FUNDAMENTA
MATHEMATICAE
230 (2015)

On finite groups of isometries of handlebodies in arbitrary dimensions and finite extensions of Schottky groups

by

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Abstract. It is known that the order of a finite group of diffeomorphisms of a 3-dimensional handlebody of genus $g > 1$ is bounded by the linear polynomial $12(g-1)$, and that the order of a finite group of diffeomorphisms of a 4-dimensional handlebody (or equivalently, of its boundary 3-manifold), faithful on the fundamental group, is bounded by a quadratic polynomial in g (but not by a linear one). In the present paper we prove a generalization for handlebodies of arbitrary dimension d, uniformizing handlebodies by Schottky groups and considering finite groups of isometries of such handlebodies. We prove that the order of a finite group of isometries of a handlebody of dimension d acting faithfully on the fundamental group is bounded by a polynomial of degree $d/2$ in g if d is even, and of degree $(d+1)/2$ if d is odd, and that the degree $d/2$ for even d is best possible. This implies analogous polynomial Jordan-type bounds for arbitrary finite groups of isometries of handlebodies (since a handlebody of dimension $d > 3$ admits S^1-actions, there does not exist an upper bound for the order of the group itself).

1. Introduction. All finite group actions in the present paper will be faithful, smooth and orientation-preserving, and all manifolds will be orientable. We study finite group actions of large order on handlebodies of dimension $d \geq 3$ and genus $g > 1$.

An orientable handlebody V_g^d of dimension d and genus g can be defined as a regular neighbourhood of a finite graph with free fundamental group of rank g embedded in the sphere S^d; alternatively, it is obtained from the ball B^d by attaching along its boundary g copies of a handle $B^{d-1} \times [0,1]$ in an orientable way, or as the boundary-connected sum of g copies of $B^{d-1} \times S^1$. The boundary of V_g^d is a closed manifold H_g^{d-1} which is the connected sum of g copies of $S^{d-2} \times S^1$.

By [Z1] the order of a finite group of diffeomorphisms of a 3-dimensional handlebody V_g^3 of genus $g > 1$ is bounded by the linear polynomial $12(g-1)$

2010 *Mathematics Subject Classification*: 57S17, 57S25, 57N16.
Key words and phrases: handlebody, finite group action, Schottky group, Jordan-type bound.

DOI: 10.4064/fm230-3-2

(see also [MMZ, Theorem 7.2], [MZ]); moreover, a finite group G acting faithfully on V_g^3 acts faithfully also on the fundamental group. On the other hand, since the closed 3-manifold H_g^3 admits S^1-actions, it has finite cyclic group actions of arbitrarily large order acting trivially on the fundamental group, and the same is true for handlebodies V_g^d of dimensions $d > 3$. However it is shown in [Z4] that if a finite group of diffeomorphisms of $H_g^3 = \partial V_g^4$ acts faithfully on the fundamental group, then the order of the group is bounded by a quadratic polynomial in g (but not by a linear one), and hence the same holds for 4-dimensional handlebodies V_g^4. As a consequence, each finite group G acting on H_g^3 or V_g^4 has a finite cyclic normal subgroup G_0 (the subgroup acting trivially on the fundamental group) such that the order of G/G_0 is bounded by a quadratic polynomial in g (see [Z4]).

There arises naturally the question (asked in [Z4]) whether there are analogous polynomial bounds for the orders of finite groups acting on handlebodies V_g^d of arbitrary dimension d. Whereas finite group actions in dimension 3 are standard by the recent geometrization of such actions after Thurston and Perelman, the situation in higher dimensions is more complicated and not well-understood. Hence one is led to consider some kind of standard actions also in higher dimensions. We will do so by uniformizing the handlebodies V_g^d by Schottky groups (groups of Möbius transformations of the ball B^d acting by isometries on its interior, the Poincaré model of hyperbolic space $\mathbb{H}^d$), thus realizing their interiors as complete hyperbolic manifolds, and then considering finite groups of isometries of such hyperbolic (Schottky) handlebodies (see Section 2 for the definition of Schottky groups).

Our main results are as follows.

Theorem 1. *Let G be a finite group of isometries of a hyperbolic handlebody V_g^d of dimension $d \geq 3$ and of genus $g > 1$ which acts faithfully on the fundamental group. Then the order of G is bounded by a polynomial of degree $d/2$ in g if d is even, and of degree $(d+1)/2$ if d is odd. The degree $d/2$ is best possible in even dimensions, whereas in odd dimensions the optimal degree is at least $(d-1)/2$.*

By hypothesis such a group G injects into the outer automorphism group of the fundamental group of V_g^d, a free group of rank g. We note that by [WZ] the optimal upper bound for the order of an arbitrary finite subgroup of the outer automorphism group $\mathrm{Out}(F_g)$ of a free group F_g of rank $g > 2$ is $2^g g!$ (i.e., exponential in g). It is shown in [Z2] that every finite subgroup of $\mathrm{Out}(F_g)$ can be induced (or realized in the sense of the Nielsen realization problem) by an isomorphic group of isometries of a handlebody V_g^d of sufficiently high dimension d.

Without the hypothesis that G acts faithfully on the fundamental group, the proof of Theorem 1 gives the following polynomial Jordan-type bound for finite groups of isometries of V_g^d.

Corollary. *Let G be a finite group of isometries of a hyperbolic handlebody V_g^d of genus $g > 1$, and let G_0 denote the normal subgroup of G acting trivially on the fundamental group. Then:*

(i) *G_0 is isomorphic to a subgroup of the orthogonal group* $\mathrm{SO}(d-2)$, *and the order of the factor group G/G_0 is bounded by a polynomial as in Theorem* 1.
(ii) *G has a normal abelian subgroup* (*a subgroup of* G_0) *whose index in G is bounded by a polynomial as in Theorem* 1.

By the classical Jordan bound, each finite subgroup G of a complex linear group $\mathrm{GL}(d, \mathbb{C})$ has a normal abelian subgroup whose index in G is bounded by a constant depending only on the dimension d (see [C] for the optimal bound for each d; see also [Z5] and its references for generalizations of the Jordan bound in the context of diffeomorphism groups of manifolds).

In more algebraic terms, Theorem 1 is equivalent to the following:

Theorem 2. *Let E be a group of Möbius transformations of S^{d-1} which is a finite effective extension of a Schottky group $\mathcal{S}_g$ of rank $g > 1$. Then the order of the factor group $E/\mathcal{S}_g$ is bounded by a polynomial in g as in Theorem* 1.

Here *effective extension* means that no element of E acts trivially on $\mathcal{S}_g$ by conjugation. By [Z2] every finite effective extension of a Schottky group can be realized by a group of Möbius transformations in some sufficiently high dimension d.

As a consequence of the geometrization of finite group actions in dimension 3, using the methods of [RZ, Section 2] every finite group G of diffeomorphisms of a 3-dimensional handlebody V_g^3 can be shown to be conjugate to a group of isometries, uniformizing V_g^3 by a suitable Schottky group (which depends on G). This is no longer true in higher dimensions; however, if G is a finite group of diffeomorphisms of a 4-dimensional handlebody V_g^4 then, uniformizing V_g^4 by a suitable Schottky group, G acts also as a group of isometries of V_g^4 inducing the same action on the fundamental group (by applying the methods of [Z4] to the boundary 3-manifold H_g^3 of V_g^4). This raises naturally the following:

Questions. (i) Is every finite group G of diffeomorphisms of a handlebody V_g^d isomorphic to a group of isometries of a hyperbolic handlebody V_g^d (inducing the same action on the fundamental group)?

(ii) Is every finite group G of diffeomorphisms of a ball B^d (i.e., a handlebody of genus zero) or of a sphere S^{d-1} isomorphic to a subgroup of the orthogonal group $\mathrm{SO}(d)$?

In general, such a finite group G of diffeomorphisms is not conjugate to a group of isometries of a handlebody resp. to a group of orthgonal maps; we note that (ii) is not true for finite groups G of homeomorphisms of B^d or S^{d-1} (see [GMZ, Section 7]).

In Section 2 we prove the first part of Theorem 1. In Section 3 we present examples of finite isometric group actions on handlebodies which show that the degree $d/2$ of the polynomial bound in Theorem 1 is best possible in even dimensions (even for finite cyclic groups G), and that a lower bound for the degree in odd dimensions is $(d-1)/2$. Note that for $d = 3$ the bound $(d+1)/2$ is not best possible since it gives a quadratic bound instead of the actual linear bound $12(g-1)$ (so maybe for all odd dimensions $d \geq 3$ the optimal degree is $(d-1)/2$, but this remains open at present).

2. Schottky groups and the proof of Theorem 1. A *Schottky group* $\mathcal{S}_g$ of rank or genus g is a group of Möbius transformations acting on the sphere $S^{d-1} = \partial B^d$ defined in the following way (analogously to the Schottky groups in dimension 2 acting on S^2, see [L], [M] or [R, p. 584]; see also [Z2]). Let $S_1, T_1, \ldots, S_g, T_g$ be spheres of dimension $d-2$ in S^{d-1} which bound disjoint balls $B_1, D_1, \ldots, B_g, D_g$ of dimension $d-1$; choose Möbius transformations $f_1, \ldots, f_g$ such that $f_i(S_i) = T_i$ and f_i maps the exterior of B_i to the interior of D_i. Then it is easy to see that $f_1, \ldots, f_g$ are free generators of a free group $\mathcal{S}_g$ of Möbius transformations. The complement in S^{d-1} of the interiors of the balls B_i and D_i is a fundamental domain for the action of $\mathcal{S}_g$ on $S^{d-1} - \Lambda(\mathcal{S}_g)$ where $\Lambda(\mathcal{S}_g)$ denotes the set of limit points of $\mathcal{S}_g$ in S^{d-1} (a Cantor set). In this definition, one may consider round spheres $S_1, T_1, \ldots, S_g, T_g$ (thus defining a so-called classical Schottky group), or just topological spheres (and it is known that non-classical Schottky groups exist); however, this is not relevant for the present paper, in particular in the examples constructed in Section 3 the Schottky subgroups will always be classical.

The group of Möbius transformations of S^{d-1} extends naturally to the interior of the ball B^d ("Poincaré extension") where it becomes the group of orientation-preserving isometries of the Poincaré model of hyperbolic space $\mathbb{H}^d$. The action of $\mathcal{S}_g$ is free and properly discontinuous on the interior $\mathbb{H}^d$ of B^d, and a fundamental domain for this action is the region of $\mathbb{H}^d$ bounded by all hyperbolic hyperplanes defined by the spheres S_i and T_i (i.e., half-spheres of dimension $d-1$ orthogonal to S^{d-1} along these spheres). The quotient $(B^d - \Lambda(\mathcal{S}_g))/\mathcal{S}_g$ is a handlebody V_g^d whose interior $\mathbb{H}^d/\mathcal{S}_g$ has the structure of a complete hyperbolic manifold, and we say that the Schottky group $\mathcal{S}_g$ *uniformizes* the handlebody V_g^d. When speaking of a finite group G of isometries of a handlebody V_g^d we mean that V_g^d can be uniformized by a Schottky group $\mathcal{S}_g$ such that G acts by hyperbolic isometries on the interior of V_g^d.

Let V_g^d be a handlebody uniformized by a Schottky group $\mathcal{S}_g$. Let G be a finite group of isometries of V_g^d which induces a faithful action on the fundamental group. The group of all lifts of elements of G to the universal covering $B^d - \Lambda(\mathcal{S}_g)$ of V_g^d defines a group E of Möbius transformations of B^d, with factor group $E/\mathcal{S}_g \cong G$, so we have a finite extension

$$1 \to \mathcal{S}_g \hookrightarrow E \to G \to 1;$$

by general covering space theory, this extension is effective since G acts faithfully on the fundamental group of V_g^d (isomorphic to the group $\mathcal{S}_g$ of covering transformations).

Lemma 1. *The extension* $1 \to \mathcal{S}_g \hookrightarrow E \to G \to 1$ *is effective if and only if* E *has no non-trivial finite normal subgroups.*

Proof. Let F be a finite normal subgroup E. Since the intersection of F with the normal torsionfree subgroup $\mathcal{S}_g$ of E is trivial, the normal subgroups F and $\mathcal{S}_g$ of E commute elementwise (any commutator $fsf^{-1}s^{-1}$ of elements $f \in F$ and $s \in \mathcal{S}_g$ is an element of both F and $\mathcal{S}_g$ and hence trivial). Hence if the extension is effective, F has to be trivial.

Conversely, suppose that every finite normal subgroup of E is trivial. The subgroup of elements of the finite extension E of $\mathcal{S}_g$ inducing by conjugation the trivial automorphism of $\mathcal{S}_g$ is clearly finite (since the centre of $\mathcal{S}_g$ is trivial), normal and hence trivial, so the extension is effective.

This completes the proof of Lemma 1.

As a consequence of Stalling's structure theorem for groups with infinitely many ends, a finite extension E of a free group is the fundamental group $\pi_1(\Gamma, \mathcal{G})$ of a finite graph of finite groups $(\Gamma, \mathcal{G})$ (see [KPS]); here Γ denotes a finite graph, and to its vertices v and edges e are associated finite vertex groups G_v and edge groups G_e, with inclusions of the edge groups into the adjacent vertex groups. The fundamental group $\pi_1(\Gamma, \mathcal{G})$ of the finite graph of finite groups $(\Gamma, \mathcal{G})$ is the iterated free product with amalgamation and HNN-extension of the vertex groups amalgamated over the edge groups, first taking the iterated free product with amalgamation over a maximal tree of Γ, and then associating an HNN-generator to each of the remaining edges. We note that each finite subgroup of $E = \pi_1(\Gamma, \mathcal{G})$ is conjugate to a vertex group of $(\Gamma, \mathcal{G})$, and that the vertex groups are maximal finite subgroups of E (see [ScW], [Se] or [Z3] for the standard theory of graphs of groups and their fundamental groups).

We will assume in the following that the graph of groups $(\Gamma, \mathcal{G})$ has no *trivial edges*, i.e. no edges with two different vertices such that the edge group coincides with one of the two vertex groups (by collapsing trivial edges, i.e. amalgamating the two vertex groups into a single vertex group); we say that such a graph of groups is in *normal form*.

We denote by

$$\chi(\Gamma, \mathcal{G}) = \sum \frac{1}{|G_v|} - \sum \frac{1}{|G_e|}$$

the *Euler characteristic* of the graph of groups $(\Gamma, \mathcal{G})$ (the sum is taken over all vertex groups G_v resp. edge groups G_e of $(\Gamma, \mathcal{G})$); then, by multiplicativity of Euler characteristics under finite coverings of graphs of groups,

$$g - 1 = -\chi(\Gamma, \mathcal{G})|G|$$

(see [ScW] or [Z3]); note that this is positive since we are assuming that $g > 1$.

The finite extension $E = \pi_1(\Gamma, \mathcal{G})$ of the Schottky group $\mathcal{S}_g$ is a group of Möbius transformations of B^d and acts as a group of hyperbolic isometries on its interior $\mathbb{H}^d$. Each finite group of isometries of hyperbolic space $\mathbb{H}^d$ has a global fixed point in $\mathbb{H}^d$ and is conjugate to a finite group of orthogonal transformations of B^d (which are exactly the isometries of $\mathbb{H}^d$ which fix the origin in B^d). In particular, each finite vertex group G_v of $E = \pi_1(\Gamma, \mathcal{G})$ has a fixed point in $\mathbb{H}^d$ and is isomorphic (conjugate) to a subgroup of the orthogonal group $\mathrm{SO}(d)$, and different vertex groups of $(\Gamma, \mathcal{G})$ have different fixed points (since the vertex groups are maximal finite subgroups of E and the action of E is properly discontinuous in $\mathbb{H}^d$); also, if a vertex group fixes a point in $\mathbb{H}^d$ then it is the maximal finite subgroup of E fixing this point.

Consider a non-closed edge e of $(\Gamma, \mathcal{G})$, i.e. with two distinct vertices v_1 and v_2, with edge group G_e and vertex groups G_1 and G_2 (which we consider as subgroups of E), with $G_e = G_1 \cap G_2$. Let $P_1 \neq P_2$ be fixed points of G_1 resp. G_2 in $\mathbb{H}^d$; then P_1 and P_2 define a hyperbolic line L which is fixed pointwise by the edge group $G_e = G_1 \cap G_2$. The line L intersects $S^{d-1} = \partial B^d$ in two points which are fixed by G_e; moreover, no subgroup of G_1 larger than G_e can fix one of these two points since otherwise it would fix pointwise the line L and hence P_2, so it would also be contained in G_2.

Now let e be a closed edge of $(\Gamma, \mathcal{G})$, i.e. an edge with only one vertex v. There are two inclusions of the edge group G_e into the vertex group G_v defining two subgroups G_e and G'_e of G_v; denoting by t an HNN-generator corresponding to the edge e, we have $t^{-1}G'_e t = G_e$ and $G_e = G_v \cap (t^{-1}G_v t)$. Note that t has infinite order, so it does not fix any point in $\mathbb{H}^d$. Let P be a fixed point of the finite subgroup G_v of E in $\mathbb{H}^d$; then $t^{-1}G_v t$ fixes the point

$t(P) \neq P$, and its subgroup $G_e = t^{-1}G'_e t$ fixes the hyperbolic line L defined by P and $t(P)$. As before, the hyperbolic line L intersects $S^{d-1} = \partial B^d$ in two points which are fixed by G_e, and G_e is the maximal subgroup of G_v fixing these two points.

Note also that, since G_e fixes a point in S^{d-1}, it is in fact isomorphic (conjugate) to a subgroup of the orthogonal group $\mathrm{SO}(d-1)$. Summarizing, we have:

Lemma 2. *Let $G_v \subset E$ be a vertex group of the graph of groups $(\Gamma, \mathcal{G})$, and let $G_e \subset G_v$ be an adjacent edge group. Then G_v has a global fixed point in $\mathbb{H}^d$, and G_e has a global fixed point in $S^{d-1} = \partial B^d$ which is not fixed by any other element of G_v. In particular, every vertex group is isomorphic to a subgroup of the orthogonal group* $\mathrm{SO}(d)$, *and every edge group is isomorphic to a subgroup of* $\mathrm{SO}(d-1)$.

We also need the following lemma which is contained in [Z4, proof of Theorem 1]; since its proof is short, we present it for the convenience of the reader. Let $\chi = \chi(\Gamma, \mathcal{G})$ denote the Euler characteristic of $(\Gamma, \mathcal{G})$; note that $-\chi > 0$ since $g > 1$, and that for any graph of groups in normal form one has $-\chi \geq 0$ unless the graph consists of a single vertex.

Lemma 3. *Let e be an edge of Γ. Denote by n the order of G and by a the order of the edge group G_e. Then*

$$\frac{n}{a} \leq 6(g-1).$$

Proof. Suppose first that e is a closed edge. If e is the only edge of $(\Gamma, \mathcal{G})$ then

$$-\chi \geq \frac{1}{a} - \frac{1}{2a} = \frac{1}{a}, \qquad g-1 = -\chi n \geq \frac{n}{2a}, \qquad \frac{n}{a} \leq 2(g-1).$$

If e is closed and not the only edge then

$$-\chi \geq \frac{1}{a}, \qquad g-1 = -\chi n \geq \frac{n}{a}, \qquad \frac{n}{a} \leq g-1.$$

Suppose that e is not closed. If e is the only edge of $(\Gamma, \mathcal{G})$ then both vertices of e are isolated and

$$-\chi \geq \frac{1}{a} - \frac{1}{2a} - \frac{1}{3a} = \frac{1}{6a}, \qquad g-1 = -\chi n \geq \frac{n}{6a}, \qquad \frac{n}{a} \leq 6(g-1).$$

If e is not closed, not the only edge and has exactly one isolated vertex then

$$-\chi \geq \frac{1}{a} - \frac{1}{2a} = \frac{1}{2a}, \qquad g-1 = -\chi n \geq \frac{n}{2a}, \qquad \frac{n}{a} \leq 2(g-1).$$

Finally, if e is not closed, not the only edge and has no isolated vertex then

$$-\chi \geq \frac{1}{a}, \qquad g-1 = -\chi n \geq \frac{n}{a}, \qquad \frac{n}{a} \leq g-1.$$

Concluding, in all cases the inequality of Lemma 3 holds.

Proof of Theorem 1. Let e be any edge of the finite graph of finite groups $(\Gamma, \mathcal{G})$ given by the G-action. By Lemma 2, G_e has a global fixed point in $S^{d-1} = \partial B^d$ and is isomorphic to a subgroup of the orthogonal group $\mathrm{SO}(d-1)$. By the classical Jordan bound for subgroups of $\mathrm{GL}(d-1, \mathbb{C})$, the edge group G_e has an abelian subgroup A_1 whose index in G_e is bounded by a constant c depending only on the dimension. We will find a polynomial upper bound in g for the order a_1 of the abelian group A_1; this will imply a polynomial bound of the same degree also for the order $a \leq c a_1$ of G_e, and finally for the order n of G, since, by Lemma 3,

$$n \leq 6(g-1)a \leq c\,6(g-1)a_1.$$

Let E_1 be the subgroup of E generated by $\mathcal{S}_g$ and A_1 (which is again an effective extension of $\mathcal{S}_g$, with factor group A_1). Then also E_1 is the fundamental group of a finite graph of finite groups in normal form, which we denote again by $(\Gamma, \mathcal{G})$. Since the finite group A_1 has a fixed point in $\mathbb{H}^d$, up to conjugation it is the vertex group G_v of some vertex v of $(\Gamma, \mathcal{G})$, and its fixed point set in S^{d-1} is a sphere S^{d_1} of dimension $d_1 \geq 0$ (since G_e has a global fixed point in S^{d-1}). Since $(\Gamma, \mathcal{G})$ has no trivial edges and E_1 has no non-trivial finite normal subgroups by Lemma 1, some edge adjacent to v has an edge group A_2 of order $a_2 < a_1$ (i.e., properly contained in A_1). By Lemma 3,

$$a_1 \leq 6(g-1)a_2.$$

By Lemma 2, the edge group A_2 has a fixed point in $S^{d-1} = \partial B^d$ which is not fixed by any other element of the vertex group A_1, hence the fixed point set of A_2 in S^{d-1} is a sphere S^{d_2} of dimension $d_2 > d_1$.

We iterate the construction and consider the subgroup E_2 of E_1 generated by $\mathcal{S}_g$ and A_2, obtaining an edge group A_3 for E_2 which fixes a sphere S^{d_3} of dimension $d_3 > d_2$ in S^{d-1}, of order

$$a_2 \leq 6(g-1)a_3.$$

Hence, after at most $d-1$ steps, we end up with a trivial edge group fixing all of S^{d-1}. Collecting, we obtain the polynomial bound

$$n \leq c\,6^d(g-1)^d$$

of degree d in g for the order of G.

To obtain a polynomial bound of the degree given in Theorem 1 we argue as follows. Suppose that the fixed point set of the normal subgroup A_2 of A_1 is a sphere S^{d_1+1} of dimension $d_2 = d_1 + 1$; note that S^{d_1+1} is invariant under the action of A_1. Let A_1' denote the subgroup of index 1 or 2 of A_1 which acts orientation-preservingly on S^{d_1+1}. Then A_1' fixes S^{d_1+1} pointwise since otherwise the fixed point set of A_1' would be a sphere of codimension at least 2 in S^{d_1+1}; this is not possible since already A_1 has fixed point set S^{d_1} of dimension d_1. Continuing now with A_1' in the place of A_1, we can assume that the dimensions d_i increase by at least 2 in each step. Hence the number of steps is at most $d/2$ if d is even, and $(d+1)/2$ if d is odd, and this gives the degree of the polynomial upper bound as stated in Theorem 1.

This completes the proof of the first part of Theorem 1; the second part on the optimality of the degree $d/2$ for even g and the lower bound $(d-1)/2$ for odd g will follow from the examples of finite group actions on handlebodies constructed in the next section.

Proof of the Corollary. The proof proceeds along the lines of the proof of Theorem 1, with the following difference. In the proof of Theorem 1 we considered the sequence of abelian subgroups $A_1, A_2, \dots$ of G; after finitely many steps, this ended with the trivial group, using the effectiveness of the corresponding extensions $E_1, E_2, \dots$ of $\mathcal{S}_g$. Without effectiveness, the sequence $A_1, A_2, \dots$ of G ends with an abelian group A_m which is a normal subgroup of the corresponding extension E_m; in particular, A_m acts trivially on $\mathcal{S}_g$ and is a subgroup of G_0. The index of A_m in G is bounded by a polynomial as in the proof of Theorem 1, hence also the index of G_0 in G is bounded by such a polynomial.

The group G_0 lifts to an isomorphic normal subgroup of the extension E of $\mathcal{S}_g$, which we also denote by G_0. The finite group G_0 has a fixed point in $\mathbb{H}^d$; we can assume that it fixes the origin $O \in B^d$ and hence is isomorphic to a subgroup of $\mathrm{SO}(d)$. Since G_0 is normal in E, it is contained (up to conjugation) in each edge group of the graph of groups $(\Gamma, \mathcal{G})$. By Lemma 2, G_0 has a global fixed point also in $S^{d-1} = \partial B^d$, hence it fixes pointwise a great sphere of dimension at least zero in S^{d-1}, and a linear subspace B of dimension at least 1 in B^d. Since G_0 commutes elementwise with $\mathcal{S}_g$, the Schottky group $\mathcal{S}_g$ acts on B. Since the action of $\mathcal{S}_g$ is properly discontinuous and $g > 1$, B has dimension at least 2. Now G_0 acts also on the orthogonal complement of B in $O \in B^d$, a linear subspace of codimension at least 2, so G_0 is isomorphic to a subgroup of the orthogonal group $\mathrm{SO}(d-2)$.

Finally, by the classical Jordan bound for linear groups, the subgroup G_0 of $\mathrm{SO}(d-2)$ contains a normal abelian subgroup whose index is bounded by a constant depending only on the dimension d. By taking the intersection of this normal abelian subgroup with all isomorphic normal subgroups of G_0 we obtain a characteristic abelian subgroup A of G_0 whose index in G_0 is also bounded by a constant depending only on the dimension d. Hence the indices of A and G_0 in G are bounded by polynomials in g of the same degree.

This completes the proof of the Corollary.

3. Examples. We construct isometric actions of finite groups G on handlebodies which realize the lower bounds for the degrees of the polynomial bounds in Theorem 1. Specifically, we prove the following:

PROPOSITION. *For a fixed $k \geq 2$ and all $m \geq 2$, the finite group $G = (\mathbb{Z}_m)^k$ admits an action, faithful on the fundamental group, on a handlebody V_g^d of genus $g = mk - k$ and dimension $d = 2k$ and $2k+1$; in particular, the order $n = m^k$ of G is given by the polynomial*

$$n = (g+k)^k/k^k = (1+g/k)^k$$

of degree $k = d/2$ in g if d is even, and $k = (d-1)/2$ if d is odd.

Proof. For $k > 1$, let $G = C_1 \times \cdots \times C_k \cong (\mathbb{Z}_m)^k$, of order $n = m^k$, be the product of k cyclic groups $C_i \cong \mathbb{Z}_m$ of order m. Choose an orthogonal action of G on the closed ball $B^{2k} \subset \mathbb{R}^{2k}$ of dimension $d = 2k$ as follows. If we decompose $\mathbb{R}^{2k} = P_1 \times \cdots \times P_k$ as the product of k orthogonal planes P_i, then each C_i acts on P_i faithfully by rotations and trivially on the $k-1$ orthogonal planes.

Define a finite graph of finite groups $(\Gamma, \mathcal{G})$ as follows. The graph Γ is star-shaped with one central vertex v with vertex group $G_v = G = C_1 \times \cdots \times C_k$ and k non-closed edges $e_1, \ldots, e_k$ each having v as a vertex, with edge groups

$$G_{e_1} = C_2 \times \cdots \times C_k, \quad G_{e_2} = C_1 \times C_3 \times \cdots \times C_k, \quad \ldots, \quad G_{e_k} = C_1 \times \cdots \times C_{k-1}$$

(i.e., exactly C_i is missing in G_{e_i}). Hence Γ has $k+1$ vertices, by definition all with vertex group $G = C_1 \times \cdots \times C_k$, and the Euler characteristic of $(\Gamma, \mathcal{G})$ is

$$\chi = (k+1)\frac{1}{m^k} - k\frac{1}{m^{k-1}}.$$

There is an obvious projection of the fundamental group $E = \pi_1(\Gamma, \mathcal{G})$ of the graph of groups $(\Gamma, \mathcal{G})$ onto G; its kernel is a free group F_g of some rank g, and we have an extension

$$1 \to F_g \hookrightarrow E \to G \to 1,$$

which by construction of $(\Gamma, \mathcal{G})$ is effective (has no non-trivial finite normal subgroups, see Lemma 1). The rank g is given by

$$g - 1 = (-\chi)n = (-\chi)m^k = mk - (k+1), \qquad g = mk - k,$$

hence

$$n = m^k = (g+k)^k/k^k,$$

which is a polynomial of degree $k = d/2$ in g and gives the maximal possibility for the degree in Theorem 1 for even dimensions d.

We realize $E = \pi_1(\Gamma, \mathcal{G})$ as a group of Möbius transformations of B^d, $d = 2k$, such that its subgroup F_g corresponds to a Schottky group $\mathcal{S}_g$. Then the quotient $(B^d - \Lambda(\mathcal{S}_g))/\mathcal{S}_g$ is a handlebody V_g^d of genus g, and E

projects to an action of the factor group $E/\mathcal{S}_g \cong G$ on V_g^d which is faithful on the fundamental group. In particular, the degree $d/2$ in Theorem 1 is best possible for even dimensions $d = 2k$.

The realization of $E = \pi_1(\Gamma, \mathcal{G})$ as a group of Möbius transformations of B^d proceeds inductively by standard combination methods (similar to those in [Z2, Section 3]). Starting with the orthogonal group G described above, we first realize the free product with amalgamation

$$G_v *_{G_{e_1}} G_{v_1} = G *_{G_e} G_1$$

where $e = e_1$ denotes the first edge of Γ, with vertices v and v_1 and vertex groups $G = G_v$ and $G_1 = G_{v_1} \cong G$. By construction, the fixed point set of the subgroup G_e of G is a 2-ball B_1 in B^d defining a hyperbolic plane in $\mathbb{H}^d$, still denoted by B_1. Let L_1 be a hyperbolic half-line in B_1 starting from its centre 0 and ending at a point R_1 in $S^{d-1} = \partial B^d$. Let V_1 be a neighbourhood of R_1 in B^d bounded by a hyperbolic hyperplane H_1 in $\mathbb{H}^d$ orthogonal to L_1; choose V_1 sufficiently small such that $f(V)$ is disjoint from V for all $f \in G - G_e$ (note that G_e fixes L_1 pointwise but that no larger subgroup of G fixes L_1 by construction of G). The reflection τ_1 in the hyperbolic hyperplane H_1 commutes elementwise with $G_e \subset G$ and, considering $G_1 = \tau_1 G \tau_1^{-1}$, we have $G \cap G_1 = G_e$. As for Schottky groups, it is now easy to see that the group of Möbius transformations generated by G and G_1 is isomorphic to the free product with amalgamation $G *_{G_e} G_1$, and that every torsionfree subgroup of finite index is in fact a Schottky group (cf. [Z2] and the combination theorems in [M]).

We iterate the construction and adjoin G_{e_2}. Let L_2 be a hyperbolic half-line starting at the centre 0 and ending at a point R_2 of $S^{d-1} = \partial B^d$ such that R_2 does not lie in $G(V_1)$. Let V_2 be a small neighbourhood of R_2 in B^d, bounded by a hyperbolic hyperplane H_2 orthogonal to L_2 which does not intersect $G(V_1)$. With $G_2 = \tau_2 G \tau_2^{-1}$ where τ_2 denotes the reflection in H_2, this realizes the free product with amalgamation

$$G_{v_2} *_{G_{e_2}} G_v *_{G_{e_1}} G_{v_1}$$

as a group of Möbius transformations. Continuing in this way, after k steps, E is realized as a group of Möbius transformations, with F_g corresponding to a Schottky group $\mathcal{S}_g$.

Finally, in odd dimensions $d = 2k+1$, we extend the orthogonal action of G on B^{2k} described above to an orthogonal action on B^{2k+1} (trivial on the last coordinate) and then proceed as before. We get a polynomial of degree $k = (d-1)/2$ in g for the order n of G, whereas Theorem 1 gives a polynomial bound of degree $(d+1)/2$. As noted in the Introduction, the optimal degree in dimension $d = 3$ is in fact 1, but for odd dimensions $d > 3$ it remains open.

This completes the proof of the Proposition, and also of Theorem 1.

The examples given in the Proposition are for finite abelian groups G. By suitably modifying the construction, one also obtains examples for finite cyclic groups as follows.

Let $d = 2k$ be a fixed even dimension, and let $p > k$ be any prime. For $i = 1, \ldots, k$, the k integers $q_i = p + ik!$ are pairwise coprime: in fact, if a prime p' divides q_i then $p' > k$; if p' divides also q_j, for some $j > i$, then p' divides $q_j - q_i = (j-i)k!$, which is a contradiction. Then $G = \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_k}$ is a cyclic group of order $n = q_1 \cdots q_k$. In analogy with the proof of the Proposition, let $(\Gamma, \mathcal{G})$ be a star-shaped graph of groups with $k+1$ vertices all with vertex group G, and with k edges where in each edge group exactly one of the factors $\mathbb{Z}_{q_i}$ of G is missing, with

$$\chi = \chi(\Gamma, \mathcal{G}) = \frac{k+1}{n} - \frac{q_1}{n} - \cdots - \frac{q_k}{n}.$$

There is an obvious surjection of $\pi_1(\Gamma, \mathcal{G})$ onto G; its kernel is a free group of rank g with

$$g - 1 = (-\chi)n = -(k+1) + q_1 + \cdots + q_k,$$
$$g = -k + kp + (1 + \cdots + k)k!,$$
$$p = (g + c_k)/k$$

for a constant c_k depending only on k. Now

$$|G| = n = q_1 \cdots q_k \geq p^k \geq (g + c_k)^k / k^k,$$

so the order of G is bounded from below by a polynomial of degree $k = d/2$ in g.

Finally, the geometric realizations of G and $E = \pi_1(\Gamma, \mathcal{G})$ are exactly as in the proof of the Proposition.

Acknowledgments. The authors were supported by a FRV grant from Università degli Studi di Trieste.

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Received 15 May 2014;
in revised form 26 January 2015