PART I

1. Introduction

The present paper is devoted to certain Markov operators which occur in diverse branches of pure and applied mathematics. Processes described by these operators arise in mathematical theory of learning [27, 29, 49], population dynamics [45], theory of stochastic differential equations [22, 26] and many others. Recently such processes have been extensively studied because of the close connection to fractals and their generalization, semifractals [2, 4, 10, 13, 14, 16–18, 25, 36, 56]. They are also used in computer graphics. If $(Z_n)_{n\geq 1}$ is a homogeneous Markov chain taking values in some metric space X and π is its transition kernel, i.e.,

$$\operatorname{prob}\{Z_{n+1}\in A\,|\,Z_n=x_n,\ldots,Z_0=x_0\}=\pi(x_n,A)$$

for $n\in\mathbb{N}$ and all Borel sets A, the corresponding Markov operator P is given by

$$P\mu(A)=\int\limits_X \pi(x,A)\,\mu(dx).$$

It is of interest to find criteria for the existence of an invariant measure for P (see [3, 8, 11, 24, 25, 28, 41, 42, 44, 51, 55, 57, 58]). Having these criteria we may study the multifractal properties of invariant measures (for details see [1, 5, 7, 9, 43, 46, 47, 50, 52, 53, 60]).

The paper is divided into two parts. The main objective of our study in Part I is the theory of general Markov operators acting on measures. It should be noted that all theorems are stated under quite general assumptions concerning the phase space X. Namely, we assume that X is a Polish space. In this case the crucial difficulty is to assure the existence of an invariant measure. The first results concerning the existence of invariant measures were proved for compact spaces (see [29]). The proofs usually go as follows. First we construct a positive invariant functional defined on the space of all continuous functions. From the Riesz theorem we deduce that it may be represented by a measure. Finally, since this functional is invariant, we conclude that the measure is also invariant. This scheme works smoothly only when the phase space X is compact. However Lasota and Yorke managed to extend it to the case when X is locally compact and σ-compact (for details see [39]). Their approach was partially based on the idea of the lower bound function technique developed for Markov operators acting on L^1-space (see [34]). They introduced the class of so-called concentrating Markov operators and showed that every operator from this class admits an invariant measure. The above mentioned result is similar in spirit to Komorowski's theorem [31]. Komorowski however considered Markov operators acting on absolutely continuous measures. Further, if we assume that a concentrating Markov operator does not increase some distance between two measures, this operator must be asymptotically stable (see [39]).

The main aim of Part I is to prove similar results for Polish spaces. In Polish spaces almost all methods developed for locally compact and σ-compact spaces break down. Therefore in our considerations we base on the concept of tightness and the well known Prokhorov theorem (see [59]).

In Part II we apply the results from Part I to some special Markov operators. We consider Markov operators generated by iterated function systems, stochastically perturbed dynamical systems and Poisson driven stochastic differential equations. We start with iterated function systems. This class of processes under the name "systems with a complete connection" was already introduced by Mihoc and Onicescu in 1935 [45]. These systems were also intensively studied as mathematical models of learning [27, 29, 49]. Today iterated function systems are considered because of their close connection with the theory of fractals. The explosion of interest in fractal sets started after the observation that some of them can be constructed by using an iteration process. More precisely, having N contractive transformations, say $S_1, \ldots, S_N$, we construct the fractal set A_* as the limit of the sequence $(F^n(A))_{n\geq 1}$, where $F(A) = \bigcup S_i(A)$ and A is a compact set. It is known that if all S_i's are contractive maps, then the set A_* exists and does not depend on the starting set A (see [25]). On the other hand, a fractal set may be obtained in the following way. Assume that every transformation S_i is associated with the probability p_i determining the frequency with which the map S_i can be chosen. Then for such system, known under the name of iterated function system, we obtain the Markov operator P, given by (6.1.2), describing the evolution of measures due to the action of the above process. It can be proved that the support of its invariant measure is equal to the fractal set for $S_1, \ldots, S_N$ (see [25]). Recently Lasota nad Myjak generalized the concept of fractal sets. Namely, the class of sets which can be defined as supports of invariant measures of asymptotically stable Markov operators contains not only fractals. This leads to the notion of a semifractal (see [36]). All sets which are equal to the support of the invariant measure corresponding to an asymptotically stable Markov operator generated by an iterated function system are called semifractals.

Our next concern will be the behaviour of stochastically perturbed dynamical systems which are a natural extension of iterated function systems. They are defined in the following way. Consider an uncountable family $(S_t)_{t\in\mathcal{P}}$ of transformations and assume that every transformation is chosen according to some density on $\mathcal{P}$. Then, analogously to the case of iterated function systems, for this process we may find criteria for the existence of an invariant measure. Such systems were considered by Lasota and Mackey and turned out a very useful tool in the theory of mathematical models of cell cycles (for details see [35]).

Finally, we will consider stochastic differential equations of the form

$$d\xi(t) = a(\xi(t))dt + \int_{\Theta} \sigma(\xi(t), \theta)\mathcal{N}_p(dt, d\theta),$$

where $\mathcal{N}_p$ is the Poisson random measure. It is well known that such equations define a semigroup of Markov operators (see [22, 38, 63]). We present a criterion for the existence of an invariant measure for this semigroup. A similar result in $\mathbb{R}^d$ was proved by J. Traple [65].

Secondly, we will study dimensions of measures. The idea of dimension of a measure is a basic tool in the study of fractals and measures generated by iterated function systems, or more generally, measures generated by Markov chains (see [53]). Various definitions of dimension have been proposed: Hausdorff dimension, box dimension, entropy dimension, correlation dimension. Closely related to the Hausdorff dimension is capacity, introduced by Kolmogorov (see [30]). This capacity however does not distinguish between a set and its closure. Ledrappier [40] has made some modification to correct this insensitivity but his version of capacity has not been completely analysed. In the present paper we give a contribution to this subject. But the results are far from being conclusive.

The outline of the paper is as follows. Chapter 2 is divided into four parts. Sections 2.1–2.3 present some preliminaries. In Section 2.1 we set up notation and terminology. Section 2.2 contains some basic facts from the theory of Markov operators. In Section 2.3 we discuss different properties of Markov operators as globally concentrating, locally concentrating, concentrating and semi-concentrating, and in Section 2.4 we look at them more closely when proving technical lemmas.

The main concept taken from Lasota and Yorke is nonexpansiveness. Chapter 3 provides a detailed exposition of it. It is worth pointing out that this assumption is essential to our proofs of the existence of an invariant measure. Nonexpansiveness may be omitted if we assume that Markov processes satisfy some ergodic conditions on compact sets (see for instance [57]).

Chapter 4 is devoted to the study of tightness. It is worth pointing out that every tight sequence of measures contains a weakly convergent subsequence. This fact, in turn, gives us a tool for proving the existence of an invariant measure.

In Chapter 5 criteria for the existence of an invariant measure are stated and proved. The main result, Theorem 5.5, ensures the existence of an invariant measure for a nonexpansive operator satisfying the semi-concentrating condition.

The second part of the paper is devoted to applications of results in Part I to some special Markov operators. The strategy for all chapters is the same. First we discuss nonexpansiveness. Next we study the existence of an invariant measure. Finally we estimate the capacity.

In Chapter 6 we are concerned with iterated function systems. Chapter 7 is devoted to stochastically perturbed dynamical systems and Chapter 8 deals with Poisson driven differential equations.

Acknowledgements. The paper was supported by the State Committee for Scientific Research (Poland) Grant No. 2 P03A 010 16 and Foundation for Polish Science. It was also supported by a Marie Curie Fellowship of the European Community program "Improving the Human Research Potential and the Socio-Economic Knowledge Base" under contract number HPMF-CT-20000-00824.

2. Preliminaries

2.1. Basic definitions and notation. Let (X, ϱ) be a Polish space, i.e., a separable, complete metric space. Throughout this paper $B(x,r)$ stands for the closed ball in X with centre at x and radius r. For every set $C \subset X$ and number $r > 0$ we denote by $\mathcal{N}^0(C,r)$ the open r-neighbourhood of the set C, i.e.,

$$\mathcal{N}^0(C,r) = \{x \in X : \varrho(C,x) < r\}$$

and by $\mathcal{N}(C,r)$ the closed r-neighbourhood of C, i.e.,

$$\mathcal{N}(C,r) = \{x \in X : \varrho(C,x) \le r\},$$

where $\varrho(C,x) = \inf\{\varrho(x,y) : y \in C\}$. For $C, C_0 \subset X$ we denote by $\operatorname{dist}(C, C_0)$ the distance of the sets C, C_0, i.e.,

$$\operatorname{dist}(C,C_0) = \min\{\inf_{x \in C} \varrho(C_0,x), \inf_{x\in C_0} \varrho(C,x)\}.$$

For $C \subset X$ we denote by $\operatorname{diam} C$ the diameter of C, i.e.,

$$\operatorname{diam} C = \sup\{\varrho(x,y) : x, y \in C\}.$$

By $\mathcal{B}(X)$ and $\mathcal{B}_b(X)$ we denote the families of all Borel sets and all bounded Borel sets, respectively.

By $\mathcal{C}_\varepsilon$, $\varepsilon > 0$, we denote the family of all closed sets C for which there exists a finite set $\{x_1, \ldots, x_n\} \subset X$ such that $C \subset \bigcup_{i=1}^n B(x_i, \varepsilon)$. By $\mathcal{C}_\varepsilon^k$, $\varepsilon > 0, k \in \mathbb{N}$, we denote the family of all $C \in \mathcal{C}_\varepsilon$ such that $C \subset \bigcup_{i=1}^k B(x_i,\varepsilon)$ for some $\{x_1, x_2, \ldots, x_k\} \subset X$.

By $\mathcal{M}_{\mathrm{fin}}$ and $\mathcal{M}_1$ we denote the sets of Borel measures (nonnegative, σ-additive) on X such that $\mu(X) < \infty$ for $\mu \in \mathcal{M}_{\mathrm{fin}}$ and $\mu(X) = 1$ for $\mu \in \mathcal{M}_1$. The elements of $\mathcal{M}_1$ are called *distributions*. By $\mathcal{M}_{\mathrm{sig}}$ we denote the family of all signed measures:

$$\mathcal{M}_{\mathrm{sig}} = \{\mu_1 - \mu_2 : \mu_1, \mu_2 \in \mathcal{M}_{\mathrm{fin}}\}.$$

We say that $\mu \in \mathcal{M}_{\mathrm{fin}}$ is *concentrated* on $A \in \mathcal{B}(X)$ if $\mu(X \setminus A) = 0$. By $\mathcal{M}_1^A$ we denote the set of all distributions concentrated on $A \in \mathcal{B}(X)$.

By $B(X)$ we denote the space of all bounded Borel measurable functions $f : X \to \mathbb{R}$. Further, by $C(X)$ we denote the subspace of all bounded continuous functions. These spaces are equipped with the supremum norm.

For X unbounded, a continuous function $V : X \to [0,\infty)$ is called a *Lyapunov function* if

$$\lim_{\varrho(x,x_0)\to\infty} V(x) = \infty \tag{2.1.1}$$

for some $x_0 \in X$.

To simplify the notation we will write

$$\langle f, \nu\rangle = \int_X f(x)\,\nu(dx) \quad \text{for } f \in B(X),\ \nu \in \mathcal{M}_{\mathrm{sig}}.$$

In the space $\mathcal{M}_{\mathrm{sig}}$ we introduce the *Fortet–Mourier norm* (see [12, 15, 20])

$$\|\nu\|_{\mathrm{FM}} = \sup\{|\langle f,\nu\rangle : f \in \mathcal{F}\}, \tag{2.1.2}$$

where $\mathcal{F} \subset C(X)$ consists of all functions such that $|f(x)| \le 1$ and $|f(x)-f(y)| \le \varrho(x,y)$ for arbitrary $x, y \in X$. It is known (see [12, 15]) that the convergence

$$\lim_{n\to\infty} \|\mu_n - \mu\|_{\mathrm{FM}} = 0 \quad \text{for } \mu_n, \mu \in \mathcal{M}_1$$

is equivalent to the weak convergence of $(\mu_n)_{n\ge 1}$ to μ .

Let $\Theta_0 \subset \mathcal{M}_1$. We call Θ_0 *tight* if for every $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $\mu(K) \ge 1-\varepsilon$ for all $\mu \in \Theta_0$.

It is well known (see [6, 12, 15]) that if the family $\{\mu_n\}_{n\ge 1}$ of distributions is tight, then there exists a subsequence $(m_n)_{n\ge 1}$ of integers and a measure $\mu_* \in \mathcal{M}_1$ such that

$$\lim_{n\to\infty} \|\mu_{m_n} - \mu_*\|_{\mathrm{FM}} = 0.$$

Let $\mu \in \mathcal{M}_1$. For given $\varepsilon > 0$ and $C \subset X$ we denote by $N_C(\varepsilon)$ the minimal number of balls with radius ε needed to cover the set C. Further, for $\varepsilon, \eta > 0$ we define

$$N(\varepsilon, \eta) = \inf\{N_C(\varepsilon) : C \subset X \text{ and } \mu(C) > 1-\eta\}.$$

Then the quantities

$$\underline{\mathrm{Cap}}_L(\mu) = \sup_{\eta>0} \liminf_{\varepsilon\to 0} \frac{\log N(\varepsilon,\eta)}{-\log\varepsilon}$$

and

$$\overline{\mathrm{Cap}}_L(\mu) = \sup_{\eta>0} \limsup_{\varepsilon\to 0} \frac{\log N(\varepsilon,\eta)}{-\log\varepsilon}$$

are called the *lower* and *upper capacity* of μ, respectively.

If the lower capacity is equal to the upper capacity we call this common value the *capacity* and denote it by $\mathrm{Cap}_L(\mu)$.

The above definitions were introduced by Ledrappier (see [40, 66]) and are closely related to the Kolmogorov dimension (see also [30]).

REMARK 2.1.1. In the definitions of the lower and upper capacity we can replace the continuous variable ε by a decreasing sequence $(\varepsilon_n)_{n\ge 1}$ with $\log\varepsilon_{n+1}/\log\varepsilon_n \to 1$ as $n \to \infty$.

2.2. Markov operators and semigroups of Markov operators. An operator $P : \mathcal{M}_{\mathrm{fin}} \to \mathcal{M}_{\mathrm{fin}}$ is called a *Markov operator* if it satisfies the following two conditions:

(i) *positive linearity*:

$$P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2$$

for $\lambda_1, \lambda_2 \ge 0$ and $\mu_1, \mu_2 \in \mathcal{M}_{\mathrm{fin}}$,

(ii) *preservation of measures*:

$$P\mu(X) = \mu(X) \quad \text{for } \mu \in \mathcal{M}_{\mathrm{fin}}.$$

It is easy to prove that every Markov operator can be extended to the space of all signed measures $\mathcal{M}_{\mathrm{sig}} = \{\mu_1 - \mu_2 : \mu_1, \mu_2 \in \mathcal{M}_{\mathrm{fin}}\}$. Namely for every $\nu \in \mathcal{M}_{\mathrm{sig}}$, $\nu = \mu_1 - \mu_2$, we set $P\nu = P\mu_1 - P\mu_2$.

A linear operator $U : B(X) \to B(X)$ is called *dual* to P if

$$\langle Uf, \mu\rangle = \langle f, P\mu\rangle \quad \text{for } f \in B(X),\ \mu \in \mathcal{M}_{\mathrm{fin}}. \tag{2.2.1}$$

Setting $\mu = \delta_x$ in (2.2.1) we obtain

$$Uf(x) = \langle f, P\delta_x \rangle \quad \text{for } f \in B(X),\ x \in X, \tag{2.2.2}$$

where $\delta_x \in \mathcal{M}_1$ is the point (Dirac) measure supported at x.

From (2.2.2) it follows immediately that U is a linear operator satisfying

$$Uf \geq 0 \quad \text{for } f \geq 0,\ f \in B(X), \tag{2.2.3}$$

$$U\mathbf{1}_X = \mathbf{1}_X, \tag{2.2.4}$$

$$Uf_n \downarrow 0 \quad \text{for } f_n \downarrow 0,\ f_n \in B(X). \tag{2.2.5}$$

Conditions (2.2.3)–(2.2.5) allow one to reverse the roles of P and U (for details see [33]). Namely we may define a Markov operator $P : \mathcal{M}_{\text{fin}} \to \mathcal{M}_{\text{fin}}$ by setting

$$P\mu(A) = \langle U\mathbf{1}_A, \mu \rangle \quad \text{for } \mu \in \mathcal{M}_{\text{fin}},\ A \in \mathcal{B}(X). \tag{2.2.6}$$

Assume now that P and U are given. If $f : X \to \mathbb{R}_+$ is a Borel measurable function, not necessarily bounded, we may assume that

$$Uf(x) = \lim_{n\to\infty} Uf_n(x),$$

where $(f_n)_{n\geq 1}$ is an increasing sequence of bounded Borel measurable functions converging pointwise to f. From the Lebesgue monotone convergence theorem it follows that Uf satisfies (2.2.1).

A Markov operator P is called a *Feller operator* if there exists a dual operator $U : B(X) \to B(X)$ satisfying (2.2.1) such that

$$Uf \in C(X) \quad \text{for } f \in C(X). \tag{2.2.7}$$

A family $\{P^t\}_{t\geq 0}$ of Markov operators is called a *semigroup* if $P^{t+s} = P^t P^s$ for all $t, s \in \mathbb{R}_+$ and P^0 is the identity operator on $\mathcal{M}_{\text{fin}}$.

2.3. Properties of Markov operators. A Markov operator P is called *nonexpansive* if

$$\|P\mu_1 - P\mu_2\|_{\text{FM}} \leq \|\mu_1 - \mu_2\|_{\text{FM}} \quad \text{for } \mu_1, \mu_2 \in \mathcal{M}_1. \tag{2.3.1}$$

Let P be a Markov operator. A measure $\mu \in \mathcal{M}_{\text{fin}}$ is called *stationary* or *invariant* if $P\mu = \mu$, and P is called *asymptotically stable* if there exists a stationary distribution $\mu_\star$ such that

$$\lim_{n\to\infty} \|P^n\mu - \mu_\star\|_{\text{FM}} = 0 \quad \text{for } \mu \in \mathcal{M}_1. \tag{2.3.2}$$

Clearly the distribution $\mu_\star$ satisfying (2.3.2) is unique.

Let $\{P^t\}_{t\geq 0}$ be a Markov semigroup. The Markov semigroup $\{P^t\}_{t\geq 0}$ is called *nonexpansive* if every Markov operator P^t, $t \geq 0$, is nonexpansive. A measure $\mu \in \mathcal{M}_{\text{fin}}$ is called *stationary* or *invariant* for the Markov semigroup $\{P^t\}_{t\geq 0}$ if $P^t\mu = \mu$ for all $t \geq 0$. The Markov semigroup $\{P^t\}_{t\geq 0}$ is called *asymptotically stable* if there exists a stationary distribution $\mu_\star$ such that

$$\lim_{t\to\infty} \|P^t\mu - \mu_\star\|_{\text{FM}} = 0 \quad \text{for } \mu \in \mathcal{M}_1. \tag{2.3.3}$$

An operator P is called *globally concentrating* if for every $\varepsilon > 0$ and every $A \in \mathcal{B}_b(X)$ there exist $B \in \mathcal{B}_b(X)$ and $n_0 \in \mathbb{N}$ such that

$$P^n\mu(B) \ge 1 - \varepsilon \quad \text{for } n \ge n_0,\ \mu \in \mathcal{M}_1^A. \tag{2.3.4}$$

An operator P is called *locally concentrating* if for every $\varepsilon > 0$ there is $\alpha > 0$ such that for every $A \in \mathcal{B}_b(X)$ there exist $C \in \mathcal{B}_b(X)$ with $\operatorname{diam} C \le \varepsilon$ and $n_0 \in \mathbb{N}$ satisfying

$$P^{n_0}\mu(C) > \alpha \quad \text{for } \mu \in \mathcal{M}_1^A. \tag{2.3.5}$$

Remark 2.3.1. One can construct a Markov operator which is locally concentrating but not globally concentrating.

An operator P is called *concentrating* if for every $\varepsilon > 0$ there exist $C \in \mathcal{B}_b(X)$ with $\operatorname{diam} C \le \varepsilon$ and $\alpha > 0$ such that

$$\liminf_{n\to\infty} P^n\mu(C) > \alpha \quad \text{for } \mu \in \mathcal{M}_1. \tag{2.3.6}$$

An operator P is called *semi-concentrating* if for every $\varepsilon > 0$ there exist $C \in \mathcal{C}_\varepsilon$ and $\alpha > 0$ such that

$$\liminf_{n\to\infty} P^n\mu(C) > \alpha \quad \text{for } \mu \in \mathcal{M}_1. \tag{2.3.7}$$

We finish this section by introducing the following notation:

$$\Omega(\mu) = \{\nu \in \mathcal{M}_1 : \exists_{(m_n)_{n\ge1}}, m_n \to \infty \text{ and } \|P^{m_n}\mu - \nu\|_{\mathrm{FM}} \to 0\} \tag{2.3.8}$$

for $\mu \in \mathcal{M}_1$.

2.4. Technical lemmas. We start with an easy lemma.

Lemma 2.4.1. *If $\|\mu_1 - \mu_2\|_{\mathrm{FM}} \le \varepsilon^2$ for $\mu_1, \mu_2 \in \mathcal{M}_1$ and some $\varepsilon > 0$, then*

$$\mu_1(\mathcal{N}^0(C,\varepsilon)) \ge \mu_2(C) - \varepsilon \quad \textit{for } C \in \mathcal{B}(X).$$

Proof. Fix $C \in \mathcal{B}(X)$. Define $f(x) = \max(\varepsilon - \varrho(C,x), 0)$. Since $f \in \mathcal{F}$ and $f(x) = 0$ for $x \notin \mathcal{N}^0(C,\varepsilon)$, while $f(x) = \varepsilon$ for $x \in C$, we have

$$\varepsilon\mu_2(C) - \varepsilon\mu_1(\mathcal{N}^0(C,\varepsilon)) \le |\langle f,\mu_1\rangle - \langle f,\mu_2\rangle| \le \|\mu_1 - \mu_2\|_{\mathrm{FM}} \le \varepsilon^2$$

and the assertion follows. ∎

Lemma 2.4.2. *Let P be a Markov operator and U its dual. Assume that there exists a Lyapunov function V, bounded on bounded sets, such that*

$$UV(x) \le aV(x) + b \quad \textit{for } x \in X, \tag{2.4.1}$$

where a,b are nonnegative constants and $a < 1$. Then P is globally concentrating.

Proof. From (2.4.1) it follows that

$$U^nV(x) \le a^nV(x) + \frac{b}{1-a} \quad \text{for } n \in \mathbb{N}. \tag{2.4.2}$$

Fix $\varepsilon > 0$. Let $A \in \mathcal{B}_b(X)$ and $\mu \in \mathcal{M}_1^A$. Set $B = \{x : V(x) \le q\}$, where $q > 2b(\varepsilon(1-a))^{-1}$. From (2.4.2) and the Chebyshev inequality we obtain

$$\begin{aligned} P^n\mu(B) &\ge 1 - \frac{1}{q}\int_X V(x)\, P^n\, \mu(dx) = 1 - \frac{1}{q}\int_X U^n V(x)\, \mu(dx) \\ &\ge 1 - \frac{1}{q}\Big(a^n \int_X V(x)\, \mu(dx) + \frac{b}{1-a}\Big) \\ &\ge 1 - \frac{\varepsilon}{2} - \frac{a^n}{q}\int_X V(x)\, \mu(dx) \ge 1 - \frac{\varepsilon}{2} - \frac{a^n}{q}\sup_{x\in A} V(x). \end{aligned}$$

Consequently, there exists an integer n_0 such that $P^n\mu(B) \ge 1-\varepsilon$ for $n \ge n_0$, $\mu \in \mathcal{M}_1^A$, and the assertion follows. ■

The proof above gives more.

COROLLARY 2.4.1. *Let P be a Markov operator and U its dual. Assume that there exists a Lyapunov function V, bounded on bounded sets, such that condition* (2.4.1) *holds. Then for every $\varepsilon > 0$ there exists $B \in \mathcal{B}_b(X)$ such that*

$$\liminf_{n\to\infty} P^n\mu(B) \ge 1-\varepsilon \quad \textit{for } \mu \in \mathcal{M}_1.$$

Proof. Fix $\varepsilon > 0$. In Lemma 2.4.1 we have proved that there exists $B \in \mathcal{B}_b(X)$ such that

$$\liminf_{n\to\infty} P^n\mu(B) \ge 1-\varepsilon/2 \quad \text{for } \mu \in \mathcal{M}_1^A,\ A \in \mathcal{B}_b(X).$$

Fix $\mu \in \mathcal{M}_1$. Let $A \in \mathcal{B}_b(X)$ be such that $\mu(A) \ge 1-\varepsilon/2$. Define the measure $\mu_* \in \mathcal{M}_1$ by

$$\mu_*(C) = \mu(C\cap A)/\mu(A) \quad \text{for } C \in \mathcal{B}(X).$$

Observe that $\mu_* \in \mathcal{M}_1^A$ and $\mu \ge (1-\varepsilon/2)\mu_*$. By the linearity of P we obtain

$$\liminf_{n\to\infty} P^n\mu(B) \ge (1-\varepsilon/2)\liminf_{n\to\infty} P^n\mu_*(B) \ge 1-\varepsilon. \ ■$$

Let P be a Markov operator. Now for every $A \in \mathcal{B}_b(X)$ and $\eta \in [0,1]$ we set

$$\mathcal{M}_1^{A,\eta} = \{\mu \in \mathcal{M}_1 : P^n\mu(A) \ge 1-\eta \text{ for } n \in \mathbb{N}\}.$$

Define the function $\varphi : \mathcal{B}_b(X) \times [0,1] \to [0,2] \cup \{-\infty\}$ by

$$\varphi(A,\eta) = \limsup_{n\to\infty}\ \sup\{\|P^n\mu_1 - P^n\mu_2\|_{\mathrm{FM}} : \mu_1,\mu_2 \in \mathcal{M}_1^{A,\eta}\}.$$

As usual, the supremum of an empty set is taken to be $-\infty$.

LEMMA 2.4.3. *Let P be a nonexpansive and locally concentrating Markov operator. Let $\varepsilon > 0$ and $\alpha > 0$ be such that, for ε, the locally concentrating property holds. If $\eta < 1/2$, then*

$$\varphi(A,\eta(1-\alpha/2)) \le (1-\alpha/2)\varphi(A,\eta) + \alpha\varepsilon/2 \tag{2.4.3}$$

for $A \in \mathcal{B}_b(X)$.

Proof. Fix $\varepsilon > 0$, $A \in \mathcal{B}_b(X)$ and $\eta < 1/2$. Let $\alpha > 0$, $n_0 \in \mathbb{N}$ and $C \in \mathcal{B}_b(X)$ be such that, for ε, the locally concentrating property holds. We see at once that if $\mathcal{M}_1^{A,\eta(1-\alpha/2)} = \emptyset$, then (2.4.3) is satisfied. Fix $\mu_1, \mu_2 \in \mathcal{M}_1^{A,\eta(1-\alpha/2)}$. As $\eta < 1/2$ we have $\mu_i \geq \mu_i^A/2$, where $\mu_i^A \in \mathcal{M}_1^A$ is of the form

$$\mu_i^A(B) = \mu_i(A \cap B)/\mu_i(A) \quad \text{for } B \in \mathcal{B}(X),\ i = 1, 2.$$

By the linearity of P we obtain

$$P^{n_0}\mu_i(C) \geq \tfrac{1}{2} \cdot P^{n_0}\mu_i^A(C) > \alpha/2 \quad \text{for } i = 1, 2.$$

Hence for $i = 1, 2$ we have

$$P^{n_0}\mu_i = (1 - \alpha/2)\overline{\mu}_i + (\alpha/2)\nu_i, \tag{2.4.4}$$

where $\nu_i \in \mathcal{M}_1^C$ is defined by

$$\nu_i(B) = P^{n_0}\mu_i(B \cap C)/P^{n_0}\mu_i(C) \quad \text{for } B \in \mathcal{B}(X)$$

and $\overline{\mu}_i$ is defined by (2.4.4). Since $\nu_1, \nu_2 \in \mathcal{M}_1^C$ and $\operatorname{diam} C \leq \varepsilon$, we check at once that $\|\nu_1 - \nu_2\|_{\mathrm{FM}} \leq \varepsilon$. From (2.4.4) we conclude that

$$\begin{aligned} P^n\overline{\mu}_i(A) &\geq \frac{1}{1-\alpha/2}\{P^{n_0+n}\mu_i(A) - \alpha/2\} \\ &\geq \frac{1}{1-\alpha/2}\{1 - \eta(1-\alpha/2) - \alpha/2\} \\ &= 1 - \eta \quad \text{for } n \in \mathbb{N} \text{ and } i = 1, 2. \end{aligned}$$

This gives $\overline{\mu}_1, \overline{\mu}_2 \in \mathcal{M}_1^{A,\eta}$ and consequently, since P is nonexpansive, we have

$$\begin{aligned} \|P^{n_0+n}\mu_1 - P^{n_0+n}\mu_2\|_{\mathrm{FM}} &\leq (1-\alpha/2)\|P^n\overline{\mu}_1 - P^n\overline{\mu}_2\|_{\mathrm{FM}} + (\alpha/2)\|P^n\nu_1 - P^n\nu_2\|_{\mathrm{FM}} \\ &\leq (1-\alpha/2)\sup\{\|P^n\mu_1 - P^n\mu_2\|_{\mathrm{FM}} : \mu_1, \mu_2 \in \mathcal{M}_1^{A,\eta}\} + \alpha\varepsilon/2. \end{aligned}$$

By the above we obtain $\varphi(A, \eta(1-\alpha/2)) \leq (1-\alpha/2)\varphi(A, \eta) + \alpha\varepsilon/2$. ∎

We denote by $\mathcal{T}_\varepsilon$, $\varepsilon > 0$, the family of all $C \in \mathcal{C}_\varepsilon$ such that there exists a positive number α satisfying

$$\liminf_{n\to\infty} P^n\mu(C) > \alpha \quad \text{for } \mu \in \mathcal{M}_1.$$

REMARK 2.4.1. It is easy to see that if P is a semi-concentrating Markov operator, then $\mathcal{T}_\varepsilon \neq \emptyset$ for every $\varepsilon > 0$.

For $\varepsilon > 0$ and $k \in \mathbb{N}$, set

$$\mathcal{T}_\varepsilon^k = \mathcal{C}_\varepsilon^k \cap \mathcal{T}_\varepsilon.$$

We are now in a position to formulate the following technical lemma.

LEMMA 2.4.4. *Let P be a nonexpansive and semi-concentrating Markov operator. Then for every $\varepsilon > 0$ there exist an integer k, a sequence $(A_1, \ldots, A_k)$, $A_i \in \mathcal{B}_b(X)$,* diam $A_i \leq \varepsilon$ *for $i = 1, \ldots, k$, and a measure $\mu_0 \in \mathcal{M}_1$ such that $\bigcup_{i=1}^k A_i \in \mathcal{T}_\varepsilon^k$ and*

$$\liminf_{n\to\infty} P^n\mu_0(A_i) > 0 \quad \textit{for } i = 1, \ldots, k.$$

Proof. Fix $\varepsilon > 0$. Set

$$k = \min\{m \in \mathbb{N} : \exists \sigma \in (0, \varepsilon/2)\ \mathcal{T}_\sigma^m \neq \emptyset\}.$$

Choose $\eta \in (0, \varepsilon/2)$ such that $\mathcal{T}_\eta^k \neq \emptyset$. Choose $C = \bigcup_{i=1}^k \widetilde{A}_i$ where $\widetilde{A}_i$ are closed balls with radius η and $\alpha > 0$ such that

$$\liminf_{n\to\infty} P^n\mu(C) > \alpha \quad \text{for } \mu \in \mathcal{M}_1. \tag{2.4.5}$$

Let $\gamma > 0$ be such that

$$\eta + \gamma < \varepsilon/2 \quad \text{and} \quad k \cdot \gamma < \alpha. \tag{2.4.6}$$

Set $\widetilde{\varepsilon} = \gamma^2$. Let $\widetilde{\alpha} > 0$, $p \in \mathbb{N}$ and $\widetilde{C} = \bigcup_{i=1}^p D_i$, where D_i are closed balls with radius $\widetilde{\varepsilon}$ chosen according to the semi-concentrating property of P for $\widetilde{\varepsilon}$. For every $\mu \in \mathcal{M}_1$ we define the set $\mathcal{S}(\mu)$ of all $j \in \{1, \ldots, p\}$ such that there exists $n \in \mathbb{N}$ satisfying $P^n\mu(D_j) \geq \widetilde{\alpha}/p$. Obviously $\mathcal{S}(\mu) \neq \emptyset$. Further, it follows easily that $j \in \mathcal{S}(\mu)$ iff there exists $n \in \mathbb{N}$ and $\nu \in \mathcal{M}_1^{D_j}$ such that

$$P^n\mu \geq (\widetilde{\alpha}/p)\nu. \tag{2.4.7}$$

Namely, it is enough to define $\nu \in \mathcal{M}_1$ by the formula

$$\nu(A) = P^n\mu(A \cap D_j)/P^n\mu(D_j) \quad \text{for } A \in \mathcal{B}(X),\ j \in \mathcal{S}(\mu).$$

We proceed to show that for every $i \in \{1, \ldots, k\}$ there exists $\mu \in \mathcal{M}_1$ such that for every $j \in \mathcal{S}(\mu)$ and $x \in D_j$ we have

$$P^n\delta_x(\widetilde{A}_i) \geq \alpha/k \quad \text{for some } n \in \mathbb{N}. \tag{2.4.8}$$

We can assume that $i = 1$. Suppose, contrary to our claim, that for every $\mu \in \mathcal{M}_1$ there exist $j \in \mathcal{S}(\mu)$ and $x \in D_j$ such that

$$P^n\delta_x(\widetilde{A}_1) < \alpha/k \quad \text{for } n \in \mathbb{N}.$$

From (2.4.5) we conclude that $P^n\delta_x(\widetilde{A}_{i_n}) \geq \alpha/k$ for all sufficiently large $n \in \mathbb{N}$, where $i_n \in \{1, \ldots, k\}$ and $i_n \neq 1$. Since $\|\delta_x - \nu\|_{\mathrm{FM}} \leq \operatorname{diam} D_j \leq \gamma^2$ for $\nu \in \mathcal{M}_1^{D_j}$, Lemma 2.4.1 now shows that

$$P^n\nu(\mathcal{N}(\widetilde{A}_{i_n}, \gamma)) \geq \alpha/k - \gamma$$

for $\nu \in \mathcal{M}_1^{D_j}$ and all sufficiently large $n \in \mathbb{N}$. Since $i_n \neq 1$, we then obtain

$$P^n\nu\Big(\bigcup_{i=2}^k \mathcal{N}(\widetilde{A}_i, \gamma)\Big) \geq \alpha/k - \gamma \quad \text{for } \nu \in \mathcal{M}_1^{D_j}.$$

From (2.4.7) we conclude that $P^{n_0}\mu \geq (\widetilde{\alpha}/p)\nu$ for some $\nu \in \mathcal{M}_1^{D_j}$ and $n_0 \in \mathbb{N}$. Consequently, by the linearity of P we have $P^{n_0+n}\mu \geq (\widetilde{\alpha}/p)P^n\nu$ for $n \in \mathbb{N}$ and

$$P^{n_0+n}\mu\Big(\bigcup_{i=2}^k \mathcal{N}(\widetilde{A}_i, \gamma)\Big) \geq (\widetilde{\alpha}/p)(\alpha/k - \gamma) \quad \text{for all sufficiently large } n \in \mathbb{N}.$$

Thus

$$\liminf_{n\to\infty} P^n\mu\Big(\bigcup_{i=2}^k \mathcal{N}(\widetilde{A}_i, \gamma)\Big) \geq (\widetilde{\alpha}/p)(\alpha/k - \gamma) \quad \text{for } \mu \in \mathcal{M}_1.$$

By the above we conclude that $\bigcup_{i=2}^k \mathcal{N}(\widetilde{A}_i, \gamma) \in \mathcal{C}_{\eta+\gamma}^{k-1}$, hence $\mathcal{T}_{\eta+\gamma}^{k-1} \neq \emptyset$, contrary to the definition of k.

Let $\mu_i \in \mathcal{M}_1$, $1 \le i \le k$, be such that for every $j \in \mathcal{S}(\mu_i)$ and $x \in D_j$ we have

$$P^n \delta_x(\widetilde{A}_i) \ge \alpha/k \quad \text{for some } n \in \mathbb{N}.$$

Fix $i \in \{1, \ldots, k\}$. For every $j \in \mathcal{S}(\mu_i)$ choose a point $x_j \in D_j$ and an integer n_j such that $P^{n_j}\delta_{x_j}(\widetilde{A}_i) \ge \alpha/k$. By Lemma 2.4.1 we obtain

$$P^{n_j}\nu(\mathcal{N}(\widetilde{A}_i, \gamma)) \ge \alpha/k - \gamma \quad \text{for } \nu \in \mathcal{M}_1^{D_j}.$$

Set $N_i = \max_{j\in\mathcal{S}(\mu_i)} n_j$ and $A_i = \mathcal{N}(\widetilde{A}_i, \gamma)$. From (2.4.6) it follows that $\operatorname{diam} A_i \le \varepsilon$. Obviously $\bigcup_{i=1}^k A_i \in \mathcal{T}_\varepsilon^k$. Define $\overline{\mu}_i \in \mathcal{M}_1$ by the formula

$$\overline{\mu}_i = \frac{\mu_i + P\mu_i + \ldots + P^{N_i}\mu_i}{N_i + 1}.$$

It is easy to check that

$$\liminf_{n\to\infty} P^n \overline{\mu}_i(A_i) \ge (\alpha/k - \gamma)\widetilde{\alpha}/(p(N_i + 1)) > 0 \tag{2.4.9}$$

for $i \in \{1, \ldots, k\}$. Write

$$\mu_0 = \frac{\overline{\mu}_1 + \ldots + \overline{\mu}_k}{k}.$$

From (2.4.9) and the linearity of P we have $\liminf_{n\to\infty} P^n\mu_0(A_i) > 0$ for $i = 1, \ldots, k$. ∎

Lemma 2.4.5. *Let P be a nonexpansive Markov operator and let $A \in \mathcal{B}(X)$. Given any $\varepsilon > 0$ suppose that* $\operatorname{diam} A \le \varepsilon^2/16$. *Moreover, assume that there exists $\mu \in \mathcal{M}_1$ such that*

$$\liminf_{n\to\infty} P^n\mu(A) > 0. \tag{2.4.10}$$

Then there exists $C \in \mathcal{C}_\varepsilon$ such that

$$P^n\nu(C) \ge 1 - \varepsilon/2 \quad \textit{for } n \in \mathbb{N} \textit{ and } \nu \in \mathcal{M}_1^A.$$

Proof. Choose $\alpha > 0$ such that $\liminf_{n\to\infty} P^n\mu(A) \ge \alpha$. If $P^n\mu(A) \ge \alpha/2$, then

$$P^n\mu \ge (\alpha/2)\nu_n, \tag{2.4.11}$$

where $\nu_n \in \mathcal{M}_1^A$ is of the form

$$\nu_n(B) = P^n\mu(B \cap A)/P^n\mu(A) \quad \text{for } B \in \mathcal{B}(X). \tag{2.4.12}$$

Define

$$\delta = \sup\{\gamma \ge 0 : \exists C_{\varepsilon/2} \in \mathcal{C}_{\varepsilon/2} \ \liminf_{n\to\infty} P^n\mu(C_{\varepsilon/2}) \ge \gamma\}. \tag{2.4.13}$$

Choose $\gamma \ge 0$ and $C_{\varepsilon/2} \in \mathcal{C}_{\varepsilon/2}$ such that $0 \le \delta - \gamma < \alpha\varepsilon/8$ and

$$\liminf_{n\to\infty} P^n\mu(C_{\varepsilon/2}) \ge \gamma.$$

We are now in a position to show that

$$P^n\nu(\mathcal{N}^0(C_{\varepsilon/2}, \varepsilon/2)) \ge 1 - \varepsilon/2 \quad \text{for } n \in \mathbb{N} \text{ and } \nu \in \mathcal{M}_1^A. \tag{2.4.14}$$

Suppose that, on the contrary, for some $n_0 \in \mathbb{N}$ and $\nu_0 \in \mathcal{M}_1^A$,

$$P^{n_0}\nu_0(\mathcal{N}^0(C_{\varepsilon/2}, \varepsilon/2)) < 1 - \varepsilon/2. \tag{2.4.15}$$

By the Ulam theorem (see [12, 15]), there exists a compact set $K \subset X \setminus \mathcal{N}^0(C_{\varepsilon/2}, \varepsilon/2)$ such that $P^{n_0}\nu_0(K) \geq \varepsilon/2$. Since P is nonexpansive, we have

$$\|P^{n_0}\nu_0 - P^{n_0}\nu\|_{\mathrm{FM}} \leq \|\nu_0 - \nu\|_{\mathrm{FM}} \leq \operatorname{diam} A \leq \varepsilon^2/16$$

for every $\nu \in \mathcal{M}_1^A$. Lemma 2.4.1 now shows that $P^{n_0}\nu(\mathcal{N}^0(K, \varepsilon/4)) \geq \varepsilon/4$. Putting $B = \mathcal{N}(K, \varepsilon/4)$ we obtain $B \in \mathcal{C}_{\varepsilon/2}$ and consequently $B \cup C_{\varepsilon/2} \in \mathcal{C}_{\varepsilon/2}$. Applying (2.4.11), the linearity of P and the fact that $\nu_n \in \mathcal{M}_1^A$ we have

$$P^{n+n_0}\mu(B) \geq (\alpha/2)P^{n_0}\nu_n(B) \geq \alpha\varepsilon/8$$

for every sufficiently large n. Since $B \cap C_{\varepsilon/2} = \emptyset$, we see that

$$\liminf_{n\to\infty} P^n\mu(B \cup C_{\varepsilon/2}) \geq \liminf_{n\to\infty} P^n\mu(B) + \liminf_{n\to\infty} P^n\mu(C_{\varepsilon/2}) \geq \alpha\varepsilon/8 + \gamma > \delta,$$

which contradicts the definition of δ. Thus (2.4.14) holds. Put $C = \mathcal{N}(C_{\varepsilon/2}, \varepsilon/2)$ and note that $C \in \mathcal{C}_\varepsilon$. ∎

3. Nonexpansiveness

We start with the following definitions. We say that a metric ϱ' is *equivalent* to ϱ if the classes of bounded sets and convergent sequences in the spaces (X, ϱ) and (X, ϱ') coincide. Obviously, if (X, ϱ) is a Polish space and ϱ, ϱ' are equivalent, then the space (X, ϱ') is still a Polish space.

We say that a Markov operator P is *essentially nonexpansive* if there exists a metric ϱ' equivalent to ϱ such that P is nonexpansive in (X, ϱ').

THEOREM 3.1. *Let P be a Markov operator. Assume that P is continuous in the weak topology. Then P is a Feller operator. Moreover, if the operator $U : B(X) \to B(X)$ given by* (2.2.2) *satisfies $U(\mathcal{F}) \subset \mathcal{F}$, then P is nonexpansive.*

Proof. Obviously U is linear. Since P is continuous in the weak topology, we see that $U(B(X)) \subset B(X)$. Further, for an arbitrary sequence $x_n \to x_0$, $x_n, x_0 \in X$, we have $\delta_{x_n} \to \delta_{x_0}$ in the weak topology, and hence $P\delta_{x_n} \to P\delta_{x_0}$. Consequently, by the definition of U we have $Uf(x_n) \to Uf(x_0)$ for $f \in C(X)$. Thus we have verified that $U(C(X)) \subset C(X)$. According to the definition of U we have

$$\langle Uf, \mu\rangle = \langle f, P\mu\rangle \quad \text{for } f \in C(X) \text{ and } \mu = \delta_x.$$

Since the linear combinations of point measures are dense in $\mathcal{M}_{\mathrm{fin}}$ (in the weak topology) and P is continuous, equality (2.2.1) holds for every $\mu \in \mathcal{M}_{\mathrm{fin}}$. Since $U(\mathcal{F}) \subset \mathcal{F}$, we have

$$\begin{aligned}\|P\mu_1 - P\mu_2\|_{\mathrm{FM}} &= \sup\{|\langle f, P\mu_1 - P\mu_2\rangle| : f \in \mathcal{F}\} = \sup\{|\langle Uf, \mu_1 - \mu_2\rangle| : f \in \mathcal{F}\} \\ &\leq \sup\{|\langle f, \mu_1 - \mu_2\rangle| : f \in \mathcal{F}\} = \|\mu_1 - \mu_2\|_{\mathrm{FM}}\end{aligned}$$

for all $\mu_1, \mu_2 \in \mathcal{M}_1$, which finishes the proof. ∎

THEOREM 3.2. *Let P be an essentially nonexpansive Markov operator. Then P is a Feller operator.*

Proof. It suffices to make the following observation. If P is essentially nonexpansive, then there exists a metric ϱ' such that P is nonexpansive in (X, ϱ'). Consequently,

P is continuous in the weak topology generated by ϱ'. Since the metric ϱ' is equivalent to ϱ, the weak topologies generated by ϱ and ϱ' coincide. An application of Theorem 3.1 finishes the proof. ■

4. Tightness criteria

The main aim of this chapter is to present tightness criteria for Markov operators. The tightness property gives us a tool for proving the existence of an invariant measure.

We start with the following lemma.

LEMMA 4.1. *If $\Theta_0 \subset \mathcal{M}_1$ is such that for every $\varepsilon > 0$ there is a set $C \in \mathcal{C}_\varepsilon$ satisfying $\mu(C) \geq 1 - \varepsilon$ for $\mu \in \Theta_0$, then Θ_0 is tight.*

Proof. Fix $\varepsilon > 0$. Let $C_k \in \mathcal{C}_{\varepsilon/2^k}$, $k \in \mathbb{N}$, be such that $\mu(C_k) \geq 1 - \varepsilon/2^k$ for $\mu \in \Theta_0$. Define $K = \bigcap_{k=1}^{\infty} C_k$. Observe that K is compact and

$$\begin{aligned}\mu(X \setminus K) &= \mu\Big(X \setminus \bigcap_{k=1}^{\infty} C_k\Big) = \mu\Big(\bigcup_{k=1}^{\infty} (X \setminus C_k)\Big)\\ &\leq \sum_{k=1}^{\infty} \mu(X \setminus C_k) \leq \sum_{k=1}^{\infty} \varepsilon/2^k = \varepsilon \quad \text{for } \mu \in \Theta_0. \ ■\end{aligned}$$

For the convenience of the reader we present the following lemma.

LEMMA 4.2. *If $(\mu_n)_{n\geq 1}$, $\mu_n \in \mathcal{M}_1, n \in \mathbb{N}$, satisfies the Cauchy condition, then $\{\mu_n\}_{n\geq 1}$ is tight.*

Proof. Fix $\varepsilon > 0$. Since $(\mu_n)_{n\geq 1}$ satisfies the Cauchy condition, there exists $n_0 \in \mathbb{N}$ such that

$$\|\mu_p - \mu_q\|_{\mathrm{FM}} \leq \varepsilon^2/4 \quad \text{for } p, q \geq n_0. \tag{4.1}$$

By the Ulam theorem (see [12, 15]) we may choose a compact set $K \subset X$ such that

$$\mu_n(K) \geq 1 - \varepsilon/2 \quad \text{for } n = 1, \ldots, n_0. \tag{4.2}$$

From Lemma 2.4.1 and conditions (4.1), (4.2) it follows that for $n \geq n_0$ we have

$$\mu_n(\mathcal{N}(K, \varepsilon/2)) \geq \mu_{n_0}(K) - \varepsilon/2 \geq 1 - \varepsilon.$$

Observe that $\mathcal{N}(K, \varepsilon/2) \in \mathcal{C}_\varepsilon$ and

$$\mu_n(\mathcal{N}(K, \varepsilon/2)) \geq 1 - \varepsilon \quad \text{for } n \in \mathbb{N}.$$

An application of Lemma 4.1 finishes the proof. ■

We say that a Markov operator $P : \mathcal{M}_{\mathrm{fin}} \to \mathcal{M}_{\mathrm{fin}}$ is *tight* if for every $\mu \in \mathcal{M}_1$ the family $\{P^n\mu\}_{n\geq 1}$ of distributions is tight.

THEOREM 4.1. *Let P be a nonexpansive and locally concentrating Markov operator. If for every $\mu \in \mathcal{M}_1$ and every $\varepsilon > 0$ there is $A \in \mathcal{B}_b(X)$ such that*

$$\liminf_{n\to\infty} P^n\mu(A) \geq 1 - \varepsilon, \tag{4.3}$$

then P is tight.

Proof. By Lemma 4.2 it is enough to show that $(P^n\mu)_{n\geq 1}$, $\mu\in\mathcal{M}_1$, satisfies the Cauchy condition. Fix $\varepsilon>0$ and $\mu\in\mathcal{M}_1$. Let $\alpha>0$ be such that, for $\varepsilon/2$, the locally concentrating property holds. Let $k\in\mathbb{N}$ be such that $4(1-\alpha/2)^k<\varepsilon$. Choose $A\in\mathcal{B}_b(X)$ satisfying

$$P^n\mu(A)\geq 1-\tfrac{1}{3}(1-\alpha/2)^k \quad \text{for } n\in\mathbb{N}.$$

Lemma 2.4.3 and an induction argument give

$$\begin{aligned}\varphi(A,\tfrac{1}{3}(1-\alpha/2)^k) &\leq (1-\alpha/2)^k\varphi(A,1/3)+\alpha\varepsilon/4\\ &\quad+\alpha\varepsilon(1-\alpha/2)/4+\ldots+\alpha\varepsilon(1-\alpha/2)^{k-1}/4\\ &\leq 2(1-\alpha/2)^k+\varepsilon/2<\varepsilon.\end{aligned}\tag{4.4}$$

It is clear that $P^m\mu, P^n\mu\in\mathcal{M}_1^{A,(1/3)\cdot(1-\alpha/2)^k}$ for $m,n\in\mathbb{N}$ and from (4.4) it follows that there exists $n_0\in\mathbb{N}$ such that

$$\|P^{n_0}P^n\mu-P^{n_0}P^m\mu\|_{\mathrm{FM}}<\varepsilon.$$

Therefore $\|P^p\mu-P^q\mu\|_{\mathrm{FM}}<\varepsilon$ for $p,q\geq n_0$. ∎

THEOREM 4.2. *Let P be a Markov operator. If for every measure $\mu\in\mathcal{M}_1$ and every $\varepsilon>0$ there is a set $C\in\mathcal{C}_\varepsilon$ satisfying $P^n\mu(C)\geq 1-\varepsilon$ for $n\in\mathbb{N}$, then P is tight.*

Proof. This is an immediate consequence of Lemma 4.1. ∎

THEOREM 4.3. *Let P be a nonexpansive and concentrating Markov operator. Then P is tight.*

Proof. By Theorem 4.2 it is enough to show that for every $\varepsilon>0$ and $\mu\in\mathcal{M}_1$ there exists $C\in\mathcal{C}_\varepsilon$ such that $P^n\mu(C)\geq 1-\varepsilon$ for $n\in\mathbb{N}$. Fix $\varepsilon>0$. Set $\widetilde{\varepsilon}=\varepsilon^2/16$. Let $\alpha>0$ and $A\in\mathcal{B}_b(X)$ with $\operatorname{diam} A\leq\varepsilon^2/16$ be chosen according to the concentrating property for $\widetilde{\varepsilon}$. It is easy to see that the assumptions of Lemma 2.4.5 are satisfied. Thus there exists $C\in\mathcal{C}_\varepsilon$ such that

$$P^n\nu(C)\geq 1-\varepsilon/2\tag{4.5}$$

for $n\in\mathbb{N}$ and $\nu\in\mathcal{M}_1^A$.

We define by induction a sequence $(n_k)_{k\geq 0}$ of integers and two sequences $(\mu_k)_{k\geq 0}$, $(\nu_k)_{k\geq 0}$ of distributions. If $k=0$ we set $n_0=0$ and $\mu_0=\nu_0=\mu$. If $k\geq 1$ and n_{k-1}, μ_{k-1}, ν_{k-1} are given we choose, according to the concentrating property, n_k such that $P^{n_k}\mu_{k-1}(A)\geq\alpha/2$ and we define

$$\nu_k(B)=\frac{P^{n_k}\mu_{k-1}(B\cap A)}{P^{n_k}\mu_{k-1}(A)},$$

$$\mu_k(B)=\frac{1}{1-\alpha/2}\left(P^{n_k}\mu_{k-1}(B)-(\alpha/2)\nu_k(B)\right) \quad \text{for } B\in\mathcal{B}(X).\tag{4.6}$$

Observe that $\nu_k\in\mathcal{M}_1^A$. Using (4.6) it is easy to verify by induction that

$$\begin{aligned}P^{n_1+\ldots+n_k}\mu&=(\alpha/2)P^{n_2+\ldots+n_k}\nu_1+(\alpha/2)(1-\alpha/2)P^{n_3+\ldots+n_k}\nu_2\\ &\quad+\ldots+(\alpha/2)(1-\alpha/2)^{k-1}\nu_k+(1-\alpha/2)^k\mu_k.\end{aligned}\tag{4.7}$$

Let $k\in\mathbb{N}$ be such that

$$(1-(1-\alpha/2)^k)(1-\varepsilon/2)\geq 1-\varepsilon.$$

Since $\nu_i \in \mathcal{M}_1^A$, $i = 1, \ldots, k$, and (4.5) holds, we have

$$\begin{aligned} P^n\mu(C) &\geq (\alpha/2)P^{n-n_1}\nu_1(C) + (\alpha/2)(1-\alpha/2)P^{n-n_1-n_2}\nu_2(C) \\ &\quad + \ldots + (\alpha/2)(1-\alpha/2)^{k-1}P^{n-n_1-\ldots-n_k}\nu_k(C) \\ &\geq (1-(1-\alpha/2)^k)(1-\varepsilon/2) \geq 1-\varepsilon \end{aligned}$$

for $n \geq n_1 + \ldots + n_k$. By the Ulam theorem (see [12, 15]), we can find a compact set $K \subset X$ such that $P^n\mu(K \cup C) \geq 1 - \varepsilon$ for $n \in \mathbb{N}$. Since $K \cup C \in \mathcal{C}_\varepsilon$, Theorem 4.2 shows that P is tight. ∎

Lemma 4.3. *Let P be a nonexpansive and semi-concentrating Markov operator. Then for every $\varepsilon > 0$ there exists $C \in \mathcal{C}_\varepsilon$ satisfying*

$$\liminf_{n\to\infty} P^n\mu(C) \geq 1-\varepsilon \quad \textit{for } \mu \in \mathcal{M}_1.$$

Proof. Fix $\varepsilon > 0$. By Lemma 2.4.4 there exist an integer k, a sequence $(A_1, \ldots, A_k)$, $A_i \in \mathcal{B}(X)$ and $\operatorname{diam} A_i \leq \varepsilon^2/16$ for $i = 1, \ldots, k$, and a measure $\mu_0 \in \mathcal{M}_1$ such that $\bigcup_{i=1}^k A_i \in \mathcal{T}^k_{\varepsilon^2/16}$ and

$$\liminf_{n\to\infty} P^n\mu_0(A_i) > 0 \quad \text{for } i = 1, \ldots, k.$$

Lemma 2.4.5 now shows that there exists a sequence $(C_1, \ldots, C_k)$, $C_i \in \mathcal{C}_\varepsilon$ for $i = 1, \ldots, k$, satisfying $P^n\nu(C_i) \geq 1 - \varepsilon/2$ for $n \in \mathbb{N}$, $\nu \in \mathcal{M}_1^{A_i}$ and $i = 1, \ldots, k$.

Set $C = \bigcup_{i=1}^k C_i$ and observe that $C \in \mathcal{C}_\varepsilon$. Moreover, we have

$$P^n\nu(C) \geq 1-\varepsilon/2 \quad \text{for } n \in \mathbb{N} \text{ and } \nu \in \textstyle\bigcup_{i=1}^k \mathcal{M}_1^{A_i}. \tag{4.8}$$

Since $\bigcup_{i=1}^k A_i \in \mathcal{T}^k_{\varepsilon^2/16}$, it follows that there exists $\widetilde{\alpha} > 0$ such that

$$\liminf_{n\to\infty} P^n\mu\Big(\bigcup_{i=1}^k A_i\Big) > \widetilde{\alpha}$$

for every $\mu \in \mathcal{M}_1$. Set $\alpha = \widetilde{\alpha}/k$ and define

$$\eta = \sup\{\gamma \geq 0 : \liminf_{n\to\infty} P^n\mu(C) \geq \gamma \text{ for all } \mu \in \mathcal{M}_1\}.$$

It is obvious that $\eta > 0$. It remains to prove that $\eta \geq 1 - \varepsilon/2$. Suppose, contrary to our claim, that $\eta < 1 - \varepsilon/2$. Hence

$$\eta > \frac{\eta}{1-\alpha} - \frac{\alpha}{1-\alpha}(1-\varepsilon/2). \tag{4.9}$$

Choose $\gamma > 0$ such that

$$\eta > \gamma > \frac{\eta}{1-\alpha} - \frac{\alpha}{1-\alpha}(1-\varepsilon/2).$$

Therefore

$$\liminf_{n\to\infty} P^n\mu(C) \geq \gamma \quad \text{for } \mu \in \mathcal{M}_1.$$

Fix $\mu \in \mathcal{M}_1$. Analysis similar to that in the proof of Theorem 4.3 shows that there exist $n_0 \in \mathbb{N}$, $\widetilde{\mu} \in \mathcal{M}_1$ and $\nu \in \bigcup_{i=1}^k \mathcal{M}_1^{A_i}$ such that

$$P^{n_0}\mu = (1-\alpha)\widetilde{\mu} + \alpha\nu.$$

By (4.8), (4.9) and the linearity of P we obtain

$$\begin{aligned}\liminf_{n\to\infty} P^{n_0+n}\mu(C) &\geq (1-\alpha)\liminf_{n\to\infty} P^n\widetilde{\mu}(C) + \alpha\liminf_{n\to\infty} P^n\nu(C)\\ &\geq (1-\alpha)\gamma + \alpha(1-\varepsilon/2) > \eta.\end{aligned}$$

Since $\mu \in \mathcal{M}_1$ is arbitrary, we see that

$$\liminf_{n\to\infty} P^n\mu(C) \geq (1-\alpha)\gamma + \alpha(1-\varepsilon) > \eta \quad \text{for } \mu \in \mathcal{M}_1,$$

which contradicts the definition of η and finishes the proof. ■

Consequently, Theorem 4.2 and Lemma 4.3 yield the following theorem.

THEOREM 4.4. *Let P be a nonexpansive and semi-concentrating Markov operator. Then P is tight.*

We finish this chapter with an easy observation.

REMARK 4.1. Theorems 4.1, 4.3 and 4.4 still hold if it is only assumed that P is essentially nonexpansive.

Proof. It is enough to observe that if ϱ' is equivalent to ϱ, then every locally concentrating, concentrating and semi-concentrating Markov operator P in (X,ϱ) is locally concentrating, concentrating and semi-concentrating in (X,ϱ'), respectively. ■

5. Invariant measures for Markov operators

The crucial fact is that tightness may be used in proving the existence of an invariant measure for Markov operators. Namely, we have the following theorem.

THEOREM 5.1. *Let P be a Markov operator. Assume that P is continuous in the weak topology. If P is tight, then P admits an invariant distribution.*

Proof. Fix $\mu \in \mathcal{M}_1$ and set

$$\overline{\mu}_n = \frac{\mu + P\mu + \ldots + P^{n-1}\mu}{n} \quad \text{for } n \in \mathbb{N}. \tag{5.1}$$

Since P is tight, the family $\{\overline{\mu}_n\}_{n\geq 1}$ is tight. From the Prokhorov theorem (see [6]) it follows that there exists a subsequence $(m_n)_{n\geq 1}$ of integers and a distribution $\overline{\mu}$ such that $\overline{\mu}_{m_n} \to \overline{\mu}$ in the weak topology. Since P is continuous, $P\overline{\mu}_{m_n} \to P\overline{\mu}$ in the weak topology. From (5.1) it follows that $\|P\overline{\mu}_{m_n} - \overline{\mu}_{m_n}\|_{\mathrm{FM}} \to 0$ as $n \to \infty$ and consequently $P\overline{\mu} = \overline{\mu}$. ■

THEOREM 5.2. *Let P be a nonexpansive and locally concentrating Markov operator. Assume that for every $\mu \in \mathcal{M}_1$ and every $\varepsilon > 0$ there is $A \in \mathcal{B}_b(X)$ such that* (4.3) *holds. Then P admits a unique invariant distribution.*

Proof. From Theorems 4.1, 5.1 and a simple observation that every nonexpansive Markov operator is continuous in the weak topology it follows that P admits an invariant distribution.

To prove uniqueness suppose, contrary to our claim, that $\overline{\mu}_1, \overline{\mu}_2 \in \mathcal{M}_1$ are two different invariant measures. Set

$$\varepsilon = \|\overline{\mu}_1 - \overline{\mu}_2\|_{\mathrm{FM}} > 0. \tag{5.2}$$

As in the proof of Theorem 4.1 let $\alpha > 0$ be such that, for $\varepsilon/2$, the locally concentrating property holds. Choose $k \in \mathbb{N}$ such that $4(1-\alpha/2)^k < \varepsilon$. Since $P^n\overline{\mu}_i = \overline{\mu}_i$, $i = 1, 2, n \in \mathbb{N}$, we conclude that $\overline{\mu}_1, \overline{\mu}_2 \in \mathcal{M}_1^{A,(1/3)(1-\alpha/2)^k}$ for some $A \in \mathcal{B}_b(X)$. From (4.4) it follows that

$$\|\overline{\mu}_1 - \overline{\mu}_2\|_{\mathrm{FM}} = \lim_{n\to\infty} \|P^n\overline{\mu}_1 - P^n\overline{\mu}_2\|_{\mathrm{FM}} \le \varphi\big(A, \tfrac{1}{3}(1-\alpha/2)^k\big) < \varepsilon,$$

contrary to (5.2). ∎

THEOREM 5.3. *Let P be a nonexpansive and concentrating Markov operator. Then P admits a unique invariant distribution. Moreover, P is asymptotically stable.*

Proof. Theorems 4.3 and 5.1 show that P admits an invariant distribution, say μ_*. To finish the proof of asymptotic stability it remains to verify (2.3.2). When an invariant distribution exists this condition is equivalent to a more symmetric relation

$$\lim_{n\to\infty} \|P^n\mu_1 - P^n\mu_2\|_{\mathrm{FM}} = 0 \quad \text{for } \mu_1, \mu_2 \in \mathcal{M}_1. \tag{5.3}$$

Fix $\mu_1, \mu_2 \in \mathcal{M}_1$ and $\varepsilon > 0$. According to the concentrating property of P we may choose $A \in \mathcal{B}_b(X)$ with $\operatorname{diam} A \le \varepsilon$ and $\alpha > 0$ such that (2.3.6) holds. As in the proof of Theorem 4.3 we define by induction a sequence $(n_k)_{k\ge1}$ of integers and four sequences of distributions $(\mu_i^k)_{k\ge0}$, $(\nu_i^k)_{k\ge0}$, $i = 1, 2$. If $k = 0$ we set $n_0 = 0$ and $\nu_i^0 = \mu_i^0 = \mu_i$. If $k \ge 1$ and $n_{k-1}, \mu_i^{k-1}, \nu_i^{k-1}$ are given we choose, according to (2.3.6), n_k such that $P^{n_k}\mu_i^{k-1}(A) \ge \alpha$ for $i = 1, 2$ and we define

$$\nu_i^k(B) = \frac{P^{n_k}\mu_i^{k-1}(B\cap A)}{P^{n_k}\mu_i^{k-1}(A)},$$

$$\mu_i^k(B) = \frac{1}{1-\alpha}\,(P^{n_k}\mu_i^{k-1}(B) - \alpha\nu_i^k(B)) \quad \text{for } B \in \mathcal{B}(X). \tag{5.4}$$

Since $\nu_i^k \in \mathcal{M}_1^A$, we obtain

$$\|\nu_1^k - \nu_2^k\|_{\mathrm{FM}} \le \operatorname{diam} A \le \varepsilon. \tag{5.5}$$

Using (5.4) it is easy to verify by induction that

$$\begin{aligned} P^{n_1+\ldots+n_k}\mu_i = \alpha P^{n_2+\ldots+n_k}\nu_i^1 &+ \alpha(1-\alpha)P^{n_3+\ldots+n_k}\nu_i^2 \\ &+ \ldots + \alpha(1-\alpha)^{k-1}\nu_i^k + (1-\alpha)^k\mu_i^k \quad \text{for } k \in \mathbb{N}. \end{aligned}$$

Since P is nonexpansive this implies

$$\begin{aligned} \|P^{n_1+\ldots+n_k}\mu_1 - P^{n_1+\ldots+n_k}\mu_2\|_{\mathrm{FM}} \le \alpha\|\nu_1^1 - \nu_2^1\|_{\mathrm{FM}} &+ \alpha(1-\alpha)\|\nu_1^2 - \nu_2^2\|_{\mathrm{FM}} \\ &+ \ldots + \alpha(1-\alpha)^{k-1}\|\nu_1^k - \nu_2^k\|_{\mathrm{FM}} \\ &+ (1-\alpha)^k\|\mu_1^k - \mu_2^k\|_{\mathrm{FM}}. \end{aligned}$$

From this, (5.5) and the obvious inequality $\|\mu_1^k - \mu_2^k\|_{\mathrm{FM}} \le 2$ it follows that

$$\|P^{n_1+\ldots+n_k}\mu_1 - P^{n_1+\ldots+n_k}\mu_2\|_{\mathrm{FM}} \le \varepsilon + 2(1-\alpha)^k.$$

Since $\varepsilon > 0$, $k \in \mathbb{N}$, $\mu_1, \mu_2 \in \mathcal{M}_1$ are arbitrary and P is nonexpansive, this implies (5.3) and finishes the proof. ∎

Above we have proved in fact the following result.

THEOREM 5.4. *Let P be a nonexpansive Markov operator. Assume that for every $\varepsilon > 0$ there is a number $\alpha > 0$ having the following property: for every $\mu_1, \mu_2 \in \mathcal{M}_1$ there exists $A \in \mathcal{B}_b(X)$ with $\operatorname{diam} A \le \varepsilon$ and $n_0 \in \mathbb{N}$ such that*

$$P^{n_0}\mu_i(A) > \alpha \quad \textit{for } i = 1, 2.$$

Then P satisfies (5.3).

THEOREM 5.5. *Let $P : \mathcal{M}_{\rm fin} \to \mathcal{M}_{\rm fin}$ be a nonexpansive and semi-concentrating Markov operator. Then*

(i) *P admits an invariant distribution*,

(ii) *$\Omega(\mu) \ne \emptyset$, $\mu \in \mathcal{M}_1$, where $\Omega(\mu)$ is given by* (2.3.8),

(iii) *$\widehat{\Omega} = \bigcup_{\mu\in\mathcal{M}_1} \Omega(\mu)$ is tight.*

Proof. (i) Theorems 4.4 and 5.1 show that P admits an invariant distribution.

(ii) Fix $\mu \in \mathcal{M}_1$. From Lemma 4.3 it follows that $\{P^n\mu\}_{n\ge1}$ is tight. By the Prokhorov theorem (see [6]) we see that $\Omega(\mu) \ne \emptyset$.

(iii) To prove the tightness of $\widehat{\Omega}$ fix $\varepsilon > 0$. Again, by Lemma 4.3 there exists a sequence $(C_k)_{k\ge1}$ of subsets of X such that

$$C_k \in \mathcal{C}_{\varepsilon/2^k} \quad \text{and} \quad \liminf_{n\to\infty} P^n\mu(C_k) \ge 1 - \varepsilon/2^k \quad \text{for } k \in \mathbb{N}.$$

Define $K = \bigcap_{k=1}^{\infty} \mathcal{N}(C_k, \varepsilon/2^k)$ and observe that K is compact. We are going to show that $\widetilde{\mu}(K) \ge 1 - \varepsilon$ for $\widetilde{\mu} \in \widehat{\Omega}$. Fix $\widetilde{\mu} \in \widehat{\Omega}$ and choose $\mu \in \mathcal{M}_1$ such that $\widetilde{\mu} \in \Omega(\mu)$. Let $(n_m)_{m\ge1}$ be a sequence of integers such that $\|P^{n_m}\mu - \widetilde{\mu}\|_{\rm FM} \to 0$ as $m \to \infty$. Then by the Aleksandrov theorem (see [6]) we have

$$\begin{aligned}\widetilde{\mu}(\mathcal{N}(C_k, \varepsilon/2^k)) &\ge \widetilde{\mu}(\mathcal{N}^0(C_k, \varepsilon/2^k)) \ge \limsup_{m\to\infty} P^{n_m}\mu(\mathcal{N}^0(C_k, \varepsilon/2^k)) \\ &\ge \limsup_{m\to\infty} P^{n_m}\mu(C_k) \ge 1 - \varepsilon/2^k \quad \text{for } k \in \mathbb{N}.\end{aligned}$$

Hence

$$\widetilde{\mu}(X \setminus K) \le \sum_{k=1}^{\infty} \widetilde{\mu}(X \setminus \mathcal{N}(C_k, \varepsilon/2^k)) \le \sum_{k=1}^{\infty} \varepsilon/2^k = \varepsilon,$$

which finishes the proof of (iii). ∎

The same conclusion as in Remark 4.1 can be drawn for theorems proved in this chapter.

REMARK 5.1. Theorems 5.2, 5.3, 5.4 and 5.5 still hold if it is only assumed that P is essentially nonexpansive.

PART II

In the second part of our paper we will consider some special Markov operators. We will apply to them our main criteria for the existence of invariant measures. Further, it is of interest to describe the properties of these invariant measures. We will only touch this problem by estimating their capacity.

6. Iterated function systems

6.1. Introduction. We are given a sequence of continuous transformations $S_i : X \to X$ for $i = 1, \ldots, N$ and a probabilistic vector $(p_1(x), \ldots, p_N(x))$, $x \in X$, i.e.,

$$p_i(x) \geq 0 \quad \text{and} \quad \sum_{i=1}^{N} p_i(x) = 1 \quad \text{for } x \in X.$$

We assume that p_i, $i = 1, \ldots, N$, are continuous functions. The pair of sequences $(S, p)_N = (S_1, \ldots, S_N; p_1, \ldots, p_N)$ is called an *iterated function system*.

Now we present an imprecise description of the process considered in this section. Choose $x_0 \in X$. If an initial point x_0 is chosen, we randomly select from the set $\{1, \ldots, N\}$ an integer in such a way that the probability of choosing k is $p_k(x_0)$, $k = 1, \ldots, N$. When a number k_0 is drawn we define $x_1 = S_{k_0}(x_0)$. Having x_1 we select k_1 according to the distribution $p_1(x_1), \ldots, p_N(x_1)$ and we define $x_2 = S_{k_1}(x_1)$ and so on. Denoting by μ_n, $n = 0, 1, \ldots$, the distribution of x_n, i.e., $\mu_n(A) = \text{prob}(x_n \in A)$ for $A \in \mathcal{B}(X)$, we define P as the transition operator such that $\mu_n = P\mu_{n-1}$ for $n \in \mathbb{N}$.

The above procedure can be easily formalized. Fix $x \in X$ and set $\mu_0 = \delta_x$. According to the definition of the dual operator U (see (2.2.1)) we have

$$Uf(x) = \langle Uf, \delta_x \rangle = \langle f, P\delta_x \rangle = \langle f, \mu_1 \rangle \quad \text{for } f \in B(X).$$

This means that $Uf(x)$ is the mathematical expectation of $f(x_1)$ if $x_0 = x$ is fixed. On the other hand, according to our description, the expectation of $f(x_1)$ is equal to

$$\sum_{i=1}^{N} p_i(x) f(S_i(x)).$$

Since x is arbitrary, we have

$$Uf(x) = \sum_{i=1}^{N} p_i(x) f(S_i(x)) \quad \text{for } x \in X. \tag{6.1.1}$$

We take this formula as the precise formal definition of our process. We check at once that U satisfies conditions (2.2.3)–(2.2.5). Hence we may define a Markov operator $P : \mathcal{M}_{\mathrm{fin}} \to \mathcal{M}_{\mathrm{fin}}$ by setting

$$P\mu(A) = \langle U\mathbf{1}_A, \mu\rangle \quad \text{for } A \in \mathcal{B}(X).$$

Therefore

$$P\mu(A) = \sum_{i=1}^{N} \int\limits_{S_i^{-1}(A)} p_i(x)\,\mu(dx) \quad \text{for } A \in \mathcal{B}(X). \tag{6.1.2}$$

Since $Uf \in C(X)$ for $f \in C(X)$, the operator P is a Feller operator.

In what follows we will study the asymptotic behaviour of P. To simplify the language we will say that the iterated function system $(S,p)_N$ is nonexpansive, essentially nonexpansive, tight, has an invariant distribution or is asymptotically stable if the Markov operator P given by (6.1.2) has the corresponding property.

6.2. Nonexpansiveness. We start with a simple lemma ensuring the nonexpansiveness of $(S,p)_N$ under rather strong assumptions on the functions S_i, p_i for $i = 1,\dots,N$.

LEMMA 6.2.1. *Let $p_i : X \to \mathbb{R}_+$, $i = 1,\dots,N$, be constants. If $S_i : X \to X$, $i = 1,\dots,N$, satisfy*

$$\sum_{i=1}^{N} p_i \varrho(S_i(x), S_i(y)) \le \varrho(x,y) \quad \textit{for } x,y \in X, \tag{6.2.1}$$

then the iterated function system $(S,p)_N$ is nonexpansive.

Proof. Let U be the corresponding dual operator given by (6.1.1). Fix $f \in \mathcal{F}$. We have

$$|Uf(x)| = \Big|\sum_{i=1}^{N} p_i f(S_i(x))\Big| \le \sum_{i=1}^{N} p_i = 1 \quad \text{for } x \in X.$$

Further, from (6.2.1) it follows that

$$\begin{aligned}|Uf(x) - Uf(y)| &= \Big|\sum_{i=1}^{N} p_i f(S_i(x)) - \sum_{i=1}^{N} p_i f(S_i(y))\Big| \\ &\le \sum_{i=1}^{N} p_i |f(S_i(x)) - f(S_i(y))| \le \sum_{i=1}^{N} p_i \varrho(S_i(x), S_i(y)) \\ &\le \varrho(x,y) \quad \text{for } x,y \in X.\end{aligned}$$

Therefore $U(\mathcal{F}) \subset \mathcal{F}$ and an application of Theorem 3.1 finishes the proof. ∎

The nonexpansiveness of iterated function systems is especially difficult to obtain. But we already know that in proving the existence of an invariant measure and its stability the nonexpansiveness can be replaced by essential nonexpansiveness (see Remarks 4.1 and 5.1). In this part of our paper we will discuss this property.

We introduce the class Φ of functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

1° φ is continuous and $\varphi(0) = 0$;
2° φ is nondecreasing and concave, i.e., $\frac{1}{2}\varphi(t_1) + \frac{1}{2}\varphi(t_2) \le \varphi\big(\frac{t_1+t_2}{2}\big)$ for $t_1, t_2 \in \mathbb{R}_+$;
3° $\varphi(t) > 0$ for $t > 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

We denote by Φ_0 the family of functions satisfying 1°, 2°. It is easy to see that for every $\varphi \in \Phi$ the function

$$\varrho_\varphi(x,y) = \varphi(\varrho(x,y)) \quad \text{for } x,y \in X$$

is again a metric on X. Moreover ϱ_φ is equivalent to ϱ.

In our considerations an important role is played by the inequality

$$\omega(t) + \varphi(r(t)) \le \varphi(t), \tag{6.2.2}$$

where $r,\omega \in \Phi_0$ are given functions. This inequality may be studied by classical methods of the theory of functional equations (see [32]). Here we will discuss only three special cases for which inequality (6.2.2) has a solution belonging to Φ.

CASE I: *Dini condition.* Assume that ω satisfies the Dini condition, i.e.,

$$\int_0^\varepsilon \frac{\omega(t)}{t}\,dt < \infty \quad \text{for some } \varepsilon > 0. \tag{6.2.3}$$

This condition is equivalent to

$$\sum_{n=1}^{\infty} \omega(c^n t) < \infty$$

for every $0 \le c < 1$ and $t \ge 0$. If $r(t) = ct$, $0 \le c < 1$, then the function

$$\varphi(t) = t + \sum_{n=0}^{\infty} \omega(c^n t)$$

is a solution of inequality (6.2.2) and belongs to Φ.

CASE II: *Hölder condition.* Assume that

$$\omega(t) \le a t^{\upsilon}, \tag{6.2.4}$$

where $a > 0$ and $\upsilon > 0$ are constants. Clearly (6.2.4) implies Dini's condition (6.2.3). But this stronger assumption allows us to replace $r(t) = ct$ where $c < 1$ by some functions tangent to the diagonal at $t = 0$. Assume namely that $r \in \Phi_0$, $r(t) < t$ and

$$0 \le r(t) \le t - t^{\epsilon+1} b \quad \text{for } 0 \le t \le \varepsilon, \tag{6.2.5}$$

where $\epsilon > 0$, $b > 0$ and $\varepsilon > 0$ are constants. From the result of W. J. Thron [64] these assumptions imply that for every $c > 0$ there is $n_0 = n_0(c)$ such that the iterates r^n of the function r satisfy

$$r^n(t) \le (\epsilon b n/2)^{-1/\epsilon} \quad \text{for } 0 \le t \le c,\ n \ge n_0.$$

From this and (6.2.4) it follows immediately that

$$\omega(r^n(t)) \le \frac{k}{n^{\upsilon/\epsilon}} \quad \text{for } 0 \le t \le c,\ n \ge n_0,$$

where k is a constant. Thus for $\upsilon > \epsilon$ the series

$$\varphi(t) = t + \sum_{n=0}^{\infty} \omega(r^n(t)) \tag{6.2.6}$$

is convergent and defines a solution of inequality (6.2.2). Since r and ω belong to Φ_0 this solution is a function from Φ.

Case III: *Lipschitz condition.* Assume that

$$\omega(t) \le at, \tag{6.2.7}$$

where $a > 0$ is a constant. This is, in fact, a special case of (6.2.4) with $\upsilon = 1$. However, in this case the existence result for solutions of inequality (6.2.2) may be sharpened a little. Namely, Schwartzman [54] has shown that any continuous function $r : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$0 < r(t) < t \quad \text{for } t > 0, \qquad \int_0^\varepsilon \frac{t\,dt}{t - r(t)} < \infty \tag{6.2.8}$$

also has the property that

$$\sum_{n=0}^{\infty} r^n(t) < \infty.$$

Thus, for $r, \omega \in \Phi_0$ satisfying (6.2.7), (6.2.8) the series (6.2.6) is convergent and it gives a solution $\varphi \in \Phi$ of (6.2.2). Clearly condition (6.2.8) is less restrictive than (6.2.5) with $0 < \epsilon < \upsilon = 1$.

We are now in a position to formulate the following lemma.

Lemma 6.2.2. *Let an iterated function system $(S,p)_N$ satisfy*

$$\sum_{i=1}^{N} |p_i(x) - p_i(y)| \le \omega(\varrho(x,y)) \quad \textit{for } x, y \in X, \tag{6.2.9}$$

$$\sum_{i=1}^{N} p_i(x)\varrho(S_i(x), S_i(y)) \le r(\varrho(x,y)) \quad \textit{for } x, y \in X. \tag{6.2.10}$$

If the pair (ω, r) satisfies the conditions formulated in one of cases I–III, *then the system $(S,p)_N$ is essentially nonexpansive.*

Proof. Since the conditions required in cases I–III are satisfied, there exists a solution $\varphi \in \Phi$ of (6.2.2). According to Theorem 3.1 it is enough to show that $U(\mathcal{F}_\varphi) \subset \mathcal{F}_\varphi$, where $\mathcal{F}_\varphi$ denotes the family of all continuous functions f such that $|f(x)| \le 1$ and $|f(x) - f(y)| \le \varphi(\varrho(x,y))$ for all $x, y \in X$. Fix $f \in \mathcal{F}_\varphi$. We have

$$|Uf(x)| = \Big|\sum_{i=1}^{N} p_i(x) f(S_i(x))\Big| \le \sum_{i=1}^{N} p_i(x) = 1 \quad \text{for } x \in X.$$

Further, from (6.2.9) it follows that

$$\begin{aligned} |Uf(x) - Uf(y)| &= \Big|\sum_{i=1}^{N} p_i(x) f(S_i(x)) - \sum_{i=1}^{N} p_i(y) f(S_i(y))\Big| \\ &\le \sum_{i=1}^{N} |p_i(x) - p_i(y)| + \sum_{i=1}^{N} p_i(x) |f(S_i(x)) - f(S_i(y))| \\ &\le \omega(\varrho(x,y)) + \sum_{i=1}^{N} p_i(x) \varphi(\varrho(S_i(x), S_i(y))) \quad \text{for } x, y \in X. \end{aligned}$$

Since φ is concave and nondecreasing, (6.2.2) and (6.2.10) imply

$$|Uf(x) - Uf(y)| \le \omega(\varrho(x,y)) + \varphi\Big(\sum_{i=1}^{N} p_i(x)\varrho(S_i(x), S_i(y))\Big)$$
$$\le \omega(\varrho(x,y)) + \varphi(r(\varrho(x,y)) \le \varphi(\varrho(x,y)) \quad \text{for } x, y \in X. \ ■$$

6.3. Invariant measures. The following theorem was proved by A. Lasota (see [33]). In his proof he used the double contraction principle. We show that this is a simple application of our criterion for the existence of an invariant measure.

THEOREM 6.3.1. *Let* $p_i : X \to \mathbb{R}_+$, $i = 1, \ldots, N$, *be constants. If* $S_i : X \to X$ *are Lipschitzean with Lipschitz constants* L_i, $i = 1, \ldots, N$, *and*

$$\sum_{i=1}^{N} p_i L_i < 1, \tag{6.3.1}$$

then the corresponding Markov operator P *given by* (6.1.2) *admits a unique invariant distribution. Moreover*, P *is asymptotically stable.*

Proof. Fix $x_0 \in X$ and define $V(x) = \varrho(x, x_0)$ for $x \in X$. Obviously V is a Lyapunov function, bounded on bounded sets. Then by (6.1.1) we have

$$UV(x) = \sum_{i=1}^{N} p_i \varrho(S_i(x), x_0) \le \sum_{i=1}^{N} p_i(\varrho(S_i(x), S_i(x_0)) + \varrho(S_i(x_0), x_0))$$
$$\le \varrho(x, x_0) \sum_{i=1}^{N} p_i L_i + \max_{1\le i\le N} \varrho(S_i(x_0), x_0) \quad \text{for } x \in X.$$

From Corollary 2.4.1 and condition (6.3.1) it follows that there exists $B \in \mathcal{B}_b(X)$ such that

$$\liminf_{n\to\infty} P^n\mu(B) \ge 1/2 \quad \text{for } \mu \in \mathcal{M}_1. \tag{6.3.2}$$

We are going to show that P is concentrating. Fix $\varepsilon > 0$. Since (6.3.1) holds, there exists $i \in \{1, \ldots, N\}$ such that $L_i < 1$. Let $m \in \mathbb{N}$ be such that $L_i^m \operatorname{diam} B \le \varepsilon$. Define $A = \operatorname{cl} S_i^m(B)$ and observe that $\operatorname{diam} A \le \varepsilon$. By induction we check at once that

$$P^m\mu(A) = \sum_{i_1,\ldots,i_m=1}^{N} p_{i_1} \cdot \ldots \cdot p_{i_m} \mu(S_{i_1}^{-1} \circ \ldots \circ S_{i_m}^{-1}(A)) \quad \text{for } \mu \in \mathcal{M}_1.$$

By the linearity of P we have

$$P^{n+m}\mu(A) \ge p_i^m \mu(S_i^{-m}(A)) \quad \text{for } \mu \in \mathcal{M}_1$$

and by (6.3.2) we obtain

$$\liminf_{n\to\infty} P^n\mu(A) \ge p_i^m/2 \quad \text{for } \mu \in \mathcal{M}_1.$$

Further, from Lemma 6.2.1 it follows that P is nonexpansive. An application of Theorem 5.3 finishes the proof. ■

LEMMA 6.3.1. *If the assumptions of Lemma* 6.2.2 *are satisfied*, *then the operator* P *given by* (6.1.2) *is semi-concentrating.*

Proof. Let P be the Markov operator corresponding to $(S,p)_N$ and given by (6.1.2) and let U be its dual. Fix $x_0 \in X$ and define $V(x) = \varrho(x, x_0)$ for $x \in X$. Then by (6.2.10) we have

$$UV(x) = \sum_{i=1}^{N} p_i(x)\varrho(S_i(x), x_0) \le \sum_{i=1}^{N} p_i(x)(\varrho(S_i(x), S_i(x_0)) + \varrho(S_i(x_0), x_0)) \le r(\varrho(x, x_0)) + \max_{1\le i\le N} \varrho(S_i(x_0), x_0) \quad \text{for } x \in X.$$

Since r is concave and $r(0) = 0$, we obtain

$$UV(x) \le r(1)V(x) + r(1) + \max_{1\le i\le N} \varrho(S_i(x_0), x_0) \quad \text{for } x \in X.$$

Obviously $r(1) < 1$. Corollary 2.4.1 now shows that there exists $B \in \mathcal{B}_b(X)$ such that

$$\liminf_{n\to\infty} P^n\mu(B) > 1/2 \quad \text{for } \mu \in \mathcal{M}_1. \tag{6.3.3}$$

Fix $\varepsilon > 0$. Choose an integer m such that $r^m(\operatorname{diam} B) < \varepsilon$. Further, let $\eta > 0$ be such that

$$(1+\eta)^m r^m(\operatorname{diam} B) \le \varepsilon. \tag{6.3.4}$$

Fix $x \in B$ and define $C \in \mathcal{C}_\varepsilon$ by

$$C = \bigcup_{i_1,\ldots,i_m=1}^{N} B(S_{i_m} \circ \ldots \circ S_{i_1}(x), \varepsilon). \tag{6.3.5}$$

By induction we may show that for every $y \in B$ and $n \in \mathbb{N}$ there exists $I_n(y) \subset \{1,\ldots,N\}^n$ such that

$$\varrho(S_{k_n} \circ \ldots \circ S_{k_1}(x), S_{k_n} \circ \ldots \circ S_{k_1}(y)) \le (1+\eta)^n r^n(\varrho(x,y)) \tag{6.3.6}$$

for $(k_1,\ldots,k_n) \in I_n(y)$ and

$$\sum_{(k_1,\ldots,k_n)\in I_n(y)} p_{k_1}(y) \cdot \ldots \cdot p_{k_n}(S_{k_{n-1}} \circ \ldots \circ S_{k_1}(y)) \ge \left(\frac{\eta}{1+\eta}\right)^n. \tag{6.3.7}$$

Fix $y \in B$. We may assume that $y \ne x$. If $n = 1$ from (6.2.10) it follows that

$$\sum p_i(y) \le \frac{1}{1+\eta}, \tag{6.3.8}$$

where the sum is taken over all $i \in \{1,\ldots,N\}$ such that $\varrho(S_i(x), S_i(y)) > (1+\eta)r(\varrho(x,y))$. Hence (6.3.6) and (6.3.7) follow. Assuming (6.3.6) and (6.3.7) to hold for n, we will prove them for $n+1$. Let $I_n(y) \subset \{1,\ldots,N\}^n$ be such that (6.3.6) and (6.3.7) hold. Fix $(k_1,\ldots,k_n) \in I_n(y)$. Then

$$\sum_{i=1}^{N} p_i(S_{k_n} \circ \ldots \circ S_{k_1}(y))\varrho(S_i \circ S_{k_n} \circ \ldots \circ S_{k_1}(x), S_i \circ S_{k_n} \circ \ldots \circ S_{k_1}(y)) \le r(\varrho(S_{k_n} \circ \ldots \circ S_{k_1}(x), S_{k_n} \circ \ldots \circ S_{k_1}(y))) \le r((1+\eta)^n r^n(\varrho(x,y))) \le (1+\eta)^n r^{n+1}(\varrho(x,y)),$$

which is due to the fact that r is concave. Hence

$$\sum p_i(S_{k_n} \circ \ldots \circ S_{k_1}(y))(1+\eta)^{n+1} r^{n+1}(\varrho(x,y)) \le (1+\eta)^n r^{n+1}(\varrho(x,y)),$$

where the sum is taken over all $i \in \{1,\dots,N\}$ such that

$$\varrho(S_i \circ S_{k_n} \circ \dots \circ S_{k_1}(x), S_i \circ S_{k_n} \circ \dots \circ S_{k_1}(y)) > (1+\eta)^{n+1} r^{n+1}(\varrho(x,y))$$

and consequently

$$\sum p_i(S_{k_n} \circ \dots \circ S_{k_1}(y)) \ge \frac{\eta}{1+\eta},$$

where the sum is taken over all $i \in \{1,\dots,N\}$ such that

$$\varrho(S_i \circ S_{k_n} \circ \dots \circ S_{k_1}(x), S_i \circ S_{k_n} \circ \dots \circ S_{k_1}(y)) \le (1+\eta)^{n+1} r^{n+1}(\varrho(x,y)).$$

From this and the assumption

$$\sum_{(k_1,\dots,k_n)\in I_n(y)} p_{k_1}(y) \cdot \ldots \cdot p_{k_n}(S_{k_n-1} \circ \dots \circ S_{k_1}(y)) \ge \left(\frac{\eta}{1+\eta}\right)^n$$

our assertion follows.

Set $\alpha = (\eta/(1+\eta))^m/2$. By induction, the definition of $I_m(y)$ and the definition of C we obtain

$$\begin{aligned} P^{n+m}\mu(C) &= \langle \mathbf{1}_C, P^{n+m}\mu\rangle = \langle U^m \mathbf{1}_C, P^n\mu\rangle \\ &= \sum_{i_1,\dots,i_m=1}^{N} \int_X p_{i_1}(y) \cdot \ldots \cdot p_{i_m}(S_{i_{m-1}} \circ \dots \circ S_{i_1}(y)) \mathbf{1}_C(S_{i_m} \circ \dots \circ S_{i_1}(y))\, P^n\mu(dy) \\ &\ge \int_B \sum_{(k_1,\dots,k_m)\in I_m(y)} p_{k_1}(y) \cdot \ldots \cdot p_{k_m}(S_{k_{m-1}} \circ \dots \circ S_{k_1}(y))\, P^n\mu(dy) \quad \text{for } \mu \in \mathcal{M}_1. \end{aligned}$$

From (6.3.3) we conclude that $\liminf_{n\to\infty} P^n\mu(C) > \alpha$ for $\mu \in \mathcal{M}_1$. Since $\varepsilon > 0$ and $\mu \in \mathcal{M}_1$ are arbitrary, P is semi-concentrating. ■

THEOREM 6.3.2. *Let an iterated function system $(S,p)_N$ satisfy conditions* (6.2.9) *and* (6.2.10). *If the pair (ω, r) satisfies the conditions formulated in one of cases* I–III *and*

$$\sum p_i(x)p_i(y) > 0 \quad \textit{for } x,y \in X, \tag{6.3.9}$$

where the summation is taken over all $i \in \{1,\dots,N\}$ such that $\varrho(S_i(x), S_i(y)) \le r(\varrho(x,y))$, then the system $(S,p)_N$ is asymptotically stable.

Proof. Let P be the Markov operator corresponding to $(S,p)_N$ and given by (6.1.2). Since the assumptions of Lemmas 6.2.2 and 6.3.1 are satisfied, P is essentially nonexpansive and semi-concentrating. From Theorem 5.5 and Remark 5.1 it follows that P admits an invariant distribution. Therefore it remains to verify (2.3.2). When an invariant distribution exists this condition is equivalent to a more symmetric relation:

$$\lim_{n\to\infty} \|P^n\mu_1 - P^n\mu_2\|_{\mathrm{FM}} = 0 \quad \text{for } \mu_1, \mu_2 \in \mathcal{M}_1.$$

Theorem 5.4 and Remark 5.1 imply that to finish the proof it is enough to show that for every $\varepsilon > 0$ there is $\alpha > 0$ with the following property: for every $\mu_1, \mu_2 \in \mathcal{M}_1$ there exist $A \in \mathcal{B}_b(X)$ with $\operatorname{diam} A \le \varepsilon$ and $n_0 \in \mathbb{N}$ such that

$$P^{n_0}\mu_i(A) > \alpha \quad \text{for } i = 1,2. \tag{6.3.10}$$

Fix $\varepsilon > 0$. According to Theorem 5.5(iii) and Remark 5.1 there is a compact set $K \subset X$ such that

$$\widetilde{\mu}(K) \geq 4/5 \quad \text{for } \widetilde{\mu} \in \widehat{\Omega} = \bigcup_{\mu \in \mathcal{M}_1} \Omega(\mu). \tag{6.3.11}$$

Choose an integer m such that

$$r^m(\operatorname{diam} K) \leq \varepsilon/3 \tag{6.3.12}$$

and define for every $x \in K$ and $(j_1, \ldots, j_m) \in \{1, \ldots, N\}$ the value

$$\Pi_{(j_1,\ldots,j_m)}(x) = p_{j_1}(x) \cdot \ldots \cdot p_{j_m}(S_{j_{m-1}} \circ \ldots \circ S_{j_1}(x)).$$

Observe that for every $x \in K$ there exists at least one sequence $(j_1, \ldots, j_m) \in \{1, \ldots, N\}^m$ such that $\Pi_{(j_1,\ldots,j_m)}(x) > 0$. For $(j_1, \ldots, j_m) \in \{1, \ldots, N\}^m$ and $x \in K$ satisfying $\Pi_{(j_1,\ldots,j_m)}(x) > 0$ we define the open neighbourhood $O_{(j_1,\ldots,j_m)}(x)$ of x by the formula

$$\begin{aligned} &O_{(j_1,\ldots,j_m)}(x) \\ &\quad = \{y \in X : \varrho(S_{j_m,\ldots,j_1}(x), S_{j_m,\ldots,j_1}(y)) < \varepsilon/3,\ \Pi_{(j_1,\ldots,j_m)}(y) > \Pi_{(j_1,\ldots,j_m)}(x)/2\}, \end{aligned}$$

where $S_{j_m,\ldots,j_1}(y) = S_{j_m} \circ \ldots \circ S_{j_1}(y)$ for $y \in X$. Define

$$O_x = \bigcap O_{(j_1,\ldots,j_m)}(x) \quad \text{for } x \in X, \tag{6.3.13}$$

where the intersection is taken over all $(j_1, \ldots, j_m) \in \{1, \ldots, N\}^m$ such that $\Pi_{(j_1,\ldots,j_m)}(x) > 0$. Since K is a compact set there is a finite covering

$$K \subset \bigcup_{i=1}^{q} O_{x_i}. \tag{6.3.14}$$

Set $G = \bigcup_{i=1}^q O_{x_i}$ and define for $i \in \{1, \ldots, q\}$,

$$\delta_i = \min\{\Pi_{(j_1,\ldots,j_m)}(x_i)/2 : \Pi_{(j_1,\ldots,j_m)}(x_i) > 0 \text{ for } (j_1, \ldots, j_m) \in \{1, \ldots, N\}^m\}.$$

Set $\delta = \min_{1 \leq i \leq q} \delta_i$. We are going to show that (6.3.10) holds with $\alpha = \delta/(2q)$. In fact, let $\mu_1, \mu_2 \in \mathcal{M}_1$. Set $\mu_0 = (\mu_1 + \mu_2)/2$. According to Theorem 5.5(i) and Remark 5.1 there exists $\widetilde{\mu} \in \mathcal{M}_1$ such that $\widetilde{\mu} \in \Omega(\mu_0)$. Consequently, there exists a sequence $(m_n)_{n \geq 1}$ such that

$$\lim_{n \to \infty} \|P^{m_n}\mu_0 - \widetilde{\mu}\|_{\mathrm{FM}} = 0. \tag{6.3.15}$$

Since (6.3.15) is equivalent to the weak convergence of $(P^{m_n}\mu_0)_{n \geq 1}$ to $\widetilde{\mu}$ and G is open, the Aleksandrov theorem (see [6]) implies

$$\liminf_{n \to \infty} P^{m_n}\mu_0(G) \geq \widetilde{\mu}(G).$$

From this, (6.3.11) and (6.3.14) it follows that there exists an integer n such that

$$P^n\mu_0(G) = (P^n\mu_1(G) + P^n\mu_2(G))/2 > 3/4.$$

Hence

$$P^n\mu_i(G) > 1/2 \quad \text{for } i = 1, 2.$$

From this and (6.3.14) it follows that there exist $s, t \in \{1, \ldots, q\}$ such that

$$P^n\mu_1(O_{x_s}) > 1/(2q) \quad \text{and} \quad P^n\mu_2(O_{x_t}) > 1/(2q).$$

Write for simplicity $O_1 = O_{x_s}$ and $O_2 = O_{x_t}$. From (6.2.10) and (6.3.12) it may be concluded that there is $(i_1, \ldots, i_m) \in \{1, \ldots, N\}^m$ such that

$$\varrho(S_{i_m} \circ \ldots \circ S_{i_1}(x_s), S_{i_m} \circ \ldots \circ S_{i_1}(x_t)) \le r^m(\varrho(x_s, x_t)) \le \varepsilon/3 \tag{6.3.16}$$

and

$$\Pi_{(i_1,\ldots,i_m)}(x_s) \cdot \Pi_{(i_1,\ldots,i_m)}(x_t) > 0.$$

Define

$$A = A_1 \cup A_2 \quad \text{where} \quad A_i = \operatorname{cl} S_{i_m} \circ \ldots \circ S_{i_1}(O_i), \quad i = 1, 2.$$

According to (6.3.13) and (6.3.16) we have $\operatorname{diam} A \le \varepsilon$. Obviously $\Pi_{(i_1,\ldots,i_m)}(y) \ge \delta$ for $y \in O_1 \cup O_2$. On the other hand, by induction we check that

$$\begin{aligned} P^{m+n}\mu_i(A) &= \langle U^{m+n}\mathbf{1}_A, \mu_i\rangle = \langle U^m \mathbf{1}_A, P^n\mu_i\rangle \\ &\ge \sum_{j_1,\ldots,j_m=1}^{N} \int_X \Pi_{(j_1,\ldots,j_m)}(y)\mathbf{1}_{A_k}(S_{j_m} \circ \ldots \circ S_{j_1}(y))\, P^n\mu_i(dy) \\ &\ge \delta P^n\mu_i(O_i) > \delta/(2q) \quad \text{for } i = 1, 2, \end{aligned}$$

which finishes the proof. ∎

We finish this section with the observation that our last theorem extends the well known result due to Barnsley *et al.* (see [3]) to Polish spaces.

THEOREM 6.3.3. *Let an iterated function system $(S, p)_N$ satisfy*

$$\sum_{i=1}^{N} |p_i(x) - p_i(y)| \le \omega(\varrho(x, y)) \quad \textit{for } x, y \in X, \tag{6.3.17}$$

$$\prod_{i=1}^{N} [\varrho(S_i(x), S_i(y))]^{p_i(x)} \le r(\varrho(x, y)) \quad \textit{for } x, y \in X. \tag{6.3.18}$$

Assume that the pair (ω, r) satisfies the Dini condition. Moreover, assume that S_i, $i = 1, \ldots, N$, are Lipschitzean and $p_i(x) \ge \delta$, $x \in X$, $i = 1, \ldots, N$, for some $\delta > 0$. Then the system $(S, p)_N$ is asymptotically stable.

Proof. We begin by recalling a standard fact. Namely, let ν be a probability measure on a Borel space I and let $f : I \to (0, \infty)$ be a bounded Borel measurable function bounded away from 0. Then

$$\lim_{q \to 0} \Big(\int_I f^q \, d\nu\Big)^{1/q} = \exp\Big(\int_I \log f \, d\nu\Big). \tag{6.3.19}$$

Moreover, if $0 < a < b$ the above convergence is uniform over all functions f such that $a \le f \le b$ and all probability Borel measures ν. The proof is left to the reader (see also [3]).

Let $(S, p)_N$ be as in the statement of our theorem. We may assume that all S_i's are Lipschitzean with Lipschitz constant $L > 1$. Further, since the pair (ω, r) satisfies the Dini condition, $r(t) = ct$ with $c < 1$. Let $\delta > 0$ be a lower bound of all p_i's and let $I = \{1, \ldots, N\}$. For $x, y \in X$ and $i \in I$ define

$$f_{x,y}(i) = \max\Big\{\frac{\varrho(S_i(x), S_i(y))}{\varrho(x, y)}, \Big(\frac{c}{L}\Big)^{1/\delta}\Big\}.$$

(Here we assume that $0/0 = 0$.) Thus $(c/L)^{1/\delta} \le f_{x,y}(i) \le L$ for $x, y \in X$ and $i \in I$. We show that

$$\sum_{i=1}^{N} p_i(x) \log f_{x,y}(i) \le \log c \quad \text{for } x, y \in X. \tag{6.3.20}$$

First observe that if $f_{x,y}(i) = \varrho(S_i(x), S_i(y))/\varrho(x, y)$ for all $i \in I$, then (6.3.20) follows from (6.3.18). On the other hand, if $f_{x,y}(i_0) = (c/L)^{1/\delta}$ for any $i_0 \in I$, then

$$p_{i_0}(x) \log f_{x,y}(i_0) \le \log c - \log L$$

and consequently

$$\sum_{i=1}^{N} p_i(x) \log f_{x,y}(i) \le \log c - \log L + \log L = \log c.$$

Choose $c_0 \in (c, 1)$ and set $\eta = c_0/c$. By (6.3.19) there exists $q_0 \in (0, 1)$ such that

$$\Big(\int_I f^{q_0}\, d\nu\Big)^{1/q_0} \le \eta \exp\Big(\int_I \log f\, d\nu\Big) \tag{6.3.21}$$

for all functions f satisfying $(c/L)^{1/\delta} \le f \le L$ and all probability Borel measures ν.

Fix $x \in X$. Let $\nu(\{i\}) = p_i(x)$ for $i \in I$. Then (6.3.21) may be rewritten as

$$\sum_{i=1}^{N} p_i(x)[f_{x,y}(i)]^{q_0} \le \eta^{q_0}\Big[\exp\Big(\sum_{i=1}^{N} p_i(x) \log f_{x,y}(i)\Big)\Big]^{q_0} \quad \text{for } y \in X.$$

Hence

$$\sum_{i=1}^{N} p_i(x)[f_{x,y}(i)]^{q_0} \le \eta^{q_0} \cdot c^{q_0} = c_0^{q_0} \quad \text{for } y \in X$$

and since $x \in X$ is arbitrary we finally obtain

$$\sum_{i=1}^{N} p_i(x)[\varrho(S_i(x), S_i(y))]^{q_0} \le c_0^{q_0}[\varrho(x, y)]^{q_0} \quad \text{for } x, y \in X. \tag{6.3.22}$$

Define the new metric $\overline{\varrho}$ by

$$\overline{\varrho}(x, y) = [\varrho(x, y)]^{q_0} \quad \text{for } x, y \in X.$$

Multiplying ω by a constant if necessary, (6.2.9) may be verified only for $x, y \in X$ such that $\varrho(x, y) \le 1$. Then

$$\sum_{i=1}^{N} |p_i(x) - p_i(y)| \le \omega(\varrho(x, y)) \le \omega(\overline{\varrho}(x, y)). \tag{6.3.23}$$

Since (6.3.22) and (6.3.23) hold, the iterated function system $(S, p)_N$ defined on $(X, \overline{\varrho})$ satisfies the assumptions of Theorem 6.3.2. Hence it is asymptotically stable. Since $\overline{\varrho}$ is equivalent to ϱ, the iterated function system $(S, p)_N$ on (X, ϱ) is also asymptotically stable. ■

6.4. Capacity of invariant measures. Let $\Omega = \{1, \ldots, N\}^\infty = \{(i_1, i_2, \ldots) : i_k \in \{1, \ldots, N\}$ for every $k \in \mathbb{N}\}$ and $\Omega_* = \bigcup_{n=1}^\infty \Omega_n$, where $\Omega_n = \{1, \ldots, N\}^n$. Observe that Ω_* (resp. Ω) is the space of all finite (resp. infinite) sequences of elements $i_k \in \{1, \ldots, N\}$. For $k \in \mathbb{N}$ we set $\Omega_{\le k} = \bigcup_{n=1}^k \Omega_n$ and $\Omega_{\ge k} = \bigcup_{n=k}^\infty \Omega_n$. For $\mathbf{i} = (i_1, \ldots, i_n) \in \Omega_*$ let $|\mathbf{i}| = n$ denote the length of $\mathbf{i}$. If $\mathbf{i} \in \Omega$ we assume that $|\mathbf{i}| = \infty$.

For $\mathbf{i} \in \Omega \cup \Omega_*$ and $m \in \mathbb{N}$, $m \le |\mathbf{i}|$, we set $\mathbf{i}|m = (i_1, \ldots, i_m)$. We say that $\mathbf{i} < \mathbf{j}$ with $\mathbf{i} \in \Omega_*$ and $\mathbf{j} \in \Omega \cup \Omega_*$ if $|\mathbf{j}| > n$ and $\mathbf{j}|n = \mathbf{i}$, where $n = |\mathbf{i}|$. Finally, for $\mathbf{i} = (i_1, \ldots, i_n) \in \Omega_*$ we write $\mathbf{i}^{-1} = (i_n, \ldots, i_1)$.

A subset $\Lambda \subset \Omega$ is called a *cylinder* if there exists $\mathbf{i} = (i_1, \ldots, i_n) \in \Omega_*$ such that

$$\Lambda = \Lambda(\mathbf{i}) = \{\mathbf{j} \in \Omega : \mathbf{j}|n = \mathbf{i}\}.$$

We denote by $\mathcal{A}$ the σ-algebra in Ω generated by such cylinders.

Given an iterated function system $(S, p)_N$ and a point $x \in X$ we denote by $\mathbb{P}_x$ the probability measure on $\mathcal{A}$ defined on the cylinder $\Lambda(\mathbf{i})$, $\mathbf{i} = (i_1, \ldots, i_n) \in \Omega_*$, by

$$\mathbb{P}_x(\Lambda(\mathbf{i})) = p_{i_1}(x) p_{i_2}(S_{i_1}(x)) \cdot \ldots \cdot p_{i_n}(S_{i_{n-1}} \circ \ldots \circ S_{i_1}(x)). \tag{6.4.1}$$

It is clear that the above formula defines the unique probability measure for realization of the Markov process starting from x for given $(S, p)_N$.

For convenience, in what follows we will write $\mathbb{P}_x(\mathbf{i})$ for $\mathbb{P}_x(\Lambda(\mathbf{i}))$ and $\mathbb{P}_x(A)$ for $\mathbb{P}_x(\Lambda(A))$, where $A \subset \Omega_*$ and $\Lambda(A) = \bigcup_{\mathbf{i} \in A} \Lambda(\mathbf{i})$. Moreover, for $\mathbf{i} = (i_1, \ldots, i_n) \in \Omega_n$ we write

$$S_{\mathbf{i}} = S_{i_n} \circ \ldots \circ S_{i_1}.$$

LEMMA 6.4.1. *For every* $\mathbf{i} = (i_1, \ldots, i_n) \in \Omega_*$, *we have*

$$\mathbb{P}_x(\mathbf{i}) = p_{i_1}(x) \mathbb{P}_{S_{i_1}(x)}((i_2, \ldots, i_n)), \tag{6.4.2}$$

$$\mathbb{P}_x(\mathbf{i}) = p_{i_n}(S_{i_{n-1}} \circ \ldots \circ S_1(x)) \mathbb{P}_x((i_1, \ldots, i_{n-1})), \tag{6.4.3}$$

$$\sum_{i=1}^{N} \mathbb{P}_x((\mathbf{i}, i)) = \mathbb{P}_x(\mathbf{i}), \tag{6.4.4}$$

$$\mathbb{P}_x(\mathbf{i}|k) \ge \mathbb{P}_x(\mathbf{i}|m) \quad \textit{if } k \le m \le n. \tag{6.4.5}$$

Proof. This follows immediately from the definition of $\mathbb{P}_x$. ∎

Using a standard martingale argument one can prove the following lemma.

LEMMA 6.4.2. *Let an iterated function system* $(S, p)_N$ *satisfy the hypotheses of Theorem* 6.3.3 *and let* $f_i : X \to \mathbb{R}_+$, $i = 1, \ldots, N$, *be bounded continuous functions such that*

$$\inf_{x \in X} f_i(x) > 0 \quad \textit{for } i = 1 \ldots, N.$$

Then for every $x \in X$ *there exists a measurable set* $\Omega_x \subset \Omega$ *with* $\mathbb{P}_x(\Omega_x) = 1$ *such that*

$$\limsup_{n \to \infty} \frac{1}{n} \log(f_{i_1}(x) f_{i_2}(S_{i_1}(x)) \ldots f_{i_n}(S_{i_{n-1}} \circ \ldots \circ S_{i_1}(x))) \le \log \Delta, \tag{6.4.6}$$

$$\liminf_{n \to \infty} \frac{1}{n} \log(f_{i_1}(x) f_{i_2}(S_{i_1}(x)) \ldots f_{i_n}(S_{i_{n-1}} \circ \ldots \circ S_{i_1}(x))) \ge \log \Gamma \tag{6.4.7}$$

for all $(i_1, i_2, \ldots) \in \Omega_x$, *where*

$$\Delta = \sup_{x \in X} \prod_{i=1}^{N} f_i(x)^{p_i(x)}, \tag{6.4.8}$$

$$\Gamma = \inf_{x \in X} \prod_{i=1}^{N} f_i(x)^{p_i(x)}. \tag{6.4.9}$$

Proof. We prove (6.4.6). Fix $x \in X$ and for arbitrary $n \in \mathbb{N}$ define the random variable $X_n : \Omega \to \mathbb{R}$ by

$$X_n(\mathbf{i}) = \log(f_{i_n}(S_{i_{n-1}} \circ \ldots \circ S_{i_1}(x))).$$

For $\mathbf{i} = (i_1, \ldots, i_n) \in \Omega_*$ we denote by $\mathcal{A}(\mathbf{i})$ the σ-algebra generated by the cylinders $\{\Lambda(\mathbf{j}) : \mathbf{j} \in \Omega_*, \mathbf{j} > \mathbf{i}\}$. Moreover, let $\mathbb{E}_x$ denote the expectation with respect to the probability measure $\mathbb{P}_x$ on Ω.

Fix $\mathbf{i} = (i_1, i_2, \ldots) \in \Omega$. We have

$$\mathbb{E}_x(X_n \mid \mathcal{A}(i_1, \ldots, i_{n-1})) = \sum_{i=1}^{N} p_i(S_{i_{n-1}} \circ \ldots \circ S_{i_1}(x)) X_n((i_1, \ldots, i_{n-1}, i, i, \ldots)).$$

By (6.4.8) we have

$$\sum_{i=1}^{N} p_i(S_{i_{n-1}} \circ \ldots \circ S_{i_1}(x)) \log(f_i(S_{i_{n-1}} \circ \ldots \circ S_{i_1}(x))) \le \log \Delta.$$

Now let $Y_n = X_n - \mathbb{E}_x(X_n \mid \mathcal{A}(i_1, \ldots, i_{n-1}))$. Then

$$\sup_{\mathbf{i} \in \Omega} |Y_n(\mathbf{i})| \le 2 \sup_{\mathbf{i} \in \Omega} |X_n(\mathbf{i})| \quad \mathbb{P}_x\text{-a.s.}$$

Set

$$M = 2 \sup_{\mathbf{i} \in \Omega} |X_n(\mathbf{i})| < \infty, \qquad Z_n = \sum_{k=1}^{n} \frac{Y_k}{k} \quad \text{for } n \in \mathbb{N}.$$

It is easy to see that $(Z_n)_{n \ge 1}$ is a martingale. Since Y_k and Y_l for $k \ne l$ are mutually orthogonal, we have

$$\mathbb{E}_x(Z_n^2) \le M^2 \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Hence $(Z_n)_{n \ge 1}$ is an $\mathbb{L}^2$-bounded martingale, and so $(Z_n)_{n \ge 1}$ is convergent a.s. (see [19]). Then by Kronecker's lemma (see [13])

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} Y_k = 0 \quad \mathbb{P}_x\text{-a.s.}$$

Thus

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} X_k - \log \Delta \le 0 \quad \mathbb{P}_x\text{-a.s.},$$

whence (6.4.6) follows immediately.

Replacing f_i with $1/f_i$ and using the same argument one can prove (6.4.7). ■

The following result may be proved in much the same way as Lemma 6.4.2.

LEMMA 6.4.3. *Let an iterated function system $(S, p)_N$ satisfy the hypotheses of Theorem 6.3.3 and let $r(t)$, $t \ge 0$, given by (6.2.10) be equal to ct with $c < 1$. Let $x_0 \in X$. Then for every $x \in X$ there exists a measurable set $\Omega_x \subset \Omega$ with $\mathbb{P}_x(\Omega_x) = 1$ such that*

(6.4.10) $$\limsup_{n \to \infty} \frac{1}{n} \log \varrho(S_{\mathbf{i}|n}(x), S_{\mathbf{i}|n}(x_0)) \le \log c \quad \textit{for all } \mathbf{i} \in \Omega_x.$$

Proof. Fix $x_0 \in X$. Let $\delta > 0$ be such that $p_i(x) \geq \delta$ for $i = 1, \ldots, N$. We may assume that S_i's are Lipschitzean with Lipschitz constant $L > 1$. For every $n \in \mathbb{N}$ define the random variable $X_n : \Omega \to \mathbb{R}$ by

$$X_n(\mathbf{i}) = \begin{cases} \max\left\{\log \dfrac{\varrho(S_{\mathbf{i}|n}(x), S_{\mathbf{i}|n}(x_0))}{\varrho(S_{\mathbf{i}|n-1}(x), S_{\mathbf{i}|n-1}(x_0))}, \left(\dfrac{c}{L}\right)^{1/\delta}\right\} & \text{if } \varrho(S_{\mathbf{i}|n-1}(x), S_{\mathbf{i}|n-1}(x_0)) \neq 0, \\ \log c & \text{if } \varrho(S_{\mathbf{i}|n-1}(x), S_{\mathbf{i}|n-1}(x_0)) = 0. \end{cases}$$

As in the proof of Lemma 6.4.2 for $\mathbf{i} = (i_1, \ldots, i_n) \in \Omega_*$ we denote by $\mathcal{A}(\mathbf{i})$ the σ-algebra generated by the cylinders $\{\Lambda(\mathbf{j}) : \mathbf{j} \in \Omega_*, \mathbf{j} > \mathbf{i}\}$. Moreover, let $\mathbb{E}_x$ denote the expectation with respect to the probability measure $\mathbb{P}_x$ on Ω.

Fix $\mathbf{i} = (i_1, i_2, \ldots) \in \Omega$. We have

$$\mathbb{E}_x(X_n \,|\, \mathcal{A}(i_1, \ldots, i_{n-1})) = \sum_{i=1}^{N} p_i(S_{i_{n-1}} \circ \ldots \circ S_{i_1}(x)) X_n((i_1, \ldots, i_{n-1}, i, i, \ldots)).$$

Using a similar argument to (6.3.20) we can show that

$$\mathbb{E}_x(X_n \,|\, \mathcal{A}(i_1, \ldots, i_{n-1})) \leq \log c.$$

The remaining part of the proof runs as in Lemma 6.4.2. We leave the details to the reader. Finally we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\infty} X_k - \log c \leq 0 \qquad \mathbb{P}_x\text{-a.s.},$$

whence (6.4.10) follows immediately. ∎

Now for $c > 0$, $x_0, x \in X$ and $n_0, n \in \mathbb{N}$, $n \geq n_0$, we define

(6.4.11) $\quad Q^n_{n_0}(c; x_0, x) = \{\mathbf{i} \in \Omega_n : \varrho(S_{\mathbf{i}|k}(x), S_{\mathbf{i}|k}(x_0)) \leq c^k \varrho(x, x_0) \text{ for } n_0 \leq k \leq n\},$

(6.4.12) $\quad Q_{n_0}(c; x_0, x) = \{\mathbf{i} \in \Omega : \varrho(S_{\mathbf{i}|k}(x), S_{\mathbf{i}|k}(x_0)) \leq c^k \varrho(x, x_0) \text{ for } k \geq n_0\}.$

LEMMA 6.4.4. *For every $c > 0$, $x_0 \in X$ and $n \in \mathbb{N}$ the functions*

(6.4.13) $\quad X \ni x \mapsto \mathbb{P}_x(Q_n(c; x_0, x)),$

(6.4.14) $\quad X \ni x \mapsto \mathbb{P}_{x_0}(Q_n(c; x_0, x))$

are Borel measurable.

Proof. Fix $c > 0$, $x_0 \in X$ and $n \in \mathbb{N}$. First observe that $Q_n(c; x_0, x)$ is measurable with respect to the σ-algebra $\mathcal{A}$ generated by all cylinders in Ω. For all $x \in X$ we have

$$\begin{aligned}\mathbb{P}_x(Q_n(c; x_0, x)) &= 1 - \mathbb{P}_x(\{\mathbf{i} \in \Omega : \varrho(S_{\mathbf{i}|m}(x), S_{\mathbf{i}|m}(x_0)) > c^m \varrho(x, x_0) \text{ for some } m \geq n\}) \\ &= 1 - \sum_{m=n}^{\infty} \sum_{\mathbf{i} \in \Omega_m} \mathbb{P}_x(\mathbf{i}) \mathbf{1}_{X_{\mathbf{i}}}(x),\end{aligned}$$

where $X_{\mathbf{i}}$, $\mathbf{i} \in \Omega_m$, stands for the set of all $y \in X$ such that $\varrho(S_{\mathbf{i}}(y), S_{\mathbf{i}}(x_0)) > c^m \varrho(y, x_0))$ and $\varrho(S_{\mathbf{i}|k}(y), S_{\mathbf{i}|k}(x_0)) \leq c^k \varrho(y, x_0))$ for $k = 1, \ldots, m-1$. Since the sets $X_{\mathbf{i}}$, $\mathbf{i} \in \Omega_*$, are Borel measurable, the function (6.4.13) is Borel measurable.

In the same manner we can see that the function (6.4.14) is Borel measurable. ∎

We are in a position to formulate the following lemma.

LEMMA 6.4.5. *Let an iterated function system $(S,p)_N$ satisfy the hypotheses of Theorem* 6.3.3 *and let a bounded set $F \subset X$ and a number $n_0 \in \mathbb{N}$ be given. Let $r(t)$, $t > 0$, given by* (6.2.10) *be equal to ct with $c < 1$. Then for every $c_0 \in (c,1)$ there exists $\beta > 0$ such that for every $x, x_0 \in F$ and $n \ge n_0$ we have*

$$\mathbb{P}_x(\mathbf{i}) \ge \beta \mathbb{P}_{x_0}(\mathbf{i}) \quad \text{for } \mathbf{i} \in Q^n_{n_0}(c_0; x_0, x).$$

Proof. Fix a bounded set $F \subset X$ and $c_0 \in (c,1)$. Since the pair (ω, r) with $r(t) = ct$ for $t > 0$ satisfies the Dini condition, we have

$$\omega_0 = \sum_{k=1}^{\infty} \omega(c_0^k \operatorname{diam} F) < \infty.$$

Fix $n_0 \in \mathbb{N}$ and let $n \ge n_0$. For $x, x_0 \in F$ and $\mathbf{i} \in Q^n_{n_0}(c_0; x_0, x)$ we have

$$\begin{aligned}\mathbb{P}_{x_0}(\mathbf{i}) &= p_{i_1}(x_0) p_{i_2}(S_{i_1}(x_0)) \cdot \ldots \cdot p_{i_n}(S_{i_{n-1}} \circ \ldots \circ S_{i_1}(x_0)) \\ &= \frac{p_{i_1}(x_0) \cdot \ldots \cdot p_{i_{n_0}}(S_{i_{n_0-1}} \circ \ldots \circ S_{i_1}(x_0))}{p_{i_1}(x) \cdot \ldots \cdot p_{i_{n_0}}(S_{i_{n_0-1}} \circ \ldots \circ S_{i_1}(x))} \cdot p_{i_1}(x) \cdot \ldots \cdot p_{i_{n_0}}(S_{i_{n_0-1}} \circ \ldots \circ S_{i_1}(x)) \\ &\quad \cdot \prod_{k=n_0+1}^{n} \biggl(\biggl(1 + \frac{p_{i_k}(S_{i_{k-1}} \circ \ldots \circ S_{i_1}(x_0)) - p_{i_k}(S_{i_{k-1}} \circ \ldots \circ S_{i_1}(x))}{p_{i_k}(S_{i_{k-1}} \circ \ldots \circ S_{i_1}(x))} \biggr) \\ &\quad \cdot p_{i_k}(S_{i_{k-1}} \circ \ldots \circ S_{i_1}(x)) \biggr)\end{aligned}$$

Let $\delta > 0$ be such that $p_i(x) \ge \delta$ for $x \in X$, $i = 1, \ldots, N$. From (6.2.9) and (6.4.11) we obtain

$$\begin{aligned}\mathbb{P}_{x_0}(\mathbf{i}) &\le \frac{(1-\delta)^{n_0}}{\delta^{n_0}} \\ &\quad \cdot \prod_{k=n_0+1}^{n} \biggl(1 + \frac{\omega(\varrho(S_{i_{k-1}} \circ \ldots \circ S_{i_1}(x_0), S_{i_{k-1}} \circ \ldots \circ S_{i_1}(x)))}{\delta} \biggr) \mathbb{P}_x(\mathbf{i}) \\ &\le \biggl(\frac{1-\delta}{\delta} \biggr)^{n_0} \prod_{k=n_0+1}^{n} \biggl(1 + \frac{\omega(c_0^{k-1} \operatorname{diam} F)}{\delta} \biggr) \mathbb{P}_x(\mathbf{i}).\end{aligned}$$

Consequently,

$$\mathbb{P}_{x_0}(\mathbf{i}) \le \biggl(\frac{1-\delta}{\delta} \biggr)^{n_0} \prod_{k=n_0+1}^{\infty} e^{\omega(c_0^{k-1} \operatorname{diam} F)/\delta} \mathbb{P}_x(\mathbf{i}) = \biggl(\frac{1-\delta}{\delta} \biggr)^{n_0} e^{\omega_0/\delta} \mathbb{P}_x(\mathbf{i}).$$

Setting $\beta = \delta^{n_0}(1-\delta)^{-n_0} e^{-\omega_0/\delta}$ we finish the proof. ■

THEOREM 6.4.1 (Upper estimate). *Let an iterated function system $(S,p)_N$ satisfy the hypotheses of Theorem* 6.3.3 *and let $r(t)$, $t \ge 0$, given by* (6.2.10) *be equal to ct with $c < 1$. Let $\mu_* \in \mathcal{M}_1$ be the unique invariant measure for $(S,p)_N$. Then*

$$\overline{\operatorname{Cap}}_L(\mu_*) \le \frac{\log p}{\log c}, \tag{6.4.15}$$

where

$$p = \inf_{x \in X} \prod_{i=1}^{N} p_i(x)^{p_i(x)}. \tag{6.4.16}$$

Proof. Fix $\eta > 0$. Let μ_* be the unique invariant distribution for $(S,p)_N$ and let K be a compact subset of X such that $\mu_*(K) \geq 1-\eta/4$. Choose $p_0 \in (0,p)$, $c_0 \in (c,1)$ and $x_0 \in K$. For $n \in \mathbb{N}$ define

$$D_n = \bigcup B(S_{\mathbf{i}}(x_0), \varepsilon_n),$$

where the union is taken over all $\mathbf{i} \in \Omega_n$ such that $\mathbb{P}_{x_0}(\mathbf{i}) \geq p_0^n$ and $\varepsilon_n = c_0^n \operatorname{diam} K$. We check at once that $N_{D_n}(\varepsilon_n) \leq p_0^{-n}$ for $n \in \mathbb{N}$.

We are going to show that

$$\liminf_{n\to\infty} \mu_*(D_n) \geq 1-\eta.$$

Since $Q_n(c_0;x_0,x) \subset Q_{n+1}(c_0;x_0,x)$ for $n \in \mathbb{N}$ and $x \in X$, Lemma 6.4.3 shows that

$$\lim_{n\to\infty} \mathbb{P}_x(Q_n(c_0;x_0,x)) = 1 \quad \text{for } x \in X.$$

From the above and Lemma 6.4.4 there exists $n_0 \in \mathbb{N}$ such that

$$\mu_*(\{x \in K : \mathbb{P}_x(Q_{n_0}(c;x_0,x)) \geq 1-\eta/4\}) \geq 1-\eta/2. \tag{6.4.17}$$

Set

$$K_0 = \{x \in K : \mathbb{P}_x(Q_{n_0}(c;x_0,x)) \geq 1-\eta/4\}.$$

Lemma 6.4.5 now shows that there exists $\beta > 0$ such that

$$\mathbb{P}_{x_0}(\mathbf{i}) \geq \beta \mathbb{P}_x(\mathbf{i}) \quad \text{for } \mathbf{i} \in Q^n_{n_0}(c_0;x_0,x),\ n \geq n_0. \tag{6.4.18}$$

By Lemma 6.4.2 (with $p_i(x)$ in place of $f_i(x)$ and p_0 in place of Γ) there exists a measurable set $\Omega_0 \subset \Omega$ with $\mathbb{P}_{x_0}(\Omega_0) = 1$ and

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{x_0}(\mathbf{i}|n) \geq \log p \quad \text{for } \mathbf{i} \in \Omega_0.$$

Hence there exists $n_1 \geq n_0$ such that

$$\mathbb{P}_{x_0}(\{\mathbf{i} \in \Omega : \mathbb{P}_{x_0}(\mathbf{i}|n) \leq p_0^n \text{ for some } n \geq n_1\}) \leq \beta\eta/4$$

and consequently, for $n \geq n_1$,

$$\mathbb{P}_{x_0}(\{\mathbf{i} \in \Omega_n : \mathbb{P}_{x_0}(\mathbf{i}) \leq p_0^n\}) \leq \beta\eta/4.$$

From (6.4.18) it follows that for $x \in K_0$,

$$\mathbb{P}_x(\{\mathbf{i} \in Q^n_{n_0}(c_0;x_0,x) : \mathbb{P}_{x_0}(\mathbf{i}) \leq p_0^n\}) \leq \eta/4 \quad \text{for } n \geq n_1$$

and consequently by the definition of K_0 we obtain

$$\mathbb{P}_x(\{\mathbf{i} \in Q^n_{n_0}(c_0;x_0,x) : \mathbb{P}_{x_0}(\mathbf{i}) \geq p_0^n\}) \geq 1-\eta/2 \tag{6.4.19}$$

for $x \in K_0$ and $n \geq n_1$. Set

$$\Omega_{n,x} = \{\mathbf{i} \in Q^n_{n_0}(c_0;x_0,x) : \mathbb{P}_{x_0}(\mathbf{i}) \geq p_0^n\}.$$

By the invariance property of μ_* and the definitions of D_n and $\Omega_{n,x}$ for all $n \in \mathbb{N}$ we have

$$\begin{aligned}\mu_*(D_n) &= \int_X \sum_{\mathbf{i}\in\Omega_n} \mathbb{P}_x(\mathbf{i}) \mathbf{1}_{D_n}(S_{\mathbf{i}}(x))\, \mu_*(dx) \\ &\geq \int_{K_0} \sum_{\mathbf{i}\in\Omega_{n,x}} \mathbb{P}_x(\mathbf{i}) \mathbf{1}_{D_n}(S_{\mathbf{i}}(x))\, \mu_*(dx).\end{aligned}$$

By (6.4.19) we obtain

$$\mu_*(D_n) \ge (1-\eta/2)(1-\eta/2) > 1-\eta \quad \text{for } n \ge n_1.$$

Since $N_{D_n}(\varepsilon_n) \le p_0^{-n}$ for $n \in \mathbb{N}$, it follows that $N(\varepsilon_n, \eta) \le p_0^{-n}$ for $n \ge n_1$. Hence

$$\limsup_{\varepsilon\to 0} \frac{\log N(\varepsilon,\eta)}{-\log\varepsilon} = \limsup_{n\to\infty} \frac{\log N(\varepsilon_n,\eta)}{-\log\varepsilon_n} \le \limsup_{n\to\infty} \frac{\log p_0^{-n}}{-\log(c_0^n d)} = \frac{\log p_0}{\log c_0}.$$

Letting $p_0 \to p$ and $c_0 \to c$ we conclude that

$$\limsup_{\varepsilon\to 0} \frac{\log N(\varepsilon,\eta)}{-\log\varepsilon} \le \frac{\log p}{\log c}.$$

Since $\eta > 0$ is arbitrary, the statement of our theorem follows. ∎

To obtain a lower estimate we will assume that $S_i : X \to X$, $i = 1,\dots,N$, are bi-lipschitzean transformations, i.e., there exist constants $l_i, L_i > 0$ such that

$$\text{(6.4.20)} \qquad l_i\varrho(x,y) \le \varrho(S_i(x), S_i(y)) \le L_i\varrho(x,y) \quad \text{for } x,y \in X.$$

Further

$$\text{(6.4.21)} \qquad r = \sup_{x\in X} \prod_{i=1}^{N} L_i^{p_i(x)} < 1,$$

$$\text{(6.4.22)} \qquad \sum_{i=1}^{n} |p_i(x) - p_i(y)| \le \omega(\varrho(x,y)) \quad \text{for } x,y \in X,$$

where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies condition (6.2.3). Finally, we assume that there exists $\delta > 0$ such that

$$\text{(6.4.23)} \qquad p_i(x) \ge \delta \quad \text{for } x \in X \text{ and } i = 1,\dots,N.$$

THEOREM 6.4.2. *Let an iterated function system $(S,p)_N$ satisfy* (6.4.20)–(6.4.23). *Then the system $(S,p)_N$ is asymptotically stable.*

Proof. It is easy to check that the assumptions of Theorem 6.3.3 are satisfied. ∎

Let an iterated function system $(S,p)_N$ be given. A finite set $\mathcal{L} \subset \Omega_*$ is called *fundamental* for $(S,p)_N$ if

$$\text{(6.4.24)} \qquad \sum_{\mathbf{i}\in\mathcal{L}} \mathbb{P}_x(\mathbf{i}) = 1 \quad \text{for every } x \in X$$

and there are no $\mathbf{i},\mathbf{j} \in \mathcal{L}$ such that $\mathbf{i} < \mathbf{j}$.

Set

$$|\mathcal{L}| = \max\{|\mathbf{i}| : \mathbf{i} \in \mathcal{L}\}.$$

LEMMA 6.4.6. *Let $\mathcal{L} \subset \Omega_*$ be a fundamental set for given $(S,p)_N$. If $\mathbf{i} = (i_1,\dots,i_n) \in \mathcal{L}$ and $n = |\mathcal{L}|$, then $(i_1,\dots,i_{n-1},i) \in \mathcal{L}$ for every $i \in \{1,\dots,N\}$.*

Proof. First observe that $\Lambda(\mathbf{i}) \cap \Lambda(\mathbf{j}) = \emptyset$ for every $\mathbf{i},\mathbf{j} \in \mathcal{L}$, $\mathbf{i} \ne \mathbf{j}$. Suppose, contrary to our claim, that there is $\mathbf{i} = (i_1,\dots,i_n) \in \mathcal{L}$ such that $(i_1,\dots,i_{n-1},i) \notin \mathcal{L}$ for some $i \in \{1,\dots,N\}$. It is easy to verify that $\Lambda(i_1,\dots,i_{n-1},i) \cap \Lambda(\mathbf{j}) = \emptyset$ for every $\mathbf{j} \in \mathcal{L}$. Since $\mathbb{P}_x$ is a probability measure and $\mathbb{P}_x(\mathbf{i}) > 0$ for every $\mathbf{i} \in \Omega_*$, we have

$$\sum_{\mathbf{i}\in\mathcal{L}} \mathbb{P}_x(\mathbf{i}) \le 1 - \mathbb{P}_x((i_1,\ldots,i_{n-1},i)) < 1,$$

which contradicts (6.4.24). ∎

REMARK 6.4.1. Note that for every $n \in \mathbb{N}$ there exists a fundamental set $\mathcal{L}$ for $(S,p)_N$ such that $\mathcal{L} \subset \Omega_{\le n}$.

LEMMA 6.4.7. *Let an iterated function system $(S,p)_N$ satisfy conditions* (6.4.20)–(6.4.23) *and let μ_* be its unique invariant distribution. Then for every fundamental set $\mathcal{L} \subset \Omega_*$ we have*

$$\mu_*(A) = \sum_{\mathbf{i}\in\mathcal{L}} \int_X \mathbb{P}_x(\mathbf{i}^{-1})\mathbf{1}_A(S_{\mathbf{i}^{-1}}(x))\,\mu_*(dx) \quad \textit{for } A \in \mathcal{B}(X). \tag{6.4.25}$$

Proof. The proof is by induction on n, where n is the smallest integer such that $\mathcal{L} \subset \Omega_{\le n}$. Suppose first that $\mathcal{L} \subset \Omega_1$. Since $p_i(x) > 0$ for $x \in X$ and $i = 1,\ldots,N$, it follows immediately that $\mathcal{L} = \{1,\ldots,N\}$ and (6.4.25) holds.

Now suppose that (6.4.25) holds for every $\mathcal{L} \subset \Omega_{\le n}$. We will prove that (6.4.25) holds for every $\mathcal{L} \subset \Omega_{\le n+1}$. By the invariance property of μ_* we have

$$\int_X f(x)\,\mu_*(dx) = \sum_{i=1}^N \int_X p_i(x) f(S_i(x))\,\mu_*(dx) \quad \text{for } f \in B(X). \tag{6.4.26}$$

Set

$$\mathcal{L}_{n+1} = \{\mathbf{i} \in \mathcal{L} : |\mathbf{i}| = n+1\} \quad \text{and} \quad \mathcal{L}_{n+1}^n = \{\mathbf{i}|n : \mathbf{i} \in \mathcal{L}_{n+1}\}. \tag{6.4.27}$$

We assume that $\mathcal{L}_{n+1} \ne \emptyset$ (otherwise there is nothing to prove). Fix $A \in \mathcal{B}(X)$. By Lemma 6.4.6, formulae (6.4.2) and (6.4.26) we have

$$\begin{aligned}
&\sum_{\mathbf{i}\in\mathcal{L}_{n+1}} \int_X \mathbb{P}_x(\mathbf{i}^{-1})\mathbf{1}_A(S_{\mathbf{i}^{-1}}(x))\,\mu_*(dx) \\
&\qquad = \sum_{\mathbf{j}\in\mathcal{L}_{n+1}^n} \sum_{i=1}^N \int_X \mathbb{P}_x((\mathbf{j},i)^{-1})\mathbf{1}_A(S_{(\mathbf{j},i)^{-1}}(x))\,\mu_*(dx) \\
&\qquad = \sum_{\mathbf{j}\in\mathcal{L}_{n+1}^n} \sum_{i=1}^N \int_X p_i(x)\mathbb{P}_{S_i(x)}(\mathbf{j}^{-1})\mathbf{1}_A(S_{\mathbf{j}^{-1}}(S_i(x)))\,\mu_*(dx) \\
&\qquad = \sum_{\mathbf{j}\in\mathcal{L}_{n+1}^n} \int_X \mathbb{P}_x(\mathbf{j}^{-1})\mathbf{1}_A(S_{\mathbf{j}^{-1}}(x))\,\mu_*(dx).
\end{aligned} \tag{6.4.28}$$

Now setting $\mathcal{L}^* = (\mathcal{L} \setminus \mathcal{L}_{n+1}) \cup \mathcal{L}_{n+1}^n$ and using (6.4.28) we obtain

$$\begin{aligned}
\sum_{\mathbf{i}\in\mathcal{L}} \int_X \mathbb{P}_x(\mathbf{i}^{-1})\mathbf{1}_A(S_{\mathbf{i}^{-1}}(x))\,\mu_*(dx) &= \sum_{\mathbf{i}\in\mathcal{L}_{n+1}} \int_X \mathbb{P}_x(\mathbf{i}^{-1})\mathbf{1}_A(S_{\mathbf{i}^{-1}}(x))\,\mu_*(dx) \\
&\quad + \sum_{\mathbf{i}\in\mathcal{L}\setminus\mathcal{L}_{n+1}} \int_X \mathbb{P}_x(\mathbf{i}^{-1})\mathbf{1}_A(S_{\mathbf{i}^{-1}}(x))\,\mu_*(dx) \\
&= \sum_{\mathbf{i}\in\mathcal{L}^*} \int_X \mathbb{P}_x(\mathbf{i}^{-1})\mathbf{1}_A(S_{\mathbf{i}^{-1}}(x))\,\mu_*(dx).
\end{aligned} \tag{6.4.29}$$

Clearly $\mathcal{L}^*$ is fundamental and $\mathcal{L}^* \subset \Omega_{\leq n}$. Hence the last term in (6.4.29) is equal to $\mu^*(A)$. Thus (6.4.25) holds for every $\mathcal{L} \subset \Omega_{\leq n+1}$. Consequently, by induction, (6.4.25) holds for every fundamental set $\mathcal{L}$. ∎

For $\mathbf{i} = (i_1, \ldots, i_k) \in \Omega_*$ we write

$$L_{\mathbf{i}} = L_{i_1} \cdot \ldots \cdot L_{i_k} \quad \text{and} \quad l_{\mathbf{i}} = l_{i_1} \cdot \ldots \cdot l_{i_k}. \tag{6.4.30}$$

Let r be given by (6.4.21). For $r_0 > r$ and $n_0, n \in \mathbb{N}$, $n \geq n_0$, we define

$$Q^n_{n_0}(r_0) = \{\mathbf{i} \in \Omega_{\geq n} : L_{\mathbf{i}|k} \leq r_0^k \text{ for } n_0 \leq k \leq n\}. \tag{6.4.31}$$

We can now rephrase Lemma 6.4.5 as follows.

LEMMA 6.4.8. *Let an iterated function system $(S,p)_N$ satisfy the hypotheses of Theorem* 6.4.2 *and let a bounded set $F \subset X$ and a number $n_0 \in \mathbb{N}$ be given. Then for every $r_0 \in (r, 1)$ there exists $\beta > 0$ such that for every $x, x_0 \in F$ and $n \geq n_0$ we have*

$$\mathbb{P}_x(\mathbf{i}) \geq \beta \mathbb{P}_{x_0}(\mathbf{i}) \quad \textit{for } \mathbf{i} \in Q^n_{n_0}(r_0) \cap \Omega_n.$$

Set

$$q = \sup_{x \in X} \prod_{i=1}^N p_i(x)^{p_i(x)}, \tag{6.4.32}$$

$$d = \inf_{x \in X} \prod_{i=1}^N l_i^{p_i(x)}. \tag{6.4.33}$$

For $d_0 \in (0, d)$ and $n \in \mathbb{N}$ define

$$J_n(d_0) = \{(i) : l_i > d_0^n\} \cup \{\mathbf{i} \in \Omega_* : |\mathbf{i}| > 1 \text{ and } l_{\mathbf{i}} \leq d_0^n < l_{\mathbf{i}||\mathbf{i}|-1}\}.$$

LEMMA 6.4.9. *Let an iterated function system $(S,p)_N$ satisfy condition* (6.4.20). *If $l_i \in (0,1)$, $i = 1, \ldots, N$, then for every $d_0 \in (0,d)$ and $n \in \mathbb{N}$ the set $J_n(d_0)$ is fundamental for $(S,p)_N$.*

Proof. Fix $d_0 \in (0,d)$ and $n \in \mathbb{N}$. It is easy to verify that $J_n(d_0) \subset \Omega_{\leq m}$, where m is the least integer such that $(\max_{1 \leq i \leq N} l_i)^m \leq d_0^n$. Consequently, $J_n(d_0)$ is a finite set. Moreover, from the definition of $J_n(d_0)$ it follows that if $\mathbf{i} \in J_n(d_0)$, $\mathbf{j} \in \Omega_*$ and $\mathbf{j} > \mathbf{i}$, then $\mathbf{j} \notin J_n(d_0)$. This implies that

$$\Lambda(\mathbf{i}) \cap \Lambda(\mathbf{j}) = \emptyset \quad \text{for } \mathbf{i}, \mathbf{j} \in J_n(d_0),\ \mathbf{i} \neq \mathbf{j}. \tag{6.4.34}$$

Finally, observe that for every $\mathbf{i} \in \Omega$ there is $k \in \mathbb{N}$ such that $\mathbf{i}|k \in J_n(d_0)$. Consequently,

$$\Omega = \bigcup_{\mathbf{i} \in J_n(d)} \Lambda(\mathbf{i}). \tag{6.4.35}$$

By (6.4.34) and (6.4.35) for all $x \in X$ we have

$$\sum_{\mathbf{i} \in J_n(d)} \mathbb{P}_x(\mathbf{i}) = \mathbb{P}_x\Big(\bigcup_{\mathbf{i} \in J_n(d)} \Lambda(\mathbf{i})\Big) = \mathbb{P}_x(\Omega) = 1. \ ∎$$

In what follows we will assume that $S_1, \dots, S_N$ satisfy the *strong Moran condition*, i.e., there exists a bounded closed subset $F \subset X$ and a constant $\sigma > 0$ such that

$$\bigcup_{i=1}^{N} S_i(F) \subset F, \tag{6.4.36}$$

$$\operatorname{dist}(S_i(F), S_j(F)) \geq \sigma \quad \text{for } i \neq j. \tag{6.4.37}$$

LEMMA 6.4.10. *Let $S_1, \dots, S_N$ satisfy the strong Moran condition. Then for every $d_0 \in (0, d)$, $n \in \mathbb{N}$ and $\mathbf{i}, \mathbf{j} \in J_n(d_0)$, $\mathbf{i} \neq \mathbf{j}$, we have*

$$\operatorname{dist}(S_{\mathbf{i}^{-1}}(F), S_{\mathbf{j}^{-1}}(F)) \geq d_0^n \sigma, \tag{6.4.38}$$

where the set F and the constant σ are given by conditions (6.4.36), (6.4.37).

Proof. Fix $d_0 \in (0, d)$, $n \in \mathbb{N}$ and $\mathbf{i}, \mathbf{j} \in J_n(d_0)$, $\mathbf{i} \neq \mathbf{j}$. Let $\mathbf{i} = (i_1, \dots, i_p)$ and $\mathbf{j} = (j_1, \dots, j_q)$. Since $J_n(d_0)$ is fundamental, there exists an integer $m \leq \min\{p, q\}$ such that $i_m \neq j_m$, but $i_k = j_k$ for $k < m$. From the strong Moran condition it follows immediately that $S_{\mathbf{i}^{-1}}(F) \subset S_{i_1} \circ \dots \circ S_{i_m}(F)$, $S_{\mathbf{j}^{-1}}(F) \subset S_{j_1} \circ \dots \circ S_{j_m}(F)$ and $\operatorname{dist}(S_{i_m}(F), S_{j_m}(F)) \geq \sigma$. Consequently,

$$\begin{aligned}\operatorname{dist}(S_{\mathbf{i}^{-1}}(F), S_{\mathbf{j}^{-1}}(F)) &\geq \operatorname{dist}(S_{i_1} \circ \dots \circ S_{i_m}(F), S_{j_1} \circ \dots \circ S_{j_m}(F)) \\ &\geq l_{i_1} \cdot \ldots \cdot l_{i_m - 1} \cdot \sigma \geq d_0^n \sigma. \ \blacksquare\end{aligned}$$

LEMMA 6.4.11. *Let an iterated function system $(S, p)_N$ satisfy* (6.4.20)–(6.4.23) *and let d be given by* (6.4.33). *Then for every $d_0 \in (0, d)$ there is $\gamma > 0$ such that*

$$\sum_{\mathbf{i} \in J_n^*(d_0)} \mathbb{P}_x(\mathbf{i}) \geq \gamma \quad \textit{for every } x \in X,$$

where

$$J_n^*(d_0) = \{\mathbf{i} \in J_n(d_0) : \mathbf{i}^{-1} \in J_n(d_0)\}. \tag{6.4.39}$$

Proof. We can assume that l_i, $i = 1, \dots, N$, given by (6.4.20) satisfy $l_1 \leq \dots \leq l_{N-1} \leq l_N$. Fix $d_0 \in (0, d)$ and observe that if $\mathbf{i} = (i_1, \dots, i_k) \in J_n(d_0)$ and $i_k = N$, then $\mathbf{i}^{-1} \in J_n(d_0)$. Moreover, for every $\mathbf{i} = (i_1, \dots, i_k) \in J_n(d_0)$ there exists a unique

$$\tau(\mathbf{i}) = (i_1, \dots, i_{k-1}, N, \dots, N) \tag{6.4.40}$$

which belongs to $J_n(d_0)$. In this way (6.4.40) defines a one-to-one map from $J_n(d_0)$ into $J_n(d_0)$. Note also that

$$(\tau(\mathbf{i}))^{-1} \in J_n(d_0) \quad \text{for every } \mathbf{i} \in J_n(d_0). \tag{6.4.41}$$

Fix $\mathbf{i} = (i_1, \dots, i_k) \in J_n(d_0)$ and choose m_0 such that $l_N^{m_0} \leq l_1$. Since $l_{\mathbf{i}} \leq d_0^n$ and $l_1 \leq l_i$ for $i = 1, \dots, N$, we have

$$|\tau(\mathbf{i})| \leq |\mathbf{i}| - 1 + m_0,$$

which means that in (6.4.40) the number N appears at most m_0 times. Now it is easy to see that for every $\mathbf{i} \in J_n(d_0)$,

$$\operatorname{card}\{\mathbf{j} \in J_n(d_0) : \tau(\mathbf{j}) = \tau(\mathbf{i})\} \leq N^{m_0}. \tag{6.4.42}$$

(Here card stands for cardinality.) By (6.4.40), (6.4.3) and (6.4.5) for all $x \in X$ we have

$$\mathbb{P}_x(\tau(\mathbf{i})) = \mathbb{P}_x((i_1, \dots, i_{k-1}, N, \dots, N)) \geq \mathbb{P}_x((i_1, \dots, i_{k-1}))\delta^{m_0} \geq \mathbb{P}_x(\mathbf{i})\delta^{m_0}, \tag{6.4.43}$$

where δ is given by (6.4.23). By Lemma 6.4.9 and (6.4.24), (6.4.43) and (6.4.42) we have

$$1 = \sum_{\mathbf{i}\in J_n(d_0)} \mathbb{P}_x(\mathbf{i}) \le \delta^{-m_0} \sum_{\mathbf{i}\in J_n(d_0)} \mathbb{P}_x(\tau(\mathbf{i})) \le N^{m_0}\delta^{-m_0} \sum_{\mathbf{i}\in J_n^*(d_0)} \mathbb{P}_x(\mathbf{i}).$$

From this, setting $\gamma = (\delta/N)^{m_0}$, we finish the proof. ∎

THEOREM 6.4.3 (Lower estimate). *Let an iterated function system $(S,p)_N$ satisfy conditions (6.4.20)–(6.4.23) and let μ_* be its unique invariant distribution. If the functions $S_1,\ldots,S_N$ satisfy the strong Moran condition, then*

$$\underline{\mathrm{Cap}}_L(\mu_*) \ge \frac{\log q}{\log d}, \tag{6.4.44}$$

where q and d are given by (6.4.32) and (6.4.33), respectively.

Proof. Consider first the case $l_i < 1$, $i = 1,\ldots,N$, where l_i satisfy (6.4.20). Let F be a closed set satisfying (6.4.36) and (6.4.37). Since F is invariant for $S_1,\ldots,S_N$, we have $\mu_* \in \mathcal{M}_1^F$. Choose $x_0 \in F$. By Lemma 6.4.2 (with p_i and q or L_i and r in place of f_i and Δ and l_i and d in place of f_i and Γ, respectively) we have

$$\limsup_{n\to\infty} \frac{1}{n}\log(\mathbb{P}_{x_0}(\mathbf{i}|n)) \le \log q \quad \mathbb{P}_{x_0}\text{-a.s.}, \tag{6.4.45}$$

$$\limsup_{n\to\infty} \frac{1}{n}\log(L_{\mathbf{i}|n}) \le \log r \quad \mathbb{P}_{x_0}\text{-a.s.}, \tag{6.4.46}$$

$$\liminf_{n\to\infty} \frac{1}{n}\log(l_{\mathbf{i}|n}) \ge \log d \quad \mathbb{P}_{x_0}\text{-a.s.} \tag{6.4.47}$$

Fix $d_0 \in (0,d)$, $r_0 \in (r,1)$ and $q_0 \in (q,1)$. Let $n_0 \in \mathbb{N}$ and $\beta > 0$ be chosen according to Lemma 6.4.8. By (6.4.45)–(6.4.47) there exists $n_1 \ge n_0$ such that

$$\mathbb{P}_{x_0}(\{\mathbf{i}\in\Omega : \mathbb{P}_{x_0}(\mathbf{i}|n) \le q_0^n \text{ for } n\ge n_1\}) \ge 1-\gamma/6, \tag{6.4.48}$$

$$\mathbb{P}_{x_0}(\{\mathbf{i}\in\Omega : L_{\mathbf{i}|n} \le r_0^n \text{ for } n\ge n_1\}) \ge 1-\gamma/6, \tag{6.4.49}$$

$$\mathbb{P}_{x_0}(\{\mathbf{i}\in\Omega : l_{\mathbf{i}|n} \ge d_0^n \text{ for } n\ge n_1\}) \ge 1-\gamma/6, \tag{6.4.50}$$

where γ is chosen according to Lemma 6.4.11. Now choose $n_* \in \mathbb{N}$ such that

$$\min\{|\mathbf{i}| : \mathbf{i}\in J_n(d_0)\} \ge n_1 \quad \text{for } n \ge n_*. \tag{6.4.51}$$

Define for $n \ge n_*$,

$$J_n^0(d_0) = \{\mathbf{i}\in J_n(d_0) : \mathbb{P}_{x_0}(\mathbf{i}^{-1}|k) \le q_0^k,\ L_{\mathbf{i}^{-1}|k} \le r_0^k \text{ and } l_{\mathbf{i}^{-1}|k} \ge d_0^k \text{ for } k\in\mathbb{N},\ n_1 \le k \le |\mathbf{i}|\}.$$

By (6.4.48)–(6.4.50) and Lemma 6.4.11 we have

$$\sum_{\mathbf{i}\in J_n^0(d_0)} \mathbb{P}_{x_0}(\mathbf{i}^{-1}) \ge \gamma/2 \quad \text{for } n \ge n_1. \tag{6.4.52}$$

Since $\mathbf{i}^{-1} \in Q_{n_1}^{|\mathbf{i}|}(r_0)$ for $\mathbf{i}\in J_n^0(d_0)$, by Lemma 6.4.8 we have

$$\beta^{-1}\mathbb{P}_{x_0}(\mathbf{i}^{-1}) \ge \mathbb{P}_x(\mathbf{i}^{-1}) \ge \beta\mathbb{P}_{x_0}(\mathbf{i}^{-1}) \quad \text{for } \mathbf{i}\in J_n^0(d_0) \text{ and } x\in F. \tag{6.4.53}$$

From (6.4.52) and (6.4.53) it follows that

$$\sum_{\mathbf{i}\in J_n^0(d_0)} \mathbb{P}_x(\mathbf{i}^{-1}) \ge \frac{\beta\gamma}{2} \tag{6.4.54}$$

and

$$\mathbb{P}_x(\mathbf{i}^{-1}) \le \beta^{-1} q_0^{|\mathbf{i}|} \tag{6.4.55}$$

for every $n \ge n_*$, $x \in F$ and $\mathbf{i} \in J_n^0(d_0)$. Since for $\mathbf{i} \in J_n^0(d_0)$ we have $l_{\mathbf{i}} \le d_0^n$ and $l_{\mathbf{i}^{-1}|k} \ge d_0^k$ for $n_1 \le k \le |\mathbf{i}|$, it follows that $|\mathbf{i}| \ge n$. Hence

$$\mathbb{P}_x(\mathbf{i}^{-1}) \le \beta^{-1} q_0^n \quad \text{for } \mathbf{i} \in J_n^0(d_0). \tag{6.4.56}$$

For $n \ge n_*$ define

$$D_n = \bigcup_{\mathbf{i}\in J_n^0(d_0)} S_{\mathbf{i}^{-1}}(F). \tag{6.4.57}$$

Since by Lemma 6.4.9 the set $J_n(d_0)$ is fundamental, Lemma 6.4.7 and inequality (6.4.54) yield

$$\begin{aligned} \mu_*(D_n) &= \sum_{\mathbf{i}\in J_n(d_0)} \int_X \mathbb{P}_x(\mathbf{i}^{-1}) \mathbf{1}_{D_n}(S_{\mathbf{i}^{-1}}(x))\, \mu_*(dx) \\ &\ge \int_F \sum_{\mathbf{i}\in J_n^0(d_0)} \mathbb{P}_x(\mathbf{i}^{-1}) \mathbf{1}_{D_n}(S_{\mathbf{i}^{-1}}(x))\, \mu_*(dx) \\ &= \int_F \sum_{\mathbf{i}\in J_n^0(d_0)} \mathbb{P}_x(\mathbf{i}^{-1})\, \mu_*(dx) \ge \frac{\beta\gamma}{2} \end{aligned} \tag{6.4.58}$$

for $n \ge n_*$. Since $\mu_* \in \mathcal{M}_1^F$, from Lemmas 6.4.7, 6.4.9 and 6.4.10, and inequality (6.4.56), for every $\mathbf{j} \in J_n^0(d_0)$ and $n \ge n_*$, we have

$$\begin{aligned} \mu_*(S_{\mathbf{j}^{-1}}(F)) &= \sum_{\mathbf{i}\in J_n(d_0)} \int_X \mathbb{P}_x(\mathbf{i}^{-1}) \mathbf{1}_{S_{\mathbf{j}^{-1}}(F)}(S_{\mathbf{i}^{-1}}(x))\, \mu_*(dx) \\ &= \sum_{\mathbf{i}\in J_n^0(d_0)} \int_F \mathbb{P}_x(\mathbf{i}^{-1}) \mathbf{1}_{S_{\mathbf{j}^{-1}}(F)}(S_{\mathbf{i}^{-1}}(x))\, \mu_*(dx) \\ &= \int_F \mathbb{P}_x(\mathbf{j}^{-1})\, \mu_*(dx) \le \frac{q_0^n}{\beta}, \end{aligned}$$

because $S_{\mathbf{i}^{-1}}(F) \cap S_{\mathbf{j}^{-1}}(F) = \emptyset$ for $\mathbf{i} \ne \mathbf{j}$, $\mathbf{i}, \mathbf{j} \in J_n^0(d_0)$. Define $\varepsilon_n = d_0^n\sigma/2$ for $n \ge n_*$, where $\sigma > 0$ is given by (6.4.37). By (6.4.38) every ball B with radius ε_n meets at most one set $S_{\mathbf{i}^{-1}}(F)$ for $\mathbf{i} \in J_n^0(d_0)$. Since $\mu_*(\bigcup_{\mathbf{i}\in J_n(d_0)} S_{\mathbf{i}^{-1}}(F)) = 1$ it follows that to cover a set of μ_*-measure greater than or equal to $1-\eta$ (with $\eta \le \beta\gamma/4$) we need at least $\beta^2\gamma q_0^{-n}/4$ balls with radius ε_n. Thus $N(\varepsilon_n, \eta) \ge \beta^2\gamma q_0^{-n}/4$ for $n \ge n_*$. Consequently,

$$\liminf_{\varepsilon\to 0} \frac{\log N(\varepsilon,\eta)}{-\log\varepsilon} \ge \liminf_{n\to\infty} \frac{\log(\beta^2\gamma q_0^{-n}/4)}{-\log\varepsilon_n} = \frac{\log q_0}{\log d_0}.$$

Thus

$$\underline{\mathrm{Cap}}_L(\mu_*) \ge \frac{\log q_0}{\log d_0}$$

and letting $q_0 \to q$ and $d_0 \to d$ we conclude that

$$\underline{\mathrm{Cap}}_L(\mu_*) \geq \frac{\log q}{\log d}.$$

Suppose now that some of the l_i's are equal to 1. Choose $\bar{l}_i < l_i$, $i = 1, \ldots, N$. Since

$$\sup_{x\in X} \prod_{i=1}^{N} \bar{l}_i^{p_i(x)} \to \sup_{x\in X} \prod_{i=1}^{N} l_i^{p_i(x)}$$

as $\bar{l}_i \to l_i$ for $i \in \{1, \ldots, N\}$, our assertion follows. ∎

In the case when $S_i : X \to X$ are similarities and $p_i : X \to \mathbb{R}_+$, $i = 1, \ldots, N$, are constants, Theorems 6.4.1 and 6.4.3 may be summarized in the following way (see also [21]).

THEOREM 6.4.4. *Let $p_i : X \to \mathbb{R}_+$, $i = 1, \ldots, N$, be constants. Assume that*

$$\varrho(S_i(x), S_i(y)) = L_i \varrho(x, y) \quad \text{for } x, y \in X \text{ and } i = 1, \ldots, N.$$

Moreover, assume that $S_1, \ldots, S_N$ satisfy the strong Moran condition. If

$$\prod_{i=1}^{N} L_i^{p_i} < 1,$$

then the unique invariant distribution μ_* for $(S, p)_N$ satisfies

$$\mathrm{Cap}_L(\mu_*) = \frac{\sum_{i=1}^{N} p_i \log p_i}{\sum_{i=1}^{N} p_i \log L_i}.$$

Proof. The upper estimate of capacity follows immediately from Theorem 6.4.1 and the lower estimate follows from Theorem 6.4.3. ∎

7. Stochastically perturbed dynamical systems

7.1. Introduction. In this section we study the stochastically perturbed dynamical system

$$x_{n+1} = S(x_n, t_n) \quad \text{for } n = 1, 2, \ldots$$

Assume that the function $S : X \times [0, T] \to X$ is Borel measurable and the t_n are random variables with values in $[0, T]$ such that

$$\mathrm{prob}(t_n < t \,|\, x_n = x) = \int_0^t p(x, u)\, du \quad \text{for } 0 \leq t \leq T,$$

where $p : X \times [0, T] \to \mathbb{R}_+$ is a Borel measurable and normalized function. We are going to derive a recurrence relation between μ_{n+1} and μ_n, where μ_n is the distribution of x_n. Let $h : X \to \mathbb{R}$ be an arbitrary bounded Borel measurable function. The mathematical expectation of $h(x_{n+1})$ is given by

$$\mathbb{E}(h(x_{n+1})) = \int_X h(x)\, \mu_{n+1}(dx).$$

On the other hand, we have

$$\mathbb{E}(h(x_{n+1})) = \mathbb{E}(h(S(x_n, t_n))) = \int_X \Big\{ \int_0^T h(S(x,t))p(x,t)\,dt \Big\} \mu_n(dx).$$

Thus if we set $h = \mathbf{1}_A$, $A \in \mathcal{B}(X)$, we obtain

$$\mu_{n+1}(A) = \int_X \Big\{ \int_0^T \mathbf{1}_A(S(x,t))p(x,t)\,dt \Big\} \mu_n(dx),$$

which is the desired recurrence relation between μ_{n+1} and μ_n. If we define an operator P by

$$P\mu(A) = \int_X \Big\{ \int_0^T \mathbf{1}_A(S(x,t))p(x,t)\,dt \Big\} \mu(dx) \quad \text{for } A \in \mathcal{B}(X), \tag{7.1.1}$$

the above equation may be rewritten as

$$\mu_{n+1} = P\mu_n.$$

A straightforward calculation shows that P is a Feller operator and its adjoint is of the form

$$Uf(x) = \int_0^T f(S(x,t))p(x,t)\,dt \quad \text{for } f \in B(X). \tag{7.1.2}$$

We make the following assumptions which we assume to hold throughout this section:

(i) The function $S : X \times [0,T] \to X$ is continuous.

(ii) The function $p : X \times [0,T] \to \mathbb{R}_+$ is a lower semi-continuous, normalized function, i.e.,

$$\int_0^T p(x,t)\,dt = 1 \quad \text{for } x \in X. \tag{7.1.3}$$

(iii) The continuity condition, i.e.,

$$\int_0^T |p(x,t) - p(y,t)|\,dt \le \omega(\varrho(x,y)) \quad \text{for } x, y \in X, \tag{7.1.4}$$

where $\omega \in \Phi_0$.

(iv) The average contractivity condition, i.e.,

$$\int_0^T \varrho(S(x,t), S(y,t))p(x,t)\,dt \le r(\varrho(x,y)) \quad \text{for } x, y \in X, \tag{7.1.5}$$

where $r \in \Phi_0$.

7.2. Nonexpansiveness. We start with an easy lemma.

LEMMA 7.2.1. *Let $S : X \times [0,T] \to X$ and $p : X \times [0,T] \to \mathbb{R}_+$ satisfy conditions* (7.1.4) *and* (7.1.5) *with ω, r such that the conditions formulated in one of cases* I–III *of Section* 6.2 *hold. Then the operator P given by* (7.1.1) *is essentially nonexpansive.*

Proof. By Theorem 3.1 it is enough to show that there exists a function $\varphi \in \Phi$ such that $U(\mathcal{F}_\varphi) \subset \mathcal{F}_\varphi$, where $\mathcal{F}_\varphi$ denotes the family of all continuous functions f such that $|f(x)| \le 1$ and $|f(x) - f(y)| \le \varphi(\varrho(x,y))$ for all $x, y \in X$.

Since ω and r satisfy the conditions in one of cases I–III, there exists $\varphi \in \Phi$ such that

$$\omega(t) + \varphi(r(t)) \le \varphi(t) \quad \text{for } t \ge 0. \tag{7.2.1}$$

Fix $f \in \mathcal{F}_\varphi$. Since $|f(x)| \le 1$ for $x \in X$ and p is a normalized function, we have

$$|Uf(x)| \le \int_0^T |f(S(x,t))| p(x,t)\,dt \le \int_0^T p(x,t)\,dt = 1 \quad \text{for } x \in X.$$

Further, for all $x, y \in X$ we have

$$\begin{aligned}|Uf(x) - Uf(y)| &\le \int_0^T |f(S(x,t))p(x,t) - f(S(y,t))p(y,t)|\,dt \\ &\le \int_0^T |f(S(y,t))|\,|p(x,t) - p(y,t)|\,dt + \int_0^T |f(S(x,t)) - f(S(y,t))| p(y,t)\,dt\end{aligned}$$

and consequently by (7.1.4) we obtain

$$|Uf(x) - Uf(y)| \le \omega(\varrho(x,y)) + \int_0^T \varphi(\varrho(S(x,t), S(y,t)))p(y,t)\,dt.$$

By Jensen's inequality and (7.1.5) we finally obtain

$$|Uf(x) - Uf(y)| \le \omega(\varrho(x,y)) + \varphi(r(\varrho(x,y))) \quad \text{for } x, y \in X.$$

Therefore $Uf \in \mathcal{F}_\varphi$ by (7.2.1). ■

7.3. Invariant measures. We start with the following lemma.

LEMMA 7.3.1. *Let the assumptions of Lemma* 7.2.1 *hold. Then*

$$\{\varepsilon > 0 : \inf_{\mu \in \mathcal{M}_1} \liminf_{n\to\infty} P^n\mu(B) > 0 \text{ for some } B \in \mathcal{C}_\varepsilon\} \ne \emptyset. \tag{7.3.1}$$

Proof. Fix $x_0 \in X$ and define $V(x) = \varrho(x, x_0)$ for $x \in X$. Obviously V is a Lyapunov function, bounded on bounded sets. Then by (7.1.5) we have

$$\begin{aligned}UV(x) &= \int_0^T V(S(x,t))p(x,t)\,dt = \int_0^T \varrho(S(x,t), x_0)p(x,t)\,dt \\ &\le \int_0^T \varrho(S(x,t), S(x_0,t))p(x,t)\,dt + \int_0^T \varrho(S(x_0,t), x_0)p(x,t)\,dt \\ &\le r(\varrho(x,x_0)) + \sup_{t\in[0,T]} \varrho(S(x_0,t), x_0) \quad \text{for } x \in X.\end{aligned}$$

Since r is a concave function and $r(0) = 0$, we obtain

$$UV(x) \le r(1)\varrho(x,x_0) + r(1) + \sup_{t\in[0,T]} \varrho(S(x_0,t), x_0).$$

If the function r satisfies the conditions in one of cases I–III of Section 6.2, then $r(1) < 1$. Finally from Lemma 2.4.2 and Corollary 2.4.1 it follows that there exists $B \in \mathcal{B}_b(X)$ such that

$$\liminf_{n\to\infty} P^n\mu(B) \ge 1/2 \quad \text{for } \mu \in \mathcal{M}_1. \ \blacksquare \tag{7.3.2}$$

LEMMA 7.3.2. *Let the assumptions of Lemma* 7.2.1 *hold. Then*

$$\inf\{\varepsilon>0:\inf_{\mu\in\mathcal{M}_1}\liminf_{n\to\infty}P^n\mu(B)>0 \text{ for some } B\in\mathcal{C}_\varepsilon\}=0 \tag{7.3.3}$$

and consequently P *is semi-concentrating.*

Proof. From Lemma 7.3.1 it follows that (7.3.1) holds. Set

$$d=\inf\{\varepsilon>0:\inf_{\mu\in\mathcal{M}_1}\liminf_{n\to\infty}P^n\mu(B)>0 \text{ for some } B\in\mathcal{C}_\varepsilon\}.$$

Suppose, contrary to our claim, that $d>0$. Since the function r satisfies the conditions in one of cases I–III, we may choose $\varepsilon>d$ such that $r(\varepsilon)<d$. Choose $\eta\in(r(\varepsilon),d)$. Since $\varepsilon>d$, we may find $\{x_1,\ldots,x_m\}\subset X$ and $\alpha>0$ such that

$$\liminf_{n\to\infty}P^n\mu\Big(\bigcup_{i=1}^m B(x_i,\varepsilon)\Big)>\alpha \quad \text{for } \mu\in\mathcal{M}_1. \tag{7.3.4}$$

Set

$$B_\varepsilon=\bigcup_{i=1}^m B(x_i,\varepsilon),\qquad C_\eta=\bigcup_{0\le t\le T}\bigcup_{i=1}^m B(S(x_i,t),\eta).$$

Fix $\mu\in\mathcal{M}_1$ and $n_0\in\mathbb{N}$ such that $P^n\mu(B_\varepsilon)\ge\alpha$ for $n\ge n_0$. Fix $i\in\{1,\ldots,m\}$. For every $x\in B(x_i,\varepsilon)$, we estimate $\int_{T_{x,i}}p(x,t)\,dt$, where $T_{x,i}=\{t\in[0,T]:\varrho(S(x,t),S(x_i,t))\le\eta\}$. We have

$$\int\limits_{[0,T]\setminus T_{x,i}}\eta\cdot p(x,t)\,dt\le\int_0^T\varrho(S(x,t),S(x_i,t))p(x,t)\,dt\le r(\varrho(x,x_i))\le r(\varepsilon).$$

Hence

$$\int\limits_{[0,T]\setminus T_{x,i}}p(x,t)\,dt\le\frac{r(\varepsilon)}{\eta}$$

and from (7.1.3) it follows that

$$\int\limits_{T_{x,i}}p(x,t)\,dt\ge\frac{\eta-r(\varepsilon)}{\eta}.$$

Therefore for $n\ge n_0$ we have

$$\begin{aligned}P^{n+1}\mu(C_\eta)&=\int_X\Big\{\int_0^T\mathbf{1}_{C_\eta}(S(x,t))p(x,t)\,dt\Big\}P^n\,\mu(dx)\\&\ge\int_{B_\varepsilon}\Big\{\int_0^T\mathbf{1}_{C_\eta}(S(x,t))p(x,t)\,dt\Big\}P^n\,\mu(dx)\\&\ge\frac{\eta-r(\varepsilon)}{\eta}\cdot P^n\mu(B_\varepsilon)\ge\frac{\eta-r(\varepsilon)}{\eta}\,\alpha.\end{aligned}$$

Since $\mu\in\mathcal{M}_1$ is arbitrary, we finally obtain

$$\inf_{\mu\in\mathcal{M}_1}\liminf_{n\to\infty}P^n\mu(C_\eta)\ge\frac{\eta-r(\varepsilon)}{\eta}\,\alpha.$$

Observe that $C_\eta\in\mathcal{C}_\gamma$ for every $\gamma>\eta$, which, since $\eta<d$, contradicts the definition of d. ■

As a consequence of Lemmas 7.2.1, 7.3.1 and 7.3.2 we obtain the following theorem.

THEOREM 7.3.1. *Let* $S : X \times [0,T] \to X$ *and* $p : X \times [0,T] \to \mathbb{R}_+$ *satisfy conditions* (7.1.4) *and* (7.1.5) *with* ω, r *such that the conditions formulated in one of cases* I–III *of Section* 6.2 *hold. Then the operator* P *given by* (7.1.1) *has an invariant distribution. Moreover, the operator* P *and the set*

$$\widehat{\Omega} = \bigcup_{\mu \in \mathcal{M}_1} \Omega(\mu), \tag{7.3.5}$$

where $\Omega(\mu)$ *is given by* (2.3.8), *are tight.*

Proof. From Lemmas 7.2.1, 7.3.1 and 7.3.2 it follows that P is essentially nonexpansive and semi-concentrating. A simple application of Theorem 5.5 (see also Remark 5.1) finishes the proof. ∎

THEOREM 7.3.2. *Let* $S : X \times [0,T] \to X$ *and* $p : X \times [0,T] \to \mathbb{R}_+$ *satisfy conditions* (7.1.4) *and* (7.1.5) *with* ω, r *such that the conditions formulated in one of cases* I–III *of Section* 6.2 *hold. Assume that for every* $x \in X$ *there exists* $\tau_x \in [0,T)$ *such that*

$$p(x,t) = 0 \quad \text{for } 0 \le t \le \tau_x, \qquad p(x,t) > 0 \quad \text{for } \tau_x < t \le T.$$

Then the operator P *given by* (7.1.1) *is asymptotically stable.*

Proof. By Theorem 7.3.1, P has an invariant distribution. Therefore it remains to verify that

$$\lim_{n\to\infty} \|P^n\mu_1 - P^n\mu_2\|_{\mathrm{FM}} = 0 \qquad \text{for } \mu_1,\mu_2 \in \mathcal{M}_1. \tag{7.3.6}$$

Since P is essentially nonexpansive, it is enough to show that for every $\varepsilon > 0$ there is $\alpha > 0$ having the following property: for every $\mu_1,\mu_2 \in \mathcal{M}_1$ there exist $A \in \mathcal{B}_b(X)$ with $\operatorname{diam} A \le \varepsilon$ and $n_0 \in \mathbb{N}$ such that

$$P^{n_0}\mu_i(A) > \alpha \qquad \text{for } i = 1,2. \tag{7.3.7}$$

Fix $\varepsilon > 0$. According to Theorem 7.3.1 there is a compact set $K \subset X$ such that

$$\widetilde{\mu}(K) \ge 4/5 \qquad \text{for } \widetilde{\mu} \in \widehat{\Omega}. \tag{7.3.8}$$

Choose $m \in \mathbb{N}$ such that $r^m(\operatorname{diam} K) \le \varepsilon/3$. For every pair (x,y), $x,y \in K$, we define open neighbourhoods $U_{x,y}$ and $V_{x,y}$ of x and y, respectively. Suppose first that $\tau_x \ge \tau_y$. From (7.1.5) it follows that there exists $\bar{t}_1 \in (\tau_x, T]$ such that

$$\varrho(S(x,\bar{t}_1), S(y,\bar{t}_1)) \le r(\varrho(x,y)).$$

Moreover, since $\bar{t}_1 \in (\tau_x,T] \subset (\tau_y,T]$ we have

$$p(x,\bar{t}_1) > 0, \qquad p(y,\bar{t}_1) > 0.$$

If $\tau_y \ge \tau_x$ we may obtain the same result by first choosing an appropriate number $\bar{t}_1 \in (\tau_y, T]$. Thus by induction for every pair (x,y) we may construct a sequence $(\bar{t}_1,\ldots,\bar{t}_m)$, $\bar{t}_i = \bar{t}_i(x,y)$, $i = 1,\ldots,m$, such that

$$\varrho(S_m(x,\bar{t}_1,\ldots,\bar{t}_m), S_m(y,\bar{t}_1,\ldots,\bar{t}_m)) \le r^m(\varrho(x,y))$$

and

$$p_m(x,\bar{t}_1,\ldots,\bar{t}_m) > 0, \qquad p_m(y,\bar{t}_1,\ldots,\bar{t}_m) > 0,$$

where the functions S_i, $i = 1, 2, \ldots$, are defined by the recurrence relations

$$\begin{aligned} S_1(x, t_1) &= S(x, t_1), \\ S_{i+1}(x, t_1, \ldots, t_{i+1}) &= S(S_i(x, t_1, \ldots, t_i), t_{i+1}), \\ p_i(x, t_1, \ldots, t_i) &= p(x, t_1) \ldots p(S_{i-1}(x, t_1, \ldots, t_{i-1})) \end{aligned}$$

for $x \in X$, $t_1, \ldots, t_{i+1} \in [0, T]$, $i = 1, 2, \ldots$ Fix $x \in X$. By the continuity of S and the lower semi-continuity of p for every $y \in K$ there exist neighbourhoods $U_{x,y}$ of x, $V_{x,y}$ of y and positive numbers $\delta = \delta(x, y)$, $\sigma = \sigma(x, y)$ such that

$$\varrho(S_m(x, \bar{t}_1, \ldots, \bar{t}_m), S_m(u, t_1, \ldots, t_m)) \le \varepsilon/3, \tag{7.3.9}$$

$$p_m(u, t_1, \ldots, t_m) \ge \sigma(x, y) \tag{7.3.10}$$

for $u \in U_{x,y}$, $|t_i - \bar{t}_i| \le \delta(x, y)$, $i = 1, \ldots, m$, and analogously

$$\varrho(S_m(y, \bar{t}_1, \ldots, \bar{t}_m), S_m(v, t_1, \ldots, t_m)) \le \varepsilon/3, \tag{7.3.11}$$

$$p_m(v, t_1, \ldots, t_m) \ge \sigma(x, y) \tag{7.3.12}$$

for $v \in V_{x,y}$, $|t_i - \bar{t}_i| \le \delta(x, y)$, $i = 1, \ldots, m$. Since K is compact and $K \subset \bigcup_{y \in K} V_{x,y}$, there exists $\{y_1, \ldots, y_{q(x)}\} \subset K$ such that

$$K \subset \bigcup_{i=1}^{q(x)} V_{x,y_i}. \tag{7.3.13}$$

Set $U_x = \bigcap_{i=1}^{q(x)} U_{x,y_i}$ and observe that U_x is an open neighbourhood of x. Therefore there exists a subset $\{x_1, \ldots, x_p\} \subset K$ such that

$$K \subset \bigcup_{i=1}^{p} U_{x_i}. \tag{7.3.14}$$

From (7.3.13) and (7.3.14) it follows that

$$\mathcal{G} = \{U_{x_i} \cap V_{x_1,y_{j_1}} \cap \ldots \cap V_{x_p,y_{j_p}} : 1 \le i \le p,\ 1 \le j_k \le q(x_k) \text{ for } k = 1, \ldots, p\}$$

covers K. Denote by M its cardinality and let $G = \bigcup \mathcal{G}$. Set

$$\delta = \min_{1 \le i \le p} \big(\min_{1 \le j \le q(x_i)} \delta(x_i, y_j) \big), \qquad \sigma = \min_{1 \le i \le p} \big(\min_{1 \le j \le q(x_j)} \sigma(x_i, y_j) \big).$$

We are going to show that P satisfies (7.3.7) with $\alpha = \sigma\delta^m/(2M)$. In fact, let $\mu_1, \mu_2 \in \mathcal{M}_1$. Set $\mu_0 = (\mu_1 + \mu_2)/2$. According to Theorem 7.3.1 there exists a measure $\overline{\mu} \in \Omega(\mu_0)$ and a sequence $(m_n)_{n \ge 1}$ such that

$$\lim_{n \to \infty} \|P^{m_n}\mu_0 - \overline{\mu}\|_{\mathrm{FM}} = 0. \tag{7.3.15}$$

Since (7.3.15) is equivalent to the weak convergence of $(P^{m_n}\mu_0)_{n \ge 1}$ to $\overline{\mu}$ and G is open, the Aleksandrov theorem (see [6]) implies

$$\liminf_{n \to \infty} P^{m_n}\mu_0(G) \ge \overline{\mu}(G).$$

From this and (7.3.8) it follows that there exists an integer n_0 such that

$$P^{n_0}\mu_0(G) = (P^{n_0}\mu_1(G) + P^{n_0}\mu_2(G))/2 > 3/4.$$

Hence

$$P^{n_0}\mu_i(G) > 1/2 \quad \text{for } i = 1, 2.$$

Consequently, there exist $U_1, U_2 \in \mathcal{G}$ such that

$$P^{n_0}\mu_i(U_i) > 1/(2M) \quad \text{for } i = 1, 2.$$

From the definition of $\mathcal{G}$ it follows that $U_1 \subset U_{x_i}$ and $U_2 \subset V_{x_i, y_{j_i}}$ for some $i \in \{1, \ldots, q\}$ and $j_i \in \{1, \ldots, q(x_i)\}$. The definition of U_x and $V_{x,y}$ implies that there exists a sequence $(\bar{t}_1, \ldots, \bar{t}_m)$ such that

$$\varrho(S_m(x_i, \bar{t}_1, \ldots, \bar{t}_m), S_m(u, t_1, \ldots, t_m)) \le \varepsilon/3,$$
$$p_m(u, t_1, \ldots, t_m) \ge \sigma$$

for $u \in U_{x_i}$ and $|t_i - \bar{t}_i| \le \delta$,

$$\varrho(S_m(y_{j_i}, \bar{t}_1, \ldots, \bar{t}_m), S_m(v, t_1, \ldots, t_m)) \le \varepsilon/3,$$
$$p_m(v, t_1, \ldots, t_m) \ge \sigma$$

for $v \in V_{x_i, y_{j_i}}$ and $|t_i - \bar{t}_i| \le \delta$, and

$$\varrho(S_m(x_i, \bar{t}_1, \ldots, \bar{t}_m), S_m(y_{j_i}, \bar{t}_1, \ldots, \bar{t}_m)) \le \varepsilon/3.$$

Now define $A = A_1 \cup A_2$, where

$$A_1 = \mathrm{cl}\{S_m(u, t_1, \ldots, t_m) : u \in U_{x_i},\ |t_i - \bar{t}_i| \le \delta,\ i = 1, \ldots, m\},$$
$$A_2 = \mathrm{cl}\{S_m(v, t_1, \ldots, t_m) : v \in V_{x_i, y_{j_i}},\ |t_i - \bar{t}_i| \le \delta,\ i = 1, \ldots, m\}$$

and observe that $\operatorname{diam} A \le \varepsilon$. Set $n = n_0 + m$. By induction we have

$$P^n\mu_i(A) \ge \int_X \Big\{ \int_0^T \ldots \int_0^T \mathbf{1}_{A_i}(S_m(u, t_1, \ldots, t_m)) p_m(u, t_1, \ldots, t_m)\, dt_1 \ldots dt_m \Big\} P^{n_0}\mu_i(du)$$

for $i = 1, 2$. Therefore, by the definition of A_1 and A_2 we obtain $P^{n_0}\mu_i(A) > \sigma\delta^m/(2M) = \alpha$ for $i = 1, 2$. ∎

Studying Ważewska's equation (see [37]) we obtain a stochastically perturbed dynamical system such that

(7.3.16) $$S : X \times [0, 1] \to X \quad \text{is continuous,}$$

(7.3.17) $$p(x, t) = 1 \quad \text{for } x \in X,\ t \in [0, 1].$$

Moreover, there exist constants $l, L \in (0, 1)$ and $a > 0$ such that

(7.3.18) $$l\varrho(x, y) \le \varrho(S(x, t), S(y, t)) \le L\varrho(x, y) \quad \text{for } x, y \in X,\ t \in [0, 1],$$

(7.3.19) $$a|t - t'| \le \varrho(S(x, t), S(y, t')) \quad \text{for } x, y \in X \text{ and } t, t' \in [0, 1].$$

THEOREM 7.3.3. *Let $S : X \times [0, 1] \to X$ and $p : X \times [0, 1] \to \mathbb{R}_+$ satisfy conditions* (7.3.16)–(7.3.19). *Then the Markov operator P given by* (7.1.1) *is asymptotically stable.*

Proof. We easily check that the assumptions of Theorem 7.3.2 are satisfied. ∎

7.4. Capacity of invariant measures. We end this chapter with the following result concerning the capacity of an invariant measure.

THEOREM 7.4.1. *Let* $S : X \times [0,1] \to X$ *and* $p : X \times [0,1] \to \mathbb{R}_+$ *satisfy conditions* (7.3.16)–(7.3.19) *and let* μ_* *be the unique invariant distribution for the Markov operator* P *given by* (7.1.1). *Then*

$$\mathrm{Cap}_L(\mu_*) = \infty.$$

Proof. Fix $x \in X$. We are going to estimate $\mu_*(B(x, al^m/2))$, $m \in \mathbb{N}$, where the constants l, a are given by (7.3.18), (7.3.19), respectively. Fix $m \in \mathbb{N}$. By the invariance property of μ_* we obtain

$$\begin{aligned}(7.4.1)\qquad \mu_*(B(x, al^m/2)) &= P^m\mu_*(B(x, al^m/2)) \\ &= \int_X \Big\{ \int_0^1 \ldots \int_0^1 \mathbf{1}_{B(x,al^m/2)}(S_m(u, t_1, \ldots, t_m))\, dt_1 \ldots dt_m \Big\} \mu_*(du),\end{aligned}$$

where S_m is defined in the proof of Theorem 7.3.2. Let $(\bar{t}_1, \ldots, \bar{t}_m)$, $\bar{t}_1, \ldots, \bar{t}_m \in [0,1]$, be such that there exists $y \in X$ such that

$$(7.4.2)\qquad S_m(y, \bar{t}_1, \ldots, \bar{t}_m) \cap B(x, al^m/2) \neq \emptyset.$$

If

$$(7.4.3)\qquad S_k(z, \bar{t}_1, \ldots, \bar{t}_{k-1}, t_k) \cap B(x, al^m/2) \neq \emptyset$$

for some $z \in X$ and $1 \le k \le m$, then $|\bar{t}_k - t_k| \le l^{m-k+1}$. Suppose, contrary to our claim, that $|\bar{t}_k - t_k| > l^{m-k+1}$ for some k, $1 \le k \le m$. Then by (7.3.19) we obtain

$$\varrho(S_k(u, \bar{t}_1, \ldots, \bar{t}_{k-1}, \bar{t}_k), S_k(z, \bar{t}_1, \ldots, \bar{t}_{k-1}, t_k)) > al^{k-1} \cdot l^{m-k+1} = al^m$$

for $u \in X$. Therefore

$$\varrho(S_m(y, \bar{t}_1, \ldots, \bar{t}_m), S_k(z, \bar{t}_1, \ldots, \bar{t}_{k-1}, t_k)) > al^m,$$

which contradicts (7.4.2) and (7.4.3). From the above and (7.4.1) it follows that

$$\mu_*(B(x, al^m/2)) \le 2^m \prod_{k=1}^m l^{m-k+1} = (2 \cdot l^{(m+1)/2})^m.$$

Fix $\eta > 0$ and observe that

$$N(al^m/2, \eta) = \inf\{N_C(al^m/2) : C \subset X,\ \mu_*(C) \ge 1 - \eta\} \ge [(1-\eta)(2 \cdot l^{(m+1)/2})^{-m}].$$

(We use $[a]$ to denote the integer part of a.) Hence

$$\underline{\mathrm{Cap}}_L(\mu_*) \ge \liminf_{m\to\infty} \frac{\log N(al^m/2, \eta)}{-\log(al^m/2)} = \infty$$

and consequently $\mathrm{Cap}_L(\mu_*) = \infty$. ■

8. Poisson driven stochastic differential equations

8.1. Introduction. In the last chapter we consider a stochastic differential equation of the form

$$(8.1.1)\qquad d\xi(t) = a(\xi(t))dt + \int_\Theta \sigma(\xi(t), \theta)\mathcal{N}_p(dt, d\theta)$$

for $t \ge 0$ with the initial condition

$$\xi(0) = \xi_0, \tag{8.1.2}$$

where $(\xi(t))_{t\ge 0}$ is a stochastic process with values in a separable Banach space $(X, \|\cdot\|)$. We make the following five assumptions:

(i) The coefficient $a : X \to X$ is Lipschitzian:

$$\|a(x) - a(y)\| \le l_a \|x - y\| \quad \text{for } x, y \in X.$$

(ii) $(\Theta, \mathcal{G}, \widetilde{p})$ is a finite measure space with $\widetilde{p}(\Theta) = 1$.

(iii) The perturbation coefficient $\sigma : X \times \Theta \to X$ is a $\mathcal{B}(X) \times \mathcal{G}/\mathcal{B}(X)$-measurable function such that

$$\|\sigma(x,\cdot) - \sigma(y,\cdot)\|_{L^2(\widetilde{p})} \le l_\sigma \|x - y\| \quad \text{for } x, y \in X.$$

(iv) There are given a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mathbb{P}})$, a sequence $(t_n)_{n\ge 0}$ of nonnegative random variables and a sequence $(\theta_n)_{n\ge 1}$ of random elements with values in Θ. The variables $\Delta t_n = t_n - t_{n-1}$ $(t_0 = 0)$ are nonnegative, independent and equally distributed with density $\lambda e^{-\lambda t}$ for $t \ge 0$. The elements θ_n are independent, equally distributed with distribution $\widetilde{p}$. The sequences $(t_n)_{n\ge 0}$ and $(\theta_n)_{n\ge 1}$ are also independent. Under this condition the mapping

$$\widetilde{\Omega} \ni \widetilde{\omega} \mapsto p(\widetilde{\omega}) = (t_n(\widetilde{\omega}), \theta_n(\widetilde{\omega}))_{n\ge 1}$$

defines a stationary Poisson point process.

(v) For every $\mu \in \mathcal{M}_1$ there is an X-valued random vector ξ_μ defined on $\widetilde{\Omega}$, independent of p and having the distribution μ.

Condition (iv) implies that for every measurable set $Z \subset (0, \infty) \times \Theta$ the variable

$$\mathcal{N}_p(Z) = \operatorname{card}\{n : (t_n, \theta_n) \in Z\}$$

is Poisson distributed. It is called the *Poisson random counting measure*.

It is easy to show that

$$\mathbb{E}(\mathcal{N}_p((0,t] \times K)) = \lambda t \widetilde{p}(K) \quad \text{for } t \in (0,\infty),\ K \in \mathcal{G},$$

where $\mathbb{E}$ denotes the expectaction on the probability space $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mathbb{P}})$.

By a solution of (8.1.1), (8.1.2) we mean a process $(\xi(t))_{t\ge 0}$ with values in X such that with probability one the following two conditions hold:

(a) The sample path is a right-continuous function such that for $t > 0$ the limit $\xi(t-) = \lim_{s\to t,\, s<t} \xi(s)$ exists and

(b) $\xi(t) = \xi_0 + \int_0^t a(\xi(s))\,ds + \int_0^t \int_\Theta \sigma(\xi(s-), \theta) \mathcal{N}_p(ds, d\theta)$ for $t \ge 0$.

It is easy to write the explicit formula for the solution of (8.1.1), (8.1.2). Consider the ordinary differential equation

$$y'(t) = a(y(t)) \quad \text{for } t \ge 0 \tag{8.1.3}$$

and denote by $y(t) = S(t,x)$, $t \in \mathbb{R}_+$, the solution of (8.1.3) satisfying the initial condition $y(0) = x$. Then for every fixed value of $p = (t_n, \theta_n)_{n\ge 1}$ the solution is given by

$$\begin{aligned} &\xi(0) = \xi_0, \qquad \xi(t_n) = \xi(t_n-) + \sigma(\xi(t_n-), \theta_n), \\ &\xi(t) = S(t - t_n, \xi(t_n)) \qquad \text{for } t \in [t_n, t_{n+1}),\ n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{8.1.4}$$

For $x \in X$ denote by $\xi_x(t)$ the solution of the initial value problem (8.1.1), (8.1.2) with $\xi_0 = x$. Then for every $t \ge 0$ and $f \in C(X)$ define

$$U^t f(x) = \mathbb{E}(f(\xi_x(t))). \tag{8.1.5}$$

The classical theory of equation (8.1.1) (see [22, 26]) ensures that $(\xi_x(t))_{t\ge 0}$ is a Markov process homogeneous in time and $(U^t)_{t\ge 0}$ is a continuous semigroup of bounded linear operators acting on $C(X)$. Obviously, this semigroup may be extended to all bounded Borel measurable functions. We check at once that the operator U^t for $t \ge 0$ satisfies conditions (2.2.3)–(2.2.5). Thus for every $t \ge 0$ there exists an operator $P^t : \mathcal{M}_{\text{fin}} \to \mathcal{M}_{\text{fin}}$ satisfying the duality condition

$$\langle f, P^t\mu\rangle = \langle U^t f, \mu\rangle \qquad \text{for } f \in B(X),\ \mu \in \mathcal{M}_{\text{fin}}.$$

Since $(U^t)_{t\ge 0}$ is a semigroup on $B(X)$, the duality condition shows that $(P^t)_{t\ge 0}$ is a semigroup on $\mathcal{M}_{\text{fin}}$.

8.2. Nonexpansiveness. For every $q \in (0,1]$ we introduce in the Banach space $(X, \|\cdot\|)$ the new metric

$$\varrho_q(x,y) = \|x-y\|^q \qquad \text{for } x, y \in X.$$

Obviously, ϱ_q is equivalent to $\varrho_{q'}$ for every $q', q \in (0,1]$. Further, for every $q \in (0,1]$ we introduce the Fortet–Mourier norm

$$\|\nu\|_{q,\text{FM}} = \sup\{|\langle f, \nu\rangle| : f \in \mathcal{F}_q\} \qquad \text{for } \nu \in \mathcal{M}_{\text{sig}},$$

where $\mathcal{F}_q$ consists of all functions f such that $|f(x)| \le 1$ and $|f(x) - f(y)| \le \varrho_q(x,y)$ for all $x, y \in X$. The following theorem provides conditions for essential nonexpansiveness of the operator P^t for $t \ge 0$.

THEOREM 8.2.1. *Let $(X, \|\cdot\|)$ be a separable Banach space. Assume that*

$$\|S(t,x) - S(t,y)\| \le e^{\alpha t}\|x-y\| \qquad \textit{for } x, y \in X,\ t \ge 0, \tag{8.2.1}$$

$$\|\tau(x,\cdot) - \tau(y,\cdot)\|_{L^1(\widetilde{p})} \le l_\tau \|x-y\| \qquad \textit{for } x, y \in X, \tag{8.2.2}$$

where $\tau(x,\theta) = x + \sigma(x,\theta)$. Moreover, assume that

$$l_\tau < \exp(-\alpha/\lambda). \tag{8.2.3}$$

Then there exists $q \in (0,1]$ such that P^t, $t \ge 0$, is nonexpansive with respect to $\|\cdot\|_{q,\text{FM}}$ and

$$\lim_{t\to\infty} \|P^t\mu_1 - P^t\mu_2\|_{q,\text{FM}} = 0 \qquad \textit{for } \mu_1, \mu_2 \in \mathcal{M}_1. \tag{8.2.4}$$

Moreover, for every $A \in \mathcal{B}_b(X)$ the above convergence is uniform over all $\mu_1, \mu_2 \in \mathcal{M}_1^A$.

Proof. By inequality (8.2.3) there exists $q \in (0,1]$ such that

$$q\alpha - \lambda + \lambda l_\tau^q < 0. \tag{8.2.5}$$

From Theorem 3.1 it follows that to finish the proof of nonexpansiveness it is enough to show that $U^t f \in \mathcal{F}_q$ for $f \in \mathcal{F}_q$ and $t > 0$. Further, when we show that for every $A \in \mathcal{B}_b(X)$,

$$\sup_{x,y\in A}\sup_{f\in\mathcal{F}_\gamma}|U^t f(x) - U^t f(y)| \to 0 \tag{8.2.6}$$

as $t \to \infty$, the proof will be finished. Indeed, if (8.2.6) holds, then (8.2.4) is satisfied for $\mu_1 = \delta_x$, $\mu_2 = \delta_y$. Since the linear combinations of point measures are dense in $\mathcal{M}_1$ (in the weak topology and also in the Fortet–Mourier distance $\|\cdot\|_{q,\mathrm{FM}}$) and P^t for $t \geq 0$ is nonexpansive with respect to $\|\cdot\|_{q,\mathrm{FM}}$, (8.2.4) will hold for all $\mu_1, \mu_2 \in \mathcal{M}_1$. Obviously, this convergence will be uniform over all measures supported in the same bounded set.

Define $\widetilde{\Omega}_n(t) = \{\widetilde{\omega} \in \widetilde{\Omega} : t_n(\widetilde{\omega}) \leq t \text{ and } t_{n+1}(\widetilde{\omega}) > t\}$ for $n \in \mathbb{N} \cup \{0\}$ and $t > 0$. Obviously $\widetilde{\mathbb{P}}(\bigcup_{n=0}^{\infty} \widetilde{\Omega}_n(t)) = 1$. Fix $f \in \mathcal{F}_q$, $t > 0$ and $A \in \mathcal{B}_b(X)$. Since the sequences $(t_n)_{n\geq 0}$, $(\theta_n)_{n\geq 1}$ are independent, for $x, y \in A$ we obtain

$$\begin{aligned}
|U^t f(x) - U^t f(y)| &= |\mathbb{E} f(\xi_x(t)) - \mathbb{E} f(\xi_y(t))| \leq \sum_{n=0}^{\infty} \int_{\widetilde{\Omega}_n(t)} \|\xi_x(t)(\widetilde{\omega}) - \xi_y(t)(\widetilde{\omega})\|^q \, \widetilde{\mathbb{P}}(d\widetilde{\omega}) \\
&\leq \sum_{n=0}^{\infty} \exp(q\alpha t) l_\tau^{nq} \|x-y\|^q \, \widetilde{\mathbb{P}}(\widetilde{\Omega}_n(t)) \\
&= \exp(q\alpha t)\|x-y\|^q \sum_{n=0}^{\infty} l_\tau^{qn} \frac{\lambda^n t^n}{n!} \exp(-\lambda t) \\
&= \exp((q\alpha - \lambda + \lambda l_\tau^q)t)\|x-y\|^q.
\end{aligned}$$

From this and (8.2.5) we obtain (8.2.6). Moreover, for every $f \in \mathcal{F}_q$ and $t \geq 0$ we have

$$|U^t f(x) - U^t f(y)| \leq \|x-y\|^q \quad \text{for } x, y \in X.$$

Since $|U^t f(x)| \leq 1$ for $f \in \mathcal{F}_q$ and $x \in X$, we obtain $U^t f \in \mathcal{F}_q$. ∎

8.3. Invariant measures. This section is devoted to the proof of the existence of an invariant measure.

LEMMA 8.3.1. *Let the assumptions of Theorem* 8.2.1 *hold. Then there exists* $A \in \mathcal{B}_b(X)$ *such that*

$$\inf_{\mu\in\mathcal{M}_1} \liminf_{t\to\infty} P^t\mu(A) > 0. \tag{8.3.1}$$

Proof. Choose $q \in (0,1]$ such that

$$q\alpha - \lambda + \lambda l_\tau^q < 0.$$

Set $V(x) = \|x\|^q$ for $x \in X$. Following the proof of inequality (17) in [22, p. 239] it is easy to conclude that $\mathbb{E}\|\xi_x(t)\|^2 < \infty$ for all $x \in X$ and $t \geq 0$. Thus

$$U^t V(0) = \mathbb{E}\|\xi_0(t)\|^q < \infty \quad \text{for all } t \geq 0.$$

Using a similar argument to that in the proof of Theorem 8.2.1 can show that

$$|U^t V(x) - U^t V(0)| \leq \exp((q\alpha - \lambda + \lambda l_\tau^q)t) V(x) \quad \text{for } x \in X \text{ and } t > 0. \tag{8.3.2}$$

Fix $s_0 > 0$. From (8.3.2) it follows that

$$U^{s_0}V(x) \le aV(x) + b \quad \text{for } x \in X, \tag{8.3.3}$$

where

$$a = \exp((q\alpha - \lambda + \lambda l_\tau^q)s_0) < 1, \quad b = U^{s_0}V(0).$$

Therefore, from Corollary 2.4.1 it follows that there exists $A_0 \in \mathcal{B}_b(X)$ such that

$$\liminf_{n\to\infty} (P^{s_0})^n\mu(A_0) > 1/2 \quad \text{for } \mu \in \mathcal{M}_1.$$

Set $A = \mathcal{N}(A_0, 1)$. We will show that

$$\liminf_{t\to\infty} P^t\mu(A) \ge 1/2 \quad \text{for } \mu \in \mathcal{M}_1.$$

Suppose, contrary to our claim, that

$$\liminf_{t\to\infty} P^t\mu(A) < 1/2 \quad \text{for some } \mu \in \mathcal{M}_1.$$

Choose a measure $\mu \in \mathcal{M}_1$ and a sequence $(s_n)_{n\ge1}$, $s_n \to \infty$ as $n \to \infty$, such that

$$\liminf_{n\to\infty} P^{s_n}\mu(A) < 1/2. \tag{8.3.4}$$

Define $m_n = [s_n/s_0]$ and $r_n = s_n - m_n s_0$ for $n \in \mathbb{N}$. Since $r_n \in [0, s_0]$, we may assume that $\lim_{n\to\infty} r_n = r$ for some $r \in [0, s_0]$. Further, by (8.2.4) we have

$$\lim_{n\to\infty} \|P^{m_n s_0 + r}\mu - P^{m_n s_0}\mu\|_{q,\mathrm{FM}} = 0.$$

Since P^t, $t \ge 0$, is nonexpansive with respect to $\|\cdot\|_{q,\mathrm{FM}}$, we obtain

$$\lim_{n\to\infty} \|P^{m_n s_0 + r_n}\mu - P^{m_n s_0 + r}\mu\|_{q,\mathrm{FM}} \le \lim_{n\to\infty} \|P^{r_n}\mu - P^r\mu\|_{q,\mathrm{FM}} = 0.$$

Therefore we have

$$\lim_{n\to\infty} \|P^{m_n s_0 + r_n}\mu - (P^{s_0})^{m_n}\mu\|_{q,\mathrm{FM}} = 0$$

and by Lemma 2.4.1 we obtain $\liminf_{n\to\infty} P^{m_n s_0 + r_n}\mu(A) \ge 1/2$ contrary to (8.3.4). ■

LEMMA 8.3.2. *If the assumptions of Theorem* 8.2.1 *hold*, *then the operator* P^{s_0} *is semi-concentrating for* $s_0 > 0$.

Proof. Fix $\varepsilon > 0$ and choose $\bar{\varepsilon} < \varepsilon$. From Lemma 8.3.1 it follows that there exists $A \in \mathcal{B}_b(X)$ such that (8.3.1) holds. Set

$$\beta_0 = \inf_{\mu\in\mathcal{M}_1} \liminf_{t\to\infty} P^t\mu(A) \tag{8.3.5}$$

and choose $\beta \in (0, \beta_0)$. According to Theorem 8.2.1 there exists $q \in (0, 1]$ such that $(P^t)_{t\ge0}$ is nonexpansive with respect to $\|\cdot\|_{q,\mathrm{FM}}$ and (8.2.4) holds. Fix $s_0 > 0$. Since convergence (8.2.4) is uniform over all $\mu_1, \mu_2 \in \mathcal{M}_1^A$, there exists $m \in \mathbb{N}$ such that

$$\|P^{s_0 m}\mu_1 - P^{s_0 m}\mu_2\|_{q,\mathrm{FM}} \le \bar{\varepsilon}^{2q} \quad \text{for } \mu_1, \mu_2 \in \mathcal{M}_1^A. \tag{8.3.6}$$

Choose $x \in A$. By the Ulam theorem (see [6, 15]) there exists a compact set $K \subset X$ such that

$$P^{s_0 m}\delta_x(K) \ge 1 - \bar{\varepsilon}.$$

From (8.3.6) and Lemma 2.4.1 it follows that

$$P^{s_0 m}\mu(\mathcal{N}_q(K, \bar{\varepsilon}^q)) \ge P^{s_0 m}\delta_x(K) - \bar{\varepsilon}^q \quad \text{for } \mu \in \mathcal{M}_1^A,$$

where $\mathcal{N}_q(K,\overline{\varepsilon}^q)$ denotes the closed $\overline{\varepsilon}^q$-neighbourhood of K in the metric ϱ_q. Therefore

$$P^{s_0 m}\mu(\mathcal{N}(K,\overline{\varepsilon})) \ge 1 - 2\overline{\varepsilon}^q \quad \text{for } \mu \in \mathcal{M}_1.$$

Set $C = \mathcal{N}(K,\overline{\varepsilon})$ and observe that $C \in \mathcal{C}_\varepsilon$. Further, from (8.3.5) it follows that for every $\mu \in \mathcal{M}_1$ there exists $n_0 \in \mathbb{N}$ such that

$$P^{s_0 n}\mu(A) \ge \beta \quad \text{for } n \ge n_0.$$

Fix $\mu \in \mathcal{M}_1$. For $n \ge n_0$ define

$$\mu_n(B) = \frac{P^{s_0 n}\mu(A \cap B)}{P^{s_0 n}\mu(A)} \quad \text{for } B \in \mathcal{B}(X)$$

and observe that

$$P^{s_0 n}\mu \ge \beta\mu_n \quad \text{for } n \ge n_0.$$

From the linearity of P we have

$$P^{s_0(n+m)}\mu \ge \beta P^{s_0 m}\mu_n \quad \text{for } n \ge n_0$$

and since $\mu_n \in \mathcal{M}_1^A$ we obtain

$$P^{s_0(n+m)}\mu(C) \ge \beta P^{s_0 m}\mu_n(C) \ge \beta(1 - 2\overline{\varepsilon}^q) \quad \text{for } n \ge n_0.$$

Therefore $\liminf_{n\to\infty} P^{s_0 n}\mu(C) \ge \beta(1 - 2\overline{\varepsilon}^q)$ for $\mu \in \mathcal{M}_1$. ∎

Combining Theorem 8.2.1 and Lemma 8.3.2 we obtain the following theorem.

THEOREM 8.3.1. *Let $(X, \|\cdot\|)$ be a separable Banach space. Assume that the functions S and τ satisfy conditions* (8.2.1), (8.2.2). *If inequality* (8.2.3) *holds*, *then the semigroup $(P^t)_{t\ge 0}$ has a unique invariant distribution. Moreover*, *$(P^t)_{t\ge 0}$ is asymptotically stable.*

Proof. Fix $s_0 > 0$. By Lemma 8.3.2 the operator P^{s_0} is semi-concentrating. Further, from Theorem 8.2.1 it follows that there exists $q \in (0,1]$ such that P^t, $t \ge 0$, is nonexpansive with respect to $\|\cdot\|_{q,\mathrm{FM}}$. Then by Theorem 5.5 and Remark 5.1, P^{s_0} has an invariant distribution μ_*. From (8.2.4) it follows that it is unique. Then for $t \ge 0$ we have

$$P^{s_0}(P^t\mu_*) = P^t(P^{s_0}\mu_*) = P^t\mu_*.$$

Since μ_* is unique, it follows that $P^t\mu_* = \mu_*$. On the other hand, by (8.2.4) we obtain

$$\lim_{t\to\infty} \|P^t\mu - \mu_*\|_{q,\mathrm{FM}} = \lim_{t\to\infty} \|P^t\mu - P^t\mu_*\|_{q,\mathrm{FM}} = 0 \quad \text{for } \mu \in \mathcal{M}_1. \ ∎$$

8.4. Capacity of invariant measures. Let $\mu \in \mathcal{M}$. Given a point $x \in X$, we call the quantity

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

the *lower pointwise dimension* of μ at x. (Here we assume that $\log 0 = -\infty$.) We start with the following lemma, whose idea is due to A. Lasota and J. Myjak (see [37]).

LEMMA 8.4.1. *Let $\mu \in \mathcal{M}_1$. If for every $\eta \in (0,1)$ and $x \in X$ there exists $r_0 > 0$ such that*

$$\mu(B(x,(3+\eta)r) \ge (2-\eta)\mu(B(x,r)) \quad \textit{for } r \le r_0, \tag{8.4.1}$$

then $\underline{d}_\mu(x) \ge \log 2/\log 3$ for all $x \in X$.

Proof. Fix $\eta \in (0,1)$ and $x \in X$. Let r_0 be such that (8.4.1) holds. Define

$$M = \frac{1}{3+\eta}, \qquad l = \frac{1}{2-\eta}, \qquad s = \frac{\log l}{\log M}, \qquad C = r_0^{-s}.$$

We claim that

$$\mu(B(x,r)) \le Cr^s \quad \text{for } r \in [M^n r_0, r_0],\ n \in \mathbb{N}. \tag{8.4.2}$$

Indeed, for $r = r_0$ we have $\mu(B(x,r_0)) \le Cr_0^s = r_0^{-s} r_0^s = 1$. Suppose now that (8.4.2) holds for some $n \in \mathbb{N}$. Let $r \in [M^{n+1} r_0, M^n r_0)$. Then $r/M \in [M^n r_0, r_0]$ and consequently by (8.4.1) we obtain

$$\mu(B(x,r)) \le l\mu(B(x,r/M)) \le lCr^s/M^s = Cr^s.$$

Thus, by induction (8.4.2) holds for every $n \in \mathbb{N}$. Since $M < 1$, this implies in turn that

$$\mu(B(x,r)) \le Cr^s \quad \text{for } r \in (0, r_0]. \tag{8.4.3}$$

Further, we obtain

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \ge \lim_{r\to 0} \frac{\log(Cr^s)}{\log r} = s.$$

Letting $\eta \to 0$ we finish the proof. ∎

LEMMA 8.4.2. *Let the assumptions of Theorem* 8.2.1 *hold. If* $a(x) \neq 0$ *for* $x \in X$, *then the unique invariant distribution* μ_* *for the semigroup* $(P^t)_{t\ge 0}$ *satisfies*

$$\underline{d}_{\mu_*}(x) \ge \log 2/\log 3 \quad \textit{for } x \in X.$$

Proof. Fix $x \in X$. From the definition of $S(t,\cdot)$ it follows that $t^{-1}[S(t,x) - x] \to a(x)$ as $t \to 0$ and consequently

$$\|S(t,x) - x\| = \|a(x)\| \cdot t + o(t),$$

where $o(t)/t \to 0$ as $t \to 0$. Since $a(x) \neq 0$, we may choose for small r a positive number t_r such that

$$\|S(t_r,x) - x\| = r + r\exp(\alpha t_r), \tag{8.4.4}$$

$$\|S(t,x) - x\| < r + r\exp(\alpha t) \quad \text{for } t < t_r. \tag{8.4.5}$$

As $\|a(x)\| > 0$ we have $\lim_{r\to 0} t_r = 0$. Consider the ball $B(x, r(1 + 2\exp(\alpha t_r)))$ and observe that from (8.2.1) and (8.4.5) it follows that for every $y \in B(x,r)$ we have

$$S(t_r, y) \in B(x, r(1 + 2\exp(\alpha t_r))).$$

Further, from the equality $P^t \mu_* = \mu_*$ for $t \ge 0$ and the definition of $(P^t)_{t\ge 0}$ we obtain

$$\mu_*(B(S(t_r,x), r\exp(\alpha t_r)) = (P^{t_r}\mu_*)(B(S(t_r,x), r\exp(\alpha t_r)) \ge \exp(-\lambda t_r)\mu_*(B(x,r)).$$

From (8.4.4) and (8.4.5) it follows that $B(S(t_r,x), r\exp(\alpha t_r)) \cap B(x,r) = \emptyset$. Hence

$$\mu_*(B(x, r(1 + 2\exp(\alpha t_r)))) \ge (1 + \exp(-\lambda t_r))\mu_*(B(x,r)).$$

Consequently, for every $\eta \in (0,1)$ there exists $r_0 > 0$ such that

$$\mu_*(B(x, (3+\eta)r)) \ge (2-\eta)\mu_*(B(x,r)) \quad \text{for } r \le r_0.$$

From Lemma 8.4.1 it follows that $\underline{d}_{\mu_*}(x) \ge \log 2/\log 3$ for $x \in X$. ∎

THEOREM 8.4.1. *Let the assumptions of Theorem* 8.2.1 *hold. If* $a(x) \neq 0$ *for* $x \in X$, *then the unique invariant distribution* μ_* *for the semigroup* $(P^t)_{t\geq 0}$ *satisfies*

$$\underline{\mathrm{Cap}}_L(\mu_*) \geq \log 2/\log 3.$$

Proof. Fix $d \in (0, \log 2/\log 3)$ and define

$$X_n = \{x \in X : \mu_*(B(x,r)) \leq r^d \text{ for } r \leq 1/n\}.$$

From Lemma 8.4.2 it follows that $\bigcup_{n=1}^{\infty} X_n = X$. Moreover, X_n is Borel measurable and $X_n \subset X_{n+1}$ for $n \in \mathbb{N}$. Therefore $\lim_{n\to\infty} \mu_*(X_n) = \mu_*(X) = 1$ and consequently there exists $n_0 \in \mathbb{N}$ such that $\mu_*(X_{n_0}) > 1/2$. Let $\eta < 1/4$ and let $C \subset X$ be such that $\mu_*(C) > 1-\eta$. Then $\mu_*(C \cap X_{n_0}) > 1/4$. Set $C_0 = C \cap X_{n_0}$. Fix $\varepsilon < 1/(2n_0)$. Note that every ball B_ε with radius ε such that $B_\varepsilon \cap C_0 \neq \emptyset$ satisfies $B_\varepsilon \subset B'_{2\varepsilon}$, where $B'_{2\varepsilon}$ is some ball with radius 2ε and centre in C_0. Since $2\varepsilon < 1/n_0$ and $C_0 \subset X_{n_0}$, we have

$$\mu_*(B_\varepsilon) \leq \mu_*(B'_{2\varepsilon}) \leq (2\varepsilon)^d$$

and consequently

$$N_{C_0}(\varepsilon) \cdot (2\varepsilon)^d \geq 1/4.$$

Since $C \subset X$ with $\mu_*(C) > 1-\eta$ and $\eta < 1/4$ are arbitrary, we obtain

$$N(\varepsilon,\eta) = \inf\{N_C(\varepsilon) : C \subset X \text{ and } \mu(C) > 1-\eta\} \geq (2\varepsilon)^{-d}/4$$

and

$$\underline{\mathrm{Cap}}_L(\mu_*) \geq \lim_{\varepsilon\to 0} \frac{\log((2\varepsilon)^{-d}/4)}{-\log\varepsilon} = d.$$

Letting $d \to \log 2/\log 3$ we finish the proof. ∎

9. Final remarks

We finish the paper with some information concerning references to the literature. All definitions of Section 2.3 can be found in [58, 59, 62]. Lemmas 2.4.1–2.4.5 have also been proved there. Nonexpansiveness has been examined in [59]. However, Theorems 3.1 and 3.2 are an extension of our former results. Further, Theorems 4.1–4.4 are a reformulation of results proved in [59, 62]. Criteria for the existence of an invariant measure (Theorems 4.1–4.4) have been formulated in [58, 59, 62]. Results presented in Section 6.2 have been proved in [58, 62]. Estimates of capacity of self-similar measures (Theorem 6.4.1 and 6.4.3) have been given in [48]. All results devoted to stochastically perturbed dynamical systems are new. In particular Theorem 7.3.2 is an extension of the main theorem of [23]. Finally, Poisson driven differential equations have been studied in [61]. In the proof of the existence of an invariant measure we used the double contraction principle. Here we show that the invariant measure can also be obtained by a simple application of our criterion.

References

[1] M. Arbeiter and N. Patzschke, *Random self-similar multifractals*, Math. Nachr. 181 (1996), 5–42.
[2] M. F. Barnsley and S. Demko, *Iterated function systems and the global construction of fractals*, Proc. Roy. Soc. London A 399 (1985), 243–275.
[3] M. F. Barnsley, S. G. Demko, J. H. Elton and J. S. Geronimo, *Invariant measures arising from iterated function systems with place dependent probabilities*, Ann. Inst. H. Poincaré 24 (1988), 367–394.
[4] M. F. Barnsley, J. Elton and D. Hardin, *Recurrent iterated function systems*, Constr. Approx. 5 (1989), 3–31.
[5] T. Bedford, *Dimension and dynamics for fractal recurrent sets*, J. London Math. Soc. 33 (1986), 98–100.
[6] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York, 1968.
[7] P. M. Centore and E. R. Vrscay, *Continuity of attractors and invariant measures for iterated function systems*, Canad. Math. Bull. 37 (1994), 315–329.
[8] E. Çinlar, *Introduction to Stochastic Processes*, Prentice-Hall, 1975.
[9] C. D. Cutler, *Connecting ergodicity and dimension in dynamical systems*, Ergodic Theory Dynam. Sysems 10 (1990), 451–462.
[10] P. Diaconis and D. Freedman, *Iterated random functions*, to be published.
[11] L. Dubins and D. Freedman, *Invariant probabilities for certain Markov processes*, Ann. Math. Statist. 37 (1966), 837–848.
[12] R. M. Dudley, *Probabilities and Metrics*, Aarhus Universitet 1976.
[13] J. H. Elton, *An ergodic theorem for iterated maps*, Ergodic Theory Dynam. Systems 7 (1987), 481–488.
[14] J. H. Elton and M. Piccioni, *Iterated function systems arising from recursive estimation problems*, Probab. Theory Related Fields 91 (1992), 103–114.
[15] S. Ethier and T. Kurtz, *Markov Processes*, Wiley, New York, 1986.
[16] K. J. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, Wiley, New York, 1990.
[17] —, *The Geometry of Fractal Sets*, Cambridge Univ. Press, Cambridge, 1985.
[18] K. J. Falconer, *Techniques in Fractal Geometry*, Wiley, Chichester, 1997.
[19] W. Feller, *An Introduction to Probability Theory and its Applications*, Wiley, New York, 1981.
[20] R. Fortet et B. Mourier, *Convergence de la répartition empirique vers la répartition théorétique*, Ann. Sci. École Norm. Sup. 70 (1953), 267–285.
[21] J. S. Geronimo and D. P. Hardin, *An exact formula for the measure dimensions associated with a class of piecewise linear maps*, Constr. Approx. 5 (1989), 89–98.
[22] I. I. Gikhman and A. V. Skorokhod, *Stochastic Differential Equations and their Applications*, Naukova Dumka, Kiev, 1982 (in Russian).

[23] K. Horbacz and T. Szarek, *Continuous iterated function systems on Polish spaces*, Bull. Polish Acad. Sci. Math. 49 (2001), 191–202.

[24] —, —, *Randomly connected dynamical systems on Banach spaces*, Stochastic Anal. Appl. 19 (2001), 519–543.

[25] J. E. Hutchinson, *Fractals and self-similarity*, Indiana Math. J. 30 (1981), 713–743.

[26] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North-Holland, Amsterdam, 1981.

[27] M. Iosifescu and R. Theodorescu, *Random Processes and Learning*, Springer, New York, 1969.

[28] R. Isaac, *Markov processes and unique stationary probability measures*, Pacific J. Math. 12 (1962), 273–286.

[29] S. Karlin, *Some random walks arising in learning models*, Pacific J. Math. 3 (1953), 725–756.

[30] A. N. Kolmogorov, *A new invariant for transitive dynamical systems*, Dokl. Akad. Nauk SSSR 119 (1958), 861–864 (in Russian).

[31] T. Komorowski, *Asymptotic periodicity of some stochastically perturbed dynamical systems*, Ann. Inst. H. Poincaré 28 (1992), 165–178.

[32] M. Kuczma, B. Choczewski and R. Ger, *Iterative Functional Equations*, Cambridge Univ. Press, New York, 1990.

[33] A. Lasota, *From fractals to stochastic differential equations*, in: Chaos—The Interplay Between Stochastic and Deterministic Behaviour (Karpacz, 1995), P. Garbaczewski *et al.* (eds.), Lecture Notes in Phys. 457, Springer, 1995, 235–255.

[34] A. Lasota and M. C. Mackey, *Chaos, Fractals and Noise. Stochastic Aspects of Dynamics*, Springer, 1994.

[35] —, —, *Cell division and stability of cellular populations*, J. Math. Biology 38 (1999), 241–261.

[36] A. Lasota and J. Myjak, *Semifractals on Polish spaces*, Bull. Polish Acad. Sci. Math. 46 (1998), 179–196.

[37] —, —, *On a dimension of measures*, ibid. 50 (2002), 221–235.

[38] A. Lasota and J. Traple, *Invariant measures related with Poisson driven stochastic differential equation*, Stochastic Process. Appl., to appear.

[39] A. Lasota and J. A. Yorke, *Lower bound technique for Markov operators and iterated function systems*, Random Comput. Dynam. 2 (1994), 41–77.

[40] F. Ledrappier, *Some relations between dimension and Lyapunov exponents*, Comm. Math. Phys. 81 (1981), 95–102.

[41] K. Łoskot and R. Rudnicki, *Limit theorems for stochastically perturbed dynamical systems*, J. Appl. Probab. 32 (1995), 459–469.

[42] G. Lu and A. Mukherjea, *Invariant measures and Markov chains*, Probab. Math. Statist. 17 (1997), 115–138.

[43] R. D. Mauldin and S. C. Williams, *Hausdorff dimension in graph direct constructions*, Trans. Amer. Math. Soc. 309 (1988), 811–829.

[44] S. P. Meyn and R. L. Tweedie, *Markov Chains and Stochastic Stability*, Springer, London, 1993.

[45] G. Mihoc and O. Onicescu, *Sur les chaînes de variables statistiques*, Bull. Soc. Math. France 59 (1935), 174–192.

[46] P. Moran, *Additive functions of intervals and Hausdorff dimension*, Proc. Cambridge Philos. Soc. 42 (1946), 15–23.

[47] J. Myjak and T. Szarek, *On Hausdorff dimension of invariant measures arising from non-contractive iterated function systems*, Ann. Mat. Pura Appl. 181 (2002), 223–237.

[48] —, —, *Some estimates of capacity of self-similar measures*, Ann. Polon. Math. 78 (2002), 141–157.

[49] M. F. Norman, *Some convergence theorems for stochastic learning models with distance diminishing operators*, J. Math. Psychology 5 (1968), 61–101.
[50] L. Olsen, *A multifractal formalism*, Adv. Math. 116 (1995), 82–195.
[51] J. C. Oxtoby and S. Ulam, *On the existence of a measure invariant under a transformation*, Ann of Math. 40 (1939), 560–566.
[52] Ya. B. Pesin, *On rigorous mathematical definitions of correlation dimension and generalized spectrum for dimensions*, J. Statist. Phys., 71 (1993), 529–547.
[53] —, *Dimension Theory in Dynamical Systems*, Univ. of Chicago Press, 1997.
[54] P. A. Schwartzman, *On the convergence of an iterative series*, Vestnik Yaroslav. Univ. 12 (1975), 155–159.
[55] R. Sine, *Convergence theorems for weakly almost periodic Markov operators*, Israel J. Math., 19 (1974), 246–255.
[56] Ö. Stenflo, *Iterated function systems controlled by a semi-Markov chain*, Theory Stochastic Process. 2 (1996), 305–313.
[57] Ł. Stettner, *Remarks on ergodic conditions for Markov processes on Polish spaces*, Bull. Polish Acad. Sci. Math. 42 (1994), 103–114.
[58] T. Szarek, *Invariant measures for iterated function systems*, Ann. Polon. Math. 75 (2000), 87–98.
[59] —, *The stability of Markov operators on Polish spaces*, Studia Math. 143 (2000), 145–152.
[60] —, *The dimension of self-similar measures*, Bull. Polish Acad. Sci. Math. 48 (2000), 293–302.
[61] —, *The pointwise dimension for invariant measures related with Poisson driven stochastic differential equations*, ibid. 50 (2002), 241–250.
[62] —, *Invariant measures for Markov operators with applications to function systems*, Studia Math. 154 (2003), 207–222.
[63] T. Szarek and S. Wędrychowicz, *Markov semigroups generated by a Poisson driven differential equation*, Nonlinear Anal. 50 (2002), 41–54.
[64] W. J. Thron, *Sequences generated by iteration*, Trans. Amer. Math. Soc. 96 (1960), 38–53.
[65] J. Traple, *Markov semigroups generated by Poisson driven differential equations*, Bull. Polish Acad. Sci. Math. 44 (1996), 161–182.
[66] L. S. Young, *Dimension, entropy and Lyapunov exponents*, Ergodic Theory Dynam. Systems 2 (1982), 109–124.

Index