilon)}$, thus it is zero if $|K| \geq |Q|$, and $h_Q(x)$ if $|K| < |Q|$. So in case (2) we get

$$\sum_{\substack{M,Q\in\mathcal{Q}\\|M|<|Q|}} \langle f_{Q,l}^{(\varepsilon)}, h_M^{(\beta)}\rangle|M|^{-1}|Q|^{-1}h_Q^{(\varepsilon)}(x)h_M^{(\beta)}(y), \tag{4.13}$$

with $\beta \neq 0$ fixed.

Finally, in case (3) we know that $\beta = 0$ and $\alpha \neq 0$, as noted before. Therefore, the inner sum of (4.11) reads

$$\sum_{\substack{K,M,Q\in\mathcal{Q}\\|K|=|M|}} \langle h_Q^{(\varepsilon)}, h_K^{(\alpha)}\rangle\langle f_{Q,l}^{(\varepsilon)}, \mathbb{1}_M\rangle|K|^{-1}|M|^{-1}|Q|^{-1}h_K^{(\alpha)}(x)\cdot\mathbb{1}_M(y)$$
$$= \sum_{\substack{M,Q\in\mathcal{Q}\\|M|=|Q|}} \langle f_{Q,l}^{(\varepsilon)}, \mathbb{1}_M\rangle|M|^{-1}|Q|^{-1}h_Q^{(\varepsilon)}(x)\cdot\mathbb{1}_M(y).$$

Expanding the y-component of the last expression into a Haar series yields

$$\sum_{\substack{\gamma\in\{0,1\}^n\\\gamma\neq 0}} \sum_{\substack{K,M,Q\in\mathcal{Q}\\|M|=|Q|}} \langle f_{Q,l}^{(\varepsilon)}, \mathbb{1}_M\rangle\langle h_K^{(\gamma)}, \mathbb{1}_M\rangle|K|^{-1}|M|^{-1}|Q|^{-1}h_Q^{(\varepsilon)}(x)h_K^{(\gamma)}(y)$$
$$= \sum_{\substack{\gamma\in\{0,1\}^n\\\gamma\neq 0}} \sum_{K,Q\in\mathcal{Q}} h_Q^{(\varepsilon)}(x)h_K^{(\gamma)}(y)|K|^{-1}|Q|^{-1} \sum_{\substack{M\subsetneq K\\|M|=|Q|}} \langle f_{Q,l}^{(\varepsilon)}, \mathbb{1}_M\rangle\langle h_K^{(\gamma)}, \mathbb{1}_M\rangle|M|^{-1}$$
$$= \sum_{\substack{\gamma\in\{0,1\}^n\\\gamma\neq 0}} \sum_{\substack{K,Q\in\mathcal{Q}\\|Q|<|K|}} h_Q^{(\varepsilon)}(x)h_K^{(\gamma)}(y)|K|^{-1}|Q|^{-1}\Big\langle f_{Q,l}^{(\varepsilon)}, \sum_{\substack{M\subsetneq K\\|M|=|Q|}} \mathbb{1}_M\langle h_K^{(\gamma)}, \mathbb{1}_M\rangle|M|^{-1}\Big\rangle.$$

Observe that with K and Q fixed, the inner sum is indeed the conditional expectation of

$h_K^{(\gamma)}$ at a finer scale. Hence, $h_K^{(\gamma)}$ is reproduced, i.e.

$$\sum_{\substack{M\subsetneq K\\ |M|=|Q|}} \mathbb{1}_M \langle h_K^{(\gamma)}, \mathbb{1}_M\rangle |M|^{-1} = h_K^{(\gamma)},$$

and we obtain

$$\sum_{\substack{\gamma\in\{0,1\}^n\\ \gamma\neq 0}} \sum_{\substack{K,Q\in\mathscr{Q}\\ |Q|<|K|}} \langle f_{Q,l}^{(\varepsilon)}, h_K^{(\gamma)}\rangle |K|^{-1}|Q|^{-1} h_Q^{(\varepsilon)}(x) h_K^{(\gamma)}(y) \tag{4.14}$$

in case (3).

Summing (4.12) and (4.13) over all $\beta\neq 0$ and adding (4.14) yields

$$\begin{aligned} K_l^{(\varepsilon)}(x,y) &= \sum_{\substack{\gamma\in\{0,1\}^n\\ \gamma\neq 0}} K_l^{(\varepsilon,\gamma)}(x,y),\\ K_l^{(\varepsilon,\gamma)}(x,y) &= \sum_{M,Q\in\mathscr{Q}} \langle f_{Q,l}^{(\varepsilon)}, h_M^{(\gamma)}\rangle |M|^{-1}|Q|^{-1} h_Q^{(\varepsilon)}(x) h_M^{(\gamma)}(y). \end{aligned} \tag{4.15}$$

We summarize the results of the preceding discussion in

Proposition 4.1. *For fixed $\varepsilon\in\{0,1\}^n\setminus\{0\}$ and every $l\in\mathbb{Z}$ and $\gamma\in\{0,1\}^n\setminus\{0\}$ let*

$$\begin{aligned} (P_l^{(\varepsilon,\gamma)}u)(x) &= \int K_l^{(\varepsilon,\gamma)}(x,y)u(y)\,dy \qquad \textit{for all } u\in L_X^p(\mathbb{R}^n),\\ K_l^{(\varepsilon,\gamma)}(x,y) &= \sum_{M,Q\in\mathscr{Q}} \langle f_{Q,l}^{(\varepsilon)}, h_M^{(\gamma)}\rangle |M|^{-1}|Q|^{-1} h_Q^{(\varepsilon)}(x) h_M^{(\gamma)}(y), \end{aligned} \tag{4.16}$$

and $f_{Q,l}^{(\varepsilon)} = \Delta_{j+l} h_Q^{(\varepsilon)}$ for all $Q\in\mathscr{Q}_j$ (see (4.4) for details). If we define

$$P_-^{(\varepsilon,\gamma)} = \sum_{l<0} P_l^{(\varepsilon,\gamma)} \quad \textit{and} \quad f_Q^{(\varepsilon)} = \sum_{l<0} f_{Q,l}^{(\varepsilon)}, \tag{4.17}$$

then the integral kernel $K_-^{(\varepsilon,\gamma)}(x,y)$ of $P_-^{(\varepsilon,\gamma)}$ is given by

$$\begin{aligned} (P_-^{(\varepsilon,\gamma)}u)(x) &= \int K_-^{(\varepsilon,\gamma)}(x,y)u(y)\,dy,\\ K_-^{(\varepsilon,\gamma)}(x,y) &= \sum_{M,Q\in\mathscr{Q}} \langle f_{Q}^{(\varepsilon)}, h_M^{(\gamma)}\rangle |M|^{-1}|Q|^{-1} h_Q^{(\varepsilon)}(x) h_M^{(\gamma)}(y). \end{aligned} \tag{4.18}$$

Furthermore, we have the following decomposition of the directional Haar projection $P^{(\varepsilon)}$:

$$P^{(\varepsilon)} = \sum_{\substack{\gamma\in\{0,1\}^n\\ \gamma\neq 0}} \Big(P_-^{(\varepsilon,\gamma)} + \sum_{l\geq 0} P_l^{(\varepsilon,\gamma)} \Big), \tag{4.19}$$

where equality holds true pointwise in $L_X^p(\mathbb{R}^n)$.

Remark 4.2. To ease notation we will drop the superscripts (ε), (γ) and (ε,γ) from all of the operators $P_l^{(\varepsilon)}$, $P_l^{(\varepsilon,\gamma)}$, $P_-^{(\varepsilon)}$, $P_-^{(\varepsilon,\gamma)}$, their respective kernels $K_l^{(\varepsilon)}$, $K_l^{(\varepsilon,\gamma)}$, $K_-^{(\varepsilon)}$, $K_-^{(\varepsilon,\gamma)}$, as well as from the mollified Haar functions $f_{Q,l}^{(\varepsilon)}$, $f_Q^{(\varepsilon)}$ and the Haar functions $h_Q^{(\varepsilon)}$, $h_Q^{(\gamma)}$. Compare Remark 2.1.

By dropping the superscripts we obtain the following generic representation for the integral kernels $K_l^{(\varepsilon,\gamma)}$ and $K_-^{(\varepsilon,\gamma)}$, abbreviated as K_l and K_-:

$$K_l(x,y) = \sum_{M,Q\in\mathscr{Q}} \langle f_{Q,l}, h_M\rangle |M|^{-1}|Q|^{-1} h_Q(x)h_M(y),$$
$$K_-(x,y) = \sum_{M,Q\in\mathscr{Q}} \langle f_Q, h_M\rangle |M|^{-1}|Q|^{-1} h_Q(x)h_M(y).$$

Note that by (4.18) and (4.16), both the Haar functions h_M share (γ) and both h_Q share the superscript (ε). Throughout this article we will work with the generic representation of the operators and will interpret every occurrence of a Haar function so that each occurrence of a Haar function might have a different superscript, i.e.

$$K_l(x,y) = \sum_{M,Q\in\mathscr{Q}} \langle f_{Q,l}^{(\alpha_Q)}, h_M^{(\beta_M)}\rangle |M|^{-1}|Q|^{-1} h_Q^{(\gamma_Q)}(x)h_M^{(\delta_M)}(y),$$
$$K_-(x,y) = \sum_{M,Q\in\mathscr{Q}} \langle f_Q^{(\alpha'_Q)}, h_M^{(\beta'_M)}\rangle |M|^{-1}|Q|^{-1} h_Q^{(\gamma'_Q)}(x)h_M^{(\delta'_M)}(y),$$

where each of the above superscripts is a vector in $\{0,1\}^n \setminus \{0\}$. In correspondence with (4.16)–(4.19) we obtain the generic operators P_l and P_- with their respective integral kernels K_l and K_-, as well as the generic mollified Haar functions $f_{Q,l}$ and f_Q.

4.3. Decomposition of P_l—the main cases. Henceforth we will use the notation of Remark 4.2. We will decompose the operator P_l guided by the different behavior of the coefficients $\langle f_{Q,l}, h_M\rangle$, $l \geq 0$, $M \in \mathscr{Q}$, and $\langle f_{Q,l}, h_M\rangle$, $l < 0$, $M \in \mathscr{Q}$. This is primarily caused by the different shape of the support of the functions $f_{Q,l}$, $l \geq 0$, and $f_{Q,l}$, $l < 0$ (compare the support inclusions in (4.20) and (4.21) below), in relation to the size of the cubes M. We remind the reader that h_Q is an abbreviation for one of $h_Q^{(\gamma)}$, $\gamma \in \{0,1\}^n \setminus \{0\}$.

4.3.1. Estimates for the coefficients. First, we want to investigate the mollified Haar functions $f_{Q,l}$, $l \in \mathbb{Z}$. To this end, let $D(Q)$ denote the set of discontinuities of the Haar function h_Q. Then

$$D_l(Q) = \{x \in \mathbb{R}^n : \operatorname{dist}(x, D(Q)) \leq C \cdot 2^{-l} \operatorname{diam}(Q)\}.$$

If $l \geq 0$, note that

$$\begin{aligned} \int f_{Q,l}(x)\,dx = 0, \qquad \operatorname{supp} f_{Q,l} \subset D_l(Q),\\ |f_{Q,l}| \leq C, \qquad \operatorname{Lip}(f_{Q,l}) \leq C \cdot 2^l (\operatorname{diam}(Q))^{-1}, \end{aligned} \tag{4.20}$$

and if $l \leq 0$, we have

$$\begin{aligned} \int f_{Q,l}(x)\,dx = 0, \qquad \operatorname{supp} f_{Q,l} \subset C \cdot 2^{|l|} Q,\\ |f_{Q,l}| \leq C \cdot 2^{-|l|(n+1)}, \qquad \operatorname{Lip}(f_{Q,l}) \leq C \cdot 2^{-|l|(n+2)} (\operatorname{diam}(Q))^{-1}, \end{aligned} \tag{4.21}$$

where the constant C does not depend on l or Q.

Recall that for $Q \in \mathscr{Q}_j$ we defined

$$f_{Q,l} = \Delta_{j+l} h_Q = h_Q * d_{j+l} = h_Q * (b_{j+l+1} - b_{j+l}).$$

Taking the sum over $l < 0$ yields

$$\sum_{l<0} f_{Q,l} = h_Q * b_j,$$

hence the mollified Haar functions f_Q defined in (4.17) are given by

$$f_Q = h_Q * b_j \quad \text{for all } Q \in \mathscr{Q}_j,$$

where $b_j(x) = 2^{jn} b(2^j x)$. The functions f_Q have the following properties, which are easily verified: there exists a $C > 0$ independent of Q such that

$$\begin{aligned} \int f_Q(x)\,dx = 0, &\qquad \operatorname{supp} f_Q \subset CQ, \\ |f_Q| \le C, &\qquad \operatorname{Lip}(f_Q) \le C(\operatorname{diam}(Q))^{-1}, \end{aligned} \tag{4.22}$$

for all $Q \in \mathscr{Q}$.

Proposition 4.3 stated below estimates the coefficients $\langle f_{Q,l}, h_M\rangle$, $l \ge 0$, and $\langle f_Q, h_M\rangle$. The different behavior of the inequalities is determined by the ratio of the diameters of the cubes Q and M.

Proposition 4.3. *For all dyadic cubes $Q, M \in \mathscr{Q}$ we have the following estimates for the coefficients $\langle f_{Q,l}, h_M\rangle$, $l \ge 0$:*

(1) *If* $\operatorname{diam}(Q) \le \operatorname{diam}(M)$, *then*

$$|\langle f_{Q,l}, h_M\rangle| \le C \cdot 2^{-l} |Q|. \tag{4.23}$$

(2) *If* $2^{-l} \operatorname{diam}(Q) \le \operatorname{diam}(M) < \operatorname{diam}(Q)$, *we get*

$$|\langle f_{Q,l}, h_M\rangle| \le C \cdot 2^{-l} \operatorname{diam}(Q)(\operatorname{diam}(M))^{n-1}. \tag{4.24}$$

(3) *If* $\operatorname{diam}(M) < 2^{-l} \operatorname{diam}(Q)$, *we obtain*

$$|\langle f_{Q,l}, h_M\rangle| \le C \cdot 2^{l} \frac{\operatorname{diam}(M)}{\operatorname{diam}(Q)} |M|. \tag{4.25}$$

The constant C does not depend on l, Q or M.

Moreover, for all dyadic cubes $Q, M \in \mathscr{Q}$ we have:

(4) *If* $\operatorname{diam}(M) \le \operatorname{diam}(Q)$, *then*

$$|\langle f_Q, h_M\rangle| \le C(\operatorname{diam}(Q))^{-1}(\operatorname{diam}(M))^{n+1}. \tag{4.26}$$

(5) *If* $\operatorname{diam}(M) > \operatorname{diam}(Q)$, *we have*

$$|\langle f_Q, h_M\rangle| \le C|Q|. \tag{4.27}$$

The constant C does not depend on Q or M.

Proof. First, we want to estimate $\langle f_{Q,l}, h_M\rangle$, so we fix $l \ge 0$ and $Q, M \in \mathscr{Q}$.

If $\operatorname{diam}(Q) \le \operatorname{diam}(M)$, then using $|D_l(Q)| \lesssim 2^{-l}|Q|$ and exploiting the boundedness of $f_{Q,l}$ and h_M implies (4.23).

If $2^{-l} \operatorname{diam}(Q) \le \operatorname{diam}(M) < \operatorname{diam}(Q)$, then the measure estimate

$$|D_l(Q) \cap M| \lesssim 2^{-l} \operatorname{diam}(Q)(\operatorname{diam}(M))^{n-1}$$

together with (4.20) yields (4.24).

If $\operatorname{diam}(M) < 2^{-l} \operatorname{diam}(Q)$, then in view of $\operatorname{Lip}(f_{Q,l}) \lesssim 2^l (\operatorname{diam}(Q))^{-1}$ and $\int h_M = 0$ in (4.20) we may infer (4.25).

Now we turn to the estimates for $\langle f_Q, h_M\rangle$, $Q, M \in \mathscr{Q}$. If $\operatorname{diam}(M) \le \operatorname{diam}(Q)$, we make use of

$$\operatorname{Lip}(f_Q) \le C(\operatorname{diam}(Q))^{-1},$$

according to (4.22), and we obtain (4.26).

For $\operatorname{diam}(M) > \operatorname{diam}(Q)$, we exploit

$$|f_Q| \le C \quad \text{and} \quad \operatorname{supp} f_Q \subset CQ$$

in (4.22) to obtain (4.27). ∎

Remark 4.4. Observe that the coefficients $\langle f_{Q,l}, h_M\rangle$ respectively $\langle f_Q, h_M\rangle$ vanish if the support of $f_{Q,l}$ respectively f_Q is contained in a set where h_M is constant (see Figure 6 on p. 44). More precisely, if we can find a $K \in \mathscr{Q}$ with $\pi(K) = M$ such that

$$\operatorname{supp} f_{Q,l} \subset K \quad \text{respectively} \quad \operatorname{supp} f_Q \subset K,$$

then certainly

$$\langle f_{Q,l}, h_M\rangle = 0 \quad \text{respectively} \quad \langle f_Q, h_M\rangle = 0.$$

Finally, note that for $\operatorname{diam}(M) > \operatorname{diam}(Q)$ the cubes Q for which $\langle f_{Q,l}, h_M\rangle \ne 0$ respectively $\langle f_Q, h_M\rangle \ne 0$ cluster in the vicinity of $D(M)$, the set of h_M's discontinuities.

4.3.2. Definition of the main cases. For each $l \ge 0$ we split the set $\mathscr{Q} \times \mathscr{Q}$ according to the cases in Proposition 4.3 into the three disjoint collections

$$\mathscr{A}_l = \{(Q, M) : \operatorname{diam}(Q) \le \operatorname{diam}(M)\}, \tag{4.28}$$

$$\mathscr{B}_l = \{(Q, M) : 2^{-l} \operatorname{diam}(Q) \le \operatorname{diam}(M) < \operatorname{diam}(Q)\}, \tag{4.29}$$

$$\mathscr{C}_l = \{(Q, M) : \operatorname{diam}(M) < 2^{-l} \operatorname{diam}(Q)\}, \tag{4.30}$$

respectively the two disjoint collections

$$\mathscr{A}_- = \{(Q, M) : \operatorname{diam}(M) \le \operatorname{diam}(Q)\}, \tag{4.31}$$

$$\mathscr{B}_- = \{(Q, M) : \operatorname{diam}(M) > \operatorname{diam}(Q)\}. \tag{4.32}$$

Then we define the integral kernels

$$A_l(x, y) = \sum_{(Q,M)\in\mathscr{A}_l} \langle f_{Q,l}, h_M\rangle h_Q(x) h_M(y) |Q|^{-1} |M|^{-1}, \tag{4.33}$$

$$B_l(x, y) = \sum_{(Q,M)\in\mathscr{B}_l} \langle f_{Q,l}, h_M\rangle h_Q(x) h_M(y) |Q|^{-1} |M|^{-1}, \tag{4.34}$$

$$C_l(x, y) = \sum_{(Q,M)\in\mathscr{C}_l} \langle f_{Q,l}, h_M\rangle h_Q(x) h_M(y) |Q|^{-1} |M|^{-1}, \tag{4.35}$$

respectively

$$A_-(x, y) = \sum_{(Q,M)\in\mathscr{A}_-} \langle f_Q, h_M\rangle h_Q(x) h_M(y) |Q|^{-1} |M|^{-1}, \tag{4.36}$$

$$B_-(x, y) = \sum_{(Q,M)\in\mathscr{B}_-} \langle f_Q, h_M\rangle h_Q(x) h_M(y) |Q|^{-1} |M|^{-1}, \tag{4.37}$$

and associate to each integral kernel the induced operator,

$$(A_l u)(x) = \int A_l(x,y)u(y)\,dy, \tag{4.38}$$

$$(B_l u)(x) = \int B_l(x,y)u(y)\,dy, \tag{4.39}$$

$$(C_l u)(x) = \int C_l(x,y)u(y)\,dy, \tag{4.40}$$

respectively

$$(A_- u)(x) = \int A_-(x,y)u(y)\,dy, \tag{4.41}$$

$$(B_- u)(x) = \int B_-(x,y)u(y)\,dy. \tag{4.42}$$

Finally, note that

$$P_l = A_l + B_l + C_l \qquad \text{for all } l \geq 0, \tag{4.43}$$

$$P_- = A_- + B_-. \tag{4.44}$$

4.4. Estimates for P_l, $l \geq 0$, and P_-. We will show that each of the operators A_l, B_l^*, C_l^* and A_-^*, B_- (see Subsection 4.3.2) can be controlled by certain weighted series of ring domain operators; for details on H_λ we refer the reader to Subsection 3.4.

Combining the results for A_l, B_l and C_l, respectively A_-^* and B_-, yields the following result.

Theorem 4.5. *Let X be a UMD-space, $1 < p < \infty$ and $n \in \mathbb{N}$. Let $L^p_X(\mathbb{R}^n)$ have type $\mathcal{T}(L^p_X(\mathbb{R}^n))$. Then there exists a constant $C > 0$ such that for all $l \geq 0$ and every $u \in L^p_X(\mathbb{R}^n)$ we have*

$$\|P_l u\|_{L^p_X(\mathbb{R}^n)} \leq C \cdot 2^{-l(1-1/\mathcal{T}(L^p_X(\mathbb{R}^n)))} \|u\|_{L^p_X(\mathbb{R}^n)}, \tag{4.45}$$

where C depends only on n, p, the UMD-constant of X and the type $\mathcal{T}(L^p_X(\mathbb{R}^n))$.

Moreover, there exists a constant $C > 0$ such that for all $u \in L^p_X(\mathbb{R}^n)$,

$$\|P_- u\|_{L^p_X(\mathbb{R}^n)} \leq C \|u\|_{L^p_X(\mathbb{R}^n)}, \tag{4.46}$$

where C depends only on n, p, the UMD-constant of X and the type $\mathcal{T}(L^p_X(\mathbb{R}^n))$.

The proof of the theorem is divided into seven parts:

- Subsection 4.4.1: Estimates for A_l.
- Subsection 4.4.2: Estimates for B_l.
- Subsection 4.4.3: Estimates for C_l.
- Subsection 4.4.4: Summary for P_l.
- Subsection 4.4.5: Estimates for A_-.
- Subsection 4.4.6: Estimates for B_-.
- Subsection 4.4.7: Summary for P_-.

Keeping in mind that

$$P_l = A_l + B_l + C_l, \quad \text{respectively} \quad P_- = A_- + B_-,$$

we will have proved the theorem once we establish the inequalities (4.47)–(4.49), summarized in Subsection 4.4.4, respectively (4.50)–(4.51), summarized in Subsection 4.4.7.

4.4.1. Estimates for A_l. In view of (4.28), (4.33) and (4.38) note that $\operatorname{diam}(Q) \leq \operatorname{diam}(M)$, and so we may utilize inequality (4.23). This setting is illustrated in Figure 6.

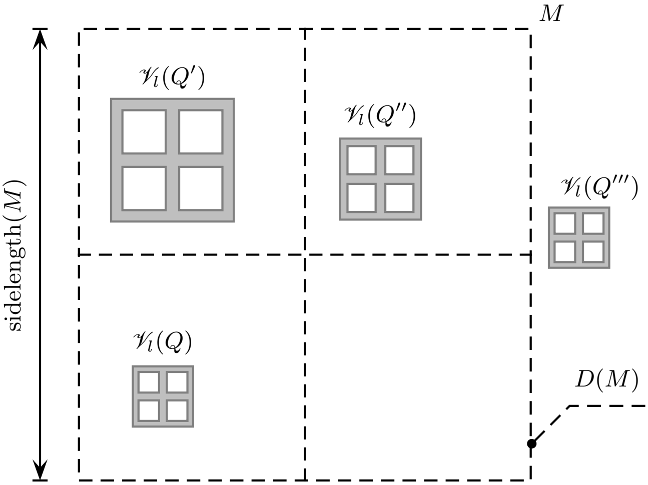

Fig. 6. The ring domains $\mathscr{V}_l(Q)$, $\mathscr{V}_l(Q')$, $\mathscr{V}_l(Q'')$, $\mathscr{V}_l(Q''')$ are contained in sets where the Haar function h_M is constant.

First, we split the set $\mathscr{A}_l$ (see (4.28)) into the disjoint collections $\mathscr{A}_{l,\lambda}$, $\lambda \geq 0$, given by

$$\mathscr{A}_{l,\lambda} = \{(Q, M) \in \mathscr{A}_l : \operatorname{diam}(Q) = 2^{-\lambda} \operatorname{diam}(M)\},$$

and define the operator $A_{l,\lambda}$ accordingly, that is,

$$A_{l,\lambda} u = \sum_{(Q,M) \in \mathscr{A}_{l,\lambda}} \langle f_{Q,l}, h_M \rangle h_Q u_M |Q|^{-1} |M|^{-1}$$

for all $u = \sum_{K \in \mathscr{Q}} u_K h_K |K|^{-1}$. Clearly,

$$A_l u = \sum_{\lambda=0}^{\infty} A_{l,\lambda} u.$$

Recalling that the coefficients $\langle f_{Q,l}, h_M \rangle$ vanish if h_M is constant on the support of $f_{Q,l}$ (see Remark 4.4) and the definition of the ring domain (3.33), we see that

$$\{Q : \langle f_{Q,l}, h_M \rangle \neq 0\} \subset \{Q : Q \cap D_\lambda(M) \neq \emptyset\} = \mathscr{V}_\lambda(M).$$

Using this fact, we have the identity

$$A_{l,\lambda} u = \sum_{M \in \mathscr{Q}} u_M |M|^{-1} \sum_{Q \in \mathscr{V}_\lambda(M)} \langle f_{Q,l}, h_M \rangle |Q|^{-1} h_Q,$$

hence glancing at inequality (4.23), utilizing the UMD-property and Kahane's contraction principle we obtain

$$\|A_{l,\lambda}u\|_{L^p_X(\mathbb{R}^n)} \lesssim 2^{-l}\Big\| \sum_{M\in\mathscr{Q}} u_M|M|^{-1} \sum_{Q\in\mathscr{V}_\lambda(M)} h_Q\Big\|_{L^p_X(\mathbb{R}^n)}$$
$$= 2^{-l}\Big\| \sum_{M\in\mathscr{Q}} u_M d_{M,\lambda}|M|^{-1}\Big\|_{L^p_X(\mathbb{R}^n)} = 2^{-l}\|H_\lambda u\|_{L^p_X(\mathbb{R}^n)}.$$

The last equality is the definition of the ring domain operator H_λ (see (3.37)). Applying the triangle inequality, using the above estimate for $A_{l,\lambda}$ and invoking Corollary 3.8 yields

$$\|A_l u\|_{L^p_X(\mathbb{R}^n)} \lesssim 2^{-l}\sum_{\lambda=0}^{\infty} 2^{-\lambda/\mathcal{C}(L^p_X(\mathbb{R}^n))}\|u\|_{L^p_X(\mathbb{R}^n)}.$$

Evaluating the geometric series we obtain the estimate

$$\|A_l u\|_{L^p_X(\mathbb{R}^n)} \le C\cdot 2^{-l}\|u\|_{L^p_X(\mathbb{R}^n)}, \tag{4.47}$$

where C depends on n, p, the UMD-constant of X and the cotype $\mathcal{C}(L^p_X(\mathbb{R}^n))$.

REMARK 4.6. Note that with $\lambda \ge 0$ fixed, the collections $\mathscr{V}_\lambda(M)$ are not disjoint as M ranges over $\mathscr{Q}$. But since the number of overlaps is bounded by a constant depending solely on the dimension n and the constant appearing in the definition of $D_\lambda(Q)$, the above proof still applies.

4.4.2. Estimates for B_l. In view of (4.29), (4.34) and (4.39) note that $2^{-l}\operatorname{diam}(Q) \le \operatorname{diam}(M) < \operatorname{diam}(Q)$, and so we may utilize inequality (4.24). This setting is visualized in Figure 7.

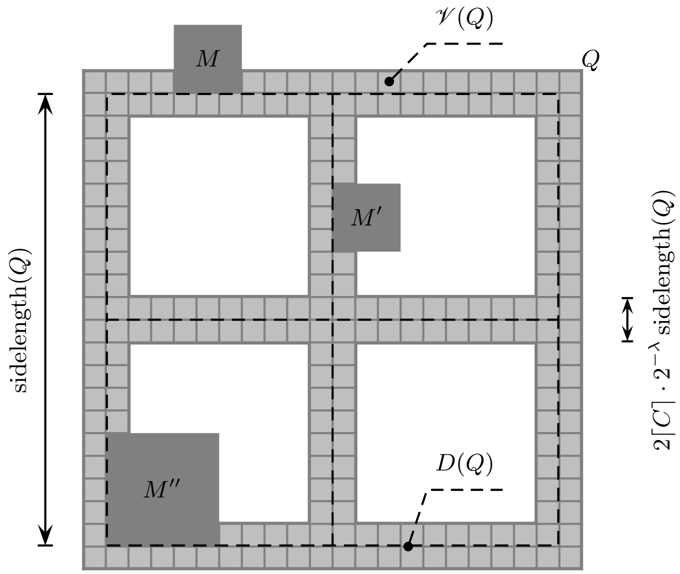

Fig. 7. The cubes M, M' and M'' intersect the ring domain $\mathscr{V}_l(Q)$.

This time we prefer to analyze B_l^*, of course with respect to the norm $\|\cdot\|_{L^q_Y(\mathbb{R}^n)}$, where $Y = X^*$ and $1/p+1/q=1$. As before, we parametrize the series according to the ratio of the sizes of Q and M. So we split the set $\mathscr{B}_l$ (see (4.29)) into the disjoint collections $\mathscr{B}_{l,\lambda}$, $\lambda \geq 0$, given by

$$\mathscr{B}_{l,\lambda} = \{(Q,M) \in \mathscr{B}_l : \operatorname{diam}(M) = 2^{-\lambda} \operatorname{diam}(Q)\},$$

and define the operator $B_{l,\lambda}$ accordingly, that is,

$$B_{l,\lambda} u = \sum_{(Q,M)\in\mathscr{B}_{l,\lambda}} \langle f_{Q,l}, h_M \rangle h_Q u_M |Q|^{-1} |M|^{-1}$$

for all $u = \sum_{K\in\mathscr{Q}} u_K h_K |K|^{-1}$.

Note that for $(Q,M) \in \mathscr{B}_{l,\lambda}$ we have

$$\{M : \langle f_{Q,l}, h_M\rangle \neq 0\} \subset \{M : M \cap D_l(Q) \neq \emptyset\} = \mathscr{V}_\lambda(Q),$$

hence we can rewrite $B^*_{l,\lambda} u$ as

$$B^*_{l,\lambda} u = \sum_{Q\in\mathscr{Q}} u_Q |Q|^{-1} \sum_{M\in\mathscr{V}_\lambda(Q)} \langle f_{Q,l}, h_M\rangle |M|^{-1} h_M.$$

Taking the norm, utilizing the UMD-property and applying Kahane's contraction principle to (4.24) yields the estimate

$$\begin{aligned} \|B^*_{l,\lambda} u\|_{L^q_Y(\mathbb{R}^n)} &\lesssim 2^{-l} \Big\| \sum_{Q\in\mathscr{Q}} u_Q |Q|^{-1} \sum_{M\in\mathscr{V}_\lambda(Q)} h_M \Big\|_{L^q_Y(\mathbb{R}^n)} \\ &= 2^{-l} \Big\| \sum_{Q\in\mathscr{Q}} u_Q \, d_{Q,\lambda} |Q|^{-1} \Big\|_{L^q_Y(\mathbb{R}^n)} = 2^{-l} \|H_\lambda u\|_{L^q_Y(\mathbb{R}^n)}. \end{aligned}$$

The last equality is the definition of the ring domain operator H_λ (see (3.37)). Recall

$$B^*_l u = \sum_{\lambda=0}^{\infty} B^*_{l,\lambda} u,$$

so applying the triangle inequality, using the above estimate for $B^*_{l,\lambda}$ and invoking Corollary 3.8 yields

$$\|B^*_l u\|_{L^q_Y(\mathbb{R}^n)} \lesssim 2^{-l} \sum_{\lambda=1}^{l} 2^\lambda \|H_\lambda u\|_{L^q_Y(\mathbb{R}^n)} \lesssim 2^{-l} \sum_{\lambda=1}^{l} 2^{\lambda(1-1/\mathcal{C}(L^q_Y(\mathbb{R}^n)))} \|u\|_{L^q_Y(\mathbb{R}^n)}.$$

Evaluating the geometric series we obtain the estimate

$$\|B^*_l u\|_{L^q_Y(\mathbb{R}^n)} \leq C \cdot 2^{-l/\mathcal{C}(L^q_Y(\mathbb{R}^n))} \|u\|_{L^q_Y(\mathbb{R}^n)}, \tag{4.48}$$

where C depends only on n, q, the UMD-constant of Y and the cotype $\mathcal{C}(L^q_Y(\mathbb{R}^n))$.

4.4.3. Estimates for C_l. In view of (4.30), (4.35) and (4.40) note that now $\operatorname{diam}(M) < 2^{-l} \operatorname{diam}(Q)$, and so we may utilize inequality (4.25). This setting is visualized in Figure 8.

As in the preceding case we aim at estimating the adjoint operator C^*_l; so with $Y = X^*$ and $1/p + 1/q = 1$, we split the collection $\mathscr{C}_l$ (see (4.30)) into the disjoint collections $\mathscr{C}_{l,\lambda}$, $\lambda \geq l+1$, given by

$$\mathscr{C}_{l,\lambda} = \{(Q,M) \in \mathscr{B}_l : \operatorname{diam}(M) = 2^{-\lambda} \operatorname{diam}(Q)\}.$$

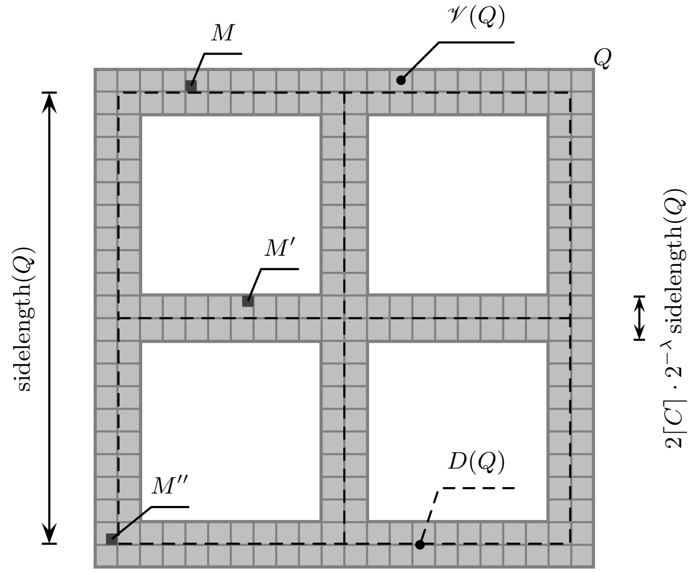

Fig. 8. The tiny cubes M, M' and M'' are contained in the cover of the ring domain $\mathscr{V}_l(Q)$.

We define the operator $C_{l,\lambda}$ accordingly, that is,

$$C_{l,\lambda} u = \sum_{(Q,M)\in\mathscr{C}_{l,\lambda}} \langle f_{Q,l}, h_M\rangle h_Q u_M |Q|^{-1}|M|^{-1}$$

for all $u = \sum_{K\in\mathscr{Q}} u_K h_K |K|^{-1}$. The adjoint operators C_l^* and $C_{l,\lambda}^*$ are given by

$$C_l^* u = \sum_{\lambda=l+1}^{\infty} \sum_{Q,M\in\mathscr{C}_{l,\lambda}} \langle f_{Q,l}, h_M\rangle |M|^{-1} h_M u_Q |Q|^{-1} = \sum_{\lambda=l+1}^{\infty} C_{l,\lambda}^* u.$$

Observe that for $(Q,M)\in\mathscr{C}_{l,\lambda}$ we have

$$\{M : \langle f_{Q,l}, h_M\rangle \neq 0\} \subset \{M : M\cap D_l(Q)\neq\emptyset\},$$

therefore

$$\Big|\sum_{\substack{(Q,M)\in\mathscr{C}_{l,\lambda}\\ \langle f_{Q,l},h_M\rangle\neq 0}} h_M\Big| \leq \Big|\sum_{M\in\mathscr{V}_l(Q)} h_M\Big| = |d_{Q,l}|.$$

We proceed by applying essentially the same steps as in the preceding cases. Using the UMD-property and subsequently Kahane's contraction principle we obtain

$$\|C_{l,\lambda}^* u\|_{L_Y^q(\mathbb{R}^n)} \lesssim 2^l 2^{-\lambda} \Big\|\sum_{Q\in\mathscr{Q}} u_Q\, d_{Q,l} |Q|^{-1}\Big\|_{L_Y^q(\mathbb{R}^n)} = 2^l 2^{-\lambda} \|H_l u\|_{L_Y^q(\mathbb{R}^n)}.$$

Hence, applying the triangle inequality and using the above estimate for $C_{l,\lambda}^*$ we get

$$\|C_l^* u\|_{L_Y^q(\mathbb{R}^n)} \lesssim \|H_l u\|_{L_Y^q(\mathbb{R}^n)}.$$

Finally, Corollary 3.8 yields

$$\|C_l^* u\|_{L^q_Y(\mathbb{R}^n)} \le C \cdot 2^{-l/\mathcal{C}(L^q_Y(\mathbb{R}^n))}, \tag{4.49}$$

where C depends only on n, q, the UMD-constant of Y and the cotype $\mathcal{C}(L^q_Y(\mathbb{R}^n))$.

4.4.4. Summary for P_l. First, note that, for $Y = X^*$ and $1/p + 1/q = 1$,

$$(L^p_X(\mathbb{R}^n))^* = L^q_Y(\mathbb{R}^n) \quad \text{and} \quad \frac{1}{\mathcal{T}(L^p_X(\mathbb{R}^n))} + \frac{1}{\mathcal{C}(L^q_Y(\mathbb{R}^n))} = 1.$$

Second, we use

$$\|B_l^* : L^q_Y(\mathbb{R}^n) \to L^q_Y(\mathbb{R}^n)\| \lesssim \|B_l : L^p_X(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)\|,$$
$$\|C_l^* : L^q_Y(\mathbb{R}^n) \to L^q_Y(\mathbb{R}^n)\| \lesssim \|C_l : L^p_X(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)\|,$$

to combine the inequalities (4.47)–(4.49) via the identity

$$P_l = A_l + B_l + C_l.$$

Thereby we obtain

$$\|P_l : L^p_X(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)\| \le C \cdot 2^{-l(1-1/\mathcal{T}(L^p_X(\mathbb{R}^n)))},$$

where $L^p_X(\mathbb{R}^n)$ has type $\mathcal{T}(L^p_X(\mathbb{R}^n))$ and C depends only on n, p, the UMD-constant of X and the type $\mathcal{T}(L^p_X(\mathbb{R}^n))$.

4.4.5. Estimates for A_-. In view of (4.31), (4.36) and (4.41) note that $\operatorname{diam}(M) \le \operatorname{diam}(Q)$, and so we may utilize inequality (4.26). In this case the size of the cube M cannot exceed the size of Q, so we may indeed use inequality (4.26). We rather want to estimate A_-^* than A_-, therefore we set $Y = X^*$ and q such that $1/p + 1/q = 1$.

First, we split the set $\mathscr{A}_-$ (see (4.31)) into the disjoint collections $\mathscr{A}_{-,\lambda}$, $\lambda \ge 0$, given by

$$\mathscr{A}_{-,\lambda} = \{(Q, M) \in \mathscr{A}_- : \operatorname{diam}(M) = 2^{-\lambda} \operatorname{diam}(Q)\},$$

and define the operator $A_{-,\lambda}$ accordingly, that is,

$$A_{-,\lambda} u = \sum_{(Q,M) \in \mathscr{A}_{-,\lambda}} \langle f_Q, h_M \rangle h_Q u_M |Q|^{-1} |M|^{-1}$$

for all $u = \sum_{K \in \mathscr{Q}} u_K h_K |K|^{-1}$. The adjoint operators A_-^* and $A_{-,\lambda}^*$ are given by

$$A_-^* u = \sum_{\lambda=0}^{\infty} \sum_{Q,M \in \mathscr{A}_{-,\lambda}} \langle f_Q, h_M \rangle u_Q h_M |Q|^{-1} |M|^{-1} = \sum_{\lambda=0}^{\infty} A_{-,\lambda}^* u.$$

Utilizing the UMD-property and subsequently Kahane's contraction principle (2.4) with respect to (4.26), we infer that

$$\|A_{-,\lambda}^* u\|_{L^q_Y(\mathbb{R}^n)} \lesssim 2^{-\lambda} \Big\| \sum_{Q \in \mathscr{Q}} \sum_{\substack{(Q,M) \in \mathscr{A}_{-,\lambda} \\ M \cap (CQ) \ne \emptyset}} u_Q |Q|^{-1} h_M \Big\|_{L^q_Y(\mathbb{R}^n)}.$$

For every $Q \in \mathscr{Q}$ we observe that

$$\Big| \sum_{\substack{(Q,M) \in \mathscr{A}_{-,\lambda} \\ M \cap (CQ) \ne \emptyset}} h_M \Big| \le 1_{CQ} \quad \text{and} \quad 1_{CQ} \le \Big| \sum_{|m| \le C_1} T_m h_Q \Big|$$

for some constant C_1. Combining the last two estimates and applying Kahane's contraction principle together with estimate (2.9), we get

$$\|A^*_{-,\lambda}u\|_{L^q_Y(\mathbb{R}^n)} \lesssim 2^{-\lambda}\Big\| \sum_{Q\in\mathscr{Q}} u_Q|Q|^{-1} \sum_{\substack{(Q,M)\in\mathscr{A}_{-,\lambda}\\ M\cap(CQ)\neq\emptyset}} h_M \Big\|_{L^q_Y(\mathbb{R}^n)} \lesssim 2^{-\lambda}\|u\|_{L^q_Y(\mathbb{R}^n)}.$$

Summing over $\lambda \geq 0$ yields

$$\|A^*_{-}u\|_{L^q_Y(\mathbb{R}^n)} \leq C\|u\|_{L^q_Y(\mathbb{R}^n)}, \tag{4.50}$$

where C depends only on n, q, the UMD-constant of Y and the cotype $\mathcal{C}(L^q_Y(\mathbb{R}^n))$.

4.4.6. Estimates for B_-. In view of (4.32), (4.37) and (4.42) note that $\operatorname{diam}(M) > \operatorname{diam}(Q)$, and so we may utilize inequality (4.27).

As usual, we split the set $\mathscr{B}_-$ (see (4.32)) into the disjoint collections $\mathscr{B}_{-,\lambda}$, $\lambda \geq 1$, given by

$$\mathscr{B}_{-,\lambda} = \{(Q,M)\in\mathscr{B}_- : \operatorname{diam}(Q) = 2^{-\lambda}\operatorname{diam}(M)\},$$

and define the operator $B_{-,\lambda}$ accordingly, that is,

$$B_{-,\lambda}u = \sum_{(Q,M)\in\mathscr{B}_{-,\lambda}} \langle f_Q, h_M\rangle h_Q u_M |Q|^{-1}|M|^{-1}$$

for all $u = \sum_{K\in\mathscr{Q}} u_K h_K |K|^{-1}$. Obviously,

$$B_- u = \sum_{\lambda=1}^{\infty} B_{-,\lambda}u.$$

For all $(Q,M)\in\mathscr{B}_{-,\lambda}$ we have the inclusions

$$\{Q : \langle f_Q, h_M\rangle \neq 0\} \subset \{Q : (CQ)\cap D(Q)\neq\emptyset\} \subset \mathscr{V}_\lambda(M).$$

Successively using the UMD-property, Kahane's contraction principle applied to (4.27) and the inclusion above, we obtain

$$\begin{aligned}\|B_{-,\lambda}u\|_{L^p_X(\mathbb{R}^n)} &\lesssim \Big\| \sum_{M\in\mathscr{Q}} u_M|M|^{-1} \sum_{Q\in\mathscr{V}_\lambda(M)} h_Q \Big\|_{L^p_X(\mathbb{R}^n)}\\ &= \Big\| \sum_{M\in\mathscr{Q}} u_M d_{M,\lambda}|M|^{-1}\Big\|_{L^p_X(\mathbb{R}^n)} = \|H_\lambda u\|_{L^p_X(\mathbb{R}^n)}.\end{aligned}$$

The last equality is the definition of H_λ (see (3.37)). The main result on ring domain operators, Corollary 3.8, yields

$$\|B_{-,\lambda}u\|_{L^p_X(\mathbb{R}^n)} \lesssim \|H_\lambda u\|_{L^p_X(\mathbb{R}^n)} \lesssim 2^{-\lambda/\mathcal{C}(L^p_X(\mathbb{R}^n))}\|u\|_{L^p_X(\mathbb{R}^n)}.$$

Hence, summation over $\lambda \geq 1$ gives

$$\|B_- u\|_{L^p_X(\mathbb{R}^n)} \leq C\|u\|_{L^p_X(\mathbb{R}^n)}, \tag{4.51}$$

where C depends only on n, p, the UMD-constant of X and the cotype $\mathcal{C}(L^p_X(\mathbb{R}^n))$.

4.4.7. Summary for P_-. First, note that for $Y = X^*$ and $1/p + 1/q = 1$ we have

$$(L^p_X(\mathbb{R}^n))^* = L^q_Y(\mathbb{R}^n) \quad\text{and}\quad \frac{1}{\mathcal{T}(L^p_X(\mathbb{R}^n))} + \frac{1}{\mathcal{C}(L^q_Y(\mathbb{R}^n))} = 1.$$

Second, we use

$$\|A_-^* : L^q_Y(\mathbb{R}^n) \to L^q_Y(\mathbb{R}^n)\| \lesssim \|A_- : L^p_X(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)\|,$$

to combine the inequalities (4.50) and (4.51) via the identity

$$P_- = A_- + B_-$$

so that we obtain

$$\|P_- : L^p_X(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)\| \le C,$$

where $L^p_X(\mathbb{R}^n)$ has type $\mathcal{T}(L^p_X(\mathbb{R}^n))$ and C depends only on n, p, the UMD-constant of X and the type $\mathcal{T}(L^p_X(\mathbb{R}^n))$.

4.5. Estimates for $P_l^{(\varepsilon)} R_{i_0}^{-1}$. Following [LMM11] we will establish estimates for $P_l^{(\varepsilon)} R_{i_0}^{-1}$, $l \in \mathbb{Z}$, by reducing them to estimates for $P_l^{(\varepsilon)}$. We exploit the fact that $(R_{i_0}^{-1})^*$ maps the mollified Haar functions $f_{Q,l}^{(\varepsilon)}$ to functions $k_{Q,l}^{(\varepsilon)}$ having similar properties. Due to the algebraic identity (4.52) below this amounts to controlling the support of the $k_{Q,l}$, besides factors depending on l. Assuming that $\varepsilon_{i_0} = 1$, we have

$$\operatorname{supp}(\mathbb{E}_{i_0} h_Q^{(\varepsilon)}) \subset Q,$$

restricting the support of the functions $k_{Q,l,i}$ defined in (4.53), and exhibiting the conditions asserted in (4.56) and (4.57).

We do not omit the superscripts (ε) this time.

It is a well known fact that one can write the inverse of the Riesz transform $R_{i_0}^{-1}$ as

$$R_{i_0}^{-1} = R_{i_0} + \sum_{\substack{1 \le i \le n \\ i \ne i_0}} \mathbb{E}_{i_0} \partial_i R_i, \tag{4.52}$$

where $\mathbb{E}_{i_0}$ is given by

$$\mathbb{E}_{i_0} f(x) = \int_{-\infty}^{x_{i_0}} f(x_1, \ldots, x_{i_0-1}, s, x_{i_0+1}, \ldots, x_n)\, ds, \qquad x = (x_1, \ldots, x_n).$$

We introduce the family of functions

$$k_{Q,l,i}^{(\varepsilon)} = \Delta_{j+l}(\mathbb{E}_{i_0} \partial_i h_Q^{(\varepsilon)}) \quad \text{ if } Q \in \mathcal{Q}_j, \tag{4.53}$$

and consider

$$\begin{aligned} P_l^{(\varepsilon)} R_{i_0}^{-1} u = &\sum_{j\in\mathbb{Z}} \sum_{Q \in \mathcal{Q}_j} \langle R_{i_0} u, \Delta_{j+l}(h_Q^{(\varepsilon)})\rangle h_Q^{(\varepsilon)} |Q|^{-1} \\ &+ \sum_{\substack{1 \le i \le n \\ i \ne i_0}} \sum_{j\in\mathbb{Z}} \sum_{Q \in \mathcal{Q}_j} \langle \mathbb{E}_{i_0} \partial_i R_i u, \Delta_{j+l}(h_Q^{(\varepsilon)})\rangle h_Q^{(\varepsilon)} |Q|^{-1}. \end{aligned} \tag{4.54}$$

Since the Riesz transforms R_i, $1 \le i \le n$, are continuous on $L^p_X(\mathbb{R}^n)$, it is obvious that the first sum of (4.54) can be treated as if it were P_l (see also (4.6)).

For the second sum of (4.54), we fix a coordinate $i \ne i_0$, rearrange the operators in the scalar product and use the functions defined in (4.53), hence

$$\sum_{j\in\mathbb{Z}} \sum_{Q \in \mathcal{Q}_j} \langle \mathbb{E}_{i_0} \partial_i R_i u, \Delta_{j+l}(h_Q^{(\varepsilon)})\rangle h_Q^{(\varepsilon)} |Q|^{-1} = \sum_{Q \in \mathcal{Q}} \langle R_i u, k_{Q,l,i}^{(\varepsilon)}\rangle h_Q^{(\varepsilon)} |Q|^{-1}.$$

Due to the continuity of the Riesz transforms $R_i : L^p_X(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)$ we may estimate the following type of operator:

$$K^{(\varepsilon)}_{l,i} u = \sum_{Q\in\mathcal{Q}} \langle u, k^{(\varepsilon)}_{Q,l,i}\rangle h^{(\varepsilon)}_Q |Q|^{-1} \tag{4.55}$$

instead of the second sum in (4.54).

In order to estimate $K^{(\varepsilon)}_{l,i}$, we need to examine the analytic properties of the functions $k^{(\varepsilon)}_{Q,l,i}$. If $l \geq 0$, then

$$\begin{gathered} \int k^{(\varepsilon)}_{Q,l,i}(x)\,dx = 0, \quad \operatorname{supp} k^{(\varepsilon)}_{Q,l,i} \subset D^{(\varepsilon)}_l(Q), \\ |k^{(\varepsilon)}_{Q,l,i}| \leq C\cdot 2^l, \quad \operatorname{Lip}(k^{(\varepsilon)}_{Q,l,i}) \leq C\cdot 2^{2l}(\operatorname{diam}(Q))^{-1}, \end{gathered} \tag{4.56}$$

and for $l \leq 0$,

$$\begin{gathered} \int k^{(\varepsilon)}_{Q,l,i}(x)\,dx = 0, \quad \operatorname{supp} k^{(\varepsilon)}_{Q,l,i} \subset C\cdot 2^{|l|}Q, \\ |k^{(\varepsilon)}_{Q,l,i}| \leq C\cdot 2^{-|l|(n+1)}, \quad \operatorname{Lip}(k^{(\varepsilon)}_{Q,l,i}) \leq C\cdot 2^{-|l|(n+2)}(\operatorname{diam}(Q))^{-1}. \end{gathered} \tag{4.57}$$

Note that the above properties of $k^{(\varepsilon)}_{Q,l,i}$ depend in particular on the coordinatewise vanishing moments of b (4.3), introduced by Δ_l in (4.4) and (4.6). Furthermore, observe that the definition of $k^{(\varepsilon)}_{Q,l,i}$ involves an integration of $h^{(\varepsilon)}_Q$ with respect to the variable x_{i_0}. Now if $\varepsilon_{i_0} = 1$, then $\mathbb{E}_{i_0} h^{(\varepsilon)}_Q$ is compactly supported in Q, but if $\varepsilon_{i_0} = 0$, then $\operatorname{supp}(\mathbb{E}_{i_0} h^{(\varepsilon)}_Q)$ is unbounded.

If we compare this with the properties (4.20) and (4.21) of $f^{(\varepsilon)}_{Q,l}$, it turns out that the properties coincide if $l \leq 0$, and that $2^{-l}k^{(\varepsilon)}_{Q,l,i}$ satisfies the same conditions as $f^{(\varepsilon)}_{Q,l}$ if $l \geq 0$. Inspecting the proof of Theorem 4.5, we note that those arguments where solely depending on the analytic properties (4.20) and (4.21) of $f^{(\varepsilon)}_{Q,l}$. With regard to (4.56) respectively (4.57), the same proofs are feasible with the functions $k^{(\varepsilon)}_{Q,l,i}$ replacing $f_{Q,l}$ if $l \leq 0$, respectively $2^{-l}\,k^{(\varepsilon)}_{Q,l,i}$ replacing $f_{Q,l}$ if $l \geq 0$. Furthermore, we have to replace P_l by $K_{l,i}$ for every $1 \leq i \leq n$.

Altogether we obtain the following theorem from the estimates of Theorem 4.5.

THEOREM 4.7. *Let X be a UMD-space, $1 < p < \infty$, $n \in \mathbb{N}$ and let $L^p_X(\mathbb{R}^n)$ have type $\mathcal{T}(L^p_X(\mathbb{R}^n))$. Furthermore, denote by R_{i_0} the Riesz transform acting in direction i_0 and let $\varepsilon_{i_0} = 1$. Then there exists a constant $C > 0$ such that for every $l \geq 0$ and all $u \in L^p_X(\mathbb{R}^n)$ we have*

$$\|P^{(\varepsilon)}_l R^{-1}_{i_0} u\|_{L^p_X(\mathbb{R}^n)} \leq C\cdot 2^{l/\mathcal{T}(L^p_X(\mathbb{R}^n))}\|u\|_{L^p_X(\mathbb{R}^n)}, \tag{4.58}$$

where C depends only on n, p, the UMD-constant of X and the type $\mathcal{T}(L^p_X(\mathbb{R}^n))$.

Moreover, there exists a constant $C > 0$ such that, for all $u \in L^p_X(\mathbb{R}^n)$,

$$\|P^{(\varepsilon)}_- R^{-1}_{i_0} u\|_{L^p_X(\mathbb{R}^n)} \leq C\|u\|_{L^p_X(\mathbb{R}^n)}, \tag{4.59}$$

where C depends only on n, p, the UMD-constant of X and the type $\mathcal{T}(L^p_X(\mathbb{R}^n))$.

5. Appendix

In order to keep the paper self-contained, we include several auxiliary results used in this work.

Lipschitz estimate for separately convex functions. We record a Lipschitz estimate for separately convex functions satisfying convenient growth estimates on the Banach space X. The resulting inequality holds without any assumptions on the underlying Banach space X.

THEOREM 5.1. *Let X be a Banach space, $n \geq 1$, $f : X^n \to \mathbb{R}$ be separately convex, and $g : X^n \to \mathbb{R}$, where $g(x) = 1 + \sum_{i=1}^n \|x_i\|_X^p$. If $0 \leq f(x) \leq g(x)$, $x \in X$, then*

$$|f(x) - f(y)| \leq C(1 + \|x\|_{X^n} + \|y\|_{X^n})^{p-1} \|x - y\|_{X^n} \tag{5.1}$$

for all $x, y \in X^n$. The constant $C > 0$ depends only on n and p.

Proof. Let $x \neq y \in X^n$, $1 \leq k \leq n$, and define

$$\begin{aligned} f_k(t) &= f(x_1, \ldots, x_{k-1}, x_k + t(y_k - x_k), x_{k+1}, \ldots, x_n), \\ g_k(t) &= g(x_1, \ldots, x_{k-1}, x_k + t(y_k - x_k), x_{k+1}, \ldots, x_n), \\ n_k(t) &= \|x_k + t(y_k - x_k)\|_X, \end{aligned}$$

for all $t \in \mathbb{R}$. We may assume that $f_k(0) \leq f_k(1)$, otherwise we would switch x_k and y_k.

Observe that $n_k(t)$ is increasing if $t \geq 2\|x_k\|/\|y_k - x_k\|$, hence $g_k(t)$ is increasing if $t \geq 2\|x_k\|/\|y_k - x_k\|$. To justify this claim, assume there exist $t_1 > t_0 > 2\|x_k\|/\|y_k - x_k\|$ such that $n_k(t_1) \leq n_k(t_0)$. The convexity of $n_k(t)$ implies $n_k(0) \geq n_k(t_0)$, so

$$\|x_k\| \geq \|x_k + t_0(y_k - x_k)\| \geq t_0\|y_k - x_k\| - \|x_k\| > \|x_k\|,$$

which is a contradiction. Thus we proved that n_k is increasing for all $t > 2\|x_k\|/\|y_k - x_k\|$, and so by continuity for all $t \geq 2\|x_k\|/\|y_k - x_k\|$ as claimed.

For $t_0 < t_1$ which will be specified later, we define the affine functions

$$\begin{aligned} \ell_1(t) &= f_k(0) + t(f_k(1) - f_k(0)), \\ \ell_2(t) &= g_k(t_0) + \frac{g_k(t_1) - g_k(t_0)}{t_1 - t_0}(t - t_0), \end{aligned}$$

and let $\bar{t}$ denote the point where $\ell_2(\bar{t}) = 0$, that is,

$$\bar{t} = t_0 - \frac{g_k(t_0)}{g_k(t_1) - g_k(t_0)}(t_1 - t_0). \tag{5.2}$$

Now we prove that if $1 \le \bar{t} < t_0 < t_1$ and $t_0 \ge 2\|x_k\|/\|y_k - x_k\|$, then

$$f_k(1) - f_k(0) \le \frac{g_k(t_1) - g_k(t_0)}{t_1 - t_0}. \tag{5.3}$$

Assume that (5.3) does not hold; then since $f_k(0) \ge 0$ and $\bar{t} \ge 1$, we have $\ell_1(t) > \ell_2(t)$ for all $t > \bar{t}$. Since $f_k(t)$ is convex we know that $f_k(t) \ge \ell_1(t)$, $t \ge \bar{t}$, and hence $f_k(t_1) \ge \ell_1(t_1) > \ell_2(t_1) = g_k(t_1)$, which contradicts $f_k(t) \le g_k(t)$, $t \in \mathbb{R}$.

Now we want to impose conditions on $t_0 < t_1$ such that $\bar{t} \ge 1$. Observe that since $n_k(t_1) > n_k(t_0)$, we obtain

$$\begin{aligned}\frac{g_k(t_1) - g_k(t_0)}{t_1 - t_0} &\ge p n_k(t_0)^{p-1} \frac{n_k(t_1) - n_k(t_0)}{t_1 - t_0} \\ &\ge p n_k(t_0)^{p-1} \left(\|y_k - x_k\| - \frac{2\|x_k\|}{t_1 - t_0} \right),\end{aligned}$$

and plugging this estimate into (5.2) yields

$$\bar{t} \ge t_0 - \frac{g_k(t_0)}{p\|x_k + t_0(y_k - x_k)\|^{p-1}(\|y_k - x_k\| - 2\|x_k\|/(t_1 - t_0))}. \tag{5.4}$$

If we impose the following constraints:

- $(t_1 - t_0)\|y_k - x_k\| \ge 2C\|x_k\|$,
- $t_0\|y_k - x_k\| \ge 2C\|x_i\|$, $1 \le i \le n$,
- $t_0\|y_k - x_k\| \ge C$,
- $t_0\|y_k - x_k\| \ge 2\|x_k\|$,

in order to estimate (5.4), we get

$$\bar{t} \ge t_0 - A_1 - A_2 - A_3,$$

where

$$\begin{aligned} A_1 &= \frac{1}{p(1 - 1/C)\|x_k + t_0(y_k - x_k)\|^{p-1}\|y_k - x_k\|} \le \frac{t_0}{p(C-1)^p}, \\ A_2 &= \sum_{i \ne k} \frac{\|x_i\|^p}{p\|x_k + t_0(y_k - x_k)\|^{p-1}\|y_k - x_k\|(1 - 1/C)} \le \frac{t_0(n-1)}{p(C-1)^p}, \\ A_3 &= \frac{\|x_k + t_0(y_k - x_k)\|}{p(1 - 1/C)\|y_k - x_k\|} \le \frac{t_0(1 + C)}{p(C-1)}. \end{aligned}$$

Using these estimates we obtain

$$\bar{t} \ge t_0 \left(1 - \frac{1}{p(C-1)^p} - \frac{n-1}{p(C-1)^p} - \frac{1 + C}{p(C-1)} \right) = t_0 \cdot \alpha. \tag{5.5}$$

If we choose C large enough so that $\alpha \ge (p-1)/(2p)$ and define

$$t_0 = \sum_{i=1}^{n} \frac{C\|x_i\|}{\|y_k - x_k\|} + \frac{C}{\|y_k - x_k\|} + \frac{1}{\alpha}, \qquad t_1 = 3t_0, \tag{5.6}$$

then t_0 and t_1 satisfy our constraints. Hence we can infer (5.5), and get $1 \le \bar{t} < t_0 < t_1$, $t_0 \ge 2\|x_k\|/\|y_k - x_k\|$. Thus (5.3) yields

$$f_k(1) - f_k(0) \le \frac{g_k(t_1) - g_k(t_0)}{t_1 - t_0}, \tag{5.7}$$

where t_0, t_1 are defined in (5.6). A straightforward computation shows that

$$\frac{g_k(t_1) - g_k(t_0)}{t_1 - t_0} \le (t_1 \|y_k - x_k\|_X + \|x_k\|_{X^n})^{p-1} \|y_k - x_k\|_X,$$

and plugging (5.6) into the latter estimate we obtain

$$\frac{g_k(t_1) - g_k(t_0)}{t_1 - t_0} \lesssim (1 + \|y_k - x_k\|_X + \|x\|_{X^n})^{p-1} \|y_k - x_k\|_X. \tag{5.8}$$

Combining (5.7) with (5.8) and recalling the definition of f_k yields

$$\begin{aligned} &|f(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) - f(x_1, \ldots, x_{k-1}, y_k, x_{k+1}, \ldots, x_n)| \\ &\qquad\qquad \lesssim (1 + \|y_k - x_k\|_X + \|x\|_{X^n})^{p-1} \|y_k - x_k\|_X. \end{aligned} \tag{5.9}$$

Using (5.9) inductively one can verify that

$$|f(x) - f(y)| \le C(1 + \|x\|_{X^n} + \|y\|_{X^n})^{p-1} \|x - y\|_{X^n},$$

where C depends only on n and p. ∎

Convolution operators on $L^p_X(\mathbb{R}^n)$. Let E and X be Banach spaces. A bounded linear operator $J : E \to X$ is a *Dunford–Pettis operator* if it is weak-to-norm sequentially continuous, which means that whenever $\{e_n\}_n \subset E$ converges to e weakly, then Te_n converges to Te in norm (see Section 2).

THEOREM 5.2. *Let E and X be Banach spaces and let $J : E \to X$ be a Dunford–Pettis operator. With $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\psi \in C_c^\infty(\mathbb{R}^n)$ fixed, define the kernel*

$$K(x, y) = \varphi(x - y)\psi(y), \qquad x, y \in \mathbb{R}^n.$$

Then if $1 < p < \infty$, the operator $T : L^p_E(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)$ given by

$$(Tu)(x) = \int_{\mathbb{R}^n} K(x, y) J(u(y))\, dy$$

is Dunford–Pettis.

REMARK 5.3. Theorem 5.2 remains valid if we replace Dunford–Pettis by compact, in both the hypothesis on J and the conclusion for T.

Proof of Theorem 5.2. Let $\varepsilon > 0$ be fixed. First note that $K \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$, hence

$$|K(x, y)| \le C_n \frac{1}{(1 + |x|)^{n+2}} \frac{1}{(1 + |y|)^{n+1}}. \tag{5.10}$$

Let B_1 denote the smallest cube centered at 0 such that

$$\frac{1}{1 + |x|} \le \varepsilon \quad \text{for all } x \notin \tfrac{1}{2} B_1,$$

and let B_2 denote the smallest cube centered at 0 such that

$$\psi(y) = 0 \quad \text{for all } y \notin B_2.$$

Choose $\eta \in C_c^\infty(\mathbb{R}^n)$ with $0 \le \eta(x) \le 1$ for all $x \in \mathbb{R}^n$, $\eta(x) = 1$ for all $x \in \frac{1}{2} B_1$, and $\eta(x) = 0$ if $x \notin B_1$. Now we split K according to η into

$$K(x, y) = \eta(x) K(x, y) + (1 - \eta(x)) K(x, y) = K_1(x, y) + K_2(x, y)$$

for all $x, y \in \mathbb{R}^n$. Notice that

$$\operatorname{supp} K_1 \subset B_1 \times B_2 \quad \text{and} \quad K_2(x,y) = 0 \quad \text{for all } x \in \tfrac{1}{2}B_1,\, y \in \mathbb{R}^n.$$

We now define two nested collections $\mathscr{P}$ and $\mathscr{Q}$ of cubes. We begin by setting $\mathscr{P}_0 = \{B_1\}$ and $\mathscr{Q}_0 = \{B_2\}$. Assuming that we have already defined $\mathscr{P}_0, \dots, \mathscr{P}_j$ and $\mathscr{Q}_0, \dots, \mathscr{Q}_j$, we proceed in the following way. We split every $P \in \mathscr{P}_j$ respectively $Q \in \mathscr{Q}_j$ into 2^n subcubes having half the diameter of P respectively Q and collect those cubes in $\mathscr{P}_{k+1}$ respectively $\mathscr{Q}_{k+1}$. Finally $\mathscr{P} = \bigcup_j \mathscr{P}_j$ and $\mathscr{Q} = \bigcup_j \mathscr{Q}_j$. We define the σ-algebra

$$\mathcal{F}_j = \sigma(\{P \times Q : P \in \mathscr{P}_j,\, Q \in \mathscr{Q}_j\})$$

and the conditional expectation

$$\mathbb{E}_j(\cdot) = \mathbb{E}(\cdot \mid \mathcal{F}_j).$$

Associated to each direction $\delta \in \{0,1\}^n \setminus \{0\}$ and cubes $P \in \mathscr{P}$ and $Q \in \mathscr{Q}$, we define Haar functions $h_P^{(\delta)}$ and $h_Q^{(\delta)}$ by

$$h_P^{(\delta)} = (h_{I_1})^{\delta_1} \otimes \cdots \otimes (h_{I_n})^{\delta_n} \quad \text{and} \quad h_Q P^{(\delta)} = (h_{J_1})^{\delta_1} \otimes \cdots \otimes (h_{J_n})^{\delta_n},$$

where $P = I_1 \times \cdots \times I_n$ with $|I_1| = \cdots = |I_n|$, $Q = J_1 \times \cdots \times J_n$ with $|J_1| = \cdots = |J_n|$, and we use the convention that $(h_K)^0 = \mathtt{1}_K$.

Recall that K_1 is smooth and supported on $B_1 \times B_2$, so $\mathbb{E}_j(K_1) \to K_1$ uniformly in $\mathbb{R}^n$. Hence, for given $\delta > 0$ we may find an integer $N_0 \geq 0$ such that

$$|K_1(x,y) - (\mathbb{E}_N K_1)(x,y)| \leq \delta \qquad \text{for all } x, y \in \mathbb{R}^n,$$

for all $N \geq N_0$. This allows us to choose N so that

$$\sup_{y \in \mathbb{R}^n} \int_{B_1} |K_1(x,y) - (\mathbb{E}_N K_1)(x,y)|^p \, dx \leq \varepsilon^p. \tag{5.11}$$

Note that $\operatorname{supp} K_1 \subset B_1 \times B_2$ as well as $\operatorname{supp}(\mathbb{E}_N K_1) \subset B_1 \times B_2$.

Now let us define the approximating operator $T_\varepsilon : L^p_E(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)$ by

$$(T_\varepsilon u)(x) = \int_{\mathbb{R}^n} (\mathbb{E}_N K_1)(x,y) J(u(y)) \, dy.$$

With $u \in L^p_E(\mathbb{R}^n)$ fixed, we see that

$$\begin{aligned} \|Tu - T_\varepsilon u\|_{L^p_X} &\leq \left\| \int_{\mathbb{R}^n} (K_1(\cdot,y) - (\mathbb{E}_N K_1)(\cdot,y)) J(u(y)) \, dy \right\|_{L^p_X} \\ &\quad + \left\| \int_{\mathbb{R}^n} K_2(\cdot,y) J(u(y)) \, dy \right\|_{L^p_X} \\ &= A + B. \end{aligned}$$

In order to estimate A we use the Minkowski inequality for integrals and Hölder's inequality to find

$$\begin{aligned} A &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |K_1(x,y) - (\mathbb{E}_N K_1)(x,y)|^p \, dx \right)^{1/p} \|J(u(y))\|_X \, dy \\ &\leq \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |K_1(x,y) - (\mathbb{E}_N K_1)(x,y)|^p \, dx \right)^{p'/p} dy \right)^{1/p'} \|Ju\|_{L^p_X}, \end{aligned}$$

where p' denotes the Hölder conjugate index to p. Recall that $\operatorname{supp} K_1 \subset B_1 \times B_2$, $\operatorname{supp}(\mathbb{E}_N K_1) \subset B_1 \times B_2$, and appeal to estimate (5.11) to obtain

$$A \leq \varepsilon |B_2|^{1/p'} \|Ju\|_{L^p_X}.$$

In a similar fashion we estimate B, but using $K_2(x,y) = 0$ if $x \in \frac{1}{2}B_1$, $y \in \mathbb{R}^n$, and estimate (5.10) to find

$$B \leq \left(\int_{\mathbb{R}^n} \left(\int_{(\frac{1}{2}B_1)^c} |K_2(x,y)|^p \, dx \right)^{p'/p} dy \right)^{1/p'} \|Ju\|_{L^p_X} \leq \varepsilon C \|Ju\|_{L^p_X},$$

where C does not depend on ε.

Considering our estimate for A and B and that J is a bounded map, we get

$$\|Tu - T_\varepsilon u\|_{L^p_X} \leq \varepsilon C \|u\|_{L^p_E},$$

with C not depending on ε. Consequently,

$$\|T_\varepsilon - T : L^p_E(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)\| \to 0 \quad \text{as } \varepsilon \text{ tends to zero.}$$

If we can show that T_ε is Dunford–Pettis for every $\varepsilon > 0$, then one can easily verify that T is Dunford–Pettis as well.

To this end, let $\varepsilon > 0$, and choose B_1 and N according to our construction above. Let $u_m \to 0$ weakly in $L^p_E(\mathbb{R}^n)$. Then certainly $\sup_m \|u_m\|_{L^p_E} \leq C$ for some $C > 0$. For each $u \in L^p_E(\mathbb{R}^n)$, we split u into $u = u^{(1)} + u^{(2)}$, where $u^{(1)} = u \cdot \mathtt{1}_{B_2}$ and $u^{(2)} = u \cdot \mathtt{1}_{(B_2)^c}$. Since $T_\varepsilon u_2 = 0$, we may assume that u_m is supported in B_2, hence

$$u_m(y) = \sum_{\delta \in \{0,1\}^n} \sum_{j=0}^{\infty} \sum_{Q \in \mathscr{Q}_j} \langle u_m, h_Q^{(\delta)} \rangle h_Q^{(\delta)}(y) |Q|^{-1},$$

where $h_Q^{(0)} = 0$ if $Q \neq B_2$, and $h_{B_2}^{(0)} = \mathtt{1}_{B_2}$. Since u_m converges to 0 weakly in $L^p_E(\mathbb{R}^n)$, one can verify that $\langle u_m, h_Q^{(\delta)} \rangle \to 0$ weakly in E for all $Q \in \mathscr{Q}$ and $\delta \in \{0,1\}^n$. This is due to the fact that $h_Q^{(\delta)} e^* \in (L^p_E(\mathbb{R}^n))^*$ whenever $e^* \in E^*$. Now since $J : E \to X$ is Dunford–Pettis, we deduce that $\|J(\langle u_m, h_Q^{(\delta)} \rangle)\|_X \to 0$ as $m \to \infty$ for all $Q \in \mathscr{Q}$ and $\delta \in \{0,1\}^n$.

Since $T_\varepsilon u_m$ is given by the finite sum

$$(T_\varepsilon u_m)(x) = \sum_{\gamma,\delta \in \{0,1\}^n} \sum_{j=0}^{N-1} \sum_{\substack{P \in \mathscr{P}_j \\ Q \in \mathscr{Q}_j}} \langle K_1, h_P^{(\gamma)} \otimes h_Q^{(\delta)} \rangle J(\langle u_m, h_Q^{(\delta)} \rangle) h_P^{(\gamma)}(x) |P|^{-1} |Q|^{-1},$$

where $h_P^{(0)} \otimes h_Q^{(0)} = 0$ if $(P \times Q) \neq (B_1 \times B_2)$ and $h_{B_1}^{(0)} \otimes h_{B_2}^{(0)} = \mathtt{1}_{B_1} \otimes \mathtt{1}_{B_2}$, we infer that $\|T_\varepsilon u_m\|_{L^p_X} \to 0$ as m tends to ∞, therefore T_ε is Dunford–Pettis.

Finally, let us verify that T is Dunford–Pettis, too. Let $u_m \to 0$ weakly in $L^p_E(\mathbb{R}^n)$ and note that

$$\|Tu_m\|_{L^p_X(\mathbb{R}^n)} \leq \|T_\varepsilon u_m\|_{L^p_X(\mathbb{R}^n)} + C\|(T - T_\varepsilon) : L^p_E(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)\|$$

for all $\varepsilon > 0$ and m, where $\sup_m \|u_m\| \leq C$. Now with ε fixed, letting $m \to \infty$ and T_ε being Dunford–Pettis implies that $\|T_\varepsilon u_m\| \to 0$, and so we obtain

$$\lim_m \|Tu_m\|_{L^p_X(\mathbb{R}^n)} \leq C\|T - T_\varepsilon : L^p_E(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)\|$$

for all $\varepsilon > 0$. We conclude the proof by recalling that $\|T - T_\varepsilon : L^p_E(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)\| \to 0$ as $\varepsilon \to 0$. ■

Fourier multipliers on $L^p_X(\mathbb{R}^n)$ and the Sobolev spaces $W^{-1,p}_X(\mathbb{R}^n)$. From now onward the Banach space X has the UMD-property. We gather some facts contributing to the proof of Theorem 1.2.

THEOREM 5.4. *Let X be a UMD-space, $n \geq 1$ and $1 < p < \infty$. If $\alpha \in \mathcal{S}(\mathbb{R}^n;\mathbb{C})$, define $\alpha_k(x) = \alpha(x/k)$ for all $x \in \mathbb{R}^n$ and every positive integer k. Then there exists a constant $C > 0$ such that*

$$\|\alpha_k u\|_{W^{-1,p}(\mathbb{R}^n;X)} \leq C\|u\|_{W^{-1,p}(\mathbb{R}^n;X)}, \tag{5.12}$$

$$\|\partial_i(\alpha_k)u\|_{W^{-1,p}(\mathbb{R}^n;X)} \leq C \cdot \frac{1}{k}\|u\|_{W^{-1,p}(\mathbb{R}^n;X)}, \tag{5.13}$$

for all $u \in W^{-1,p}(\mathbb{R}^n;X)$, $k > 0$. The constant C does not depend on k.

Proof. Note that in UMD-spaces

$$\|u\|_{W^{-1,p}(\mathbb{R}^n;X)} = \|\mathcal{F}^{-1}(\langle\xi\rangle^{-1}\mathcal{F}u)\|_{L^p(\mathbb{R}^n;X)},$$

where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$ and $\mathcal{F}$ denotes the Fourier transform. Since

$$\mathcal{F}^{-1}\big(\langle\xi\rangle^{-1}\mathcal{F}(\alpha_k u)\big)(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\eta}\mathcal{F}\alpha_k(\eta)\langle\eta\rangle^N T_{m_\eta}\big(\mathcal{F}^{-1}(\langle\xi\rangle^{-1}\mathcal{F}u)\big)(x)\,d\eta,$$

where

$$T_{m_\eta}f = \mathcal{F}^{-1}\big(m_\eta(\xi)\mathcal{F}f(\xi)\big), \qquad m_\eta(\xi) = \langle\xi\rangle\langle\xi+\eta\rangle^{-1}\langle\eta\rangle^{-N},$$

we obtain

$$\|\alpha_k u\|_{W^{-1,p}(\mathbb{R}^n;X)} \leq \|\mathcal{F}\alpha_k(\eta)\langle\eta\rangle^N\|_{L^1(\mathbb{R}^n;\mathbb{R})}\sup_{\eta\in\mathbb{R}^n}\big\|T_{m_\eta}\big(\mathcal{F}^{-1}(\langle\xi\rangle^{-1}\mathcal{F}u(\xi))\big)\big\|_{L^p(\mathbb{R}^n;X)}.$$

Observe that $\langle\xi+\eta\rangle\langle\eta\rangle \geq c\langle\xi\rangle$ for a constant $c > 0$, hence $|\partial^\beta_\xi m_\eta(\xi)| \leq A\langle\xi\rangle^{-|\beta|}$ for all multi-indices β. Note that the constant A does not depend on η, if $N = N(\beta)$ is chosen sufficiently large. Setting $N = n+2$ will be good enough for our purposes. Thus we know by [McC84, Theorem 1.1] that

$$\|T_{m_\eta} : L^p(\mathbb{R}^n;X) \to L^p(\mathbb{R}^n;X)\| \leq C,$$

where C does not depend on η, hence

$$\|\alpha_k u\|_{W^{-1,p}(\mathbb{R}^n;X)} \leq C\|\mathcal{F}\alpha_k(\eta)\langle\eta\rangle^{n+2}\|_{L^1(\mathbb{R}^n;\mathbb{R})}\|\mathcal{F}^{-1}(\langle\xi\rangle^{-1}\mathcal{F}u(\xi))\|_{L^p(\mathbb{R}^n;X)}.$$

Since $\alpha \in \mathcal{S}(\mathbb{R}^n;\mathbb{C})$, one can check that

$$\|\mathcal{F}\alpha_k(\eta)\langle\eta\rangle^{n+2}\|_{L^1(\mathbb{R}^n;\mathbb{R})} \leq C_n,$$

thus we proved inequality (5.12).

Now we prove inequality (5.13) by using (5.12). Define $\beta = \partial_i\alpha$, and $\beta_k(x) = \beta(x/k)$ for all $x \in \mathbb{R}^n$ and every positive integers k. Then clearly $\partial_i\alpha_k = \frac{1}{k}\beta_k$, and since β_k is in $\mathcal{S}(\mathbb{R}^n;\mathbb{C})$, we may use estimate (5.12) with α and α_k replaced by β and β_k, yielding

$$k\|(\partial_i\alpha_k)u\|_{W^{-1,p}(\mathbb{R}^n;X)} = \|\beta_k u\|_{W^{-1,p}(\mathbb{R}^n;X)} \leq C\|u\|_{W^{-1,p}(\mathbb{R}^n;X)}$$

for all positive integers k. ■

Theorem 5.5. *Let E and X be Banach spaces, assume that X has the UMD-property, and let $J : E \to X$ be a Dunford–Pettis operator. Let R_i denote the Riesz transform with respect to direction i, and let $\psi \in C_c^\infty(\mathbb{R}^n)$. Then*

$$R_i(\psi \cdot Ju) = (R_i T_1)(u) + T_2\big(\mathcal{F}^{-1}(\langle\xi\rangle^{-1}\xi_i \cdot \mathcal{F}(\psi \cdot Ju))\big) \tag{5.14}$$

for all $u \in L^p_E(\mathbb{R}^n)$, where

$$T_1 : L^p_E(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n) \quad \text{is Dunford–Pettis, and}$$
$$T_2 : L^p_X(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n) \quad \text{is bounded.}$$

Remark 5.6. Theorem 5.5 remains valid if we replace Dunford–Pettis by compact, in both the hypothesis on J and the conclusion for T_1.

Proof of Theorem 5.5. If $u \in L^p_E(\mathbb{R}^n)$, then $Ju = (x \mapsto J(u(x))) \in L^p_X(\mathbb{R}^n)$. Let us choose a smooth cut-off function $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $0 \le \varphi \le 1$, $\varphi(x) = 1$ if $|x| \le 1/2$ and $\varphi(x) = 0$ if $|x| \ge 1$. Observe that

$$\begin{aligned} R_i(\psi \cdot Ju) &= \mathcal{F}^{-1}(\xi_i|\xi|^{-1} \cdot \mathcal{F}(\psi \cdot Ju)) \\ &= R_i\mathcal{F}^{-1}(\varphi \cdot \mathcal{F}(\psi \cdot Ju)) + \mathcal{F}^{-1}\big((1-\varphi(\xi))\xi_i|\xi|^{-1} \cdot \mathcal{F}(\psi \cdot Ju)\big) \\ &= R_i(\mathcal{F}^{-1}(\varphi) * (\psi \cdot Ju)) + \mathcal{F}^{-1}\big((1-\varphi(\xi))|\xi|^{-1}\langle\xi\rangle\langle\xi\rangle^{-1}\xi_i \cdot \mathcal{F}(\psi \cdot Ju)\big) \\ &= (R_i T_1)(u) + T_2\big(\mathcal{F}^{-1}(\langle\xi\rangle^{-1}\xi_i\mathcal{F}(\psi \cdot Ju))\big), \end{aligned}$$

where

$$\begin{aligned} (T_1 u)(x) &= \int_{\mathbb{R}^n} \mathcal{F}^{-1}(\varphi)(x-y)\psi(y)J(u(y))\,dy, && u \in L^p_E(\mathbb{R}^n), \\ (T_2 v)(x) &= \mathcal{F}^{-1}(m \cdot \mathcal{F}v)(x), && v \in L^p_X(\mathbb{R}^n). \end{aligned}$$

The smooth function m is given by $m(\xi) = (1-\varphi(\xi))\langle\xi\rangle|\xi|^{-1}$ and satisfies

$$|\partial_\xi^\alpha m(\xi)| \le A_\alpha\langle\xi\rangle^{-|\alpha|} \quad \text{for all multi-indices } \alpha \text{ and } \xi \in \mathbb{R}^n,$$

and is therefore a Fourier multiplier.

The representation of the operator T_1 fits the hypothesis of Theorem 5.2, from which we deduce that T_1 is Dunford–Pettis.

Since m satisfies the above differential inequalities, we know from [McC84, Theorem 1.1] that T_2 is bounded. ■

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