## How many clouds cover the plane?

by

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**Abstract.** The plane can be covered by n+2 clouds iff  $2^{\aleph_0} \leq \aleph_n$ .

According to Komjáth [1], a subset C of the Euclidean plane  $\mathbb{R}^2$  is a cloud around  $\mathbf{a}$  if  $\mathbf{a} \in \mathbb{R}^2$  and whenever  $\ell \subseteq \mathbb{R}^2$  is a line and  $\mathbf{a} \in \ell$ , then  $C \cap \ell$  is finite. The following is proved in [1].

THEOREM 1 (Komjáth). Suppose that  $1 \leq n < \omega$ . If  $2^{\aleph_0} \leq \aleph_n$ , then  $\mathbb{R}^2$  can be covered by n + 2 clouds.

It should be remarked that a somewhat stronger conclusion was proved: whenever  $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{n+1} \in \mathbb{R}^2$  are n+2 distinct noncollinear points, then each  $\mathbf{a}_i$  has a cloud  $C_i$  around it such that  $\mathbb{R}^2 = C_0 \cup C_1 \cup \ldots \cup C_{n+1}$ . A converse to Theorem 1 was proved in [1] for n = 1. That is, if  $\mathbb{R}^2$  can be covered by 3 clouds, then the Continuum Hypothesis holds. The converse for n > 1 was left open. However, Komjáth [2] did prove that it is relatively consistent that  $2^{\aleph_0} = \aleph_4$  and  $\mathbb{R}^2$  cannot be covered by 3 clouds.

The following converse to Theorem 1 is the main new result in this note.

THEOREM 2. Suppose that  $1 \leq n < \omega$ . If  $\mathbb{R}^2$  can be covered by n+2 clouds, then  $2^{\aleph_0} \leq \aleph_n$ .

There are many results closely related to Theorems 1 and 2. Simms [4] presents a thorough historical account of these theorems. We will make use of the following one which Simms attributes to Kuratowski [3].

THEOREM 3 (Kuratowski). Suppose that  $n < \omega$  and that X is any set. Then  $|X| \leq \aleph_n$  iff there are  $D_0, D_1, \ldots, D_{n+1} \subseteq X^{n+2}$  which cover  $X^{n+2}$ such that  $D_i \cap \ell$  is finite whenever  $i \leq n+1$  and  $\ell \subseteq X^{n+2}$  is a line parallel to the *i*th coordinate axis.

In this theorem we have referred to  $\ell$  as a line parallel to the *i*th coordinate axis; such an  $\ell$  is a set for which

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$$\ell = \{(a_0, a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n, a_{n+1}) \in X^{n+2} : x \in X\}$$

for some  $a_0, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n+1} \in X$ .

Proof (1) of Theorem 2. Suppose that  $C_0, C_1, \ldots, C_{n+1}$  cover  $\mathbb{R}^2$ , where each  $C_i$  is a cloud around  $\mathbf{a}_i$ . Without loss of generality, we assume that each  $\mathbf{a}_i \neq \mathbf{0}$ . We will show that  $2^{\aleph_0} \leq \aleph_n$ .

For  $2 \leq m < \omega$ , consider  $\mathbb{R}^m$  as a vector space over  $\mathbb{R}$ , and let  $\mathbf{P}_m(\mathbb{R})$  be real projective *m*-space, with  $\mathbb{R}^m$  being embedded in  $\mathbf{P}_m(\mathbb{R})$  by identifying the point  $(x_0, x_1, \ldots, x_{m-1}) \in \mathbb{R}^m$  with the point in  $\mathbf{P}_m(\mathbb{R})$  having homogeneous coordinates  $[x_0, x_1, \ldots, x_{m-1}, 1]$ .

For i < n + 2, let  $\infty_i \in \mathbf{P}_{n+2}(\mathbb{R})$  be the point at infinity on (every line parallel to) the *i*th coordinate axis of  $\mathbb{R}^{n+2}$ , and let  $\mathbf{e}_i \in \mathbb{R}^{n+2}$  be the *i*th standard basis vector. Let  $S : \mathbf{P}_{n+2}(\mathbb{R}) \to \mathbf{P}_{n+2}(\mathbb{R})$  be a collineation for which  $S(\infty_i) = \mathbf{e}_i$  and  $S(\mathbf{0}) = \mathbf{0}$ , and let  $T : \mathbb{R}^{n+2} \to \mathbb{R}^2$  be the unique linear transformation such that  $T(\mathbf{e}_i) = \mathbf{a}_i$ . If  $\mathbf{x} \in \mathbb{R}^{n+2}$ , then  $TS(\mathbf{x})$  is defined as long as  $\mathbf{x}$  is not in the hyperplane  $H \subseteq \mathbb{R}^{n+2}$ . Since  $\mathbf{0} \notin H$ , let  $I = (-\varepsilon, \varepsilon)$  be an open interval such that  $I^{n+2} \cap H = \emptyset$ , and then let  $D_i = (TS)^{-1}(C_i) \cap I^{n+2}$  for each i < n + 2. Clearly, the  $D_i$ 's cover  $I^{n+2}$ since the  $C_i$ 's cover  $\mathbb{R}^2$ .

Let  $\ell$  be a line of  $\mathbf{P}_{n+2}(\mathbb{R})$  such that  $\infty_i \in \ell$  and  $\ell$  meets  $I^{n+2}$ . Since  $T(\mathbf{e}_i) \neq \mathbf{0}, TS$  is one-one on  $\ell \cap I^{n+2}$ , and then, since TS preserves collinearity, there is a unique line  $\ell'$  of  $\mathbb{R}^2$  which TS maps  $\ell$  into. Then  $\mathbf{a}_i = TS(\infty_i) \in \ell'$ , so  $D_i \cap \ell$  is finite. Kuratowski's theorem now applies, so  $2^{\aleph_0} = |I| \leq \aleph_n$ .

Komjáth [1] also defines a set  $C \subseteq \mathbb{R}^2$  to be a *circle around* **a** if  $\mathbf{a} \in \mathbb{R}^2$  and every half-line from **a** meets C at no more than one point. Let us say that a cloud around **a** is an *n*-cloud if it meets each line through **a** at no more than n points. Thus, every 1-cloud is a circle, and every circle is a 2-cloud.

Komjáth [1] proves that  $\mathbb{R}^2$  can be covered by countably many circles. In fact, he shows that whenever  $\mathbf{a}_0, \mathbf{a}_1, \ldots$  are countably many noncollinear points, then each  $\mathbf{a}_i$  has a circle  $C_i$  around it such that  $\mathbb{R}^2 = C_0 \cup C_1 \cup \ldots$ His proof works equally well with 1-clouds. He conjectures that finitely many circles do not suffice to cover  $\mathbb{R}^2$ , and he remarks that he has proved that this is so when the circles are around distinct points. The following affirms Komjáth's conjecture.

THEOREM 4. Finitely many circles cannot cover  $\mathbb{R}^2$ .

*Proof.* The proof is practically already in the proof of Theorem 2. For a contradiction, suppose that  $C_0, C_1, \ldots, C_{n+1}$  are finitely many circles,

<sup>(&</sup>lt;sup>1</sup>) This presentation of the proof owes much to Jan Mycielski who saw what was really going on with my original proof.

around  $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{n+1}$  respectively, which cover  $\mathbb{R}^2$ . We allow the possibility that not all the  $\mathbf{a}_i$  are distinct, but we do assume that each  $\mathbf{a}_i \neq \mathbf{0}$ . Proceed just as in the proof of Theorem 2 to obtain  $TS : \mathbf{P}_{n+2}(\mathbb{R}) \to \mathbb{R}^2$ and  $D_0, D_1, \ldots, D_{n+1} \subseteq I^{n+2} \subseteq \mathbb{R}^{n+2}$ . It is clear that each line which is parallel to the *i*th coordinate axis meets  $D_i$  in at most 2 points. We show this leads to a contradiction.

Let  $N \subseteq I$  be a set having exactly k = 2n + 5 elements. For each  $i \leq n+1$ , there are exactly  $k^{n+1}$  lines parallel to the *i*th coordinate axis which meet  $N^{n+2}$ ; hence,  $|D_i \cap N^{n+2}| \leq 2k^{n+1}$ . Therefore,  $k^{n+2} = |N^{n+2}| \leq (n+2) \cdot 2k^{n+1}$ , so that  $k \leq 2(n+2)$ , which is a contradiction.

The proof of Theorem 1 is very robust, being easily adapted to apply to objects other than clouds. For example, let us say that a subset  $C \subseteq \mathbb{R}^2$ is a *spray around* **a** if the intersection of C with any circle (in the classical Euclidean sense) centered at **a** is finite. Then the following can be proved along the lines of the proof of Theorem 1.

THEOREM 5. Suppose that  $1 \leq n < \omega$ . If  $2^{\aleph_0} \leq \aleph_n$ , then  $\mathbb{R}^2$  can be covered by n+2 sprays.

We do not know if there is a converse to Theorem 5 in the style of Theorem 2.

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