Turning Borel sets into clopen sets effectively

by

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Abstract. We present the effective version of the theorem about turning Borel sets in Polish spaces into clopen sets while preserving the Borel structure of the underlying space. We show that under some conditions the emerging parameters can be chosen in a hyperarithmetical way and using this we obtain some uniformity results.

1. Introduction. One of the topics of effective descriptive set theory is the refinement of well-known theorems in recursive-theoretic terms. This “effective” version of a theorem is stronger than the original one, and as is often the case, it provides a uniformity result which does not seem to follow from the original statement. A typical example is Suslin’s Theorem, which states that every bi-analytic subset of a Polish space is Borel, and its refinement, the Suslin–Kleene Theorem, which provides a recursive (and thus continuous) function \( u : \mathcal{N} \times \mathcal{N} \to \mathcal{N} \), where \( \mathcal{N} = \omega^\omega \), such that whenever \( \alpha \) and \( \beta \) are codes of complementary analytic sets, say \( A \) and \( \mathcal{X} \setminus A \), then \( u(\alpha, \beta) \) is a Borel code of \( A \) (cf. [5, 7B.4] and [6]).

In this article we prove the effective version of the following well-known theorem of classical descriptive set theory: if \((\mathcal{X}, \mathcal{T})\) is a Polish space and \( A \) is a Borel subset of \( \mathcal{X} \), then

\[ \text{(*) there is a Polish topology } \mathcal{T}_\infty \text{ on } \mathcal{X} \text{ which extends } \mathcal{T}, \text{ yields the same Borel sets as } \mathcal{T}, \text{ and } A \text{ is clopen in } (\mathcal{X}, \mathcal{T}_\infty) \text{ (Theorem 2.1)}. \]

Our proof will not follow the usual proof of the latter theorem. We will instead present a different proof, perhaps new. We will see that there are certain advantages of our approach which allow us to proceed with the effective version. Let us give a brief description of the usual proof. One starts with a Polish space \( \mathcal{X} \), defines

\[ \mathcal{S} = \{ A \subseteq \mathcal{X} \mid A \text{ is Borel and satisfies (*) above} \} \]

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and shows that $\mathcal{S}$ is a $\sigma$-algebra which contains the open sets and so it contains every Borel set. The effective version of this proof requires the notion of the effective $\sigma$-field (cf. [5, Section 7B]) and a rather messy encoding of Polish topologies as we proceed with the induction. Moreover this approach seems to provide little information on the best choice for the parameters which will emerge.

On the other hand our approach is based on the following result of Lusin–Suslin: every Borel subset of a Polish space is the injective continuous image of a closed subset of $\mathcal{N}$ ([11, 13.7]). So let us assume that $\mathcal{X}$ is a Polish space and that $\mathcal{A}$ is a Borel subset of $\mathcal{X}$. Then there are closed sets $F_1, F_2 \subseteq \mathcal{N}$ and continuous functions $\pi_1, \pi_2 : \mathcal{N} \to \mathcal{X}$ such that $\pi_i$ is injective on $F_i$, $i = 1, 2$, $\pi_1[F_1] = \mathcal{A}$ and $\pi_2[F_1] = \mathcal{X} \setminus \mathcal{A}$. Now we define a distance function $d_{\mathcal{A}}$ on $\mathcal{A}$ in such a way that $\pi_1$ becomes an isometry, i.e. $d_{\mathcal{A}}(x, y) = p_{\mathcal{N}}(\alpha, \beta)$, where $\alpha, \beta \in F_1$ with $\pi_1(\alpha) = x$ and $\pi_1(\beta) = y$. Similarly we define the distance function $d_{\mathcal{A}^c}$ on the complement of $\mathcal{A}$ and then we consider the direct sum $(\mathcal{A}, d_{\mathcal{A}}) \oplus (\mathcal{X} \setminus \mathcal{A}, d_{\mathcal{A}^c})$. The latter space has all the required properties.

The advantage of the latter proof is that it reduces the problem from Borel sets to closed sets where the verification of Cauchy-completeness is obvious. Moreover it is straightforward to effectivize, for the effective version of the Lusin–Suslin Theorem has been proved by Moschovakis (cf. [5, 4A.7]). Finally, as we will see, this approach provides very clear information about the emerging parameters.

In the rest of this section we recall the basic definitions and notation. We assume that the reader is familiar with recursion theory and effective descriptive set theory (cf. [5, Chapter 3]). In the next section we prove our main theorem (Theorem 2.1), and afterwards we examine the problem of choosing the emerging parameters in a $\Delta^1_1$ way (cf. Theorems 3.9 and 3.11). We conclude this article with a related uniformity result about choosing the extended topology $\mathcal{T}_\infty$ in a Borel way (cf. Theorem 4.2). This is the only result of this article whose statement is purely classical in the sense that it involves no notions from effective theory. Its proof however is a corollary of almost all the preceding effective results.

Notation and definitions. By $\omega$ we mean the least infinite ordinal, which we identify with the set of natural numbers, and by $\omega_1$ the least uncountable ordinal. We fix once and for all a recursive encoding $\langle \cdot \rangle$ of all

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One may express some reservations on whether our suggested proof is indeed independent of the original one, because one can derive the Lusin–Suslin Theorem from the theorem about turning Borel sets into clopen sets (cf. [1]). There are however straightforward proofs of the Lusin–Suslin Theorem: see for Example 1G.5 or—for a very different proof—4A.7 in [5]. Therefore the proof that we are now suggesting is indeed independent of the usual proof. Nevertheless—as it becomes clear from our comments—these two theorems are “equivalent” in the sense that we can deduce one from the other.
Turning Borel sets into clopen sets effectively

finite sequences of naturals by a natural number. The number 0 will be the
code of the empty sequence. We denote by $\text{Seq}$ the recursive set of all
codes of finite sequences in $\omega$. If $s \in \text{Seq}$ is a code of $u = (u_0, \ldots, u_{n-1})$ we define
$
\text{lh}(s) = n
$
and if $i < n$ we put $(s)_i = u_i$. If $s \in \text{Seq}$ and $i \geq \text{lh}(s)$ or if $s \not\in \text{Seq}$ and $i$ is arbitrary we define $(s)_i$ to be 0. Finally we fix the following enumeration of the rational numbers:

$$q_s = (-1)^{(s)_0} \frac{(s)_1}{(s)_2 + 1}$$

for $s \in \omega$. By $O$ we mean the usual Kleene’s canonical set of ordinal notation.

We denote by $\mathcal{N}$ the space $\omega^\omega$ of all infinite sequences of naturals with
the product topology. The space $\mathcal{N}$ is the Baire space. The members of $\mathcal{N}$
will be denoted by lowercase Greek letters such as $\alpha, \beta$ etc. We fix the usual
distance function $p_\mathcal{N}$ on $\mathcal{N}$ defined by

$$p_\mathcal{N}(\alpha, \beta) = ((\text{least } k \text{ with } \alpha(k) \neq \beta(k)) + 1)^{-1}$$

for $\alpha \neq \beta$. We may view every $\alpha \in \mathcal{N}$ as a code of an infinite sequence
in $\mathcal{N}$. To be more specific, we define $(\alpha)_i(n) = \alpha(\langle i, n \rangle)$ for all $i \in \omega$ and so
$\alpha$ gives rise to the sequence $((\alpha)_i)_{i \in \omega}$. Of course we may apply the inverse
procedure: if $(\alpha_i)_{i \in \omega}$ is a sequence in $\mathcal{N}$ there is some $\alpha \in \mathcal{N}$ such that
$(\alpha)_i = \alpha_i$ for all $i \in \omega$. For $\alpha, \beta \in \mathcal{N}$ we will denote by $\langle \alpha, \beta \rangle$ the unique
$\gamma \in \mathcal{N}$ such that $(\gamma)_0 = \alpha$, $(\gamma)_1 = \beta$ and $\gamma(t) = 0$ if $t \neq \langle i, n \rangle$ for all $i = 0, 1$
and all $n \in \omega$.

We will often identify relations with the sets they define and write $P(x)$
instead of $x \in P$.

A topological space is a Polish space if it is separable and is metrizable
by a complete distance function. We will call such a distance function a suitable
distance function. We employ the standard hierarchy $(\Sigma^0_\xi)_{\xi < \omega_1}$ of
Borel sets in Polish spaces (cf. [1]).

Suppose now that $(\mathcal{X}, d)$ is a complete and separable metric space.
A sequence $(x_n)_{n \in \omega}$ is a recursive presentation of $(\mathcal{X}, d)$ if

(I) the set $\{x_n \mid n \in \omega\}$ is dense in $\mathcal{X}$,

(II) the relations $P_\prec, P_\leq \subseteq \omega^4$ defined by

$$P_\prec(i, j, k, m) \iff d(x_i, x_j) < k(m + 1)^{-1},$$
$$P_\leq(i, j, k, m) \iff d(x_i, x_j) \leq k(m + 1)^{-1}$$

are recursive.

We say that $(\mathcal{X}, d)$ admits a recursive presentation, or that $(\mathcal{X}, d)$ is recursively presented, if there is a sequence $(x_n)_{n \in \omega}$ in $\mathcal{X}$ which satisfies the conditions (I) and (II).
For every complete space \((X, d)\) with a recursive presentation \((x_n)_{n \in \omega}\) we consider the set
\[
B(x_n, m, k) = \{ x \in X \mid d(x, x_n) < k(m + 1)^{-1} \}.
\]
The latter is either an empty set or the ball with center \(x_n\) and radius \(m(m + 1)^{-1}\). For \(s \in \omega\) define
\[
N(X, s) = B(x(s)_0, (s)_1, (s)_2).
\]
The family \(\{N(X, s) \mid s \in \omega\}\) is the associated neighborhood system of \(X\) (with respect to \((x_n)_{n \in \omega}\) and \(d\)) and it is clear that it forms a basis for the topology of \(X\). We say that a Polish space \(X\) is recursively presented if there is a suitable distance function \(d\) such that the corresponding space \((X, d)\) is recursively presented. When we refer to a recursively presented Polish space we will always assume that we are given a suitable distance function and a recursive presentation. Standard examples of recursively presented Polish spaces are the Baire space \(\mathcal{N}\), the space of real numbers and \(\omega\).

The property of being recursively presented clearly carries over to finite products and sums of spaces, i.e., if \(\mathcal{X}_1, \ldots, \mathcal{X}_n\) are recursively presented Polish spaces, then both \(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n\) and \(\mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_n\) are recursively presented. We fix once and for all a scheme for passing from the recursive presentations of finitely many spaces to recursive presentations of their product and sum.

Suppose that \((\mathcal{X}, d)\) is complete and recursively presented. We say that \(A \subseteq \mathcal{X}\) is in \(\Sigma^0_1\) (or that \(A\) is a \(\Sigma^0_1\) set, or that \(A\) is semirecursive) if there is a recursive function \(f : \omega \to \omega\) such that \(A = \bigcup_{s \in \omega} N(\mathcal{X}, f(s))\). In other words \(\Sigma^0_1\) sets are the unions of a recursive collection from the family of our fixed open neighborhoods. This definition suggests that the property of being a \(\Sigma^0_1\) set depends on the way we have encoded the basic neighborhoods \(N(\mathcal{X}, s)\); however this is not the case (see [5, 3C.12]). (This property does depend of course on the distance function and the recursive presentation.) The set \(A\) is in \(\Pi^0_1\) if \(\mathcal{X} \setminus A\) is in \(\Sigma^0_1\). Inductively we define the family of \(\Sigma^0_{n+1}\) subsets of \(\mathcal{X}\) as the family of all sets which are the projection along \(\omega\) of a \(\Pi^0_n\) subset of \(\mathcal{X} \times \omega\), and \(\Pi^0_{n+1}\) sets as the complements of sets in \(\Sigma^0_{n+1}\). The set \(A\) is in \(\Sigma^1_1\) if it is the projection along \(\mathcal{N}\) of a \(\Pi^0_0\) subset of \(\mathcal{X} \times \mathcal{N}\). The set \(A\) is in \(\Pi^1_1\) if its complement is in \(\Sigma^1_1\), and \(A\) is in \(\Delta^1_1\) if it is both in \(\Sigma^1_1\) and \(\Pi^1_1\). Of course all these definitions coincide with the usual ones of Kleene when \(\mathcal{X} = \omega\) or \(\mathcal{X} = \mathcal{N}\).

A function \(f : \mathcal{X} \to \mathcal{Y}\) between recursively presented Polish spaces is recursive (or \(\Sigma^1_0\)-recursive) if the relation \(Rf \subseteq \mathcal{X} \times \omega\) defined by
\[
Rf(x, s) \iff f(x) \in N(\mathcal{Y}, s)
\]
is in \(\Sigma^0_1\). Similarly the function \(f\) is \(\Delta^1_1\)-recursive if the set \(Rf\) is in \(\Delta^1_1\).
An important notion is the one of relativization with respect to some parameter. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are recursively presented Polish spaces and that $y \in \mathcal{Y}$. A subset $A$ of $\mathcal{X}$ is in $\Sigma^0_1(y)$ if there is some $P \subseteq \mathcal{Y} \times \mathcal{X}$ such that $A$ is the $y$-section of $P$, i.e., $A = P_y := \{ x \in \mathcal{X} \mid P(x, y) \}$. Similarly one defines the classes of sets $\Sigma^0_n(y)$, $\Pi^0_n(y)$, $\Sigma^1_1(y)$ and $\Pi^1_1(y)$. The class $\Delta^1_1(y)$ is $\Sigma^1_1(y) \cap \Pi^1_1(y)$. By replacing $\Delta^1_1$ with $\Delta^1_1(y)$ one defines the class of $\Delta^1_1(y)$-recursive functions and similarly one defines the class of $\Sigma^1_1(y)$-recursive or simply $y$-recursive functions.

For any pointclass $\Gamma$ and for any recursively presented Polish space $\mathcal{X}$, a point $x \in \mathcal{X}$ is in $\Gamma$ or it is a $\Gamma$ point exactly when the relation $U \subseteq \omega$ defined by $U(s) \Leftrightarrow x \in N(\mathcal{X}, s)$ is in $\Gamma$. Suppose that $x \in \Gamma$; we say that

- $x$ is recursive (resp. recursive in $y$) if $\Gamma = \Sigma^0_1$ (resp. $\Sigma^0_1(y)$),
- $x$ is hyperarithmetic (resp. hyperarithmetical in $y$) if $\Gamma = \Delta^1_1$ (resp. $\Delta^1_1(y)$),
- $x$ is arithmetical (resp. arithmetical in $y$) if $\Gamma = \Sigma^n_1$ (resp. $\Sigma^n_1(y)$) for some $n \geq 1$.

If $x$ is hyperarithmetical in $y$ and $y$ is hyperarithmetical in $x$ we say that $x$ and $y$ have the same hyperdegree. We will also consider the cases where the above $x$ and $y$ are in fact subsets of $\omega$ by identifying a set $P \subseteq \omega$ with its characteristic function $\chi_P$, which is a member of the recursively presented Polish space $2^\omega$. We say for example that $P$ is hyperarithmetical in $Q$ if $\chi_P \in \Delta^1_1(\chi_Q)$. The latter is equivalent to saying that the function $\chi_P : \omega \rightarrow 2$ is $\Delta^1_1(\chi_Q)$-recursive.

The relativization applies to recursive presentations as well. Consider a point $\epsilon \in N$. A sequence $(x_n)_{n \in \omega}$ in a complete metric space $(\mathcal{X}, d)$ is an $\epsilon$-recursive presentation of $(\mathcal{X}, d)$ if the above conditions (I) and (II) are satisfied with the modification that the relations $P_<$ and $P_\leq$ are now $\epsilon$-recursive. One can then repeat all previous definitions by replacing everywhere the term “recursive” with “$\epsilon$-recursive”. For example a subset $A$ of an $\epsilon$-recursively presented Polish space $\mathcal{X}$ is $\epsilon$-semirecursive if there is an $\epsilon$-recursive function $f : \omega \rightarrow \omega$ such that $A = \bigcup_{s \in \omega} N(\mathcal{X}, f(s))$. We will denote the class of $\epsilon$-semirecursive sets by $\Sigma^0_1(\epsilon)$. There is a potential double meaning for $\Sigma^0_1(\epsilon)$, because every recursive presentation is also an $\epsilon$-recursive presentation for any $\epsilon \in N$. One can check however (with the help of [5, 3C.5]) that the $\epsilon$-recursive unions of basic neighborhoods $N(\mathcal{X}, s)$—where $\mathcal{X}$ is recursively presented—are exactly the $\epsilon$-sections of $\Sigma^0_1$ subsets of $N \times \mathcal{X}$. So no conflict arises.

It is not true that every Polish space is recursively presented but it is easy to see that every Polish space is recursively presented in some parameter $\epsilon$. All theorems about recursively presented Polish spaces are transferred to $\epsilon$-recursively presented Polish spaces. The latter claim is the Relativiza-
tion Principle. This principle is fundamental for the applications of effective descriptive set theory to “classical mathematics” such as Theorem 4.2. For more information refer to [5, 3I].

A function \( f : (\mathcal{X}, d_{\mathcal{X}}) \to (\mathcal{Y}, d_{\mathcal{Y}}) \) is an isometry if \( d_{\mathcal{Y}}(f(x), f(y)) = d_{\mathcal{X}}(x, y) \) for all \( x, y \in \mathcal{X} \) and \( f \) is surjective. If the spaces \( (\mathcal{X}, d_{\mathcal{X}}) \) and \( (\mathcal{Y}, d_{\mathcal{Y}}) \) are complete and recursively presented, and the function \( f \) is recursive, it is not hard to verify that a set \( A \) is in \( \Gamma \) exactly when \( f[A] \) is in \( \Gamma \), where \( \Gamma \) is any of the classes \( \Sigma^0_n(\alpha), \Pi^0_n(\alpha), \Sigma^1_1(\alpha), \Pi^1_1(\alpha) \) and \( \Delta^1_1(\alpha) \).

We are going to use some fundamental theorems of effective descriptive set theory including (but not restricted to) the Theorem on Restricted Quantification (4D.3), the Effective Perfect Set Theorem (4F.1), the Strong \( \Delta \)-Selection Principle (4D.6) and the theorem about the existence of \( \Delta^1_1 \) points inside \( \Pi^1_1 \) non-meager sets (4F.20). (All these references are from [5].) We should mention explicitly the following result.

**Theorem 1.1** (cf. [5, 4A.7]). Every \( \Delta^1_1 \) subset of a recursively presented Polish space is the recursive injective image of a \( \Pi^0_1 \) subset of \( \mathcal{N} \).

In order to estimate the emerging parameters we need some fundamental results of higher recursion theory, which are due to Kleene and Spector. We give a brief presentation of these results. The reader may refer to [5] and to [8] for more information about their proofs, as well as to the original articles. (Below we give more detailed references.)

A set \( P \subseteq \omega \) is \( \Pi^1_1 \)-complete if \( P \) is in \( \Pi^1_1 \) and for all sets \( Q \subseteq \omega \) in \( \Pi^1_1 \) there exists a recursive function \( f : \omega \to \omega \) such that for all \( n \in \omega \) we have

\[
Q(n) \iff P(f(n)).
\]

We shall give a typical example of a \( \Pi^1_1 \)-complete set (although this is not straightforward to verify), but let us assume for the moment the \( \Pi^1_1 \)-complete sets do exist and let us examine some basic facts about them. It is clear from the definition that every subset of \( \omega \) in \( \Sigma^1_1 \cup \Pi^1_1 \) is recursive in every \( \Pi^1_1 \)-complete set. It is also easy to see that a \( \Pi^1_1 \)-complete set is not a \( \Sigma^1_1 \) set, for otherwise every \( \Pi^1_1 \) subset of \( \omega \) would be in \( \Sigma^1_1 \) as well.

Another important fact about \( \Pi^1_1 \)-complete sets is that they have the same hyperdegree. This follows from the following result of Spector.

**Theorem 1.2** (Spector, cf. [9]). If \( A \) and \( B \) are \( \Pi^1_1 \) subsets of \( \omega \) and \( B \) is not in \( \Delta^1_1 \), then \( A \in \Delta^1_1(B) \). (Equivalently any two sets in \( \Pi^1_1 \backslash \Delta^1_1 \) have the same hyperdegree.)

We will present a proof of the latter theorem using some tools from [5]. Suppose that \( \mathcal{X} \) is a recursively presented Polish space and that \( P \) is a non-empty subset of \( \mathcal{X} \). A norm on \( P \) is any function \( \varphi : P \to \text{Ordinals} \). A \( \Pi^1_1 \)-norm on \( P \) is a norm \( \varphi \) (on \( P \)) for which there exist two relations \( \leq_\Sigma \)
and \( \leq_\Pi \) in \( \Sigma^1_1 \) and \( \Pi^1_1 \) respectively such that for all \( x \in \mathcal{X} \) and all \( y \in P \) we have

\[
[P(x) \& \varphi(x) \leq \varphi(y)] \iff x \leq_\Sigma y \iff x \leq_\Pi y.
\]

The following is an important fact about \( \Pi^1_1 \) sets.

**Theorem 1.3** (cf. \([5 \text{ IVB.2}]\)). Every \( \Pi^1_1 \) set admits a \( \Pi^1_1 \)-norm.

Now we are ready to prove Spector’s Theorem.

**Proof of Theorem 1.2.** Consider a \( \Pi^1_1 \)-complete set \( P \subseteq \omega \). It suffices to show that every \( C \subseteq \omega \) in \( \Pi^1_1 \setminus \Delta^1_1 \) has the same hyperdegree as \( P \).

Suppose that \( C \subseteq \omega \) is in \( \Pi^1_1 \setminus \Delta^1_1 \) and that \( f: \omega \to \omega \) is a recursive function such that \( n \in C \iff f(n) \in P \) for all \( n \in \omega \). Clearly \( C \) is recursive in \( P \). Consider a \( \Pi^1_1 \)-norm \( \varphi \) on \( P \). The claim is that the set \( \{ \varphi(f(n)) \mid n \in C \} \) is unbounded in \( \{ \varphi(m) \mid m \in P \} \). Indeed if this were not the case, there would be some \( m_0 \in P \) such that \( \varphi(f(n)) \leq \varphi(m_0) \) for all \( n \in C \). Using the key property of a \( \Pi^1_1 \)-norm we have

\[ n \in C \iff n \in C \& \varphi(f(n)) \leq \varphi(m_0) \iff f(n) \leq_\Sigma m_0. \]

The latter implies that \( C \) is in \( \Sigma^1_1 \), which contradicts the fact that \( C \) is not in \( \Delta^1_1 \).

Therefore for all \( m \in P \) there exists \( n \in C \) such that \( \varphi(m) < \varphi(f(n)) \).

From this and from the key property of a \( \Pi^1_1 \)-norm it follows that

\[ m \in P \iff (\exists n)[n \in C \& \varphi(m) \leq \varphi(f(n))] \]

\[ \iff (\exists n)[n \in C \& m \leq_\Sigma f(n)] \iff (\exists n)[n \in C \& m \leq_\Pi f(n)]. \]

The latter two equivalences show that \( P \) is in \( \Sigma^1_1(C) \cap \Pi^1_1(C) = \Delta^1_1(C). \)

We now present a typical example of a \( \Pi^1_1 \)-complete set. We consider the standard enumeration of the partial recursive functions from \( \omega \) to \( \omega \). By \( \{ e \} \) we mean the \( e \)th term of this enumeration.

**Definition 1.4** (Kleene, cf. \([2 \text{ and } 3]\)). Define the condition \( \mathcal{A}(X,Y) \), where \( X \subseteq \omega \) and \( Y \subseteq \omega \times \omega \), to hold exactly when

\[
\text{(III) } 1 \in X,
\text{(IV) } (\forall y)[y \in X \to 2^y \in X \& (y, 2^y) \in Y],
\text{(V) } (\forall e)\{ \text{if for all } n \in \omega, \{ e \}(n) \text{ is defined and } \{ e \}(n) \in X \text{ then } 3 \cdot 5^n \in X \& (\forall n)(\{ e \}(n), 3 \cdot 5^n) \in Y\},
\text{(VI) } (\forall x, y, z)[(x, y) \in Y \& (y, z) \in Y \to (x, z) \in Y].
\]

Kleene’s \( O \) and the relation \( <_O \) are defined as follows:

\[
x \in O \iff (\forall X, Y)[\mathcal{A}(X,Y) \to x \in X],
(x, y) \in <_O \iff (\forall X, Y)[\mathcal{A}(X,Y) \to (x, y) \in Y]
\]

for all \( x, y \in \omega \). It is not hard to verify that \( \mathcal{A}(O, <_O) \) holds, so that \( (O, <_O) \) is the least pair \( (X,Y) \) satisfying conditions \((\text{III})\)–\((\text{VI})\).
Kleene’s $O$ serves as an ordinal notation system (cf. [7]). However we will not pursue this direction here, since what we are interested in is the following.

**Theorem 1.5 (Kleene, cf. [3] or [8]).** Kleene’s $O$ is a $\Pi^1_1$-complete set.

So from Theorems 1.2 and 1.5 and the comments about $\Pi^1_1$-complete sets we have:

(VII) every subset of $\omega$ in $\Pi^1_1$ is recursive in $O$,

(VIII) every subset of $\omega$ in $\Pi^1_1 \setminus \Delta^1_1$ has the same hyperdegree as $O$.

Conditions (VII) and (VIII) are all that we are going to need about Kleene’s $O$. Therefore one may replace $O$ in all of its upcoming occurrences with any $\Pi^1_1$-complete set (2).

### 2. The effective version.

In this section we present our main theorem, which is the effective version of the theorem about turning a Borel subset of a Polish space into a clopen set.

**Theorem 2.1.** Suppose that $(\mathcal{X}, T)$ is a recursively presented Polish space, $d$ is a suitable distance function for $(\mathcal{X}, T)$, and $A$ is a $\Delta^1_1$ subset of $\mathcal{X}$. Then there exist an $\varepsilon_A \in \mathcal{N}$, which is recursive in $O$, and a Polish topology $T_\infty$ with suitable distance function $d_\infty$, which extends $T$ and has the following properties:

1. The Polish space $(\mathcal{X}, T_\infty)$ is $\varepsilon_A$-recursively presented.
2. The set $A$ is a $\Delta^0_1(\varepsilon_A)$ subset of $(\mathcal{X}, d_\infty)$.
3. If $B \subseteq \mathcal{X}$ is a $\Delta^1_1(\alpha)$ subset of $(\mathcal{X}, d)$, where $\alpha \in \mathcal{N}$, then $B$ is a $\Delta^1_1(\varepsilon_A, \alpha)$ subset of $(\mathcal{X}, d_\infty)$.
4. If $B \subseteq \mathcal{X}$ is a $\Delta^1_1(\varepsilon_A, \alpha)$ subset of $(\mathcal{X}, d_\infty)$, where $\alpha \in \mathcal{N}$, then $B$ is a $\Delta^1_1(\varepsilon_A, \alpha)$ subset of $(\mathcal{X}, d)$.

**Proof.** From Theorem 1.1 there are $\Pi^0_1$ sets $F_1, F_2 \subseteq \mathcal{N}$ and recursive functions $\pi_1, \pi_2 : \mathcal{N} \rightarrow \mathcal{X}$ such that $\pi_i$ is one-to-one on $F_i$, $i = 1, 2$, and $A = \pi_1[F_1]$, $A^c = \pi_2[F_2]$; here $A^c$ stands for the complement of $A$ in $\mathcal{X}$. We may assume that $A \neq \emptyset, \mathcal{X}$ for otherwise the result is immediate. So $F_1, F_2 \neq \emptyset$.

Now we define a distance function on $A$ such that the function $\pi_1$ becomes an isometry. Let $p_{\mathcal{N}}$ be the usual distance function on the Baire space $\mathcal{N}$, and $x, y \in A$ and $\alpha, \beta$ be the unique members of $F_1$ such that

$(2)$ Another (perhaps more natural) example of a $\Pi^1_1$-complete set is Spector’s $W$ (cf. [9]), which is defined as follows:

$W = \{ e \in \omega \mid \{ e \} \text{ is total and the set } \{(n, m) \mid \{ e \}(\langle n, m \rangle) = 0\} \text{ is a well-ordering}\}$. 
$x = \pi_1(\alpha)$ and $y = \pi_1(\beta)$. Put

$$d_A(x, y) = d_A(\pi_1(\alpha), \pi_1(\beta)) = p_N(\alpha, \beta).$$

We do the same for $A^c$, i.e. for $x, y \in A^c$ we define

$$d_{A^c}(x, y) = d_{A^c}(\pi_2(\alpha), \pi_2(\beta)) = p_N(\alpha, \beta)$$

where $\alpha, \beta$ are the unique members of $F_2$ such that $x = \pi_2(\alpha)$ and $y = \pi_2(\beta)$.

Let $p_{F_i}$ be the restriction of the distance function $p_N$ on $F_i \times F_i$, $i = 1, 2$. Since the sets $F_i$ are closed the metric spaces $(F_i, p_{F_i})$ are separable and complete, $i = 1, 2$. Also since the functions $\pi_i|_{F_i}$ are isometries we see that the spaces $(A, d_A), (A^c, d_{A^c})$ are separable and complete as well.

Now we are going to define $\varepsilon_1, \varepsilon_2 \in \mathcal{N}$ in which the metric spaces $(A, d_A), (A^c, d_{A^c})$ admit a recursive presentation, respectively. Recall the set $\text{Seq} \subseteq \omega$ of the codes of all finite sequences of $\omega$. For $s \in \text{Seq}$ we define

$$N_s = \{ \alpha \in \mathcal{N} \mid \alpha(i) = (s)_i \forall i < \text{lh}(s) \}.$$  

We adopt the notation $s \upharpoonright k$ for the natural number $\langle (s)_0, \ldots, (s)_{\text{lh}(s) - 1}, k \rangle$, where $s \in \text{Seq}$ and $k \in \omega$. Define

$$\varepsilon_1(s) = 1 \iff \text{Seq}(s) \& F_1 \cap N_s \neq \emptyset$$

and $\varepsilon_1(s) = 0$ otherwise. Notice that for all $s$ with $\varepsilon_1(s) = 1$ there always exists some $k \in \omega$ such that $\varepsilon_1(s \upharpoonright k) = 1$.

We now define a sequence $(\alpha_s)_{s \in \omega}$ which is dense in $F_1$. First for $\varepsilon_1(s) = 1$ define $\alpha_s(i) = (s)_i$ for all $i < \text{lh}(s)$, and for $i \geq \text{lh}(s)$,

$$\alpha_s(i) = (\mu k)[\varepsilon_1(\langle \alpha_s(0), \ldots, \alpha_s(i - 1), k \rangle) = 1].$$

It is clear that for all $s$ with $\varepsilon_1(s) = 1$ we have $\alpha_s \in N_s$ and

$$N_{(\alpha_s(0), \ldots, \alpha_s(i - 1))} \cap F_1 \neq \emptyset$$

for all $i \in \omega$. Since $F_1$ is closed it follows that $\alpha_s \in N_s \cap F_1$ for all $s \in \text{Seq}$ such that $N_s \cap F_1 \neq \emptyset$. Therefore the sequence $(\alpha_s)_{s : \varepsilon_1(s) = 1}$ is dense in $F_1$.

Now pick the least natural $s$, call it $s_0$, for which $\varepsilon_1(s) = 1$, and define $\alpha_s = \alpha_{s_0}$ for all $s$ for which $\varepsilon_1(s) = 0$.

We will prove that the sequence $(\alpha_s)_{s \in \omega}$ is an $\varepsilon_1$-recursive presentation of $(F_1, p_{F_1})$. First notice that the relation

$$P(s, t, i) \iff \alpha_s(i) = \alpha_t(i)$$

is recursive in $\varepsilon_1$. To see this, notice that the definitions above can be reformulated so that $s$ becomes a variable, i.e. there is a $\Sigma_1^0(\varepsilon_1)$-recursive function $f : \omega \times \omega \to \omega$ such that $f(s, i) = \alpha_s(i)$ for all $s, i$.

Now we claim that the relations $Q \subseteq \omega \times \omega$ and $R \subseteq \omega \times \omega$ defined by

$$Q(s, t, i) \iff P(s, t, i) \& (\forall j < i)\neg P(s, t, j)$$

$$\iff i \text{ is the least } k \text{ for which } \alpha_s(k) \neq \alpha_t(k)$$
and

\[ R(s, t) \Leftrightarrow \alpha_s = \alpha_t \]

are recursive in \( \varepsilon_1 \). This is clear for \( Q \), since \( P \) is recursive in \( \varepsilon_1 \). We prove the claim for \( R \). If \( \varepsilon_1(s) = \varepsilon_1(t) = 1 \) then the equality between \( \alpha_s \) and \( \alpha_t \) can be verified by only finite values; in particular

\[ \alpha_s = \alpha_t \iff [s \sqsubseteq t \& (\forall i)[lh(s) \leq i < lh(t) \Rightarrow \alpha_s(i) = \alpha_t(i)]] \vee \]
\[ [t \sqsubseteq s \& (\forall i)[lh(t) \leq i < lh(s) \Rightarrow \alpha_s(i) = \alpha_t(i)]]. \]

Similarly if \( \varepsilon_1(s) = 1 \) and \( \varepsilon_1(t) = 0 \) then

\[ \alpha_s = \alpha_t \Leftrightarrow \alpha_s = \alpha_{s_0}. \]

If \( \varepsilon_1(s) = \varepsilon_1(t) = 0 \) then clearly \( \alpha_s = \alpha_{s_0} = \alpha_t \). From these remarks it follows that the relation \( R \) is recursive in \( \varepsilon_1 \). We now compute

\[ P_<(s, t, m, k) \Leftrightarrow p_{F1}(\alpha_s, \alpha_t) < m/k + 1 \]
\[ \Leftrightarrow [R(s, t) \& m > 0] \]
\[ \vee (\exists i)[\alpha_s(i) \neq \alpha_t(i) \& (\forall j < i)[\alpha_s(j) = \alpha_t(j)] \]
\[ \& k + 1 < (i + 1)m] \]
\[ \Leftrightarrow [R(s, t) \& m > 0] \vee (\exists i)[Q(s, t, i) \& k + 1 < (i + 1)m] \]
\[ \Leftrightarrow [R(s, t) \& m > 0] \vee (\forall i)[Q(s, t, i) \rightarrow k + 1 < (i + 1)m]. \]

Thus the relation \( P_< \) is recursive in \( \varepsilon_1 \). It is clear that \( P_\leq \) is recursive in \( \varepsilon_1 \) as well. Therefore the metric space \((F_1, p_{F_1})\) is recursively presented in \( \varepsilon_1 \). Since \( \pi_1 \) is an isometry between \((F_1, p_{F_1})\) and \((A, d_A)\) it follows that the sequence \((\pi_1(\alpha_s))_{s \in \omega}\) is an \( \varepsilon_1 \)-recursive presentation of \((A, d_A)\). Similarly we define \( \varepsilon_2 \) for the metric space \((A^c, d_{A^c})\) with the \( \varepsilon_2 \)-recursive presentation \((\pi_1(\alpha_s))_{s \in \omega}\). We take

\[ \varepsilon_A = \langle \varepsilon_1, \varepsilon_2 \rangle \]

and we check that \( \varepsilon_A \) is recursive in \( O \). Define

\[ P_1(s) \Leftrightarrow F_1 \cap N(\mathcal{X}, s) \neq \emptyset \quad \text{and} \quad P_2(s) \Leftrightarrow F_2 \cap N(\mathcal{X}, s) \neq \emptyset. \]

Notice that

\[ \varepsilon_A(\langle i, s \rangle) = 1 \iff [i = 0 \& \varepsilon_1(s) = 1] \vee [i = 1 \& \varepsilon_2(s) = 1] \]
\[ \iff [i = 0 \& P_1(s)] \vee [i = 1 \& P_2(s) = 1]. \]

From this it follows that \( \varepsilon_A \) is recursive in a \( \Sigma^1_1 \) subset of \( \omega \). From the property \( [\text{VII}] \) mentioned in the Introduction we deduce that \( \varepsilon_A \) is recursive in Kleene’s \( O \).

We now define the topology \( T_\infty \) on \( \mathcal{X} \) as the direct sum of \((A, d_A)\) and \((A^c, d_{A^c})\), i.e. \( V \in T_\infty \) if and only if \( V \cap A \) is \( d_A \)-open and \( V \cap A^c \) is \( d_{A^c} \)-open. As we mentioned in the Introduction, \((\mathcal{X}, T_\infty)\) is Polish and recursively
presented in $\varepsilon_A$. A suitable dense sequence is
\[ x(i,s) = \begin{cases} 
\pi_1(\alpha_s) & \text{if } i = 0, \\
\pi_2(\beta_s) & \text{if } i = 1,
\end{cases} \]
and $x_t = \alpha_{s_0}$ in any other case.

Clearly $A$ is $\Delta^0_1(\varepsilon_A)$ in $\langle \mathcal{X}, d_\infty \rangle$, where $d_\infty$ is the distance function which arises from the direct sum of $(A, d_A)$ and $A^c, d_{A^c})$. We prove that the topology $T_\infty$ extends $T$. Let $V$ be in $T$. We will show that $V \cap A$ is $d_A$-open. The proof for $V \cap A^c$ is similar. Let $y = \pi_1(\alpha) \in V \cap A$, with $\alpha \in F_1$. Since $\pi_1 : \mathcal{N} \to (\mathcal{X}, d)$ is continuous there is some $k \in \omega$ such that for all $\beta \in \mathcal{N}$ with $p_{\mathcal{N}}(\alpha, \beta) < 1/(k+1)$ we have $\pi_1(\beta) \in V$. We claim that the $d_A$-ball of center $y$ and radius $1/(k+1)$ is contained in $V$. Let $z = \pi_1(\beta)$, with $\beta \in F_1$, be such that $d_A(y, z) < 1/(k+1)$. Then
\[ p_{\mathcal{N}}(\alpha, \beta) = d_A(\pi_1(\alpha), \pi_1(\beta)) = d_A(y, z) < \frac{1}{k+1}, \]
hence $z = \pi_1(\beta) \in V$.

Now we deal with the rest of the assertions. It is not hard to see that every recursive function $\pi : \mathcal{N} \to (\mathcal{X}, d)$ is $\Delta^1_1(\varepsilon_A)$-recursive as a function from $\mathcal{N}$ to $\langle \mathcal{X}, d_\infty \rangle$ (by considering the standard recursive presentation of $\mathcal{N}$ as an $\varepsilon_A$-recursive presentation). It follows from [5, 4A.7 and 4D.7] that every $\Delta^1_1$ subset of $(\mathcal{X}, d)$ is a $\Delta^1_1(\varepsilon_A)$ subset of $(\mathcal{X}, d_\infty)$. This settles assertion (3).

To prove assertion (4), suppose $B$ is a $\Delta^1_1(\varepsilon_A)$ subset of $(\mathcal{X}, d_\infty)$. The set $\pi^{-1}_1[B \cap A] \subseteq F_1 \subseteq \mathcal{N}$ is in $\Delta^1_1(\varepsilon_A)$ since $\pi_1$ is recursive as a function from $\mathcal{N}$ to $(\mathcal{X}, d)$ and therefore $\Delta^1_1(\varepsilon_A)$-recursive as a function from $\mathcal{N}$ to $(\mathcal{X}, d_\infty)$. Since $\pi_1$ is one-to-one on $\pi^{-1}_1[B \cap A]$, again from [5, 4D.7] we deduce that $B \cap A = \pi^{-1}_1[B \cap A]$ is a $\Delta^1_1(\varepsilon_A)$ subset of $(\mathcal{X}, d)$. Similarly we prove that $B \cap A^c$ is a $\Delta^1_1(\varepsilon_A)$ subset of $(\mathcal{X}, d)$ as well. Therefore $B = (B \cap A) \cup (B \cap A^c)$ is a $\Delta^1_1(\varepsilon_A)$ subset of $(\mathcal{X}, d)$.

\section*{3. Computing the parameter.} As we have seen, one can choose the parameter $\varepsilon$ of Theorem [2.1] to be recursive in $O$. Since we are dealing mostly with $\Delta^1_1$ sets, a natural question to ask is whether we can choose a hyperarithmetical such $\varepsilon$. We will see that the latter is not necessarily true even if we start with a $\Sigma^0_1$ set which we want to turn into $\Delta^0_1$. Nevertheless we will show that in some cases, choosing a hyperarithmetical such $\varepsilon$ is possible (cf. Theorem [3.9]). Moreover we will see that, under some specific assumptions about the set we start with and about our underlying space, we can characterize the case where the choice of a hyperarithmetical $\varepsilon$ is possible (cf. Theorem [3.11]).
**Definition 3.1.** Suppose that \((X, \mathcal{T})\) is a recursively presented Polish space, \(d\) is a suitable distance function, and \(A\) is a \(\Delta^1_1\) subset of \(X\).

(1) We say that \(\varepsilon \in \mathcal{N}\) is a **good parameter for** \(A\) if all conclusions of Theorem 2.1 are satisfied if we take \(\varepsilon_A\) as \(\varepsilon\). The latter means that there is a Polish topology \(\mathcal{T}_\infty\) on \(X\) which extends \(\mathcal{T}\); the space \((X, \mathcal{T}_\infty)\) is \(\varepsilon\)-recursively presented; the set \(A\) is \(\Delta^0_1(\varepsilon)\) in \((X, d_\infty)\), where \(d_\infty\) is a suitable distance function; every \(B\) which is a \(\Delta^1_1(\alpha)\) subset of \((X, d)\) is a \(\Delta^1_1(\varepsilon, \alpha)\) subset of \((X, d_\infty)\); and every \(B\) which is a \(\Delta^1_1(\varepsilon, \alpha)\) subset of \((X, d_\infty)\) is a \(\Delta^1_1(\alpha, \varepsilon)\) subset of \((X, d)\).

(2) We say that the class \(\Delta^1_1\) is **dense in** \(A\) if whenever some \(V \in \mathcal{T}\) intersects \(A\) then \(V\) intersects \(A\) in a \(\Delta^1_1\) point of \((X, d)\), i.e.

\[
(\forall V \in \mathcal{T}) [V \cap A \neq \emptyset \Rightarrow (\exists x \in \Delta^1_1[(X, d)]) [x \in V \cap A]].
\]

The next proposition gives a necessary condition for a \(\Delta^1_1\) set to admit a good parameter in \(\Delta^1_1\).

**Proposition 3.2.** If a \(\Delta^1_1\) subset \(A\) of a recursively presented Polish space \((X, \mathcal{T})\) admits a good parameter in \(\Delta^1_1\) then the class \(\Delta^1_1\) is dense both in \(A\) and in the complement \(X \setminus A\) with respect to \(\mathcal{T}\).

**Proof.** Let \(\varepsilon\) be a good parameter for \(A\) in \(\Delta^1_1\), \(\mathcal{T}_\infty\) be the corresponding extension of the topology \(\mathcal{T}\), and \(d\) and \(d_\infty\) be the corresponding suitable distance functions. Suppose that \(V \in \mathcal{T}\) intersects \(A\). Since \(\mathcal{T}_\infty\) extends \(\mathcal{T}\) we find that \(V \in \mathcal{T}_\infty\). From the choice of \(\varepsilon\) and \(\mathcal{T}_\infty\) we also deduce that \(A \in \mathcal{T}_\infty\). Thus \(V \cap A\) is a non-empty \(\mathcal{T}_\infty\)-open set. Since \((X, \mathcal{T}_\infty)\) is recursively presented in \(\varepsilon\), the set \(V \cap A\) contains a point, say \(x_0\), which is recursive in \(\varepsilon\) with respect to \((X, d_\infty)\). It follows that \(\{x_0\}\) is a \(\Delta^1_1(\varepsilon)\) subset of \((X, d_\infty)\), and therefore \(\{x_0\}\) is a \(\Delta^1_1(\varepsilon)\) subset of \((X, d)\). Thus \(x_0\) is a \(\Delta^1_1(\varepsilon)\) point of \((X, d)\). Since \(\varepsilon \in \Delta^1_1\) we conclude that \(x_0\) is a \(\Delta^1_1\) point of \((X, d)\).

The proof for \(X \setminus A\) is similar. ■

**Remark 3.3.** It follows from the previous proposition that a \(\Delta^1_1\) set \(A\) may not admit a good parameter in \(\Delta^1_1\). To see this take \(X = \mathcal{N}\) and \(A\) any non-empty \(\Pi^0_1\) set with no \(\Delta^1_1\) members.

Our next goal is to find sufficient conditions under which a given \(\Delta^1_1\) set admits a good parameter in \(\Delta^1_1\). We will work inside a special category of spaces.

**Definition 3.4.** A recursively presented Polish space \((X, \mathcal{T})\) is **recursively zero-dimensional** if there is a distance function \(d\) on \(X\) which generates the topology \(\mathcal{T}\), witnesses that \((X, \mathcal{T})\) is recursively presented with recursive presentation the sequence \((r_i)_{i \in \omega}\), and the relation \(I \subseteq X \times \omega \times \omega\) defined by

\[
I(x, i, s) \iff d(x, r_i) < q_s
\]
is recursive. By replacing “recursive” with “ε-recursive” one defines the notion of an ε-recursively zero-dimensional space.

It is well-known that every zero-dimensional Polish space is topologically isomorphic to a closed subset of the Baire space (cf. [1, 7.8]). The next lemma is the effective analogue of this statement.

**Lemma 3.5.** For every recursively zero-dimensional Polish space $\mathcal{X}$ there is a recursive injection $f: \mathcal{X} \to \mathbb{N}$ such that the set $\mathcal{Y} := f[\mathcal{X}]$ is in $\Pi^0_1(\varepsilon)$ for some $\varepsilon \in \Delta^0_2$. Moreover the inverse function $f^{-1}: \mathcal{Y} \to \mathcal{X}$ is computed by a semirecursive subset of $\mathcal{N} \times \omega \times \omega$ on $\mathcal{Y}$, i.e. there is a semirecursive $R \subseteq \mathcal{N} \times \omega \times \omega$ such that for all $\alpha \in \mathcal{Y}$ we have

$$d(f^{-1}(\alpha), r_i) < q_s \iff R(\alpha, i, s).$$

In particular the inverse function $f^{-1}$ is continuous.

**Proof.** Fix a suitable distance function $d$ on $\mathcal{X}$. By replacing $d$ with $d/(1 + d)$ we may assume that $d \leq 1$. First we claim that there is a recursive relation $A \subseteq \mathcal{X} \times \text{Seq}$ such that:

1. $A_0 = \mathcal{X}$,
2. $A_s = \bigcup_{n \in \omega} A_{s \cdot n}$,
3. $A_{s \cdot n} \cap A_{s \cdot m} = \emptyset$ for all $n \neq m$, and
4. each $A_s$ has diameter less than $1/2^{\text{lh}(s)}$,

where $A_s = \{x \in \mathcal{X} \mid A(x, s)\}$. Notice that each $A_s$ is a clopen set and that we do not exclude the case $A_s = \emptyset$.

We now prove the claim. For all $i, n \in \omega$ we denote by $N(i, n)$ the open $d$-ball with center $r_i$ and radius $1/(n + 1)$, so that the relation $I \subseteq \mathcal{X} \times \omega \times \omega$ defined by

$$I(x, i, n) \iff x \in N(i, n)$$

is recursive. Using Kleene’s Recursion Theorem we obtain a recursive function $b: \mathcal{X} \times \text{Seq} \to \mathbb{N}$ such that $b(x, 0) = 1$ for all $x \in \mathcal{X}$ and

$$b(x, s \cdot k) = 1 \iff b(x, s) = 1 \& x \in N(k, 2^{\text{lh}(s)} + 2) \& b(r_k, s) = 1$$

for all $x \in \mathcal{X}$, $s \in \text{Seq}$ and $k \in \omega$, so that if we define

$$B_s = \{x \in \mathcal{X} \mid b(x, s) = 1\}$$

then $B_0 = \mathcal{X}$ and

$$B_{s \cdot k} = \begin{cases} B_s \cap N(k, 2^{\text{lh}(s)} + 2) & \text{if } r_k \in B_s, \\ \emptyset & \text{otherwise} \end{cases}$$

for all $s \in \text{Seq}$ and $k \in \omega$. It is clear that the diameter of $B_s$ is less than $2^{-\text{lh}(s)}$, and $B_s$ is a clopen set. Also, $B_s = \bigcup_{n \in \omega} B_{s \cdot n}$. To see this let $x \in B_s$; since $B_s$ is open we can choose $r_k \in B_s$ with $d(x, r_k) < (2^{\text{lh}(s)} + 2 + 1)^{-1}$. 


Then \( x \in B_s \cap N(k, 2^{\text{lh}(s)+2}) = B_{s,k} \). We apply the Recursion Theorem one more time to get a recursive set \( A \subseteq \mathcal{X} \times \text{Seq} \) such that \( A_0 = \mathcal{X} \) and

\[
A_{s,k} = \left( B_{s,k} \setminus \bigcup_{i<k} B_{s,i} \right) \cap A_s
\]

for all \( s \in \text{Seq} \) and \( k \in \omega \), where \( A_s \) is the \( s \)th section of \( A \). It is clear that conditions (1)–(4) are satisfied for this family \( (A_s)_{s \in \text{Seq}} \).

Having proved our claim we define \( f : \mathcal{X} \to \mathcal{N} \) as follows:

\[
f(x)(n) = \text{the unique } i \text{ for which there is (a unique) } s \in \text{Seq} \text{ such that } x \in A_{s,i} \text{ and } \Delta(s) = n,
\]

for all \( x \in \mathcal{X} \) and \( n \in \omega \). It is easy to check that \( f(x)|n \) is the unique \( s \in \text{Seq} \) of length \( n \) such that \( x \in A_s \), and that \( f(x)(n) \) is the unique \( i \) such that \( x \in A_{f(x)(n-1)}(i) \). Clearly the function \( f \) is recursive and injective.

We now define \( \varepsilon(s) = 1 \) exactly when there exists \( i \) such that \( r_i \in A_s \), and 0 otherwise. It is clear that \( \varepsilon \in \Delta^0_2 \). Since each \( A_s \) is clopen we see that \( \varepsilon(s) = 1 \) exactly when \( A_s \neq \emptyset \). Moreover one can verify that

\[
\alpha \in f[\mathcal{X}] \iff (\forall n)[A_{\alpha|n} \neq \emptyset] \iff (\forall n)[\varepsilon(\alpha|n) = 1].
\]

So the set \( \mathcal{Y} := f[\mathcal{X}] \) is a \( \Pi^0_1(\varepsilon) \) subset of \( \mathcal{N} \).

Finally we prove the assertion about \( f^{-1} \). Suppose that \( \alpha \in \mathcal{Y} \) and \( x = f^{-1}(\alpha) \). We claim that

\[
d(x, r_i) < q_s \iff (\exists n, j)[r_j \in A_{\alpha|n} \& d(r_j, r_i) < q_s - 1/2^n].
\]

For the left-to-right direction we choose natural numbers \( n, j \) such that \( 2/2^n < q_s - d(x, r_i) \) and \( r_j \in A_{\alpha|n} \). (The set \( A_{\alpha|n} \) is non-empty since \( \alpha \in \mathcal{Y} \).) As both \( r_j \) and \( x = f^{-1}(\alpha) \) are members of \( A_{\alpha|n} \) we have \( d(r_j, x) < 1/2^n \). Then

\[
d(r_j, r_i) \leq d(r_j, x) + d(x, r_i) < 1/2^n + d(x, r_i) < q_s - 1/2^n.
\]

For the right-to-left direction we use \( x = f^{-1}(\alpha) \in A_{\alpha|n} \) for all \( n \) to compute

\[
d(x, r_i) \leq d(x, r_j) + d(r_j, r_i) < 1/2^n + q_s - 1/2^n = q_s.
\]

Thus the equivalence is proved. Take now

\[
R(\alpha, i, s) \iff (\exists n, j)[r_j \in A_{\alpha|n} \& d(r_j, r_i) < q_s - 1/2^n],
\]

and we are done.

It would be interesting to see if the parameter \( \varepsilon \) in the previous proof can be chosen to be recursive.

**Remark 3.6.** Suppose that \( \mathcal{X} \) is a recursively zero-dimensional Polish space and that \( \mathcal{Y} \) and \( f : \mathcal{X} \to \mathcal{Y} \) are as in the previous lemma.

(1) Consider a recursive presentation for \( \mathcal{X} \), say \((x_i)_{i \in \omega} \), and define

\[
R(i, j) \iff x_i \neq x_j.
\]
The metric space $(\mathcal{Y}, p_N)$ admits a recursive presentation. To see this define $y_i = f(x_i)$ for all $i \in \omega$ and notice that the sequence $(y_i)_{i \in \omega}$ is dense in $\mathcal{Y}$. Moreover for all $i, j$ with $x_i \neq x_j$ there is exactly one triple $(u, n, m)$, where $u$ is a finite sequence of naturals and $n \neq m \in \omega$, such that $x_i, x_j \in A_u$, $x_i \in A_{u^+(n)}$ and $x_j \in A_{u^-(m)}$. So the least natural $k$ for which $f(x_i)(k) \neq f(x_j)(k)$ is exactly the length of the finite sequence $u$. We define $g(i, j) = \text{the length of } u$ if $R(i, j)$, and $g(i, j) = 0$ otherwise. The function $g$ is recursive and from the definition of $p_N$ it is clear that $p_N(y_i, y_j) = p_N(f(x_i), f(x_j)) = (g(i, j) + 1)^{-1}$ for $x_i \neq x_j$.

(2) Consider the parameter $\varepsilon \in \Delta^0_2$ of Theorem 3.11 of the proof of the previous lemma, so that $\mathcal{Y}$ is in $\Pi^0_1(\varepsilon)$. For every $A \subseteq \mathcal{X}$, if $A$ is in $\Pi^0_1(\varepsilon)$ the set $f[A]$ is a $\Pi^0_1(\varepsilon)$ subset of $\mathcal{N}$, and if $A$ is in $\Sigma^0_{n+1}$ the set $f[A]$ is a $\Sigma^0_{n+1}$ subset of $\mathcal{N}$ for all $n \geq 1$. To see this consider for all $k \in \omega$ the space $\mathcal{X} \times \omega^k$ (for $k = 0$ we just mean the space $\mathcal{X}$) and the function $\tilde{f} : \mathcal{X} \times \omega^k \to \mathcal{N} \times \omega^k$ defined by $\tilde{f}(x, \bar{z}) = (f(x), \bar{z})$. Since $\mathcal{Y}$ is a $\Pi^0_1(\varepsilon)$ subset of $\mathcal{N}$ one can show that for every $k \in \omega$ and for every $A \subseteq \mathcal{X} \times \omega^k$ in $\Pi^0_1$ the set $\tilde{f}[A]$ is a $\Pi^0_1(\varepsilon)$ subset of $\mathcal{N} \times \omega^k$. From this and the fact that $\varepsilon \in \Delta^0_2$ one can show, by induction on $n \geq 1$, that for all $k \in \omega$ and all $A \subseteq \mathcal{X} \times \omega^k$ in $\Sigma^0_{n+1}$ the set $\tilde{f}[A]$ is a $\Sigma^0_{n+1}$ subset of $\mathcal{N} \times \omega^k$.

The proof of Theorem 1.1 (cf. [5]) shows that every $\Delta^1_1$ subset of $\mathcal{N}$ is in fact the injective image under the projection of a $\Pi^0_1$ subset of $\mathcal{N} \times \mathcal{N}$. Thus in the case of $\Delta^1_1$ subsets of $\mathcal{N}$ the function $\pi$ of Theorem 1.1 can be chosen to be an open mapping, i.e. to carry open sets to open sets. But since $\pi$ is injective on $F$ and not necessarily injective on $\mathcal{N}$, we cannot conclude that the inverse function $\pi^{-1} : A \to F$ is continuous. For reasons that will become clear later on, the continuity of the inverse function will be necessary for our purposes. This property though puts a limitation on the $\Delta^1_1$ set we start with.

Remark 3.7. Suppose that $A \subseteq \mathcal{N}$ is in $\Delta^1_1$ and that there is a $\Pi^0_1$ subset of $\mathcal{N}$, say $F$, and a recursive function $\pi : F \to \mathcal{N}$ such that $\pi[F] = A$, $\pi$ is injective on $F$ and the inverse function $\pi^{-1} : A \to F$ is continuous. Then $A$ is a $G_\delta$ subset of $\mathcal{N}$.

To see why the latter holds define the distance function $d$ on $A$ by $d(\alpha, \beta) = p_N(\pi^{-1}(\alpha), \pi^{-1}(\beta))$ for all $\alpha, \beta \in A$. Since both functions $\pi$ and $\pi^{-1}$ are continuous it follows that the topology on $A$ induced by $d$ is exactly $\{V \cap A \mid V \text{ open in } \mathcal{N}\}$, i.e. the relative topology of $\mathcal{N}$ on $A$. Moreover since $F$ is closed it follows that $(A, d)$ is Cauchy-complete. Therefore the set $A$ with the relative topology induced by $\mathcal{N}$ is a Polish space. It follows from Theorem 3.11 in [1] that $A$ is a $G_\delta$ set.

What is perhaps more important is that the previous remark has a converse.
Lemma 3.8. For every $A \subseteq \mathcal{N}$ in $\Pi^0_2$ there is an $F \subseteq \mathcal{N} \times \mathcal{N}$ in $\Pi^0_1$ such that

$$A(\alpha) \iff (\exists \beta) F(\alpha, \beta) \iff (\exists \text{ unique } \beta) F(\alpha, \beta)$$

for all $\alpha \in \mathcal{N}$, and the function $\text{pr}^{-1} : A \to F$, $\text{pr}^{-1}(\alpha) = (\alpha, \beta)$, where $\beta$ is such that $F(\alpha, \beta)$, is continuous.

As usual there is a similar version of Lemma 3.8 with respect to some parameter $\alpha$. For reasons of exposition we refrain from stating this parameterized version.

Proof of Lemma 3.8. Since $A$ is in $\Pi^0_2$, using [5, 3C.4] there is a recursive $R \subseteq \mathcal{N} \times \omega \times \omega$ such that

$$A(\alpha) \iff (\forall n) (\exists m) R(\alpha, n, m)$$

for all $\alpha \in \mathcal{N}$. Define

$$F(\alpha, \beta) \iff (\forall n)[R(\alpha, n, \beta(n)) \land (\forall k < \beta(n)) \neg R(\alpha, n, k)]$$

for all $\alpha, \beta \in \mathcal{N}$. It is clear that $F$ is in $\Pi^0_1$ and that for all $\alpha \in \mathcal{N}$ there is at most one $\beta$ such that $F(\alpha, \beta)$. Moreover

$$A(\alpha) \iff (\exists \beta) F(\alpha, \beta)$$

for all $\alpha, \beta \in \mathcal{N}$. Now we prove that the inverse function $\text{pr}^{-1} : A \to F$, $\text{pr}^{-1}(\alpha) = (\alpha, \beta)$, is continuous. Clearly it suffices to prove that the function $g : A \to \mathcal{N}$, $g(\alpha) = \beta$, where $\beta$ is such that $F(\alpha, \beta)$, is continuous. So take any $\alpha$ and $\beta$ with $F(\alpha, \beta)$ and $N \in \omega$. For all $n \in \omega$ we consider the set

$$V_n = \{ \alpha' \mid R(\alpha', n, \beta(n)) \land (\forall k < \beta(n)) \neg R(\alpha', n, k) \}.$$

Since $F(\alpha, \beta)$ and $R$ is recursive the set $V = \bigcap_{n < N} V_n$ is an open neighborhood of $\alpha$. It is then clear that for all $\alpha' \in V$ and all $\beta'$ with $F(\alpha', \beta')$ we have $\beta'|N = \beta|N$. ■

We are now ready for the next theorem.

Theorem 3.9. Suppose that $(\mathcal{X}, T)$ is a recursively zero-dimensional Polish space with no isolated points and that $A$ is a countable $\Delta^1_1$ subset of $\mathcal{X}$. Then there exists an $\varepsilon \in \Delta^1_1$ and a Polish topology $T_{\infty}$ with suitable distance function $d_{\infty}$, which extends $T$ and has the following properties:

1. The Polish space $(\mathcal{X}, T_{\infty})$ is $\varepsilon$-recursively presented.
2. The set $A$ is a $\Delta^1_1(\varepsilon)$ subset of $(\mathcal{X}, d_{\infty})$.
3. Every $B \subseteq \mathcal{X}$ is a $\Delta^1_1(\alpha)$ subset of $(\mathcal{X}, d)$, where $d$ is a suitable distance function for $(\mathcal{X}, T)$ and $\alpha \in \mathcal{N}$, exactly when $B$ is a $\Delta^1_1(\varepsilon, \alpha)$ subset of $(\mathcal{X}, d_{\infty})$.

Proof. For notational purposes we interchange $A$ with $\mathcal{X} \setminus A$, for it is the case of a co-countable $\Delta^1_1$ set which is interesting. So let us assume that
\( \mathcal{X} \setminus A \) is countable. Since \( A \) has a countable complement and it is \( \Delta_1^1 \) there is a \( \Delta_1^1 \) sequence \((x_n)_{n \in \omega} \) in \( \mathcal{X} \) such that \( \mathcal{X} \setminus A = \{x_n \mid n \in \omega\} \). Using this it is easy to find some \( \varepsilon_D \in \Delta_1^1 \) such that \( \mathcal{X} \setminus A \) is in \( \Sigma_2^0(\varepsilon_D) \) and so \( A \) is in \( \Pi_2^0(\varepsilon_D) \).

To see this we remark that \( x \notin A \iff (\exists n)(\forall s)[x \in N(\mathcal{X}, s) \to x_n \in N(\mathcal{X}, s)] \).

Define then \( \varepsilon_D((n, s)) = 1 \iff x_n \in N(\mathcal{X}, s) \), and 0 otherwise.

We first show how one can reduce the problem to subspaces of \( \mathcal{N} \). We consider the embedding \( f : \mathcal{X} \to \mathcal{N} \) of Lemma 3.5 and an \( \varepsilon_0 \in \Delta_2^0 \) such that the set \( \mathcal{Y} := f[\mathcal{X}] \) is in \( \Pi_0^0(\varepsilon_0) \). We take \( \varepsilon^* = (\varepsilon_0, \varepsilon_D) \in \Delta_1^1 \) and using (the parameterized version of) Remark 3.6 we deduce that \( f[A] \) is a \( \Pi_2^0(\varepsilon^*) \) subset of \( \mathcal{N} \).

We consider the set \( \mathcal{Y} \) with the usual distance function \( p_{\mathcal{N}} \) of the Baire space. From Remark 3.6 the space \( (\mathcal{Y}, p_{\mathcal{N}}) \) is recursively presented in some arithmetical parameter, which we may assume that we have included in the previous \( \varepsilon^* \). Moreover the space \( (\mathcal{Y}, p_{\mathcal{N}}) \) has no isolated points for otherwise \( (\mathcal{X}, d) \) would have had isolated points. The set \( \mathcal{Y} \setminus f[A] = f[\mathcal{X} \setminus A] \) is a countable subset of \( \mathcal{Y} \) and since \( \mathcal{Y} \) has no isolated points it follows that \( f[A] \) is co-meager in \( \mathcal{Y} \).

Thus we have an \( \varepsilon^* \in \Delta_1^1 \) such that the space \( (\mathcal{Y}, p_{\mathcal{N}}) \) is recursively presented in \( \varepsilon^* \), the set \( f[A] \) is a \( \Pi_2^0(\varepsilon^*) \) subset of \( \mathcal{N} \), and \( f[A] \) is co-countable and co-meager in \( \mathcal{Y} \). We will show that there is a Polish topology \( \mathcal{S}_\infty \) on \( \mathcal{Y} \) with suitable distance function \( p_\infty \) and an \( \varepsilon \in \Delta_1^1 \) such that all conclusions are satisfied for this \( \varepsilon \) and for \( \mathcal{Y} \), \( p_\infty \), \( \mathcal{S}_\infty \) and \( f[A] \) in place of \( \mathcal{X} \), \( d_\infty \), \( \mathcal{T}_\infty \) and \( A \) respectively. Then we can return to \( \mathcal{X} \) using the function \( f \).

More specifically we define \( d_\infty(x, y) = p_\infty(f(x), f(y)) \) for all \( x, y \in \mathcal{X} \), and we define an \( \varepsilon \)-recursive presentation for \( (\mathcal{X}, d_\infty) \) via the \( \varepsilon \)-recursive presentation of \( (\mathcal{Y}, p_\infty) \). Therefore we turn \( f \) into an \( \varepsilon \)-recursive isometry. All conclusions are then satisfied.

We now go back to the proof of Theorem 2.1 and we show that the \( \varepsilon_1 \) and \( \varepsilon_2 \) defined there are in fact in \( \Delta_1^1 \). Since \( f[A] \) is a \( \Pi_2^0(\varepsilon^*) \) subset of \( \mathcal{N} \), from the parameterized version of Lemma 3.8 there is an \( F_1 \subset \mathcal{N} \times \mathcal{N} \) in \( \Pi_0^0(\varepsilon^*) \) and a function \( \pi_1 : \mathcal{N} \times \mathcal{N} \to \mathcal{N} \) such that:

- \( \pi_1 \) is \( \varepsilon^* \)-recursive,
- \( \pi_1 \) is injective on \( F_1 \),
- \( \pi_1[F_1] = f[A] \), and
- the function \( \pi_1^{-1} : f[A] \to F_1 \) is continuous.

In order to match the notation of the proof of Theorem 2.1 we view \( F_1 \) as a subset of \( \mathcal{N} \).

We now show that the \( \varepsilon_1 \) of the proof of Theorem 2.1 is in \( \Delta_1^1 \). Define the relation \( R_1 \subset \text{Seq} \) by

\[
R_1(s) \iff N_s \cap F_1 \neq \emptyset,
\]
where \( N_s = \{ \alpha \in \mathcal{N} \mid \alpha(i) = (s)_i \ \forall i < \lh(s) \} \) for all \( s \in \text{Seq} \). We need to show that \( R_1 \) is \( \Delta^1_1 \). Since \( \varepsilon^* \in \Delta^1_1 \) and \( F_1 \) is in \( \Pi^0_1(\varepsilon^*) \) it is clear that \( R_1 \) is in \( \Sigma^1_1 \). We now claim that

\[
R_1(s) \Leftrightarrow (\exists \alpha \in \Delta^1_1)[\alpha \in N_s \cap F_1].
\]

By the Theorem on Restricted Quantification [5, 4D.3] it follows from the latter equivalence that \( R_1 \) is in \( \Pi^1_1 \). The right-to-left implication is clear, so assume that \( R_1(s) \). Pick some \( \alpha_0 \in N_s \cap F_1 \). In particular \( \alpha_0 = \pi_1^{-1}(y_0) \) for some \( y_0 \in f[A] \). Since the function \( \pi_1^{-1} : f[A] \to F_1 \) is continuous there is some basic open \( \text{and thus recursive) neighborhood } V \subseteq \mathcal{N} \text{ such that } y_0 \in V \) and \( \pi_1^{-1}(y) \in N_s \cap F_1 \) for all \( y \in V \cap f[A] \). Since \( f[A] \) is co-meager in \( \mathcal{Y} \) and \( V \cap \mathcal{Y} \) is non-empty and open in \( \mathcal{Y} \) we see that \( V \cap f[A] \) is non-meager in \( \mathcal{Y} \). Moreover \( V \cap f[A] \) is a \( \Pi^0_1(\varepsilon^*) \)—and thus \( \Pi^1_1(\varepsilon^*) \)—subset of \( \mathcal{Y} \). From the parameterized version of [5, 4F.20] there is some \( y \in V \cap f[A] \) which is a \( \Delta^1_1(\varepsilon^*) \) point of \( \mathcal{Y} \). Since \( \varepsilon^* \in \Delta^1_1 \) it is easy to check that \( y \) is a \( \Delta^1_1 \) point of \( \mathcal{N} \) and so \( \alpha = \pi_1^{-1}(y) \in N_s \cap F_1 \) is also in \( \Delta^1_1 \). Thus the equivalence is proved.

Now we deal with the complement of \( f[A] \) in \( \mathcal{Y} \) and subsequently with \( \varepsilon_2 \). From Theorem 1.1 there is a set \( F_2 \subseteq \mathcal{N} \) in \( \Pi^0_1 \) and a recursive function \( \pi_2 : \mathcal{N} \to \mathcal{X} \) such that \( \pi_2[F_2] = \mathcal{Y} \setminus f[A] \) and \( \pi_2 \) is injective on \( F_2 \). Since \( \mathcal{Y} \setminus f[A] \) is countable, so is \( F_2 \). It follows from the Effective Perfect Set Theorem [5, 4F.1] that \( F_2 \) consists of \( \Delta^1_1 \) points. Hence

\[
N_s \cap F_2 \neq \emptyset \Leftrightarrow (\exists \alpha \in \Delta^1_1)[\alpha \in N_s \cap F_2].
\]

From this it follows that \( \varepsilon_2 \in \Delta^1_1 \). Now we define \( \varepsilon = (\varepsilon^*, \varepsilon_1, \varepsilon_2) \in \Delta^1_1 \) and we continue as in the proof of Theorem 2.1.

Notice that in the previous proof there is no need to choose the function \( \pi_2 \) so that its inverse function is continuous. The reason is our countability hypothesis which ensures that the corresponding closed set \( F_2 \) consists of \( \Delta^1_1 \) points. Suppose for the moment that we drop the countability hypothesis in favor of \( A \) and its complement being \( G_\delta \) and \( \Delta^1_1 \) sets. Assume moreover that \( \Delta^1_1 \) is dense in both \( A \) and \( \mathcal{X} \setminus A \). Then from (the parameterized version of) Lemma 3.8 we can always choose the above functions \( \pi_1 \) and \( \pi_2 \) so that their inverses are continuous functions and then we can repeat the previous proof. The only problem with this idea is that we have to make sure that our application of Lemma 3.8 stays within the level of hyperarithmetic parameters. Under the countability hypothesis that is clear, for a countable \( \Delta^1_1 \) set is easily in \( \Sigma^0_2(\varepsilon^*) \) for some hyperarithmetic \( \varepsilon^* \). But now it is not so clear that the \( G_\delta \) sets \( A \) and \( \mathcal{X} \setminus A \) we start with (which are also in \( \Delta^1_1 \)) are in fact in \( \Pi^0_2(\varepsilon^*) \) for some hyperarithmetic \( \varepsilon^* \). The latter assertion is true thanks to a deep result of Louveau (cf. [4]).
Theorem 3.10 (Louveau). Suppose that $\mathcal{X}$ is a recursively presented Polish space and that $A \subseteq \mathcal{X}$ is $\Delta^1_1$ and $\Pi^0_\xi$ for some $\xi < \omega_1$. Then there exists an $\varepsilon \in \Delta^1_1$ such that $A$ is in $\Pi^0_\xi(\varepsilon)$.

We can now proceed to the next theorem.

Theorem 3.11. Suppose that $(\mathcal{X}, \mathcal{T})$ is a recursively zero-dimensional Polish space and that $A$ is a $\Delta^1_1$ subset of $\mathcal{X}$ which is also in $\Delta^0_2$. Then:

1. The set $A$ admits a good parameter in $\Delta^1_1$ if and only $\Delta^1_1$ is dense in both $A$ and $\mathcal{X} \setminus A$.
2. If $A$ does not admit a good parameter in $\Delta^1_1$, then $O$ is hyperarithmetically in every good parameter for $A$.

Proof. From Louveau’s Theorem there is an $\varepsilon_D \in \Delta^1_1$ such that both $A$ and $\mathcal{X} \setminus A$ are in $\Pi^0_1(\varepsilon_D)$. We consider the embedding $f : \mathcal{X} \to \mathcal{N}$ of Lemma 3.5 and an $\varepsilon_0 \in \Delta^0_2$ such that the set $\mathcal{Y} := f[\mathcal{X}]$ is in $\Pi^0_1(\varepsilon_0)$. Consider also an arithmetical $\varepsilon'_0$ such that the space $(\mathcal{Y}, p_{\mathcal{Y}})$ admits an $\varepsilon'_0$-recursive presentation and define $\varepsilon^* = \langle \varepsilon_D, \varepsilon_0, \varepsilon'_0 \rangle$. It is clear that $\varepsilon^* \in \Delta^1_1$, the space $(\mathcal{Y}, p_{\mathcal{Y}})$ is $\varepsilon^*$-recursively presented and the sets $f[A]$ and $\mathcal{Y} \setminus f[A]$ are in $\Pi^0_2(\varepsilon^*)$.

As pointed out in the proof of Theorem 3.9 every good parameter for $f[A]$ (as a subset of $\mathcal{Y}$) is also a good parameter for $A$, for we can define the new distance function on $\mathcal{X}$ in such a way that the function $f$ becomes an isometry and recursive with respect to this good parameter. Using the same method one can see that the converse is also true modulo the parameter $\varepsilon^*$, i.e. if $\varepsilon$ is a good parameter for $A$ then $\langle \varepsilon^*, \varepsilon \rangle$ is a good parameter for $f[A]$. Thus $A$ admits a good parameter in $\Delta^1_1$ exactly when $f[A]$ does.

We apply the parameterized version of Lemma 3.8 to get sets $F_i \subseteq \mathcal{N}$ in $\Pi^0_1(\varepsilon^*)$ and $\varepsilon^*$-recursive functions $\pi_i : \mathcal{N} \to \mathcal{X}$, $i = 1, 2$, such that:

- $\pi_i$ is injective on $F_i$ for $i = 1, 2$,
- $\pi_1[F_1] = f[A]$, $\pi_2[F_2] = \mathcal{Y} \setminus f[A]$, and
- the inverses $\pi_1^{-1} : f[A] \to F_1$ and $\pi_2^{-1} : \mathcal{Y} \setminus f[A] \to F_2$ are continuous.

We are now ready to prove (1). The left-to-right implication is Proposition 3.2 so we prove the converse. Consider the previous functions $\pi_i$ and the sets $F_i$, $i = 1, 2$. We repeat the steps of the proof of Theorem 2.1. Since $\Delta^1_1$ is dense in $A$ and $\mathcal{X} \setminus A$ and since $f$ is recursive and injective, it is easy to check that $\Delta^1_1$ is dense both in $f[A]$ and in $\mathcal{Y} \setminus f[A]$. Using the density argument as in the previous proof we can see that the parameters $\varepsilon_1$ and $\varepsilon_2$ (with the notation of the proof of Theorem 2.1) are in fact in $\Delta^1_1$. Thus the fraction $\varepsilon = \langle \varepsilon^*, \varepsilon_1, \varepsilon_2 \rangle$ is a good parameter for $f[A]$ which is in $\Delta^1_1$ and so from the previous comments $A$ admits a good parameter in $\Delta^1_1$. 

Turning Borel sets into clopen sets effectively 137
Now let us prove (2). Since $A$ (and hence $f[A]$) does not admit a good parameter in $\Delta^1_1$, at least one of the relations $R_1, R_2 \subseteq \text{Seq}$ defined by

$$R_i(s) \iff N_s \cap F_i \neq \emptyset,$$

where $i = 1, 2$ and $N_s = \{ \alpha \in \mathcal{N} \mid \alpha(j) = (s)_j \ \forall j < \text{lh}(s) \}$, say $R_1$, is not in $\Delta^1_1$. (For otherwise both $\varepsilon_1$ and $\varepsilon_2$ would be in $\Delta^1_1$ and so as above $f[A]$ would admit a good parameter in $\Delta^1_1$.) It is clear that $R_1$ is a $\Sigma^1_1$ set. From the property (VIII) of the Introduction it follows that $R_1$ has the same hyperdegree as the one of Kleene’s $O$ and hence so does its characteristic function $\varepsilon_1$. Now let $\varepsilon$ be any good parameter for $A$. We will prove that $R_1$ is in $\Delta^1_1(\varepsilon)$.

Notice that $\langle \varepsilon^*, \varepsilon \rangle$ is a good parameter for $f[A]$. We claim that

$$R_1(s) \iff (\exists \alpha \in \Delta^1_1(\varepsilon))[\alpha \in N_s \cap F_1].$$

To see this, assume that $N_s \cap F_1 \neq \emptyset$; therefore $\pi_1[N_s \cap F_1] \neq \emptyset$. Since the function $\pi_1^{-1} : f[A] \rightarrow F_1$ is continuous the set $\pi_1[N_s \cap F_1]$ is open in $f[A]$. Hence there is an open $V \subseteq \mathcal{N}$ such that $\pi_1[N_s \cap F_1] = V \cap f[A] \neq \emptyset$. The set $V \cap f[A]$ is open in the extended topology, which is $\langle \varepsilon^*, \varepsilon \rangle$-recursively presented, and therefore it contains a point, say $y_0$, which is recursive in $\langle \varepsilon^*, \varepsilon \rangle$. It follows that the singleton $\{y_0\}$ is a $\Delta^1_1(\langle \varepsilon^*, \varepsilon \rangle)$ subset of $(\mathcal{Y}, p_\infty)$, where $p_\infty$ is a suitable distance function for the extended topology. Therefore $\{y_0\}$ is a $\Delta^1_1(\langle \varepsilon^*, \varepsilon \rangle)$ subset of $(\mathcal{Y}, p_\mathcal{N})$. Since $y_0 \in V \cap f[A] = \pi_1[N_s \cap F_1]$ there is some $\alpha_0 \in N_s \cap F_1$ such that $y_0 = \pi_1(\alpha_0)$. Using the fact that $\pi_1$ is injective on $F_1$ we deduce for $\beta \in \mathcal{N}$ that

$$\beta = \alpha_0 \iff \beta \in F_1 \& \pi_1(\beta) \in \{y_0\}.$$

Since the function $\pi_1$ is recursive it follows from the above equivalence that the point $\alpha_0$ is in $\Delta^1_1(\langle \varepsilon^*, \varepsilon \rangle) = \Delta^1_1(\varepsilon)$ and so the claim has been proved.

From the Theorem on Restricted Quantification [5, 4D.3] the set $R_1$ is in $\Pi^1_1(\varepsilon)$. Since $R_1$ is in $\Sigma^1_1$ we conclude that $R_1$ is in $\Delta^1_1(\varepsilon)$. ■

**Corollary 3.12.** Suppose that $(\mathcal{X}, T)$ is a recursively zero-dimensional Polish space and that $A$ is a $\Delta^1_1$ subset of $\mathcal{X}$ which is also in $\Delta^0_1$. Then $A$ admits a good parameter which either is in $\Delta^1_1$ or has the same hyperdegree as $O$.

**Proof.** This is immediate from Theorems 2.1 and 3.11. ■

It would be interesting to see if the previous theorem can be extended to sets which are above the level of $\Delta^0_2$ sets.

Before we close this section it is perhaps worth mentioning that the idea of using the $\Pi^0_1$ sets $F_i$ and the recursive functions $\pi_i$ is not always the best way to compute the complexity of the parameter, for in some cases it is easier to use directly the properties of the set we start with and of the underlying space.
Proposition 3.13. Suppose that \((\mathcal{X}, \| \cdot \|)\) is a separable vector space (real or complex). Consider the induced distance function \(d(x, y) = \|x - y\|\), \(x, y \in \mathcal{X}\), and assume that the metric space \((\mathcal{X}, d)\) is recursively presented. Then every basic open ball of \(\mathcal{X}\) admits a good parameter in \(\Delta^1_1\).

Proof. It is enough to prove the conclusion for \(A = \{x \in \mathcal{X} \mid d(x, 0) < 1\}\). Define

\[
d_A(x, y) = d(x, y) + \frac{1}{d(x, A^c)} - \frac{1}{d(y, A^c)}
\]

for all \(x, y \in A\). Then \(d_A\) is a distance function on \(A\) which generates the same topology as the restriction of the topology of the norm on \(A\) (cf. the proof of Theorem 3.11 in [1]). Moreover the metric space \((A, d_A)\) is complete. It is easy to find a \(\Delta^1_1\) sequence \((x_n)_{n \in \omega}\) in \(A\) such that the set \(\{x_n \mid n \in \omega\}\) is dense in \(A\). Then \((x_n)_{n \in \omega}\) is a presentation of \((A, d_A)\) which is recursive in some hyperarithmetical parameter.

Now we deal with the complement of \(A\). Let \(d_{A^c}\) be the restriction of the distance function \(d\) to \(\mathcal{X} \setminus A\). Since the latter set is closed it follows that \((\mathcal{X} \setminus A, d_{A^c})\) is complete and separable. Moreover it admits a presentation which is recursive in some hyperarithmetical parameter, for there is a \(\Delta^1_1\) sequence \((y_n)_{n \in \omega}\) with \(\|y_n\| > 1\) for all \(n \in \omega\), which is dense in \(\mathcal{X} \setminus A\). Finally consider the direct sum \((A, d_A) \oplus (\mathcal{X} \setminus A, d_{A^c})\).

4. Uniformity results. We conclude this article with the uniformity result mentioned in the Introduction.

Definition 4.1. We denote by \(Df(\omega)\) the set of all distance functions on \(\omega\) and by \([([\omega, d])\) the completion of the metric space \((\omega, d)\), where \(d \in Df(\omega)\). It is clear that every complete and separable metric space which is an infinite set is isometric to a space of the form \([([\omega, d])\) for some \(d \in Df(\omega)\), so the set \(Df(\omega)\) can characterize up to isometry all complete and separable infinite metric spaces. Every \(d \in Df(\omega)\) can be encoded by a member of \(\mathcal{N}\), for \(d\) is just a double sequence of real numbers. In our case it is enough to work with distance functions on \(\omega\) which take only rational values; these are a bit shorter to encode.

Suppose that \(d \in Df(\omega)\) takes only rational values. We define \(\beta_d \in \mathcal{N}\) as follows: \(\beta_d(i, j, m, n) = 1 \iff d(i, j) = m/(n + 1)\), and \(\beta_d(t)\) is 0 in any other case. We say that \(\beta \in \mathcal{N}\) encodes the complete and separable space \((\mathcal{X}, d^X)\) if there is a distance function \(d\) with rational values such that \(\beta = \beta_d\) and \((\mathcal{X}, d^X)\) is isometric to \([([\omega, d])\).

We will also use the following encoding of \(y\)-recursive functions from \(\mathcal{N}\) to \(\mathcal{N}\), where \(y\) varies through a recursively presented Polish space \(\mathcal{Y}\), as given in [5, 7A]. For all recursively presented Polish spaces \(\mathcal{X}\) consider the \(G^X \subseteq \omega \times \mathcal{X}\) as defined in [5, 3H.1], which is in \(\Sigma^0_1\) and is universal for
Consider now sequences \((\alpha, \pi)\) such that for all \(\pi \in \Sigma^0_1 \upharpoonright \mathcal{X}\). Fix now a recursively presented Polish space \(\mathcal{Y}\). For every function \(\pi : \mathcal{N} \to \mathcal{N}\) which is \(y\)-recursive for some \(y \in \mathcal{Y}\) there is some \(e \in \omega\) such that for all \(s \in \omega\) we have \(\pi(\alpha) \in \mathcal{N}(\mathcal{N}, s) \iff G^{\mathcal{Y} \times \mathcal{N} \times \omega}(e, y, \alpha, s)\). We consider such a number \(e\) to be a code of the function \(\pi\). We define the set \(\text{Cod}^\mathcal{Y} \subseteq \omega \times \mathcal{Y}\) by

\[\text{Cod}^\mathcal{Y}(e, y) \iff (\exists \alpha)(\exists \beta)(\forall s)[\beta \in \mathcal{N}(\mathcal{N}, s) \iff G^{\mathcal{Y} \times \mathcal{N} \times \omega}(e, y, \alpha, s)]\]

It is easy to verify that \(\text{Cod}^\mathcal{Y}\) is a \(\Pi^1_1\) set and so the set of codes of (total) \(y\)-recursive functions is in \(\Pi^1_1(y)\).

**Theorem 4.2.** Suppose that \(\mathcal{Z}\) is a Polish space, \(\mathcal{X}\) is a closed subset of \(\mathcal{N}\) and that \(P\) is a Borel subset of \(\mathcal{Z} \times \mathcal{X}\) such that every section \(P_z\) is neither a finite nor a co-finite set. Assume moreover that

(1) for all \(z \in \mathcal{Z}\) the \(z\)-section \(P_z\) is either countable or co-countable.

Then there is a Borel-measurable function \(f : \mathcal{Z} \to \mathcal{N}\) and a family \((d_z^\mathcal{X})_{z \in \mathcal{Z}}\) of distance functions on \(\mathcal{X}\) such that for all \(z \in \mathcal{Z}\) the following are true:

(a) the metric space \((\mathcal{X}, d_z^\mathcal{X})\) is complete and separable,

(b) \(f(z)\) encodes the space \((\mathcal{X}, d_z^\mathcal{X})\),

(c) the topology induced by \(d_z^\mathcal{X}\) extends the original one,

(d) the section \(P_z\) is \(d_z^\mathcal{X}\)-clopen, and

(e) a subset of \(\mathcal{X}\) is Borel in the original topology exactly when it is

Borel in the topology induced by \(d_z^\mathcal{X}\).

**Proof.** Fix for the moment some \(z \in \mathcal{Z}\) and let \(F_i \subseteq \mathcal{N}\) be closed and \(\pi_i : \mathcal{N} \to \mathcal{X}\) be continuous such that \(\pi_i\) is injective on \(F_i\) for \(i = 1, 2\), \(\pi_1[F_1] = P_z\) and \(\pi_2[F_2] = \mathcal{X} \setminus P_z\). As usual we define the distance functions \(d_i\) and \(d_2\) on \(P_z\) and \(\mathcal{X} \setminus P_z\) respectively so that the \(\pi_i\)'s become isometries. The space \((\mathcal{X}, p_z) : = (P_2, d_2) \oplus (\mathcal{X} \setminus P_z, d_2)\) satisfies the conclusions (c)–(e). Consider now sequences \((\alpha^k_n)_{n \in \omega}\) of distinct terms which are dense in \(F_i\), \(i = 1, 2\). It follows that the sequence \((\pi_1(\alpha^k_0), \pi_2(\alpha^k_0), \pi_1(\alpha^k_1), \pi_2(\alpha^k_2), \ldots)\) is dense in \((\mathcal{X}, p_z)\). We define \(d \equiv d(z) \in \text{Df}(\omega)\) by \(d(2i, 2j) = p_N(\alpha^k_i, \alpha^k_j)\), \(d(2i + 1, 2j + 1) = p_N(\alpha^k_2, \alpha^k_2)\) and \(d(2i, 2j + 1) = d(2i + 1, 2j) = 2\). It is clear that \([\omega, d(z)]\) is isometric to \((\mathcal{X}, p_z)\). Notice that \(d(z)\) takes only rational values. We consider the \(\beta_d(z)\) which encodes \(d(z)\) as in Definition 4.1. We will show that there is a Borel way of choosing this \(\beta_d(z)\).

Let us assume for simplicity that \(\mathcal{Z}\) is recursively presented, \(\mathcal{X}\) is recursively zero-dimensional and \(P\) is in \(\Delta^1_1\), so that every section \(P_z\) is in \(\Delta^1_1(z)\). We will apply the Strong \(\Delta\)-Selection Principle \([5, 4D.6]\), where the underlying pointclass is \(\Gamma = \Pi^1_1\). From our hypothesis (1) and the construction in the proof of Theorem 3.9 it follows that for all \(z \in \mathcal{Z}\) there are \(\varepsilon \in \Delta^1_1(z)\), sets \(F_i \subseteq \mathcal{N}\) in \(\Pi^1_1(\varepsilon, z)\) and \((\varepsilon, z)\)-recursive functions \(\pi_i : \mathcal{N} \to \mathcal{X}\), \(i = 1, 2\), such

\(^{(3)}\) The term “finite” also applies to the empty set.
that \( \pi_1[F_1] = P_z, \pi_2[F_2] = \mathcal{X} \setminus P_z \) and \( \pi_i \) is injective on \( F_i, i = 1, 2 \). Moreover there are \( \Delta^1_1(z) \) sequences \( (\alpha^n_i)_{n<\omega} \), which are dense in \( F_i \) for \( i = 1, 2 \) and—as we may assume—consist of distinct terms.

We consider the space \( \text{Tr} \) of trees on the natural numbers (cf. \([1, 4.32]\)). This is easily a recursively presented Polish space. Moreover every \( F \subseteq \mathcal{N} \) in \( \Pi^1_1(z) \) is the body [\( T \)] of some \( z \)-recursive tree \( T \); this is immediate from \([3, 4A.1]\). We define the relation \( R_1 \subseteq \mathcal{Z} \times \omega \times \text{Tr} \times \mathcal{N}^2 \) as follows:

\[
R_1(z, e, T, \varepsilon, \alpha) \iff T \text{ is an } (\varepsilon, z) \text{-recursive tree}
\]

\[\& e \text{ encodes a total } (\varepsilon, z) \text{-recursive function } \]

\[\pi : \mathcal{N} \to \mathcal{N} \text{ which is injective on } [T]
\]

\[\& \pi[[T]] = P_z \& ((\alpha)_i)_{i<\omega} \text{ is dense in } [T].\]

Similarly we define the relation \( R_2 \) by replacing \( P_z \) with \( \mathcal{X} \setminus P_z \). From our previous comments and from the fact that \( P_z \neq \emptyset, \mathcal{X} \) it follows that for all \( z \in \mathcal{Z} \) there are \( (e_i, T_i, \varepsilon_i, \alpha_i) \in \Delta^1_1(z) \) such that \( R_i(z, e_i, T_i, \varepsilon_i, \alpha_i) \) for \( i = 1, 2 \).

Suppose for the moment that we have proved that the \( R_i \)'s are II^1_1 \) sets. Then from the Strong \( \Delta \)-Selection Principle there are \( \Delta^1_1 \)-recursive functions \( g = (g_1, g_2, g_3, g_4), h = (h_1, h_2, h_3, h_4) : \mathcal{Z} \to \omega \times \text{Tr} \times \mathcal{N}^2 \) such that for all \( z \in \mathcal{Z} \) we have \( R_1(z, g(z)) \) and \( R_2(z, h(z)) \). Let \( (\alpha^n_i)_{i<\omega} \) and \( (\beta^n_i)_{i<\omega} \) be the sequences which arise from \( g_4(z) \) and \( h_4(z) \) respectively. As above we define \( d \equiv d(z) \in Df(\omega) \) as follows: \( d(2i, 2j) = p_\mathcal{N}(\alpha^n_i, \alpha^n_j), d(2i + 1, 2j + 1) = p_\mathcal{N}(\alpha^n_i, \beta^n_j) \) and \( d(2i, 2j + 1) = d(2i + 1, 2j) = 2 \). Finally we define \( f(z) = \beta_d(z) \) for all \( z \in \mathcal{Z} \). It is clear that \( f \) is \( \Delta^1_1 \)-recursive and thus it is Borel-measurable. According to the previous comments this function \( f \) satisfies the conclusions \((a)-(e)\).

So it remains to verify that the sets \( R_1 \) and \( R_2 \) are indeed in II^1_1 \). We consider the set \( \text{Cod}^{\mathcal{N} \times \mathcal{Z}} \equiv \text{Cod} \) as in the comments following Definition 4.1. We also define the relation \( \text{Gr} \subseteq \omega \times \mathcal{N} \times \mathcal{N} \times \mathcal{N}^2 \) by

\[\text{Gr}(e, \varepsilon, z, \alpha, \beta) \iff (\forall s)[\beta \in N(N, s) \iff G^{\mathcal{N} \times \mathcal{Z} \times \omega}(e, \varepsilon, z, \alpha, s)].\]

In other words, when \( \text{Cod}(e, \varepsilon, z) \) holds, \( \text{Gr}(e, \varepsilon, z, \alpha, \beta) \) means that \( \beta \) is the image of \( \alpha \) under the \( (\varepsilon, z) \)-recursive function encoded by \( e \). Clearly \( \text{Gr} \) is in \( \Delta^1_1 \). It is now easy to see that \( e \) encodes an \( (\varepsilon, z) \)-recursive function \( \pi : \mathcal{N} \to \mathcal{N} \) which is injective on \( [T] \), and \( \pi[[T]] = P_z \) exactly when

\[
\text{Cod}(e, \varepsilon, z) \& (\forall \alpha, \beta)[\alpha \in [T] \& \text{Gr}(e, \varepsilon, z, \alpha, \beta) \to \beta \in P_z],
\]

\[
(\forall \alpha_1, \alpha_2, \beta_1, \beta_2)(\forall i = 1, 2)[\text{Gr}(e, \varepsilon, z, \alpha_i, \beta_i) \& \alpha_1 \neq \alpha_2 \to \beta_1 \neq \beta_2],
\]

\[
(\forall \beta)(\exists \alpha \in \Delta^1_1(\varepsilon, z, \beta))[\beta \in P_z \to \alpha \in [T] \& \text{Gr}(e, \varepsilon, z, \alpha, \beta)].
\]

(Here we are using the fact that for all \( \beta \in P_z \) the unique \( \alpha \in [T] \) for which \( \pi(\alpha) = \beta \) is in \( \Delta^1_1(\varepsilon, z, \beta) \).) Using a similar method with universal sets one
can prove that the relation $Q_1 \subseteq \mathcal{N} \times \mathcal{Z} \times \text{Tr}$ defined by
\[ Q_1(\varepsilon, z, T) \iff T \text{ is } (\varepsilon, z)\text{-recursive} \]
is in $\Delta^1_1$. Moreover it is easy to check using the definition that the relation $Q_2 \subseteq \mathcal{N} \times \text{Tr}$ defined by $Q_2(\alpha, T) \iff ((\alpha)_s)_{s \in \omega} \text{ is dense in } [T]$ is in $\Pi^1_1$. It follows that $R_1$ is in $\Pi^1_1$. Similarly one shows that $R_2$ is in $\Pi^1_1$. ■

Remark 4.3. Our assumption about $P_z$ being either countable or co-countable was in order to be able to apply the proof of Theorem 3.9. The analogous assumption can be added to the statement of Theorem 4.2 in order to be able to apply the proof of Theorem 3.11 as well. To be more specific the conclusion of Theorem 4.2 is still true if the hypothesis (\*) in its statement is replaced by the following. For all $z \in \mathcal{Z}$ one of the following conditions applies:

- $P_z$ is a $\Delta^0_2$ set and $\Delta^1_1(z)$ is dense both in $P_z$ and in $\mathcal{X} \setminus P_z$,
- $P_z$ is countable,
- $P_z$ is co-countable.

We point out that the condition “$\Delta^1_1(z)$ is dense both in $P_z$ and in $\mathcal{X} \setminus P_z$” is met if there is a $\sigma$-finite Borel measure $\mu$ on $\mathcal{X}$ such that when $P_z$ meets an open set $V$ in an uncountable set, then $\mu(P_z \cap V) > 0$, and similarly for $\mathcal{X} \setminus P_z$ (provided that $P \in \Delta^1_1$). To see the latter, one needs to use the following result of Tanaka and Sacks, which is the counterpart of the Thomason–Hinman Theorem for measure (cf. [5, 4F.20] and the discussion following it): if $\mu$ is a $\sigma$-finite Borel measure on a recursively presented Polish space and $P \subseteq \mathcal{X}$ is a $\Pi^1_1$ set of positive $\mu$-measure, then there exists some $x \in \Delta^1_1$ such that $x \in P$.

One may ask about a uniformity result which follows just from the proof of Theorem 2.1, i.e. without any additional assumptions about the $z$-sections $P_z$. We can still get a uniformizing function $f$ but we would not be able to infer that it is Borel-measurable.

As usual the symbol $\Pi^1_1$ stands for the class of co-analytic sets and $\Sigma^1_2$ for the class of sets which are continuous images of $\Pi^1_1$ sets. A set $A$ is in $\Delta^1_1$ if both $A$ and its complement are in $\Sigma^1_2$.

**Theorem 4.4.** Suppose that $\mathcal{Z}$ is a Polish space, $\mathcal{X}$ is a closed subset of $\mathcal{N}$, and $P$ is a Borel subset of $\mathcal{Z} \times \mathcal{X}$ such that every section $P_z$ is neither finite nor co-finite. Then there is a $\Delta^1_2$-measurable function $f : \mathcal{Z} \to \mathcal{N}$ such that for all $z \in \mathcal{Z}$ the conclusions (a)–(e) of Theorem 4.2 are satisfied.

**Proof.** Consider the sets $R_1$ and $R_2$ of the proof of Theorem 4.2 As is shown there, these sets are $\Pi^1_1$ sets and for all $z \in \mathcal{Z}$, since $P_z \neq \emptyset$, $\mathcal{X}$, it follows from the proof of Theorem 2.1 that there are $(e_i, T_i, \varepsilon_i, \alpha_i)$ (not necessarily in $\Delta^1_1(z)$) such that $R_i(z, e_i, T_i, \varepsilon_i, \alpha_i)$ for $i = 1, 2$. Now we apply
the Uniformization Property of $\Pi^1_1$ in order to get functions
\[ \begin{align*}
g &= (g_1, g_2, g_3, g_4), \\
h &= (h_1, h_2, h_3, h_4) : Z \to \omega \times \text{Tr} \times \mathcal{N}^2
\end{align*} \]
whose graphs are $\Pi^1_1$ sets such that for all $z \in Z$ we have $R_1(z, g(z))$ and $R_2(z, h(z))$. It follows easily that the functions $g$ and $h$ are $\Delta^1_2$-measurable and from the closure properties of $\Delta^1_2$ we deduce that the arising function $f$ is $\Delta^1_2$-measurable as well.

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