FUNDAMENTA
MATHEMATICAE
219 (2012)

Waraszkiewicz spirals revisited

by

Pavel Pyrih and **Benjamin Vejnar** (Praha)

Abstract. We study compactifications of a ray with remainder a simple closed curve. We give necessary and sufficient conditions for the existence of a bijective (resp. surjective) mapping between two such continua. Using those conditions we present a simple proof of the existence of an uncountable family of plane continua no one of which can be continuously mapped onto any other (the first such family, so called Waraszkiewicz's spirals, was created by Z. Waraszkiewicz in the 1930's).

1. Introduction. In the 1930's Z. Waraszkiewicz constructed an uncountable family of plane continua no one of which can be mapped onto any other by a continuous mapping [War]. This family consists of continua which can be obtained as compactifications of a ray with remainder a simple closed curve.

Using the same construction Z. Waraszkiewicz [War2] proved that there is no universal continuum (a continuum which can be mapped onto all continua), solving a problem posed by H. Hahn [Hahn, p. 357].

Unfortunately both proofs are very technical and long. A nice short proof of the second result was given by D. P. Bellamy (the proof was never published; a modification of Bellamy's original proof can be found in [MaTy, pp. 49–50]).

In the present paper a simple proof of Waraszkiewicz's first result is given (notice that the existence of an uncountable family with no common preimage does not imply the incomparability of members of the family, i.e. the second result does not imply the first one).

Notice that since Waraszkiewicz's results, a lot of other attempts has been made to construct an uncountable family with some additional properties. We mention D. P. Bellamy [Bel] for chainable continua, M. M. Awartani

2010 *Mathematics Subject Classification*: Primary 54F15; Secondary 54F50.
Key words and phrases: continuum, spiral, continuous map, incomparability, uncountable family.

DOI: 10.4064/fm219-2-1

[Awa] for chainable compactifications of a ray, P. Minc [Min] for dendroids and C. Islas [Isl] for planar fans.

The existence of a common model for some classes of continua (a continuum which can be mapped onto all members of a given class) is discussed in R. L. Russo [Rus] (using Waraszkiewicz's original method from [War2]).

Our proof consists of several steps. First, we reformulate the original topological problem into the language of real functions. Next, we prove that it is enough to obtain a special combinatorial structure. Finally, we construct such a structure.

2. Preliminaries. A *continuum* is a non-empty compact connected metrizable space. By a *path component* of a continuum we mean a maximal subset S of the continuum such that for any pair of points x and y from S there always exists a continuous mapping of the unit interval into S which sends 0 to x and 1 to y.

An *arc* is a space homeomorphic to the closed unit interval $[0,1]$. A *ray* is a space homeomorphic to $[0,1)$. A *simple closed curve* is a space homeomorphic to the unit circle. A *spiral* is a continuum obtained by compactification of a ray with remainder a simple closed curve.

We denote by $\mathbb{S}$ the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ in the complex plane. A *lift* of a continuous mapping $\varphi\colon \mathbb{S} \to \mathbb{S}$ is a continuous function $j\colon \mathbb{R} \to \mathbb{R}$ such that $e^{2\pi i j(s)} = \varphi(e^{2\pi i s})$ for any $s \in \mathbb{R}$.

An *almost disjoint* system $\mathcal{S}$ on a set S is a family of subsets of S such that the intersection of any two distinct elements of $\mathcal{S}$ is finite.

We say that two continua are (*continuously*) *comparable* if there is a mapping from one onto the other. Otherwise we call them *mutually incomparable.*

3. Spirals. We give a complete characterization of comparability of two spirals using real functions only. Consequently, we give a simple necessary condition for comparability of two spirals.

Notation 1. We denote by $\mathbb{H}$ the ray $[0,\infty)$ and by $\overline{\mathbb{H}}$ the one-point compactification $\mathbb{H} \cup \{\infty\}$. With every continuous function $f\colon \mathbb{H} \to \mathbb{R}$ we associate a continuum

$$W_f = \{(e^{2\pi i f(t)}, t) \in \mathbb{S} \times \mathbb{H} \colon t \geq 0\} \cup (\mathbb{S} \times \{\infty\})$$

considered as a subspace of $\mathbb{S} \times \overline{\mathbb{H}}$.

Theorem 2. *Let $f, g\colon \mathbb{H} \to \mathbb{R}$ be two continuous functions which do not have a finite limit at infinity. Then W_f can be continuously mapped onto W_g if and only if there exist $k \in \mathbb{Z}$ and continuous functions $h\colon \mathbb{H} \to \mathbb{H}$ and $j\colon \mathbb{R} \to \mathbb{R}$ such that*

- *h is onto,*
- $\lim_{t\to\infty} h(t) = \infty$*,*
- *$j(s+1) - j(s) = k$ for every $s \in \mathbb{R}$,*
- *$j(\mathbb{R})$ contains an interval of length one,*
- $\lim_{t\to\infty}(j \circ f(t) - g \circ h(t)) = 0$*.*

Moreover W_f is homeomorphic to W_g if and only if there is a homeomorphism $h\colon \mathbb{H} \to \mathbb{H}$ and a homeomorphism $j\colon \mathbb{R} \to \mathbb{R}$ such that all the five conditions hold with $k = 1$.

Proof. Let us start with the direct implication. We suppose there is a continuous onto mapping $\varphi\colon W_f \to W_g$. Since neither f nor g has a finite limit at infinity, it follows that the continua W_f and W_g each have two path components, one of which is dense and the other not. This implies that φ maps the dense path component $\{(e^{2\pi i f(t)}, t)\colon t \geq 0\}$ of W_f onto the dense path component $\{(e^{2\pi i g(t)}, t)\colon t \geq 0\}$ of W_g, and $\mathbb{S} \times \{\infty\}$ onto $\mathbb{S} \times \{\infty\}$. We denote by $p\colon \mathbb{S} \times \mathbb{H} \to \mathbb{H}$ the projection onto the second coordinate and define $h\colon \mathbb{H} \to \mathbb{H}$ by $h(t) = p \circ \varphi(e^{2\pi i f(t)}, t)$. We observe that

$$\varphi(e^{2\pi i f(t)}, t) = (e^{2\pi i g\circ h(t)}, h(t)).$$

The function h is continuous and onto. Moreover $h(t)$ converges to infinity as t goes to infinity.

There exists a continuous function $j'\colon \mathbb{R} \to \mathbb{R}$ which is a lift of φ restricted to $\mathbb{S} \times \{\infty\}$. Hence $\varphi(e^{2\pi i s}, \infty) = (e^{2\pi i j'(s)}, \infty)$ for every $s \in \mathbb{R}$. The difference $j'(s+1) - j'(s)$ is a fixed integer k. Since $\varphi(\mathbb{S} \times \{\infty\}) = \mathbb{S} \times \{\infty\}$, we see that $j'([0,1])$ contains an interval of length one.

Now we are going to prove that all the cluster points of the function $j' \circ f - g \circ h$ are integers, i.e. if $t_n \to \infty$ and $j' \circ f(t_n) - g \circ h(t_n)$ converges to $r \in \mathbb{R} \cup \{-\infty, \infty\}$ then $r \in \mathbb{Z}$. Towards a contradiction, suppose there is a cluster point $r \in \mathbb{R} \cup \{-\infty, \infty\}$ of $j' \circ f - g \circ h$ which is not an integer.

Suppose first that $r \in \mathbb{R}$. Then there is a sequence (t_n) in $\mathbb{H}$ which converges to infinity and for which $j' \circ f(t_n) - g \circ h(t_n) \to r$. By passing to a subsequence we can assume that $e^{2\pi i f(t_n)} \to e^{2\pi i s}$ for some $s \in \mathbb{R}$. Then $e^{2\pi i j' \circ f(t_n)} \to e^{2\pi i j'(s)}$. Since φ is continuous we find that $\varphi(e^{2\pi i f(t_n)}, t_n) \to \varphi(e^{2\pi i s}, \infty)$. Thus $(e^{2\pi i g\circ h(t_n)}, h(t_n)) \to (e^{2\pi i j'(s)}, \infty)$ and hence $e^{2\pi i g\circ h(t_n)} \to e^{2\pi i j'(s)}$, so $e^{2\pi i (g\circ h(t_n) - j'\circ f(t_n))}$ converges to 1. But it converges to $e^{-2\pi i r}$, thus $r \in \mathbb{Z}$. This is a contradiction.

If $r \in \{-\infty, \infty\}$ we can find a sequence (t_n) in $\mathbb{H}$ converging to infinity such that $e^{2\pi i (j'\circ f(t_n) - g\circ h(t_n))} = -1$ for every n, and we obtain a contradiction as in the previous part.

Since $j' \circ f - g \circ h$ is a continuous function whose cluster points are integers, it follows that there is a unique integer cluster point m and thus it is the limit of this function at infinity.

Finally we put $j = j' - m$. It follows that $j(s+1) - j(s) = k$ for every $s \in \mathbb{R}$, $j(\mathbb{R})$ contains an interval of length one and

$$\lim_{t\to\infty} (j \circ f(t) - g \circ h(t)) = m - m = 0.$$

Moreover if φ is a homeomorphism then so are h and j, and we have $k = 1$.

For the reverse implication suppose we are given k, h and j as in the statement. We define $\varphi\colon W_f \to W_g$ by

$$\begin{aligned} \varphi(e^{2\pi i f(t)}, t) &= (e^{2\pi i g\circ h(t)}, h(t)) \quad && \text{if } t \in \mathbb{H}, \\ \varphi(e^{2\pi i s}, \infty) &= (e^{2\pi i j(s)}, \infty) && \text{if } s \in \mathbb{R}. \end{aligned}$$

This is a correctly defined mapping since $j(s+1) - j(s)$ is an integer. The mapping φ is onto, because h is onto and $j(\mathbb{R})$ contains an interval of length one. Clearly φ is continuous at all points of the form $(e^{2\pi i f(t)}, t)$ for $t \in \mathbb{H}$, because h is continuous. Moreover the restriction of φ to $\mathbb{S} \times \{\infty\}$ is continuous, because j is continuous.

It remains to show that the φ image of a sequence $(e^{2\pi i f(t_n)}, t_n)$ converging to $(e^{2\pi i s}, \infty)$ converges to $\varphi(e^{2\pi i s}, \infty)$. Thus we have to show that $(e^{2\pi i g\circ h(t_n)}, h(t_n)) \to (e^{2\pi i j(s)}, \infty)$. Clearly $h(t_n) \to \infty$, because $h(t) \to \infty$ as $t \to \infty$. We know that $e^{2\pi i (j\circ f(t_n) - g\circ h(t_n))} \to 1$ and $e^{2\pi i j\circ f(t_n)} \to e^{2\pi i j(s)}$. Hence $e^{2\pi i g\circ h(t_n)} \to e^{2\pi i j(s)}$. Thus φ maps continuously W_f onto W_g. Moreover if h and j are homeomorphisms and $k = 1$ we can easily observe that φ is a one-to-one mapping and thus a homeomorphism, being a bijection of two compact spaces. ∎

Corollary 3. *Let $f, g\colon \mathbb{H} \to \mathbb{R}$ be two continuous functions which do not have a finite limit at infinity. If W_f can be continuously mapped onto W_g then there exist a continuous function $h\colon \mathbb{H} \to \mathbb{H}$ converging to infinity and $k \in \mathbb{Z}$ such that*

$$\sup_{t\in\mathbb{H}} |kf(t) - g \circ h(t)| < \infty.$$

Proof. By Theorem 2 there exist $k \in \mathbb{Z}$, $h\colon \mathbb{H} \to \mathbb{H}$ and $j\colon \mathbb{R} \to \mathbb{R}$ with the properties given there. We can assume that the function $s \mapsto j(s) - ks$ is periodic and continuous and hence bounded. Let l be a number such that $|j(s) - ks| + 1 \le l$ for every $s \in \mathbb{R}$. Since $j \circ f(t) - g \circ h(t) \to 0$ as $t \to \infty$, there exists $t_0 \in \mathbb{H}$ such that $|j \circ f(t) - g \circ h(t)| \le 1$ for every $t > t_0$. Thus for $t > t_0$ we obtain

$$|kf(t) - g \circ h(t)| \le |kf(t) - j \circ f(t)| + |j \circ f(t) - g \circ h(t)| \le l - 1 + 1 = l.$$

Moreover the function $|kf(t) - g \circ h(t)|$ is continuous and hence bounded on $[0, t_0]$. Thus the required inequality holds. ∎

Example 4. Let $f\colon \mathbb{H} \to \mathbb{R}$ be the identity function and let $g\colon \mathbb{H} \to \mathbb{R}$ be a piecewise linear function whose break points are at positive integers and for which $g(2n-2) = 0$ and $g(2n-1) = 2n-1$ for every positive integer n. We claim that the corresponding spirals W_f and W_g are incomparable.

Indeed, suppose first that W_f can be continuously mapped onto W_g. Then by Corollary 3 there is a continuous mapping $h\colon \mathbb{H} \to \mathbb{H}$ converging to infinity and $k \in \mathbb{Z}$ such that

$$\sup_{t\in\mathbb{H}} |kf(t) - g\circ h(t)| < \infty.$$

Hence $l := \sup_{t\in\mathbb{H}} |kt - g\circ h(t)| < \infty$. There is an increasing sequence $(t_i)_{i=1}^{\infty}$ of positive numbers converging to infinity such that $h(t_i)$ is always an even integer. Thus $|kt_i - g\circ h(t_i)| \le l$ and hence $|kt_i| \le l$ for every i. Consequently, $k = 0$. Let $(u_i)_{i=1}^{\infty}$ be an increasing sequence converging to infinity for which $h(u_i)$ is always an odd integer and $u_1 > t_0$. Then $|g\circ h(u_i)| \le l$ for every i and hence $|h(u_i)| \le l$ for every i. This contradicts the fact that h converges to infinity.

Now suppose that W_g can be continuously mapped onto W_f. By Corollary 3 there is a continuous mapping $h\colon \mathbb{H} \to \mathbb{H}$ converging to infinity and $k \in \mathbb{Z}$ such that

$$\sup_{t\in\mathbb{H}} |kg(t) - f\circ h(t)| < \infty.$$

Hence $l := \sup_{t\in\mathbb{H}} |kg(t) - h(t)| < \infty$. Let $(t_i)_{i=1}^{\infty}$ be an increasing sequence of even positive integers converging to infinity. Then $|kg(t_i) - h(t_i)| \le l$ and thus $|h(t_i)| \le l$ for every i. This again contradicts the fact that h converges to infinity.

4. Peak points. We define a notion of peak point. We prove two simple lemmas about the behavior of this notion with respect to composition of functions and with respect to near functions.

Definition 5. Let $I \subseteq \mathbb{R}$ be an interval, $f\colon I \to \mathbb{R}$ a continuous function, $x \in I$ and $v \in \mathbb{R}$. We say that f has a *peak of height* v at x if there exists an interval $[a,b] \subseteq I$ containing x such that $f(t) \le f(x)$ for all $t \in [a,b]$, $f(a) \le f(x) - v$ and $f(b) \le f(x) - v$.

Lemma 6. *Let $I \subseteq \mathbb{R}$ be an interval, $g\colon I \to \mathbb{R}$ a continuous function and $h\colon I \to I$ a continuous onto function. If g has a peak of height v at y, then there is an $x \in I$ such that $g\circ h$ has a peak of height v at x and $g(y) = g\circ h(x)$.*

Proof. Since h is onto, there is an $x \in I$ such that $h(x) = y$. Clearly $g(y) = g\circ h(x)$. Since y is a peak point of height v, there exists an interval $[a,b] \subseteq I$ containing y such that $g(t) \le g(y)$ for $t \in [a,b]$, $g(a) \le g(x) - v$ and $g(b) \le g(x) - v$. There is an interval $[c,d] \subseteq I$ containing y for which $h(\{c,d\}) = \{a,b\}$ and $h([c,d]) = [a,b]$. Clearly for any $t \in [c,d]$ we get

$g \circ h(t) \leq g(y) = g \circ h(x)$, and $g \circ h(c)$ as well as $g \circ h(d)$ are less than $g(y) - v = g \circ h(x) - v$. Thus x is a peak point of $g \circ h$ of height v. ■

LEMMA 7. *Let $I \subseteq \mathbb{R}$ be an interval, $u > 0$ and $f, g\colon I \to \mathbb{R}$ two continuous functions such that $|f - g| \leq u$. If f has a peak point x of height v then g has a peak point y of height $v - 2u$ such that $|f(x) - g(y)| \leq u$.*

Proof. There is an interval $[a, b] \subseteq I$ containing x such that $f(t) \leq f(x)$ for every $t \in [a, b]$, $f(a) \leq f(x) - v$ and $f(b) \leq f(x) - v$. We take a point $y \in [a, b]$ at which g attains its maximum on $[a, b]$. Then $|f(x) - g(y)| \leq u$. It follows that $g(a) \leq f(a) + u \leq f(x) - v + u \leq g(y) + u - v + u = g(y) - (v - 2u)$. Similarly $g(b) \leq g(y) - (v - 2u)$. Hence g has a peak point y of height $v - 2u$. ■

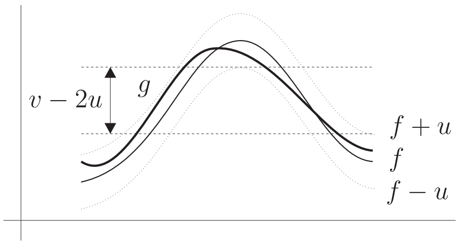

Fig. 1. Peak point of height $v - 2u$ for the mapping g

5. Reduction to a discrete case. We now associate a function with every subset of positive integers. Then we give a condition under which the spirals corresponding to two such functions are not comparable.

NOTATION 8. For an infinite set M of positive integers $m_1 < m_2 < \cdots$, we define a continuous piecewise linear function $f_M\colon \mathbb{H} \to \mathbb{R}$ whose only break points are at positive integers and $f(2i - 2) = 0$ and $f(2i - 1) = m_i$ for any positive integer i.

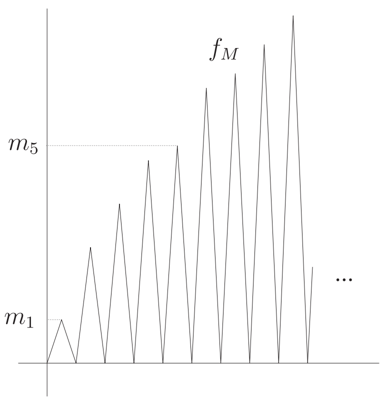

Fig. 2. Piecewise linear mapping f_M

PROPOSITION 9. *Let M, N be two infinite sets of positive integers and suppose that W_{f_M} can be continuously mapped onto W_{f_N}. Then there exist positive integers k and l such that for every $n \in N$ there is $m \in M$ for which $|km - n| \leq l$.*

Proof. Denote $M = \{m_1 < m_2 < \cdots\}$ and $N = \{n_1 < n_2 < \cdots\}$. By Corollary 3 there exist a continuous function $h \colon \mathbb{H} \to \mathbb{H}$ converging to infinity, $k \in \mathbb{Z}$ and a positive integer l' such that $|kf_M(t) - f_N \circ h(t)| \leq l'$ for any $t \in \mathbb{H}$. Since f_M as well as $f_N \circ h$ are unbounded non-negative functions we see that $k > 0$. Fix a positive integer q greater than t_0 and $2l'$. Let l be a positive integer greater than

$$\max\{l', |km_1 - n_1|, \ldots, |km_1 - n_q|\}.$$

For any positive integer $j > q$ the function f_N has a peak point $2j - 1$ of height n_j and $f_N(2j - 1) = n_j$. By Lemma 6 the function $f_N \circ h$ has a peak point x of height n_j and $f_N \circ h(x) = n_j$. Now, Lemma 7 applied to $g = kf_M$ and $f = f_N \circ h$ on the interval $I = (0, \infty)$ shows that kf has a peak point y of height $n_j - 2l'$ such that $|kf_M(y) - f_N \circ h(x)| \leq l'$. Note that $n_j - 2l' \geq j - 2l' > q - 2l' > 0$. We can see that the only peak points of positive height of kf_M are odd positive integers. Hence, there exists a positive integer i such that $2i - 1 = y$. Consequently, $kf_M(2i - 1) = km_i$ and thus $|km_i - n_j| \leq l' \leq l$. ∎

6. An uncountable family. We now construct an uncountable system of infinite subsets of positive integers such that for any pair of distinct elements the conclusion of Proposition 9 is not satisfied.

PROPOSITION 10. *There exists an uncountable system $\mathcal{S}$ of infinite subsets of positive integers such that for any pair of distinct sets $M, N \in \mathcal{S}$ and for any positive integers k and l there is a point $n \in N$ such that $|km - n| > l$ for every $m \in M$.*

Proof. Let $\mathcal{S}$ be an uncountable almost disjoint system of infinite subsets of $\{1!, 2!, 3!, \ldots\}$. Suppose that $M, N \in \mathcal{S}$ are two distinct sets and k, l are positive integers. Let $n = j!$ be an element of $N \setminus M$ such that j is greater than k and $l + 1$. Now for any $m = i! \in M$ we find that if $i < j$ then

$$|km - n| \geq n - km = j! - ki! \geq j! - (j-1)(j-1)! = (j-1)! \geq j - 1 > l,$$

and if $i > j$ then

$$|km - n| \geq km - n = ki! - j! \geq i! - j! \geq (j+1)! - j! \geq j > l. \ ∎$$

COROLLARY 11. *There exists an uncountable collection of incomparable plane continua.*

Proof. We take an uncountable system $\mathcal{S}$ from Proposition 10. It follows from Proposition 9 that the collection $\{W_{f_M} : M \in \mathcal{S}\}$ consists of pairwise incomparable continua. All of them are planar because they are subcontinua of $\mathbb{S} \times \overline{\mathbb{H}}$. ∎

7. Remarks. We note that the system $\mathcal{S}$ in Proposition 10 can be made of size continuum (by well known simple constructions without using the axiom of choice). Moreover in the proof of Proposition 10 we do not need an almost disjoint system: it suffices to take a system of subsets of positive integers such that the difference of any two distinct sets is infinite.

Acknowledgements. The paper is an outgrowth of joint research of the Open Problem Seminar, Charles University, Prague. This research was supported by the grant MSM 0021620839.

The authors would like to thank very much the referees for their helpful suggestions to modify this paper.

References

[Awa] M. M. Awartani, *An uncountable collection of mutually incomparable chainable continua*, Proc. Amer. Math. Soc. 118 (1993), 239–245.

[Bel] D. P. Bellamy, *An uncountable collection of chainable continua*, Trans. Amer. Math. Soc. 160 (1971), 297–304.

[Hahn] H. Hahn, *Problem 52*, Fund. Math. 15 (1930), 357.

[Isl] C. Islas, *An uncountable collection of mutually incomparable planar fans*, Topology Proc. 31 (2007), 151–161.

[MaTy] T. Maćkowiak and E. D. Tymchatyn, *Continuous mappings on continua II*, Dissertationes Math. 225 (1984).

[Min] P. Minc, *An uncountable collection of dendroids mutually incomparable by continuous functions*, Houston J. Math. 36 (2010), 1185–1205.

[Rus] R. L. Russo, *Universal continua*, Fund. Math. 105 (1979), 41–60.

[War] Z. Waraszkiewicz, *Une famille indénombrable de continus plans dont aucun n'est l'image d'un autre*, Fund. Math. 18 (1932), 118–137.

[War2] Z. Waraszkiewicz, *Sur un problème de M. H. Hahn*, Fund. Math. 22 (1934), 180–205.

Pavel Pyrih, Benjamin Vejnar
Faculty of Mathematics and Physics
Charles University
Sokolovská 83
CZ-186 75 Praha 8, Czech Republic
E-mail: pyrih@karlin.mff.cuni.cz
vejnar@karlin.mff.cuni.cz

Received 26 May 2011;
in revised form 5 August 2012