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Ideal limits of sequences of continuous functions

by

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Abstract. We prove that for every Borel ideal, the ideal limits of sequences of continuous functions on a Polish space are of Baire class one if and only if the ideal does not contain a copy of Fin $\times$ Fin. In particular, this is true for $F_{\sigma\delta}$ ideals. In the proof we use Borel determinacy for a game introduced by C. Laflamme.

1. Introduction. A family of sets $J \subset P(\omega)$ is an *ideal* if it is closed under taking finite unions and subsets. Throughout this paper we assume that $[\omega]^{<\omega} \subset J$, $\omega \notin J$, and we write $\mathrm{Fin} = [\omega]^{<\omega}$. In a metric space $\langle X, \varrho\rangle$ a sequence $\{x_n\}_{n<\omega}$ is *J-convergent* to x if $(\forall\varepsilon > 0)$ $\{n : \varrho(x_n, x) \geq \varepsilon\} \in J$; we write $J\text{-}\lim x_n = x$. For functions $f, f_n : X \to Y$, where Y is a metric space, define $J\text{-}\lim f_n = f$ iff $J\text{-}\lim f_n(x) = f(x)$ for each $x \in X$. $J\text{-}\lim \mathcal{F}$ will denote the set of all J-limits of sequences of functions from $\mathcal{F}$. For every topological space X let $C(X)$ be the family of all continuous real-valued functions, and let $B_\alpha(X)$ be the family of all real-valued functions on X of Baire class $\alpha < \omega_1$. An ideal J is called a *P-ideal* if for each family $\{A_n : n \in \omega\} \subset J$ there is $A \in J$ with $A_n \subset^* A$ (i.e. $|A \setminus A_n| < \omega$) for each n. Let $J^* = \{A : \omega \setminus A \in J\}$.

It is known that J-limits can be very irregular.

Fact 1 (folklore). *Let J be a maximal ideal. Let $f_n : P(\omega) \to \mathbb{R}$ be such that $f_n(A) = 1$ if $n \in A$ and $f_n(A) = 0$ if $n \notin A$. Then f_n is continuous for every n, and $J\text{-}\lim f_n = \chi_{J^*}$, so the limit is nonmeasurable and without Baire property.*

In [Ka] Katětov proved that for each $\alpha < \omega_1$ there is a Borel ideal $\mathcal{N}^\alpha$ such that $\mathcal{N}^\alpha\text{-}\lim C(X) = B_\alpha(X)$. In particular, $\mathcal{N}^2 = \mathrm{Fin} \times \mathrm{Fin} = \{A \subset \omega \times \omega : \forall_n^\infty\ (\{k : (n, k) \in A\} \text{ is finite})\}$. From Proposition 3.6 of [DM] it follows that if the ideal J is of Borel additive or multiplicative class α then

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J-$\lim C(\mathbb{R}) \subset B_\alpha(\mathbb{R})$. In [KSW] the authors proved that for J_d, the ideal of sets of density 0 (and some of its generalizations), J_d-$\lim C(X) = B_1(X)$ for every complete metric space X. In this paper we prove that for a Polish space X and a Borel ideal J, J-$\lim C(X) = B_1(X)$ iff J does not contain an isomorphic copy of $\mathrm{Fin} \times \mathrm{Fin}$ iff J and J^* can be separated by an F_σ-set.

Our investigation is closely related to the separability of J and J^* by F_σ-sets. So the next section is devoted to this subject.

2. A Laflamme game and F_σ-separation. For an ideal J on the integers define an infinite game $G(J)$ as follows: player I in the nth move plays an element C_n of the ideal, and then player II plays a finite set F_n of integers with $F_n \cap C_n = \emptyset$. Player I wins when $\bigcup_n F_n \in J$. Otherwise player II wins. This game was investigated by C. Laflamme [La], who denoted it by $\mathcal{G}(J^*, [\omega]^{<\omega}, (J^*)^+)$. An ideal J *contains an ideal isomorphic to* $\mathrm{Fin} \times \mathrm{Fin}$ if there is a bijection $h : \omega \times \omega \to \omega$ such that for each $A \in \mathrm{Fin} \times \mathrm{Fin}$ we have $h[A] \in J$.

Lemma 2 (essentially Laflamme [La]). *Player I has a winning strategy in $G(J)$ iff J contains an ideal isomorphic to* $\mathrm{Fin} \times \mathrm{Fin}$.

Proof. From [La] we know that player I has a winning strategy iff J^* is *not a weak P-filter*, i.e. if there is a decreasing sequence $\{B_n : n \in \omega\} \subset J^*$ such that $A \in J$ whenever $A \subset^* B_n$ for each n, or equivalently, if there is a sequence $\{A_n : n \in \omega\}$ of pairwise disjoint elements of the ideal such that if A has finite intersection with each A_n, then $A \in J$. We may assume that all A_n are infinite and $\bigcup_n A_n = \omega$, so J contains an ideal isomorphic to $\mathrm{Fin} \times \mathrm{Fin}$. ∎

A set $Z \subset [\omega]^{<\omega}$ is called *J^*-universal* if it contains a subset of each element of J^*. We say that J^* is *ω-diagonalized by J^*-universal sets* if there are J^*-universal sets $\{Z_n\}_n$ such that for all $Y \in J^*$ there is an $n \in \omega$ such that $x \cap Y \neq \emptyset$ for all but finitely many $x \in Z_n$. Laflamme [La] proved that player II has a winning strategy iff J^* is ω-diagonalized by J^*-universal sets.

Now we will characterize Borel ideals J such that J and J^* can be *separated by an F_σ-set*, i.e. there is an F_σ-set $F \subset P(\omega)$ with $J^* \subset F$ and $J \cap F = \emptyset$. Solecki [So2] proved that if J is an $F_{\sigma\delta}$-ideal then J and J^* can be separated by an F_σ-set, and if $J = \mathrm{Fin} \times \mathrm{Fin}$ then J and J^* cannot be separated by an F_σ-set.

Proposition 3. *If J is a Borel ideal*, *then $G(J)$ is a determined game*, *i.e. one of the players has a winning strategy.*

Proof. Let $A = P(\omega)$, and define a pruned tree

$$T = \{(a_0, a_1, \ldots, a_n) \in A^{<\omega} : a_{2i} \in J \,\&\, a_{2i+1} \in [\omega]^{<\omega} \,\&\, a_{2i} \cap a_{2i+1} = \emptyset\}.$$

Let $F : [T] \to P(\omega)$ with $F((a_0, a_1, \ldots)) = \bigcup_i a_{2i+1}$. Then $G(J)$ can be denoted by $G(T, F^{-1}(J))$ according to the terminology of [Ke]. The domain of F, $[T]$, is a subspace of A^ω with the product topology with the discrete topology on A, and the codomain of F is taken with the standard Cantor topology. Then F is Borel measurable, because the set $F^{-1}[\{Z : k \in Z\}] = \{(a_0, a_1, \ldots) : k \in \bigcup_i a_{2i+1}\} = \bigcup_i \{(a_0, a_1, \ldots) : k \in a_{2i+1}\}$ is open in A^ω. Since J is Borel, it follows from Martin's theorem on Borel determinacy (see [Ke]) that $G(T, F^{-1}(J))$ is determined. ∎

THEOREM 4. *Assume that J is a Borel ideal. Then J and J^* can be separated by an F_σ-set iff J does not contain an isomorphic copy of* Fin × Fin.

Proof. First assume that II has a winning strategy in $G(J)$. Let $\{Z_n^k : k \in \omega\}_{n \in \omega}$ be J^*-universal sets ω-diagonalizing J^*. Let $F = \{A \subset \omega : (\exists n)\forall_k^\infty\ Z_n^k \cap A \neq \emptyset\}$. If $Z = \{a_1, \ldots, a_l\}$ then $\{A : Z \cap A \neq \emptyset\} = \{A : a_1 \in A \vee \cdots \vee a_l \in A\}$ is clopen, so F is F_σ such that $J^* \subset F$ and $J \cap F = \emptyset$.

If J contains an isomorphic copy of Fin × Fin then, by the result of Solecki, J and J^* cannot be separated by an F_σ-set. If J does not contain an isomorphic copy of Fin × Fin then I does not have a winning strategy in $G(J)$, so II has a winning strategy, thus J and J^* can be separated by an F_σ-set. ∎

3. Ideal convergence. The main result of this section is the following theorem.

THEOREM 5. *Let X be an uncountable Polish space*, *and let J be a Borel ideal. Then the following are equivalent*:

(1) $J\text{-}\lim C(X) = B_1(X)$.
(2) *J does not contain an isomorphic copy of* Fin × Fin.
(3) *J and J^* can be separated by an F_σ-set.*

LEMMA 6 (well known). *Let X be a complete metric space. Assume that $f : X \to \mathbb{R}$ has no point of continuity. Then there are reals α, β with $\alpha < \beta$ and an open nonempty set U such that $A = f^{-1}[(-\infty, \alpha)] \cap U$ and $B = f^{-1}[(\beta, \infty)] \cap U$ are dense in U.*

Proof. Let $\{(a_n, b_n) : n \in \omega\}$ be an open basis for $\mathbb{R}$. Then $X = \bigcup_n (f^{-1}[(a_n, b_n)] \setminus \mathrm{Int}(f^{-1}[(a_n, b_n)]))$. So there are n and a nonempty open set W such that $f^{-1}[(a_n, b_n)] \setminus \mathrm{Int}(f^{-1}[(a_n, b_n)])$ is nowhere meager in W. So $\{x \in W : f(x) \geq b_n\}$ or $\{x \in W : f(x) \leq a_n\}$, say the former, is dense in a nonempty open set $V \subset W$. Let $W_k = \{x \in V : f(x) < b_n - 1/k\}$. There are k and a nonempty open set $U \subset V$ such that W_k is nowhere meager in U. Then let $\alpha = b_n - 1/k$, and let β be any number with $\alpha < \beta < b_n$. ∎

Lemma 7. *Let X be a complete metric space. Assume that player II has a winning strategy in $G(J)$. If $f \in J\text{-}\lim C(X)$, then f has a point of continuity.*

Proof. Assume that player II has a winning strategy $S : J^{<\omega} \to [\omega]^{<\omega}$, and $f \in J\text{-}\lim C(X)$ does not have a point of continuity. Then there are reals α, β with $\alpha < \beta$ and an open nonempty set U such that $A = f^{-1}[(-\infty, \alpha)] \cap U$ and $B = f^{-1}[(\beta, \infty)] \cap U$ are dense in U.

Let $x_0 \in A$, and let $I_0 \subset U$ be a closed ball with $x_0 \in \mathrm{Int}(I_0)$. Since $J\text{-}\lim f_n(x_0) = f(x_0) < \alpha$, the set $C_0 = \{n : f_n(x_0) \ge \alpha\}$ belongs to J. Let $K_0 = S(\langle C_0 \rangle)$. There is a closed ball I_1 of radius smaller than 1 and $x_0 \in \mathrm{Int}(I_1) \subset I_1 \subset \mathrm{Int}(\{x \in I_0 : (\forall i \in K_0)\ f_i(x) < \alpha\}$.

For k even, let $x_k \in I_k \cap A$. So $J\text{-}\lim f_n(x_k) < \alpha$, and we define $C_k = \{n : f_n(x_k) \ge \alpha\} \in J$ and $K_k = S(C_0, C_2, \ldots, C_k)$. There is a closed ball I_{k+1} such that $x_k \in \mathrm{Int}(I_{k+1}) \subset I_{k+1} \subset \mathrm{Int}\{x \in I_k : (\forall i \in K_k)\ f_i(x) < \alpha\}$, and the radius of I_{k+1} is less than $1/(k+1)$.

For k odd, let $x_k \in I_k \cap B$. So $J\text{-}\lim f_n(x_k) > \beta$, and define $C_k = \{n : f_n(x_k) \le \beta\} \in J$ and $K_k = S(C_1, C_3, \ldots, C_k)$. There is a closed ball I_{k+1} such that $x_k \in \mathrm{Int}(I_{k+1}) \subset I_{k+1} \subset \mathrm{Int}\{x \in I_k : (\forall i \in K_k)\ f_i(x) > \beta\}$, and the radius of I_{k+1} is less than $1/(k+1)$.

Then let $\overline{x} \in \bigcap_k I_k$. We have

$$\{n : f_n(\overline{x}) < \alpha\} \supset \overline{C} = \bigcup_k S(C_0, C_2, \ldots, C_{2k}) \notin J,$$

$$\{n : f_n(\overline{x}) > \beta\} \supset C' = \bigcup_k S(C_1, C_3, \ldots, C_{2k+1}) \notin J.$$

So $J\text{-}\lim f_n(\overline{x})$ does not exist. ∎

Proposition 8. *Let X be a complete metric space. Assume that player II has a winning strategy in $G(J)$. Then $J\text{-}\lim C(X) = B_1(X)$.*

Proof. Our previous theorem implies that the restriction of any $f \in J\text{-}\lim C(X)$ to each nonempty perfect set has a point of continuity, so by Baire's theorem (see [Ku]), it is of Baire class one. ∎

Proof of Theorem 5. We know that (2) and (3) are equivalent. Proposition 3 implies that one of the players has a winning strategy.

If J contains an ideal isomorphic to $\mathrm{Fin} \times \mathrm{Fin}$, then $B_2(X) \subset J\text{-}\lim C(X)$ (see [Ka]). So $J\text{-}\lim C(X) \ne B_1(X)$ since X is an uncountable Polish space.

If J does not contain an ideal isomorphic to $\mathrm{Fin} \times \mathrm{Fin}$ then, by Lemma 2, I does not have a winning strategy, so II has one. Then, by Proposition 8, $J\text{-}\lim C(X) = B_1(X)$. ∎

Observe that the implications (2)→(1) and (3)→(1) hold for any complete metric space, not necessarily uncountable Polish.

REMARK. As a consequence, we see that $J\text{-}\lim C(X) = B_1(X)$ for every $F_{\sigma\delta}$-ideal J and complete metric space X. In particular, this is true for analytic P-ideals (see [So1]), F_σ-ideals (see [Maz]), and for the ideal $\mathrm{NWD}(\mathbb{Q}) = \{A \subset \mathbb{Q} : A \text{ is nowhere dense in } \mathbb{Q}\}$. However, for these three types of ideals we do not need to go through Martin's theorem on Borel determinacy. It is possible to construct simple strategies for player II using some characterizations of these types of ideals.

Recall that if J is a maximal ideal then J-limits may be irregular (see Fact 1 above). However, if we assume additionally that a given J-limit is Borel, then it will be automatically of Baire class one for some maximal ideals. The proof of the following theorem is essentially included in [JR].

PROPOSITION 9. *Assume the Continuum Hypothesis. There is a maximal P-ideal J (i.e. J^* is a P-point) such that for any sequence $\langle f_n : \mathbb{R} \to \mathbb{R} : n \in \omega\rangle$ of continuous functions, if $J\text{-}\lim f_n$ is Borel measurable, then there is an $A \in J^*$ with $J\text{-}\lim f_n = \lim_{n\in A} f_n$ (and so $J\text{-}\lim f_n$ is of Baire class one).*

Proof. Let $\{\langle\langle f_n\ \alpha : n \in \omega\rangle, g^\alpha\rangle : \alpha < 2^\omega\}$ be a well ordering of all pairs $\langle\langle f_n : n \in \omega\rangle, g\rangle$, where $\langle f_n : n \in \omega\rangle$ is a sequence of continuous real functions on the real line and g is a Borel measurable function. Let $P(\omega) = \{A_\alpha : \alpha < 2^\omega\}$. Define a transfinite sequence $\{X_\alpha : \alpha < 2^\omega\} \subset [\omega]^\omega$ decreasing with respect to $\subset^*$. On limit stages λ we choose X_λ such that $X_\lambda \subset^* X_\alpha$ for $\alpha < \lambda$. On a successor stage, first let $X'_{\alpha+1} \subset X_\alpha$ be an infinite set such that $X'_{\alpha+1} \subset A_\alpha$ or $X'_{\alpha+1} \subset \omega \setminus A_\alpha$. If, for each $x \in \mathbb{R}$, $\langle f_n^\alpha(x)\rangle_{n\in X'_{\alpha+1}}$ converges to $g^\alpha(x)$, then let $X_{\alpha+1} = X'_{\alpha+1}$. If there exists an $\overline{x} \in \mathbb{R}$ such that $\langle f^\alpha(\overline{x})\rangle_{n\in X'_{\alpha+1}}$ does not converge to $g^\alpha(\overline{x})$, then let $X_{\alpha+1} \subset X'_{\alpha+1}$ be such that $(\exists\varepsilon > 0)(\forall n \in X_{\alpha+1})\ |f_n^\alpha(\overline{x}) - g^\alpha(\overline{x})| > \varepsilon$.

Then $F = \{Y : (\exists\alpha)\ Y \supset^* X_\alpha\}$ is a P-point and F^* has the required properties. ∎

Observe that in the case above $J\text{-}\lim C(\mathbb{R})$ cannot be equal to $B_1(\mathbb{R})$, because some J-limits are nonmeasurable. Moreover, we have actually proved that, instead of assuming that $J\text{-}\lim f_n$ is Borel measurable, we can assume that $J\text{-}\lim f_n \in \mathcal{F}$ for a given family $\mathcal{F}$ of functions of size continuum.

The following result was suggested by A. Louveau.

PROPOSITION 10. *Assume that $J\text{-}\lim C(X) = B_1(X)$ for any Polish space X. Then $J\text{-}\lim B_\alpha(X) = B_{\alpha+1}(X)$ for any Polish space X.*

Proof. Assume that $\{f_n : n \in \omega\} \subset B_\alpha(X)$ and $f = J\text{-}\lim f_n$. Let $\{(V_k, W_k) : k \in \omega\}$ be the family of all pairs of open basic sets in $\mathbb{R}$ with $\mathrm{cl}(V_k) \subset W_k$. Then $f_n^{-1}[\mathrm{cl}(V_k)] \in \Pi^0_\alpha(X)$ and $f_n^{-1}[W_k]^c \in \Pi^0_\alpha(X)$, and they are disjoint for each $k, n \in \omega$. By a separation result (see [Ke, Theorem

22.16]), there is a family $\mathcal{E} = \{U_n^k : k, n \in \omega\} \subset \Delta_\alpha^0(X)$ with $f_n^{-1}[\mathrm{cl}(V_k)] \subset U_n^k \subset f_n^{-1}[W_k]$. Observe that $f^{-1}(U) = \bigcup\{W : W \in \mathcal{E} \,\&\, W \subset f^{-1}(U)\}$. By a theorem of Kuratowski (see [Ke, Theorem 22.18]), there is a Polish topology $\tau' \subset \Sigma_\alpha^0(X,\tau)$ such that $\mathcal{E} \subset \Delta_1^0(X,\tau')$. So $\{f_n\}_n \subset C(X,\tau')$, and thus by the assumption we have $J\text{-}\lim C(X,\tau') = B_1(x,\tau')$. Now, since $f \in B_1(X,\tau')$, we have $f^{-1}[U] \in \Sigma_2^0(X,\tau') \subset (\Pi_\alpha^0(X,\tau))_\sigma \subset \Sigma_{\alpha+1}^0(X,\tau)$. So $f \in B_{\alpha+1}(X,\tau)$. ∎

In [KSW] the authors observed that for the ideal J_d of sets of density zero, $J_d\text{-}\lim C(X) = B_1(X)$ for any space. This is based on the observation that for any bounded sequence $\{x_n\}_n$ of reals, if $J_d\text{-}\lim x_n = x$ then $(\sum_{i=1}^n x_i)/n \to x$. We generalize this result to nonpathological ideals (see [Far]).

A map $\varPhi : P(\omega) \to [0,\infty]$ is *a submeasure* on ω if $\varPhi(\emptyset) = 0$ and $\varPhi(A) \le \varPhi(A \cup B) \le \varPhi(A) + \varPhi(B)$ for all $A, B \subset \omega$. It is *lower semicontinuous* if for all $A \subset \omega$ we have $\varPhi(A) = \lim_n \varPhi(A \cap n)$. For any lower semicontinuous submeasure on ω, let $\|\cdot\|_\varPhi : P(\omega) \to [0,\infty]$ be the submeasure defined by $\|A\|_\varPhi = \lim_n \varPhi(A \setminus n)$. Let $\mathrm{Exh}(\varPhi) = \{A \subset \omega : \|A\|_\varPhi = 0\}$. It is clear that $\mathrm{Exh}(\varPhi)$ is an ideal (not necessarily proper in general) for an arbitrary submeasure $\varPhi$.

A submeasure $\varPhi$ is *nonpathological* if

$$\varPhi(A) = \sup\{\mu(A) : \mu \le \varPhi,\ \mu \text{ is a measure}\}$$

for each A.

LEMMA 11. *Assume that $J = \mathrm{Exh}(\varPhi)$ for some nonpathological lower semicontinuous submeasure $\varPhi$. Then there is a sequence $\{\langle a_1^k, \ldots, a_{n_k}^k\rangle : k \in \omega\}$, where all a_i^k are nonnegative real numbers, such that for any bounded sequence $\{x_n\}_n$ of reals with $J\text{-}\lim x_n = x$ we have*

$$\frac{\sum_{i=1}^{n_k} a_i^k x_i}{\sum_{i=1}^{n_k} a_i^k} \to x.$$

Proof. There is a $\gamma > 0$ such that $\|\omega\|_\varPhi > \gamma$. For each k we have $\varPhi(\omega \setminus k) > \gamma$, so there is $n_k > k$ with $\varPhi([k, n_k]) > \gamma$. Since $\varPhi$ is nonpathological, there is a measure $\mu_k \le \varPhi$ such that $\mu_k([k, n_k]) > \gamma$. Define

$$a_i^k = \begin{cases} 0, & i < k, \\ \mu_k(\{i\}), & k \le i \le n_k. \end{cases}$$

Assume that $J\text{-}\lim x_n = x$ and $|x_n| < K$ for each n. Let $\varepsilon > 0$. Then there is $A \in J$ such that $\{x_n\}_{n \notin A} \to x$. There is N such that $\varPhi(A \setminus k) < (\varepsilon\gamma)/(2K)$ for each $k > N$. There is M such that for each $i > M$ and $i \notin A$ we have $|x - x_i| < \varepsilon/2$. Let $k > \max(N, M)$. Then

$$\begin{aligned}\left|x-\frac{\sum_{i=1}^{n_k}a_i^k x_i}{\sum_{i=1}^{n_k}a_i^k}\right| &\le \frac{\sum_{i=1}^{n_k}a_i^k|x-x_i|}{\sum_{i=1}^{n_k}a_i^k}\\ &\le \frac{\sum_{i=1,\,i\notin A}^{n_k}a_i^k|x-x_i|}{\sum_{i=1}^{n_k}a_i^k}+\frac{\sum_{i=1,\,i\in A}^{n_k}a_i^k|x-x_i|}{\sum_{i=1}^{n_k}a_i^k}\\ &\le \frac{\sum_{i=1,\,i\notin A}^{n_k}a_i^k(\varepsilon/2)}{\sum_{i=1}^{n_k}a_i^k}+\frac{\sum_{i=1,\,i\in A}^{n_k}a_i^k K}{\sum_{i=1}^{n_k}a_i^k}\\ &\le \varepsilon/2+\frac{\mu_k([k,n_k]\cap A)K}{\mu_k([k,n_k])}\le \varepsilon/2+\frac{\Phi(A\setminus k)K}{\gamma}\le\varepsilon. \ \blacksquare\end{aligned}$$

COROLLARY 12. *Assume that* $J = \mathrm{Exh}(\Phi)$ *for some nonpathological lower semicontinuous submeasure* Φ. *Then* $J\text{-}\lim C(X) = B_1(X)$ *for any topological space* X.

Proof. Let $\{f_n : X \to \mathbb{R} : n \in \omega\}$ be a sequence of continuous functions with $J\text{-}\lim f_n = f$. Assume that $f_n[X] \subset (0,1)$ for each n and that $f[X] \subset (0,1)$. Then the functions

$$g_k : X \to (0,1), \qquad g_k(x) = \frac{\sum_{i=1}^{n_k} a_i^k f_i(x)}{\sum_{i=1}^{n_k} a_i^k},$$

are continuous, and $\lim_{k\to\infty} g_k(x) = f(x)$ for each x. $\blacksquare$

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