FUNDAMENTA
MATHEMATICAE
203 (2009)

Decompositions of the plane and the size of the continuum

by

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Abstract. We consider a triple $\langle E_0, E_1, E_2\rangle$ of equivalence relations on $\mathbb{R}^2$ and investigate the possibility of decomposing the plane into three sets $\mathbb{R}^2 = S_0 \cup S_1 \cup S_2$ in such a way that each S_i intersects each E_i-class in finitely many points. Many results in the literature, starting with a famous theorem of Sierpiński, show that for certain triples the existence of such a decomposition is equivalent to the continuum hypothesis. We give a characterization in ZFC of the triples for which the decomposition exists. As an application we show that the plane can be covered by three *sprays* regardless of the size of the continuum, thus answering a question of J. H. Schmerl.

1. Introduction. In 1919, Sierpiński [7] proved that the continuum hypothesis (CH) is equivalent to the existence of a subset S of the plane such that each horizontal line intersects S in countably many points and each vertical line intersects S in co-countably many points. Later in [8] he proved that CH is equivalent to the statement that the three-dimensional euclidean space can be decomposed into three sets S_i $(i \in 3)$ in such a way that each line parallel to the x_i axis intersects S_i in finitely many points. After that, many mathematicians (see for example [1–6, 9, 10]) have found generalizations of these theorems in different directions. In [2] Erdős asked the following question which he attributed to Sierpiński: does there exist a set of three directions d_i $(i = 1, 2, 3)$ in the plane, together with a decomposition of the plane into three corresponding sets S_i, such that every line in direction d_i intersects S_i in a finite set? A few years later Davies [1] showed that this is equivalent to CH and that it is irrelevant which directions you choose as long as they are different. More recently Komjáth [4] proved a similar equivalence where instead of three directions and lines in those directions one has three points and lines that pass through those points. Finally, in [6], Schmerl asks if the same equivalence holds when we consider

2000 *Mathematics Subject Classification*: Primary 03E50; Secondary 03E05, 51M05.
Key words and phrases: continuum hypothesis, Sierpiński's theorem, sprays.

DOI: 10.4064/fm203-1-6

three points and circles centered at those points. He remarks that CH does imply that there is such a decomposition but the question remains whether CH is indeed necessary in this case.

We find Schmerl's question very interesting because, as far as we know, all results related to Sierpiński's theorem deal essentially with linear objects in some euclidean space. However, the combinatorics of the problem is pretty much the same (whether it concerns lines or circles), at least at first sight. To state some context let us fix three equivalence relations E_i, $i \in 3$, on the plane with the property that for any $x \in \mathbb{R}^2$ and for $i, j \in 3$ with $i \neq j$ the set $[x]_i \cap [y]_j$ is finite (where $[x]_i$ denotes the equivalence class of x mod E_i). We shall say that a decomposition of the plane $\mathbb{R}^2 = \bigcup_{i\in 3} F_i$ is $\langle E_0, E_1, E_2\rangle$-*acceptable* if $[x]_i \cap F_i$ is finite for every $x \in \mathbb{R}^2$ and every $i \in 3$. In Davies' theorem, E_i is the equivalence relation whose classes are precisely the lines with direction d_i, and he proves that the existence of an acceptable decomposition is equivalent to CH. In Komjáth's theorem the equivalence classes mod E_i are the lines that pass through a fixed point p_i, and again he proves that CH is equivalent to the existence of an acceptable decomposition of the plane. Finally, in Schmerl's question each E_i is the relation "being at the same distance from p_i" for some fixed centers p_i $(i \in 3)$. Intuitively, one should expect the same kind of theorem, but surprisingly only one of the implications is (trivially) true, namely that CH implies the existence of an acceptable decomposition.

Our main goal in this note is to characterize the triples for which there is an acceptable decomposition of the plane. Then we will use that characterization to explain the role of CH in the theorems of Davies, Komjáth and Sierpiński. Finally, in the last section we show how one can use that characterization to get in ZFC an acceptable decomposition of the plane for a specific triple, thus answering Schmerl's question.

2. Notation and terminology. We think of $\mathfrak{c} = |\mathbb{R}|$ as an initial ordinal. Let $T = \bigcup_{n\in\omega} \mathfrak{c}^n$ be the set of finite sequences of ordinals in $\mathfrak{c}$. We have two natural orders on T, the (partial) tree order $\subseteq$ and the lexicographic order $\leq$. In both orders we have the same minimum element Λ, the empty sequence. Given any $\sigma \in T$ and $\alpha \in \mathfrak{c}$ we write $\sigma^\frown\alpha = \sigma \cup \{\langle|\sigma|, \alpha\rangle\}$. For $\sigma \in T \setminus \{\Lambda\}$ we write $\sigma + 1$ for the successor of σ in the lexicographic order of $\mathfrak{c}^{|\sigma|}$; that is,

$$\sigma + 1 = (\sigma\restriction(|\sigma| - 1))^\frown(\sigma(|\sigma| - 1) + 1).$$

We shall write $\sigma \wedge \tau$ for the infimum in the tree order of σ and τ; thus for $\sigma \neq \tau$ we have

$$\sigma \wedge \tau = \sigma\restriction|\sigma \wedge \tau| = \tau\restriction|\sigma \wedge \tau| \quad \text{and} \quad \sigma(|\sigma \wedge \tau|) \neq \tau(|\sigma \wedge \tau|).$$

Whenever we say that M is an *elementary submodel of the universe* (and write $M \prec V$) we really mean that $(M, \in)$ is an elementary submodel of $(H(\theta), \in)$, where $H(\theta)$ is the set of all sets of hereditary cardinality less than θ and $\theta = (2^{\mathfrak{c}})^+$. All the objects that we will consider (essentially points in $\mathbb{R}^2$, subsets of $\mathbb{R}^2$ and the collection of all subsets of $\mathbb{R}^2$) are in $H(\theta)$ and all the statements ϕ for which we shall use elementarity will be bounded so that in fact $M \models \phi$ if and only if ϕ is true in the universe.

As mentioned in the introduction, we will consider triples $\langle E_0, E_1, E_2\rangle$ of equivalence relations on $\mathbb{R}^2$. For $x \in \mathbb{R}^2$, the equivalence class of x mod E_i will be denoted by $[x]_i$. All triples will have the property that $[x]_i \cap [x]_j$ is finite whenever $i \neq j$. We say that a decomposition $\mathbb{R}^2 = \bigcup_{i\in 3} F_i$ is $\langle E_0, E_1, E_2\rangle$-*acceptable* (or that the decomposition is *acceptable for* $\langle E_0, E_1, E_2\rangle$) if for every $x \in \mathbb{R}^2$ and $i \in 3$, the set $[x]_i \cap F_i$ is finite.

3. Twisted triples. We wish to characterize the triples $\langle E_0, E_1, E_2\rangle$ for which there exists an acceptable decomposition of the plane. Our main notion is the following:

Definition 3.1. We say that the triple $\langle E_0, E_1, E_2\rangle$ is *twisted* if for every $a \in \mathbb{R}^2$, for all $M, N \prec V$ such that $\langle E_0, E_1, E_2\rangle \in M \cap N$ and $N \in M$ and whenever $\{i, j, k\} = 3$, we have

$$|\{x \in [a]_k : [x]_i \in (M \setminus N),\ [x]_j \in (N \setminus M)\}| < \aleph_0.$$

Our goal is to show that twisted triples are exactly the ones that admit acceptable decompositions. But first we need a coherent way to cover the plane with countable elementary submodels of the universe.

We fix $M_\Lambda \prec V$ such that $\mathbb{R}^2 \cup \{E_0, E_1, E_2\} \subseteq M_\Lambda$ and $|M_\Lambda| = \mathfrak{c}$. Now we can find inductively (on the length of $\sigma \in T$) models $M_\sigma \prec V$ such that:

(i) the sequence $\langle M_{\sigma^\frown\alpha} : \alpha \in \mathrm{cof}(|M_\sigma|)\rangle$ is a continuous (increasing) elementary chain,
(ii) $M_\sigma \subseteq \bigcup\{M_{\sigma^\frown\alpha} : \alpha \in \mathrm{cof}(|M_\sigma|)\}$,
(iii) $\{E_0, E_1, E_2\} \cup \{M_\tau : \tau + 1 \subseteq \sigma\} \subseteq M_{\sigma^\frown 0}$,
(iv) $|M_\tau| > |M_\sigma|$ whenever $\tau \subset \sigma$ and M_τ is uncountable.

We actually do not need to define $M_{\sigma^\frown\alpha}$ if M_σ is countable or if $\alpha \geq \mathrm{cof}(|M_\sigma|)$. On the other hand, note that conditions (ii) and (iv) imply that $\mathrm{rk}(x)$ is well defined as follows:

Definition 3.2. For $x \in M_\Lambda$ we define $\mathrm{rk}(x)$ (the *rank of* x) as the minimum $\sigma \in T$ (in the lexicographic order) such that M_σ is countable and $x \in M_\tau$ for all $\tau \subseteq \sigma$.

It is easy to see (using the continuity of the chains) that $\mathrm{rk}(x)$ is always a finite sequence of ordinals which are either successor ordinals or 0. Moreover, if $\sigma_x = \mathrm{rk}(x)$, $\sigma_y = \mathrm{rk}(y)$, $\sigma_x < \sigma_y$ and $n = |\sigma_x \wedge \sigma_y|$, then $\sigma_y(n)$ is a successor ordinal, say $\alpha + 1$, and we define

$$\Delta(x, y) = (\sigma_x \wedge \sigma_y)^\frown \alpha.$$

The reason for this definition is that $M_{\Delta(x,y)}$ is in some sense the first model that witnesses the fact that $\mathrm{rk}(x)$ and $\mathrm{rk}(y)$ are different. Note that $\Delta(x, y)+1 \subseteq \mathrm{rk}(y)$ and that conditions (i) and (iii) imply that $M_{\Delta(x,y)} \in M_\sigma$ whenever σ is a proper extension of $\Delta(x, y) + 1$.

Observation 3.3. Given $x \in \mathbb{R}^2$ and $\sigma \in T$ note that if $x \in M_\sigma$ then by elementarity $\{[x]_0, [x]_1, [x]_2\} \subseteq M_\sigma$, so that $\mathrm{rk}([x]_i) \leq \mathrm{rk}(x)$ for all $i \in 3$; on the other hand, if $x \notin M_\sigma$ then by elementarity $|\{[x]_0, [x]_1, [x]_2\} \cap M_\sigma| \leq 1$ since $[x]_i \cap [x]_j$ is finite for $i \neq j$.

This simple fact will help us prove the following:

Lemma 3.4. *If $\langle E_0, E_1, E_2\rangle$ is twisted then for each $x \in \mathbb{R}^2$ there is an $i \in 3$ such that* $\mathrm{rk}([x]_i) = \mathrm{rk}(x)$.

Proof. First note that if $i \neq j$ and $\mathrm{rk}([x]_i) = \mathrm{rk}([x]_j)$ then necessarily $\mathrm{rk}(x) = \mathrm{rk}([x]_i)$, so we may assume without loss of generality that

$$\mathrm{rk}([x]_0) < \mathrm{rk}([x]_1) < \mathrm{rk}([x]_2);$$

we shall prove that in that case $\mathrm{rk}(x) = \mathrm{rk}([x]_2)$.

By Observation 3.3 we easily see that $\Delta([x]_0, [x]_2) = \Delta([x]_0, x)$ and $\Delta([x]_1, [x]_2) = \Delta([x]_1, x)$, and from these we get

$$\Delta([x]_0, [x]_2) + 1 \subseteq \mathrm{rk}([x]_2) \wedge \mathrm{rk}(x),$$
$$\Delta([x]_1, [x]_2) + 1 \subseteq \mathrm{rk}([x]_2) \wedge \mathrm{rk}(x).$$

Again by Observation 3.3 we have $\Delta([x]_0, [x]_2)+1 \subset \Delta([x]_1, [x]_2)$ and hence $M_{\Delta([x]_0,[x]_2)} \in M_{\Delta([x]_1,[x]_2)}$. We also have $[x]_0 \in M_{\Delta([x]_0,[x]_2)} \setminus M_{\Delta([x]_1,[x]_2)}$ and $[x]_1 \in M_{\Delta([x]_1,[x]_2)} \setminus M_{\Delta([x]_0,[x]_2)}$. But this means that, if σ is a proper extension of $\mathrm{rk}([x]_2) \wedge \mathrm{rk}(x)$, then $M_{\Delta([x]_0,[x]_2)}, M_{\Delta([x]_1,[x]_2)} \in M_\sigma$, and therefore $x \in M_\sigma$ if and only if $[x]_2 \in M_\sigma$. Here we have used the fact that $\langle E_0, E_1, E_2\rangle$ is twisted and therefore x belongs to a finite set definable in terms of $[x]_2$, $M_{\Delta([x]_0,[x]_2)}$ and $M_{\Delta([x]_1,[x]_2)}$. This shows that $\mathrm{rk}(x) = \mathrm{rk}([x]_2)$. ∎

Lemma 3.5. *If $\langle E_0, E_1, E_2\rangle$ is twisted then for every $a \in \mathbb{R}^2$, whenever $\{i, j, k\} = 3$, the set*

$$X = \{x \in [a]_k : \mathrm{rk}([x]_i) < \mathrm{rk}([x]_j) < \mathrm{rk}([x]_k) = \mathrm{rk}(x)\}$$

is finite.

Proof. As we noticed in the proof of the previous lemma, $x \in X$ implies that $[x]_i \in M_{\Delta([x]_i,[x]_k)} \setminus M_{\Delta([x]_j,[x]_k)}$, $[x]_j \in M_{\Delta([x]_j,[x]_k)} \setminus M_{\Delta([x]_i,[x]_k)}$ and

$M_{\Delta([x]_i,[x]_k)} \in M_{\Delta([x]_j,[x]_k)}$. Thus it is clear that

$$X \subseteq \bigcup_{\substack{\sigma+1\subseteq \mathrm{rk}([a]_k)\\ \tau+1\subseteq \mathrm{rk}([a]_k)}} \{x \in [a]_k : [x]_i \in (M_\sigma \setminus M_\tau), [x]_j \in (M_\tau \setminus M_\sigma)\},$$

and since $\langle E_0, E_1, E_2\rangle$ is twisted, the latter is a finite union of finite sets. ∎

Now we need a way to order in type ω all the elements of M_Λ of the same rank. This is easily done by fixing an injective enumeration

$$M_\sigma = \{t_n^\sigma : n \in \omega\}$$

for each σ for which M_σ is countable, and defining:

Definition 3.6. For $x \in M_\Lambda$ we define $\deg(x)$ (the *degree* of x) as the unique $n \in \omega$ for which $x = t_n^{\mathrm{rk}(x)}$.

The following is true for any triple $\langle E_0, E_1, E_2\rangle$, twisted or not.

Lemma 3.7. *For every $a \in \mathbb{R}^2$ and for all $i, k \in 3$ with $i \neq k$, the set*

$$X = \{x \in [a]_k : \mathrm{rk}([x]_i) = \mathrm{rk}([x]_k) \text{ and } \deg([x]_i) < \deg([x]_k)\}$$

is finite.

Proof. Let $\sigma = \mathrm{rk}([a]_k)$ and $n = \deg([a]_k)$. Note that if $x \in X$ then there is an $m < n$ (namely, $m = \deg([x]_i)$) such that $x \in t_m^\sigma \cap t_n^\sigma$ and $t_m^\sigma \cap t_n^\sigma$ is finite. Hence X is contained in a finite union of finite sets. ∎

Finally, we are ready to prove the main result of this section.

Theorem 3.8. *The following are equivalent*:

(1) *The triple $\langle E_0, E_1, E_2\rangle$ is twisted.*
(2) *There is an $\langle E_0, E_1, E_2\rangle$-acceptable decomposition of $\mathbb{R}^2$.*

Proof. Suppose first that the triple $\langle E_0, E_1, E_2\rangle$ is twisted and for each $k \in 3$ define F_k as the set of all $x \in \mathbb{R}^2$ such that:

(i) $\mathrm{rk}(x) = \mathrm{rk}([x]_k)$,
(ii) for all $i \in 3 \setminus \{k\}$, if $\mathrm{rk}([x]_i) = \mathrm{rk}([x]_k)$ then $\deg([x]_i) < \deg([x]_k)$.

Given $x \in \mathbb{R}^2$ we know by Lemma 3.4 that $I := \{i \in 3 : \mathrm{rk}(x) = \mathrm{rk}([x]_i)\}$ in not empty. If we let $k \in I$ be such that $\deg([x]_i) \leq \deg([x]_k)$ for all $i \in I$, then it is immediate from the construction that $x \in F_k$. Hence $\mathbb{R}^2 = F_0 \cup F_1 \cup F_2$.

On the other hand, using Lemmas 3.5 and 3.7 we see that for any $a \in \mathbb{R}^2$ and $k \in 3$ the set $[a]_k \cap F_k$ is finite, so that the decomposition $\mathbb{R}^2 = F_0 \cup F_1 \cup F_2$ is indeed $\langle E_0, E_1, E_2\rangle$-acceptable.

Now suppose (2) holds and $\langle E_0, E_1, E_2\rangle$ is not twisted. This means that there are $M, N \prec V$ with $\langle E_0, E_1, E_2\rangle \in M \cap N$ and $N \in M$ such that for some $a \in \mathbb{R}^2$ and some $\{i, j, k\} = 3$ the set

$$X = \{x \in [a]_k : [x]_i \in (M \setminus N),\ [x]_j \in (N \setminus M)\}$$

is infinite. On the other hand, using elementarity and the fact that $N \in M$, we can find $F_0, F_1, F_2 \in M \cap N$ such that $\mathbb{R}^2 = F_0 \cup F_1 \cup F_2$ is $\langle E_0, E_1, E_2\rangle$-acceptable. Now for each $x \in X$ we have $x \notin M$ because $[x]_j \notin M$ and therefore $x \notin F_i$, since otherwise x would be in a finite set definable from F_i and $[x]_i$ which are both in M. Similarly, $x \notin N$ and hence $x \notin F_j$. But then $X \subseteq F_k$, which contradicts the fact that $[a]_k \cap F_k$ is finite. ∎

4. The role of CH**.** In principle, the fact that a given triple is twisted (or untwisted) should depend only on the geometry of the triple. However, the presence of CH trivializes things as we now show.

THEOREM 4.1. *Under CH every triple $\langle E_0, E_1, E_2\rangle$ is* (*trivially*) *twisted.*

Proof. Let $a \in \mathbb{R}^2$, let $M, N \prec V$ be such that $\langle E_0, E_1, E_2\rangle \in M \cap N$ and $N \in M$, and let $\{i, j, k\} = 3$. It is well known that either $N \cap \omega_1$ is countable or $\omega_1 \subseteq N$. Under CH this implies that either $\mathbb{R}^2 \cap N$ is countable or $\mathbb{R}^2 \subseteq N$. If $\mathbb{R}^2 \cap N$ is countable then since $N \in M$ we have $\mathbb{R}^2 \cap N \subseteq M$, so there is no $x \in \mathbb{R}^2$ for which $[x]_j \in N \setminus M$. On the other hand, if $\mathbb{R}^2 \subseteq N$ then there is no $x \in \mathbb{R}^2$ for which $[x]_i \in M \setminus N$. In either case the set

$$\{x \in [a]_k : [x]_i \in M \setminus N,\ [x]_j \in N \setminus M\}$$

is empty, and hence $\langle E_0, E_1, E_2\rangle$ is twisted. ∎

As an immediate consequence of Theorems 3.8 and 4.1 we get the following result which is already known (for instance, it is a special case of Theorem 2 in [3], for $\theta = 1$ and $(r, s) = (2, 1)$):

COROLLARY 4.2. *Under CH every triple $\langle E_0, E_1, E_2\rangle$ admits an acceptable decomposition.*

Under ¬CH the situation is quite different. Say we want to show that a certain triple is not twisted. Now we can just take $M, N \prec V$ with $|M| = \aleph_0$ and $|\mathbb{R}^2 \cap N| = \aleph_1 < \mathfrak{c}$ and it is usually the case that there is $x \in \mathbb{R}^2$ for which, say, $[x]_0 \in M \setminus N$ and $[x]_1 \in N \setminus M$. Of course, there may still be cases where this does not happen (e.g. if all the classes are finite) and again the triple will be twisted for trivial reasons. But for the triples we have mentioned (the ones associated with Davies', Komjáth's and Sierpiński's theorems and the triple associated with Schmerl's question) we are in that situation.

The question now is whether you can "move" the point x in such a way that $[x]_2$ remains constant while $[x]_0$ and $[x]_1$ change in a "definable way" (not depending on $[x]_2$) so that they do not get out of M and N respectively. In Sierpiński's theorem (here we are working in $\mathbb{R}^3$ instead of $\mathbb{R}^2$ but that does not make any difference) this task is trivial because here $[x]_i$ is just the line that passes through x and is parallel to the e_i axis, so we can just add any rational number to the third coordinate of x and we are done. Similarly, in Davies's theorem, $[x]_i$ is the line in direction d_i that passes through x; now we just need to move x in direction d_2 by a rational amount. The task is not as immediate in Komjáth's theorem and it requires a rather clever trigonometric argument.

We illustrate in more detail these ideas with the following result which is half of the answer to Schmerl's question. According to Schmerl [6], a subset $S \subseteq \mathbb{R}^2$ is a *spray around* $c \in \mathbb{R}^2$ if the intersection of S with any circle centered at c is finite.

THEOREM 4.3. *Let c_i for $i \in 3$ be three distinct points on $\mathbb{R}^2$ that lie on the same line. Then the following are equivalent*:

(1) *CH.*
(2) $\mathbb{R}^2$ *can be covered by three sprays around the points c_i $(i \in 3)$.*

Proof. We define E_i as the set of all $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$ such that $\|x - c_i\| = \|y - c_i\|$. Now (1) implies (2) by Corollary 4.2. Assume then that $\mathfrak{c} > \aleph_1$. We are done if we can prove that $\langle E_0, E_1, E_2\rangle$ is not twisted.

For $i \in 2$ fix rational intervals $I_i \subseteq \mathbb{R}$ such that whenever C_i is a circle centered at c_i with radius $r_i \in I_i$ then $C_0 \cap C_1 \neq \emptyset$.

Now let $M, N \prec V$ with $|M| = \aleph_0$ and $|\mathbb{R} \cap N| = \aleph_1 < \mathfrak{c}$ be such that $\{c_0, c_1, c_2\} \subseteq M \cap N$ and $N \in M$. Since $\mathbb{R} \setminus N \neq \emptyset$ and $N \in M$ we deduce by elementarity that there is an $r_0 \in I_0 \cap (M \setminus N)$. On the other hand, $|I_1 \cap N| = \aleph_1 > |M|$ so there is an $r_1 \in I_1 \cap (N \setminus M)$. Let $x \in \mathbb{R}^2$ be such that $\|x - c_i\| = r_i$ for $i \in 2$. Then

$$(t-1)\|x - c_2\|^2 = tr_0^2 - r_1^2 + t(\|c_2\|^2 - \|c_0\|^2) + \|c_1\|^2 - \|c_2\|^2$$

where t is the unique real number for which $c_1 - c_2 = t(c_0 - c_2)$. What is important is that $t \in M \cap N$, so that for every large enough $n \in \omega$ we have $r_{0,n} := \sqrt{r_0^2 + 1/n} \in I_0 \cap (M \setminus N)$, $r_{1,n} := \sqrt{r_1^2 + t/n} \in I_1 \cap (N \setminus M)$ and any point $x_n \in \mathbb{R}^2$ for which $\|x_n - c_i\| = r_{i,n}$ for $i \in 2$ (which exists by the definition of I_0 and I_1) will satisfy $\|x_n - c_2\| = \|x - c_2\|$. Therefore $\langle E_0, E_1, E_2\rangle$ is not twisted. ■

5. The plane can be covered by three sprays. The goal of this section is to show (in ZFC) that if $c_0, c_1, c_2 \in \mathbb{R}^2$ are the vertices of an equilateral triangle, then the plane can be covered by three sprays S_0, S_1 and S_2 around c_0, c_1 and c_2 respectively. We suspect that the same remains true for any triangle (as long as the c_i's do not lie on the same line) and we have checked a couple of examples, but we have not found a reasonable way to prove it simultaneously because in one step of the proof we require a computer algebra system to check for the irreducibility of a certain polynomial.

We may assume without loss of generality (applying first a circle preserving transformation if necessary) that $c_0 = (-1,0)$, $c_1 = (1,0)$ and $c_2 = (0,\sqrt{3})$. Suppose that three circles C_0, C_1 and C_2 centered at c_0, c_1 and c_2 respectively have a point in common. If we use α_0, α_1 and α_2 for the squares of their radii we obtain the relation

$$\alpha_0^2 + \alpha_1^2 + \alpha_2^2 - \alpha_0\alpha_1 - \alpha_0\alpha_2 - \alpha_1\alpha_2 - 4\alpha_0 - 4\alpha_1 - 4\alpha_2 + 16 = 0.$$

Now suppose that C_0' and C_1' are also circles centered at c_0 and c_1 respectively, have radii β_0 and β_1, and C_0', C_1' and C_2 also have a point in common. Then we have a similar equation (involving β_0, β_1 and α_2) and we can eliminate α_2 to get a relation of the form

$$p(\alpha_0, \beta_0, \alpha_1, \beta_1) = 0$$

where $p = p(X,Y,Z,W)$ is a polynomial of degree 4. We want to think of X, Y as variables and regard Z, W as parameters. It is not worth it to write down here what p exactly is; instead we state without proof the facts about p that we will be needing. First of all one can use a computer algebra system (e.g. MAPLE) to check that $p(X,Y,1,2)$ is absolutely irreducible as a polynomial in two variables. On the other hand, one can prove that for any such polynomial, the set of $(z,w) \in \mathbb{R}^2$ for which $p(X,Y,z,w)$ is not absolutely irreducible is either the whole $\mathbb{R}^2$ or is contained in a curve. Since we already know that the former is not the case, we obtain the following:

Lemma 5.1. *If $A \subseteq \mathbb{R}$ is infinite then there exist $z, w \in A$ such that $p(X,Y,z,w)$ is absolutely irreducible as a polynomial in X, Y.*

Now suppose that we have two pairs of parameters (z,w) and (z',w') such that both $p(X,Y,z,w)$ and $p(X,Y,z',w')$ are irreducible. If $(z,w) \neq (z',w')$ we can check by simple inspection that these two polynomials are not constant multiples of each other. This implies the following:

Lemma 5.2. *Suppose that $z, z', w, w' \in \mathbb{R}$, $(z,w) \neq (z',w')$ and that both $p(X,Y,z,w)$ and $p(X,Y,z',w')$ are absolutely irreducible. Then the system*

$$p(X,Y,z,w) = 0,$$
$$p(X,Y,z',w') = 0$$

has finitely many solutions.

We are ready to prove that $\langle E_0, E_1, E_2\rangle$ is twisted, where for each $i \in 3$, E_i is the equivalence relation in $\mathbb{R}^2$ whose classes are circles centered at c_i. If $x \in \mathbb{R}^2$ and $i \in 3$ we define $r_i(x) := \|x - c_i\|^2$. Fix $M, N \prec V$ with $\langle E_0, E_1, E_2\rangle \in M \cap N$ and $N \in M$. Since the triangle $c_0c_1c_2$ is equilateral, it is enough to prove that

$$B(a) := \{x \in [a]_2 : [x]_0 \in (M \setminus N),\ [x]_1 \in (N \setminus M)\}$$

is finite for any $a \in \mathbb{R}^2$. Suppose for contradiction that there is $a \in \mathbb{R}^2$ for which $B(a)$ is infinite. Then $A := \{r_1(x) : x \in B(a)\}$ is an infinite subset of $\mathbb{R}$ and therefore by Lemma 5.1 there are $\alpha_1, \beta_1 \in A$ such that $p(X, Y, \alpha_1, \beta_1)$ is absolutely irreducible. By the definition of A there are $x, x' \in [a]_2$ such that $\alpha_1 = r_1(x)$ and $\beta_1 = r_1(x')$, and we let $\alpha_0 = r_0(x)$ and $\beta_0 = r_0(x')$. Now let G be the set of all $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ such that:

(i) $\alpha, \beta \in N$,
(ii) there are $z, z' \in \mathbb{R}^2$ such that $r_0(z) = \alpha_0$, $r_0(z') = \beta_0$, $r_1(z) = \alpha$, $r_1(z') = \beta$ and $r_2(z) = r_2(z')$,
(iii) the polynomial $p(X, Y, \alpha, \beta)$ is absolutely irreducible.

Note that $G \in M$ since it is definable in terms of N, α_0 and β_0 which are all in M. Also note that $(\alpha_1, \beta_1) \in G$ so that $G \neq \emptyset$. By elementarity there is $(\alpha_1', \beta_1') \in G \cap M$ and since $(\alpha_1, \beta_1) \notin M$ we have $(\alpha_1, \beta_1) \neq (\alpha_1', \beta_1')$. Finally, by Lemma 5.2 we deduce that the system

$$p(X, Y, \alpha_1, \beta_1) = 0,$$
$$p(X, Y, \alpha_1', \beta_1') = 0$$

has finitely many solutions. But this system is definable in N and therefore (α_0, β_0), being one of its solutions, must belong to N, a contradiction. Hence $\langle E_0, E_1, E_2\rangle$ is twisted.

Using Theorem 3.8 we have just proved the following (no assumptions on the size of the continuum):

THEOREM 5.3. *If $c_0, c_1, c_2 \in \mathbb{R}^2$ are the vertices of an equilateral triangle, then the plane can be covered by three sprays S_0, S_1 and S_2 around c_0, c_1 and c_2 respectively.*

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Received 1 October 2008;
in revised form 10 December 2008