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An operator invariant for handlebody-knots

by

Kai Ishihara (Yamaguchi) and **Atsushi Ishii** (Tsukuba)

Abstract. A handlebody-knot is a handlebody embedded in the 3-sphere. We improve Luo's result about markings on a surface, and show that an IH-move is sufficient to investigate handlebody-knots with spatial trivalent graphs without cut-edges. We also give fundamental moves with a height function for handlebody-tangles, which helps us to define operator invariants for handlebody-knots. By using the fundamental moves, we give an operator invariant.

1. Introduction. A handlebody-knot is a handlebody embedded in the 3-sphere S^3. Since to each knot corresponds a genus one handlebody-knot obtained by taking a regular neighborhood, handlebody-knot theory is a generalization of knot theory. Handlebody-knots are also closely related to spatial graphs, finite graphs embedded in S^3. Handlebody-knots appear in spatial graph theory as neighborhood equivalence classes of spatial graphs, introduced by Suzuki [14].

A handlebody-link is a disjoint union of handlebodies embedded in S^3. Any handlebody-link is a regular neighborhood of a spatial trivalent graph. Furthermore, any handlebody-link is a regular neighborhood of a spatial trivalent graph without cut-edges, where a cut-edge is an edge such that the number of connected components of the graph increases when we remove the edge. When a handlebody-link H is a regular neighborhood of a spatial trivalent graph K, we say that H is represented by K.

Two spatial trivalent graphs related by an IH-move represent an equivalent handlebody-link, where an IH-move is a local spatial move for spatial trivalent graphs. The second author [2] showed that two spatial trivalent graphs represent an equivalent handlebody-link if and only if they are related by a finite sequence of IH-moves:

$$\{\text{handlebody-link}\} = \{\text{spatial trivalent graph}\}/\text{IH-moves}.$$

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This was derived from Luo's result [8] about markings on a surface. In this paper, we improve Luo's result and show that two spatial trivalent graphs without cut-edges represent an equivalent handlebody-link if and only if they are related by a finite sequence of IH-moves such that all trivalent graphs appearing in the sequence have no cut-edges:

$$\{\text{handlebody-link}\} = \{\text{spatial trivalent graph without cut-edges}\}/\text{IH-moves}.$$

We remark that the cycle double cover conjecture [13, 15] states that any graph without cut-edges has a cycle double cover, which was used to normalize the Yamada polynomial [3].

A diagram of a handlebody-link is a diagram of a spatial trivalent graph which represents the handlebody-link. Kishimoto, Moriuchi, Suzuki and the second author [6] gave a table of genus two handlebody-knots up to six crossings, and classified them according to the crossing number and irreducibility. Koda [11] investigated the symmetry group of a genus two handlebody-knot.

The fundamental group of the exterior of a handlebody-link is an invariant of the handlebody-link, although it does not work for handlebody-links with homeomorphic exteriors. In [2, 4, 7], quandle cocycle invariants are defined for handlebody-links. Since the discovery of the Jones polynomial, many link invariants have been defined, which include so-called quantum invariants. It is important for handlebody-knot theory to define quantum invariants for handlebody-links. In this paper, we give a sufficient condition needed to define operator invariants for handlebody-links by improving Luo's result. We note that a quantum invariant is an operator invariant. This is the first step to topological field theory for handlebody-knots. We give an operator invariant for handlebody-links by using the condition obtained.

Throughout this paper we work in the piecewise linear category.

2. A marking on a surface with boundary. Let $\Sigma_{g,r}$ be a compact orientable surface of genus g with r boundary components. A *marking* $m = \bigcup_{i=1}^{3g+r-3} m_i$ is a disjoint union of $3g+r-3$ pairwise non-parallel, essential, non-boundary parallel unoriented simple closed curves in $\mathrm{int}(\Sigma_{g,r})$. From a marking m, we can obtain a new marking $m' = \bigcup_{i=1}^{3g+r-3} m'_i$ where $m'_j = m_j$ for $j \neq i$ and $m'_i \cap m_i$ consists of two points of different intersection signs. We call this operation a *type II move*. An arc α in $\mathrm{int}(\Sigma_{g,r})$ is called a *wave* with respect to the marking m if $\alpha \cap m = \alpha \cap m_i = \partial\alpha$ for some i, and α approaches its end points from the same side of m_i. For any given markings m, n in a compact orientable surface without boundary, Luo [8] showed that m can be transformed into m' by a finite sequence of type II moves and isotopies so that n contains no waves with respect to m'. Here we extend his theorem to markings on a compact orientable surface $\Sigma_{g,r}$ with boundary.

THEOREM 2.1. *If m and n are two markings on $\Sigma_{g,r}$, then there is a marking m' obtained from m by a finite sequence of type II moves and isotopies so that n contains no waves with respect to m'.*

Proof. The proof is as in [8]. We use the minimum number $W(n|m)$ of waves with respect to m which are contained in n up to isotopy, and the geometric intersection number $I(m,n)$ between m and n. The theorem can be shown by induction on the complexity $(W(n|m), I(m,n))$ in the lexicographic order. Here we recall Luo's arguments. He showed that when $W(n|m) > 0$, we can obtain a new marking m' by a type II move so that $W(n|m') < W(n|m)$, or we can obtain a new marking m' by a finite sequence of type II moves so that $W(n|m') \leq W(n|m)$ and $I(m', n) < I(m, n)$. Since no marking intersects $\partial\Sigma_{g,r}$, and $\partial\Sigma_{g,r}$ is fixed on each type II move, we can show the theorem in the same way as in [8]. ∎

For two markings m, n on $\Sigma_{g,r}$, we say that m, n *determine the same handlebody structure* if there exists an embedding from $\Sigma_{g,r}$ to the boundary ∂H_g of the genus g handlebody H_g such that each image of m and n bounds a union of disjoint proper disks in H_g.

COROLLARY 2.2. *If m, n are two markings on $\Sigma_{g,r}$ which determine the same handlebody structure, then they are related by a finite sequence of type II moves.*

3. The type II move without separating curves

THEOREM 3.1. *Suppose that two markings m, n on $\Sigma_{g,r}$ are related by a finite sequence of type II moves and isotopies, and both m and n consist of non-separating simple closed curves in $\Sigma_{g,r}$. Then there is a sequence $m = m^{(0)}, m^{(1)}, \ldots, m^{(k)} = n$ of markings such that, for each $i \in \{1, \ldots, k\}$, the marking $m^{(i)}$ consists of non-separating simple closed curves and is obtained from $m^{(i-1)}$ by a single type II move and isotopies.*

COROLLARY 3.2. *Suppose that m, n are two markings on $\Sigma_{g,r}$ which determine the same handlebody structure, and both m and n consist of non-separating simple closed curves in $\Sigma_{g,r}$. Then there is a sequence $m = m^{(0)}, m^{(1)}, \ldots, m^{(k)} = n$ of markings such that, for each $i \in \{1, \ldots, k\}$, the marking $m^{(i)}$ consists of non-separating simple closed curves and is obtained from $m^{(i-1)}$ by a single type II move and isotopies.*

Lemma 3.3 below is needed for the proof of Theorem 3.1. Here we consider markings on punctured spheres. By the well known Jordan curve theorem, each simple closed curve contained in a marking separates the boundary components of the punctured sphere into two non-empty sets of components. Throughout this section, we write AmB when a simple closed curve m separates the set of boundary components $A \cup B$ into A and B.

LEMMA 3.3.

(a) *Let $\Sigma_{0,4}$ be a 4-punctured sphere with boundary components a, b, c, d, and m a marking on $\Sigma_{0,4}$. Suppose that each of two markings m' and m'' is obtained from m by a single type II move, and $m' \cap m''$ contains no isotopically removable points. If $\{a, b\}m''\{c, d\}$, then there is a finite sequence $m' = m^{(0)}, m^{(1)}, \dots, m^{(k)} = m''$ of markings on $\Sigma_{0,4}$ such that $m^{(i)}$ is obtained from $m^{(i-1)}$ by a single type II move, and for each $i \in \{1, \dots, k\}$, either $\{a, b\}m^{(i)}\{c, d\}$ or $\{a, d\}m^{(i)}\{b, c\}$.*

(b) *Let $\Sigma_{0,5}$ be a 5-punctured sphere with boundary components a, b, c, d, e, and $m_1 \cup m_2$ a marking on $\Sigma_{0,5}$. Suppose that each of $m'_1 \cup m_2$ and $m_1 \cup m'_2$ is obtained from $m_1 \cup m_2$ by a single type II move, and neither $m'_1 \cap m_1$ nor $m_2 \cap m'_2$ contains isotopically removable points. If $\{a, b\}m_1\{c, d, e\}$, $\{a, b, c\}m_2\{d, e\}$, $\{a, d, e\}m'_1\{b, c\}$ and $\{a, b, e\}m'_2\{c, d\}$, then there is a simple closed curve n in $\Sigma_{0,5}$ such that $\{a, e\}n\{b, c, d\}$ and $n \cap m'_i = \emptyset$, and $n \cap m_i$ consists of two points of different intersection signs, for $i \in \{1, 2\}$.*

Proof. (a) Each marking on $\Sigma_{0,4}$ is a simple closed curve which separates the four boundary components of $\Sigma_{0,4}$ into two pairs (see Figure 1, left). Since m, m'' are related by a single type II move, we have either $\{a, d\}m\{b, c\}$ or $\{a, c\}m\{b, d\}$. If $\{a, d\}m\{b, c\}$, then we put $m^{(1)} = m$, $m^{(2)} = m''$, and $k = 2$. If $\{a, c\}m\{b, d\}$, then $m'' = t^k(m')$, where t is a right-hand or left-hand half Dehn twist along m, and k is a non-negative integer. Put $m^{(i)} = t^i(m')$ for each $i \in \{0, 1, \dots, k\}$. Then $m^{(i)}$ is obtained from $m^{(i-1)}$ by a single type II move, and for each $i \in \{1, \dots, k\}$, either $\{a, b\}m^{(i)}\{c, d\}$ or $\{a, d\}m^{(i)}\{b, c\}$.

(b) Let $\Sigma_{0,3}$ be a 3-punctured sphere which is obtained from $\Sigma_{0,5}$ by cutting open along $m_1 \cup m_2$ and has the boundary component c. Since $m'_1 \cap m'_2$ contains no isotopically removable points, there are two intersection points p_1, p_2 of m'_1 and m'_2 in $\Sigma_{0,3}$. For each $i \in \{1, 2\}$, there is an arc α_i in $m'_i \cap \Sigma_{0,3}$ whose boundary is $p_1 \cup p_2$. Let n be a simple closed curve in $\Sigma_{0,5}$ which is isotopic to the curve $\mathrm{cl}(m'_1 - \alpha_1) \cup \mathrm{cl}(m'_2 - \alpha_2)$, and does not intersect $m'_1 \cup m'_2$ as shown in Figure 1 (right). Then $\{a, e\}n\{b, c, d\}$ and $n \cap m_i$ consists of two points of different intersection signs because of the type II move between m'_i and m_i for $i \in \{1, 2\}$. ∎

Proof of Theorem 3.1. Throughout this proof, by a finite sequence of markings which connects two markings, we mean a finite sequence of markings from one to another such that any successive two are related by a single type II move and isotopies. By the assumption of Theorem 3.1, there is a finite sequence of markings which connects m and n. We assume that there are some markings in the sequence which contain separating simple

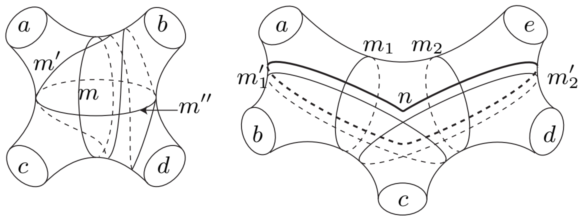


Fig. 1. Markings on punctured spheres

closed curves in $\Sigma_{g,r}$. For each marking of the sequence, we consider the number of separating simple closed curves contained in the marking. Let $m^{(0)}, m^{(1)}, \dots, m^{(k)}$ be a subsequence of the sequence that is a maximal part among the sequence, where $k \geq 2$. In other words, $m^{(0)}$ and $m^{(k)}$ each contain exactly $N-1$ separating simple closed curves, and $m^{(i)}$ contains exactly N separating simple closed curves for each $i \in \{1, \dots, k-1\}$. We will show that there is a finite sequence of markings which connects $m^{(k-2)}$ and $m^{(k)}$ such that each marking of the sequence except $m^{(k-2)}$ contains at most $N-1$ separating simple closed curves. Then we can reduce the total length of maximal parts of the sequence without changing the start m and the target n. By reducing the length repeatedly, we obtain a finite sequence of markings which connects m and n so that each marking consists of non-separating simple closed curves in $\Sigma_{g,r}$.

A single type II move replaces only one curve of a marking. Without loss of generality, we can put

$$\begin{aligned} m^{(k-2)} &= m_1 \cup \dots \cup m'_p \cup \dots \cup m_l, \\ m^{(k-1)} &= m_1 \cup \dots \cup m_l, \\ m^{(k)} &= m_1 \cup \dots \cup m''_q \cup \dots \cup m_l, \end{aligned}$$

and assume that $m'_p \cap m''_q$ contains no isotopically removable points. Note that m_q is separating and m''_q is non-separating in $\Sigma_{g,r}$. There are three cases for m_p and m_q: 1) $p = q$; 2) $p \neq q$ and the closure of some component of $\Sigma_{g,r} - m^{(k-1)}$ contains both m_p and m_q as boundary components; 3) $p \neq q$ and the closure of no component of $\Sigma_{g,r} - m^{(k-1)}$ contains both m_p and m_q as boundary components. Note that each of m_p, m'_p, m_q, m''_q connects exactly two pants components of $\Sigma_{g,r} - m^{(k-1)}$, because the pairs m_p, m'_p and m_q, m''_q are each related by a single type II move.

In the first case, the connected surface obtained from two components of $\Sigma_{g,r} - m^{(k-1)}$ by gluing along m_p is homeomorphic to the 4-punctured sphere. Let $\Sigma_{0,4}$ be the closure of that punctured sphere. Without loss of generality, we can put labels a, b, c, d on the four boundary components of $\Sigma_{0,4}$ so that $\{a, c\} m_p \{b, d\}$ and $\{a, b\} m''_p \{c, d\}$. By putting $m = m_p$,

$m' = m'_p$ and $m'' = m''_p$ in Lemma 3.3(a), we obtain a finite sequence $m_{p,0} = m'_p, m_{p,1}, \ldots, m_{p,k'} = m''_p$ of markings on $\Sigma_{0,4}$ connecting m'_p and m''_p such that for each $i \in \{1, \ldots, k'\}$, either $\{a,b\}m_{p,i}\{c,d\}$ or $\{a,d\}m_{p,i}\{b,c\}$. When there exists a component of the closure of $\Sigma_{g,r} - \Sigma_{0,4}$ which contains simple closed curves x and y, we write $x \sim y$. Since m_p is separating and m''_p is non-separating in $\Sigma_{g,r}$, we have $a \sim c$ or $b \sim d$. Hence $m_{p,i}$ is a non-separating simple closed curve in $\Sigma_{g,r}$ for each $i \in \{1, \ldots, k'\}$. Put $m^{(k,i)} = (m^{(k-1)} - m_p) \cup m_{p,i}$ for each $i \in \{0, 1, \ldots, k'\}$. Then $m^{(k,0)}, \ldots, m^{(k,k')}$ is a finite sequence of markings on $\Sigma_{g,r}$ connecting $m^{(k-2)}$ and $m^{(k)}$, and the number of separating simple closed curves of $m^{(k,i)}$ is $N-1$ for each $i \in \{1, \ldots, k'\}$.

In the second case, the connected surface obtained from some components of $\Sigma_{g,r} - m^{(k-1)}$ by gluing along $m_p \cup m_q$ is not homeomorphic to the 2-punctured torus, because m_q is a separating simple closed curve in $\Sigma_{g,r}$, and thus is homeomorphic to a 5-punctured sphere. Let $\Sigma_{0,5}$ be that sphere. Without loss of generality, we can put labels a, b, c, d, e on the five boundary components of $\Sigma_{0,5}$ so that $\{a,b\}m_p\{c,d,e\}$, $\{a,b,c\}m_q\{d,e\}$, $\{a,d,e\}m'_p\{b,c\}$ and $\{a,b,e\}m''_q\{c,d\}$. By putting $m_1 = m_p$, $m_2 = m_q$, $m'_1 = m'_p$ and $m'_2 = m''_q$ in Lemma 3.3(b), we obtain a simple closed curve n in $\Sigma_{0,5}$ such that $\{a,e\}n\{b,c,d\}$ and $n \cap m'_i = \emptyset$, and $n \cap m_i$ consists of two points of different intersection signs, for $i \in \{1,2\}$. When there exists (resp. does not exist) a component of the closure of $\Sigma_{g,r} - \Sigma_{0,5}$ which contains simple closed curves x and y, we write $x \sim y$ (resp. $x \nsim y$). We suppose that n is separating in $\Sigma_{g,r}$, and show that m_p is non-separating and m'_p is separating in $\Sigma_{g,r}$, which contradicts the assumption that the number of separating simple closed curves of $m^{(k-2)}$ is at most N. Since m''_q is non-separating in $\Sigma_{g,r}$, we have

$$a \sim c, \quad a \sim d, \quad b \sim c, \quad b \sim d, \quad e \sim c \quad \text{or} \quad e \sim d.$$

Since m_q and n are separating in $\Sigma_{g,r}$, we have $b \sim c$, which implies that m_p is non-separating in $\Sigma_{g,r}$. Since m_q and n are separating in $\Sigma_{g,r}$, we have

$$b \nsim e, \quad b \nsim d, \quad c \nsim e, \quad c \nsim d, \quad a \nsim b \quad \text{and} \quad a \nsim c,$$

which implies that m'_p is separating in $\Sigma_{g,r}$. Hence n is a non-separating simple closed curve in $\Sigma_{g,r}$. Put $m^{(k,1)} = (m^{(k-2)} - m_q) \cup n$ and $m^{(k,2)} = (m^{(k)} - m_p) \cup n$. Then $m^{(k-2)}, m^{(k,1)}, m^{(k,2)}, m^{(k)}$ is a finite sequence of markings on $\Sigma_{g,r}$ connecting $m^{(k-2)}$ and $m^{(k)}$, and the number of separating simple closed curves of $m^{(k,i)}$ is at most $N-1$ for each $i \in \{1,2\}$.

In the third case, $m_p \cap m_q = \emptyset$. Put $m^{(k,1)} = (m^{(k-2)} - m_q) \cup m''_q$. Then $m^{(k-2)}, m^{(k,1)}, m^{(k)}$ is a finite sequence of markings on $\Sigma_{g,r}$ connecting $m^{(k-2)}$ and $m^{(k)}$, and the number of separating simple closed curves of $m^{(k,1)}$ is at most $N-1$. ■

4. A handlebody-tangle. A *handlebody-link* [2] is a disjoint union of handlebodies embedded in the 3-sphere S^3. Two handlebody-links are *equivalent* if one can be transformed into the other by an isotopy of S^3. A *handlebody-tangle* is a disjoint union of handlebodies embedded in the 3-ball B^3 such that the intersection of the handlebodies and the boundary of B^3 consists of disks, which we call the *end disks* of the handlebody-tangle. Two handlebody-tangles are *equivalent* if one can be transformed into the other by an isotopy of B^3 fixed on the boundary of B^3. We note that a handlebody-tangle with no end disks corresponds to a handlebody-link.

A *uni-trivalent graph* is a finite graph with each vertex of valency 1 or 3. In this paper, a uni-trivalent graph may contain circle components. A *trivalent tangle* is a uni-trivalent graph embedded in B^3 such that the intersection of the graph and the boundary of B^3 is the union of all univalent vertices of the graph, where we call a univalent vertex an *end point* of the trivalent tangle. When a handlebody-tangle H is a regular neighborhood of a trivalent tangle T such that every end disk of H contains just one univalent vertex of T, we say that H is *represented* by T (see Figure 2).

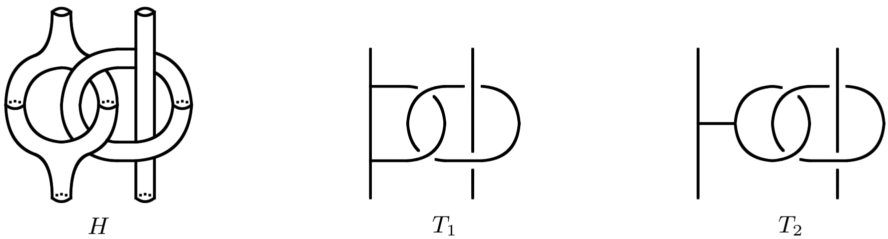


Fig. 2. T_1 and T_2 represent H

A *cut-edge* of a trivalent tangle T is an edge whose ends are trivalent vertices such that the number of connected components of T increases when we remove the edge. An *IH-move* is a local change of a trivalent tangle as described in Figure 3, where the replacement is applied in a disk embedded in the interior of B^3. Since a type II move corresponds to an IH-move, Corollaries 2.2 and 3.2 imply the following theorems (see also [2]).

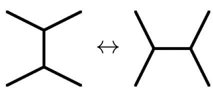


Fig. 3. An IH-move

THEOREM 4.1. *Two trivalent tangles with the same end points represent an equivalent handlebody-tangle if and only if they are related by a finite sequence of IH-moves and isotopies of B^3 fixed on the boundary of B^3.*

Theorem 4.2. *Two trivalent tangles with the same end points and no cut-edges represent an equivalent handlebody-tangle if and only if they are related by a finite sequence of IH-moves and isotopies of B^3 fixed on the boundary of B^3 such that all trivalent tangles appearing in the sequence have no cut-edges.*

A diagram of a trivalent tangle is a projection image of the trivalent tangle into the disk B^2 together with "over and under" information at each transversal double point, which is the only admissible multiple point on the diagram. Any two diagrams of equivalent spatial trivalent graphs are related by a finite sequence of moves R1–5 depicted in Figure 4 [18] (see also [9]). Since we may apply an IH-move in a small disk by an isotopy of B^3 fixed on the boundary of B^3, we have the following corollaries.

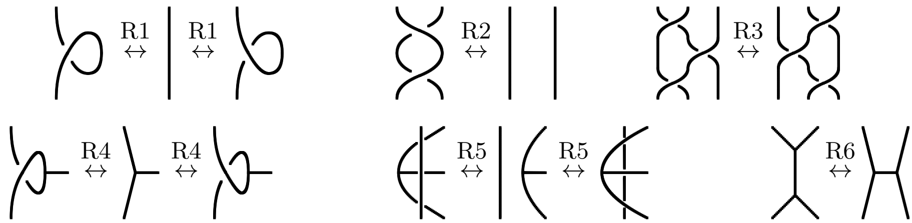


Fig. 4

Corollary 4.3. *Two trivalent tangles with the same end points represent an equivalent handlebody-tangle if and only if their diagrams are related by a finite sequence of moves* R1–6.

Corollary 4.4. *Two trivalent tangles with the same end points and no cut-edges represent an equivalent handlebody-tangle if and only if their diagrams are related by a finite sequence of moves* R1–6 *such that all diagrams appearing in the sequence are diagrams of trivalent tangles without cut-edges.*

In these corollaries, two diagrams are regarded as the same if they can be transformed into each other by an isotopy of B^2 fixed on the boundary of B^2.

Theorem 4.2 is a generalization of Theorem 7.1 in [10], where Koda showed the statement of Theorem 4.2 for spatial theta-curves by showing an actual deformation for a bypass. Kishimoto and the second author [5] introduced the *IH-complex* $\mathcal{C}_{\mathrm{IH}}$ of spatial trivalent graphs, which is the simplicial complex defined by the following conditions:

- The vertex set of $\mathcal{C}_{\mathrm{IH}}$ consists of all spatial trivalent graphs.
- A family of $n+1$ vertices $\{K_0, K_1, \ldots, K_n\}$ spans an n-simplex if and only if K_i, K_j are related by a single IH-move for any $i, j \in \{0, 1, \ldots, n\}$ such that $i \neq j$.

By Theorem 4.2, we have the following corollary.

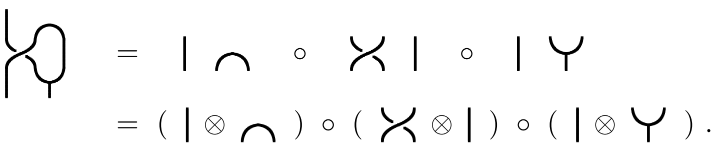
COROLLARY 4.5. *The full subcomplex of $\mathcal{C}_{\mathrm{IH}}$ spanned by spatial trivalent graphs without cut-edges is connected in each component of $\mathcal{C}_{\mathrm{IH}}$.*

5. Sliced diagrams. In this section and the next, we mainly follow the notation and definitions of [12, Chapter 3]. We consider trivalent tangles in I^3 whose end points lie on $I^2 \times \partial I$, and whose diagrams are depicted in I^2, where I is a closed interval. The last coordinate gives a height function. Two trivalent tangle diagrams are assumed to be the same if they can be transformed into each other by an isotopy preserving the order of crossings, vertices, maxima, and minima with respect to the height function. We note that the box I^3 may be resized as necessary.

Fig. 5

Fig. 6

For trivalent tangle diagrams D_1, D_2, we define their *tensor product* $D_1 \otimes D_2$ by the left diagram in Figure 5. For trivalent tangle diagrams D_1, D_2 such that the number of bottom ends of D_1 coincides with that of top ends of D_2, we define the *composition* $D_1 \circ D_2$ by the right diagram in Figure 5. The *elementary trivalent tangle diagrams* are those shown in Figure 6. Any trivalent tangle diagram can be expressed as a composition of tensor products of copies of elementary trivalent tangle diagrams. For example, we have



A *trivial tangle diagram* is a tensor product of copies of the leftmost diagram in Figure 6. We note that the empty diagram is regarded as a trivial tangle diagram. A classical tangle is a trivalent tangle without trivalent vertices. A trivial tangle diagram is a classical tangle diagram.

A *sliced diagram* is a composition of tensor products of trivial tangle diagrams and one elementary trivalent tangle diagram. In other words, a sliced diagram is a composition of trivalent tangle diagrams each of which contains at most one non-trivial elementary trivalent tangle diagram. The left diagram in Figure 7 is a sliced diagram, while the right one is not.

Fig. 7

Theorem 5.1.

(1) *Two sliced diagrams are related by an isotopy of the plane if and only if they are related by a finite sequence of moves* Pi*,* Pii*,* Piii*,* Piv *depicted in Figure* 8*.*
(2) *Two sliced diagrams represent an equivalent handlebody-tangle if and only if they are related by a finite sequence of moves depicted in Figure* 8*.*

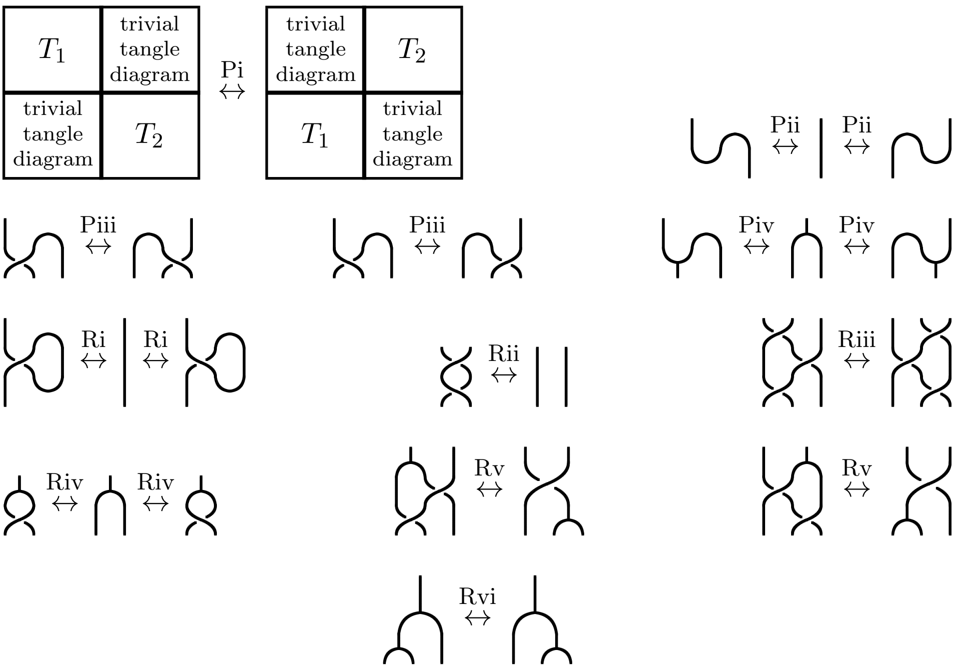
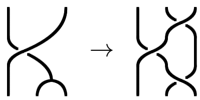


Fig. 8

Proof. The theorem is true for classical tangles [1, 16, 17] (see [12, Chapter 3]).

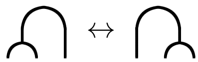
(1) If two sliced diagrams are related by a finite sequence of moves Pi, Pii, Piii, Piv, then they are related by an isotopy of the plane. Let D_1, D_2



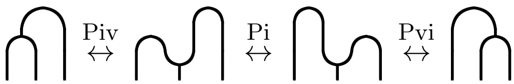
D_1 $\qquad$ D_1'

Fig. 9

be sliced diagrams which are related by an isotopy φ of the plane. Let D_1' be a classical sliced diagram obtained from D_1 by replacing every trivalent vertex by a crossing and adding an edge between the free end of the crossing and the top or bottom of the tangle diagram as shown in Figure 9, where we allow new crossings on the edge. Let D_2' be a classical sliced diagram obtained from D_1' by the isotopy φ. We remark that we can recover D_2 from D_2' by removing the added edges. Since the classical sliced diagrams D_1', D_2' are related by an isotopy of the plane, they are related by a finite sequence of moves Pi, Pii, Piii. When we remove the added edges, the moves Pi, Pii, Piii become Pi, Pii, Piii, Piv and the move



This last move and Piv are obtained from Piii by removing one of the four edges at a crossing. The last move is derived from Pi and Piv as follows:

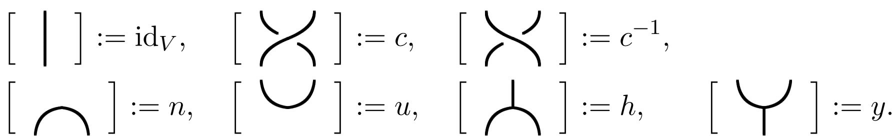


Therefore the sliced diagrams D_1, D_2 are related by a finite sequence of moves Pi, Pii, Piii, Piv.

(2) This follows from (1) and Corollary 4.3. ■

We note that moves based on [18] contain the following trivalent tangle diagrams as elementary trivalent tangle diagrams:

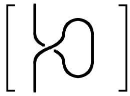
6. An operator invariant. Let V be a module over a ring R. For an automorphism $c : V \otimes V \to V \otimes V$ and linear maps $n : V \otimes V \to R$, $u : R \to V \otimes V$, $h : V \otimes V \to V$, $y : V \to V \otimes V$, to every sliced diagram D we assign a linear map $[D]$ as follows. We set



For a sliced diagram D, we define the linear map $[D]$ by using

$$[D_1 \otimes D_2] = [D_1] \otimes [D_2], \qquad [D_1 \circ D_2] = [D_1] \circ [D_2],$$

recursively. For example,

 $= (\mathrm{id}_V \otimes n)(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes u),$

where we denote $f \circ g$ by fg.

By Theorem 5.1, we have the following.

THEOREM 6.1. *If an automorphism* $c : V \otimes V \to V \otimes V$ *and linear maps* $n : V \otimes V \to R$, $u : R \to V \otimes V$, $h : V \otimes V \to V$, $y : V \to V \otimes V$ *satisfy the equalities*

$$
\begin{aligned}
&(\mathrm{id}_V \otimes n)(u \otimes \mathrm{id}_V) = \mathrm{id}_V = (n \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes u),\\
&(\mathrm{id}_V \otimes n)(c \otimes \mathrm{id}_V) = (n \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c^{-1}),\\
&(\mathrm{id}_V \otimes n)(c^{-1} \otimes \mathrm{id}_V) = (n \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c),\\
&(\mathrm{id}_V \otimes n)(y \otimes \mathrm{id}_V) = h = (n \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes y),\\
&(\mathrm{id}_V \otimes n)(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes u) = \mathrm{id}_V = (\mathrm{id}_V \otimes n)(c^{-1} \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes u),\\
&(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c),\\
&hc = h = hc^{-1},\\
&(h \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V) = c(\mathrm{id}_V \otimes h),\\
&(\mathrm{id}_V \otimes h)(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c) = c(h \otimes \mathrm{id}_V),\\
&h(h \otimes \mathrm{id}_V) = h(\mathrm{id}_V \otimes h),
\end{aligned}
$$

then $[D]$ *is an invariant of a handlebody-tangle* H *represented by* D. *In particular, if* H *is a handlebody-link,* $[D](1)$ *is an invariant which takes values in* R.

Put $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$ and $R := \mathbb{Z}[\omega]/(\omega^m - 1)$. We remark that ω^{is} is well-defined for $i \in \mathbb{Z}$ and $s \in \mathbb{Z}_m$. Let V be a free module over R with a basis $\{v_s^i \mid s \in \mathbb{Z}_m, i = 1, 2\}$. We define the automorphism $c : V \otimes V \to V \otimes V$ and the linear maps $n : V \otimes V \to R$, $u : R \to V \otimes V$, $h : V \otimes V \to V$, $y : V \to V \otimes V$ by

$$
\begin{aligned}
&c(v_s^i \otimes v_t^j) = \omega^{(1-\delta_{ij})st} v_t^j \otimes v_s^i,\\
&n(v_s^i \otimes v_t^j) = \delta_{ij}\delta_{s(-t)}, \qquad u(1) = \sum_{i\in\{1,2\}} \sum_{s\in\mathbb{Z}_m} v_s^i \otimes v_{-s}^i,\\
&h(v_s^i \otimes v_t^j) = \delta_{ij} v_{s+t}^i, \qquad y(v_s^i) = \sum_{t\in\mathbb{Z}_m} v_{s-t}^i \otimes v_t^i,
\end{aligned}
$$

where δ_{ij} is the Kronecker symbol. These linear maps satisfy the equalities in Theorem 6.1. We verify some of the equalities here:

$$(\mathrm{id}_V \otimes n)(u \otimes \mathrm{id}_V)(v_s^i) = (\mathrm{id}_V \otimes n)\Big(\sum_{j\in\{1,2\}}\sum_{t\in\mathbb{Z}_m} v_t^j \otimes v_{-t}^j \otimes v_s^i\Big)$$
$$= \sum_{j\in\{1,2\}}\sum_{t\in\mathbb{Z}_m} \delta_{ji}\delta_{(-t)(-s)} v_t^j = v_s^i = \mathrm{id}_V(v_s^i),$$

$$\begin{aligned}(\mathrm{id}_V \otimes n)(c \otimes \mathrm{id}_V)(v_s^i \otimes v_t^j \otimes v_u^k) &= (\mathrm{id}_V \otimes n)(\omega^{(1-\delta_{ij})st} v_t^j \otimes v_s^i \otimes v_u^k)\\ &= \omega^{(1-\delta_{ij})st}\delta_{ik}\delta_{s(-u)} v_t^j\\ &= \omega^{-(1-\delta_{jk})tu}\delta_{ik}\delta_{s(-u)} v_t^j\\ &= (n \otimes \mathrm{id}_V)(\omega^{-(1-\delta_{jk})tu} v_s^i \otimes v_u^k \otimes v_t^j)\\ &= (n \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c^{-1})(v_s^i \otimes v_t^j \otimes v_u^k),\end{aligned}$$

$$(\mathrm{id}_V \otimes n)(y \otimes \mathrm{id}_V)(v_s^i \otimes v_t^j) = (\mathrm{id}_V \otimes n)\Big(\sum_{u\in\mathbb{Z}_m} v_{s-u}^i \otimes v_u^i \otimes v_t^j\Big)$$
$$= \sum_{u\in\mathbb{Z}_m} \delta_{ij}\delta_{u(-t)} v_{s-u}^i = \delta_{ij} v_{s+t}^i = h(v_s^i \otimes v_t^j),$$

$$\begin{aligned}&(\mathrm{id}_V \otimes n)(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes u)(v_s^i)\\ &\quad= (\mathrm{id}_V \otimes n)(c \otimes \mathrm{id}_V)\Big(\sum_{j\in\{1,2\}}\sum_{t\in\mathbb{Z}_m} v_s^i \otimes v_t^j \otimes v_{-t}^j\Big)\\ &\quad= (\mathrm{id}_V \otimes n)\Big(\sum_{j\in\{1,2\}}\sum_{t\in\mathbb{Z}_m} \omega^{(1-\delta_{ij})st} v_t^j \otimes v_s^i \otimes v_{-t}^j\Big)\\ &\quad= \sum_{j\in\{1,2\}}\sum_{t\in\mathbb{Z}_m} \omega^{(1-\delta_{ij})st}\delta_{ij}\delta_{st} v_t^j = v_s^i = \mathrm{id}_V(v_s^i),\end{aligned}$$

$$\begin{aligned}&(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V)(v_s^i \otimes v_t^j \otimes v_u^k)\\ &\quad= (c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(\omega^{(1-\delta_{ij})st} v_t^j \otimes v_s^i \otimes v_u^k)\\ &\quad= (c \otimes \mathrm{id}_V)(\omega^{(1-\delta_{ij})st}\omega^{(1-\delta_{ik})su} v_t^j \otimes v_u^k \otimes v_s^i)\\ &\quad= \omega^{(1-\delta_{ij})st}\omega^{(1-\delta_{ik})su}\omega^{(1-\delta_{jk})tu} v_u^k \otimes v_t^j \otimes v_s^i\\ &\quad= (\mathrm{id}_V \otimes c)(\omega^{(1-\delta_{ik})su}\omega^{(1-\delta_{jk})tu} v_u^k \otimes v_s^i \otimes v_t^j)\\ &\quad= (\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V)(\omega^{(1-\delta_{jk})tu} v_s^i \otimes v_u^k \otimes v_t^j)\\ &\quad= (\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(v_s^i \otimes v_t^j \otimes v_u^k),\end{aligned}$$

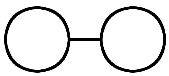
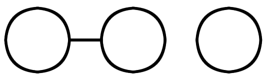
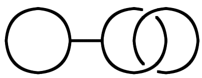
$$\begin{aligned}hc(v_s^i \otimes v_t^j) &= h(\omega^{(1-\delta_{ij})st} v_t^j \otimes v_s^i)\\ &= \omega^{(1-\delta_{ij})st}\delta_{ji} v_{t+s}^j = \delta_{ij} v_{s+t}^i = h(v_s^i \otimes v_t^j),\end{aligned}$$

$$\begin{aligned}
(h\otimes \mathrm{id}_V)(\mathrm{id}_V\otimes c)(c\otimes \mathrm{id}_V)(v_s^i\otimes v_t^j\otimes v_u^k) \\
&= (h\otimes \mathrm{id}_V)(\mathrm{id}_V\otimes c)(\omega^{(1-\delta_{ij})st}v_t^j\otimes v_s^i\otimes v_u^k)\\
&= (h\otimes \mathrm{id}_V)(\omega^{(1-\delta_{ij})st}\omega^{(1-\delta_{ik})su}v_t^j\otimes v_u^k\otimes v_s^i)\\
&= \omega^{(1-\delta_{ij})st}\omega^{(1-\delta_{ik})su}\delta_{jk}v_{t+u}^j\otimes v_s^i\\
&= \omega^{(1-\delta_{ij})s(t+u)}\delta_{jk}v_{t+u}^j\otimes v_s^i\\
&= c(\delta_{jk}v_s^i\otimes v_{t+u}^j)\\
&= c(\mathrm{id}_V\otimes h)(v_s^i\otimes v_t^j\otimes v_u^k),
\end{aligned}$$

$$\begin{aligned}
h(h\otimes \mathrm{id}_V)(v_s^i\otimes v_t^j\otimes v_u^k) &= h(\delta_{ij}v_{s+t}^i\otimes v_u^k) = \delta_{ij}\delta_{ik}v_{s+t+u}^i = \delta_{ij}\delta_{jk}v_{s+t+u}^i\\
&= h(\delta_{jk}v_s^i\otimes v_{t+u}^j) = h(\mathrm{id}_V\otimes h)(v_s^i\otimes v_t^j\otimes v_u^k).
\end{aligned}$$

By Theorem 6.1, we obtain an invariant associated with the linear maps c, n, u, h, y. In Table 1, we evaluate the invariant for some handlebody-links. We note that this invariant is a kind of linking invariant and gives the same value for any handlebody-knot of fixed genus.

Table 1

	$2m^2$
	$4m^3$
	$2m\sum_{s,t\in\mathbb{Z}_m}\omega^{2st}+2m^3$

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Kai Ishihara
Department of Mathematics and Information Sciences
Faculty of Education
Yamaguchi University
1677-1 Yoshida
Yamaguchi 753-8513, Japan
E-mail: kisihara@yamaguchi-u.ac.jp

Atsushi Ishii
Institute of Mathematics
University of Tsukuba
1-1-1 Tennodai
Tsukuba, Ibaraki 305-8571, Japan
E-mail: aishii@math.tsukuba.ac.jp

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