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Relations approximated by continuous functions in the Vietoris topology

by

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Dedicated to the memory of Professor J. Pelant

Abstract. Let X be a Tikhonov space, $C(X)$ be the space of all continuous real-valued functions defined on X, and $\mathrm{CL}(X \times \mathbb{R})$ be the hyperspace of all nonempty closed subsets of $X \times \mathbb{R}$. We prove the following result: Let X be a locally connected locally compact paracompact space, and let $F \in \mathrm{CL}(X \times \mathbb{R})$. Then F is in the closure of $C(X)$ in $\mathrm{CL}(X \times \mathbb{R})$ with the Vietoris topology if and only if: (1) for every $x \in X$, $F(x)$ is nonempty; (2) for every $x \in X$, $F(x)$ is connected; (3) for every isolated $x \in X$, $F(x)$ is a singleton set; (4) F is upper semicontinuous; and (5) F forces local semiboundedness. This gives an answer to Problem 5.5 in [HM] and to Question 5.5 in [Mc2] in the realm of locally connected locally compact paracompact spaces.

1. Introduction. Let X be a Tikhonov space, $C(X)$ be the space of all continuous real-valued functions defined on X, and $\mathrm{CL}(X \times \mathbb{R})$ be the hyperspace of all nonempty closed subsets of $X \times \mathbb{R}$, where $\mathbb{R}$ is the space of real numbers.

The fundamental result concerning approximation of relations by continuous functions is due to Cellina [Ce], who studied approximation of relations in the Hausdorff metric (see also [Be3], [Ho1], [Ho2]). Beer [Be3] extended Cellina's result given for multifunctions with convex values to continuous starshaped valued multifunctions.

In [HM] and [HMP] the authors studied approximation of relations in the Vietoris and the locally finite topologies. In fact, they gave a satisfactory solution for the approximation problem in the locally finite topology, by proving the following:

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Let X be a countably paracompact normal q-space and $F \in \mathrm{CL}(X \times \mathbb{R})$. Then:

(a) If $\dim X = 0$ then F is in the closure of $C(X)$ in $\mathrm{CL}(X \times \mathbb{R})$ with the locally finite topology if and only if F is the graph of a usco map which maps isolated points into singletons.

(b) If X is locally connected then F is in the closure of $C(X)$ with the locally finite topology if and only if F is the graph of a cusco map which maps isolated points into singletons.

A *q-space* is a space in which every point has a sequence (U_n) of neighbourhoods such that if $x_n \in U_n$ for each n, then (x_n) has a cluster point. This concept was introduced in [Mi] and has been useful, among other things, for studying function spaces (see [MN]). The class of q-spaces includes the first countable spaces and Čech-complete spaces.

The paper [HMP] gives the solution of Problem 5.5 in [HM] and Question 5.5 in [Mc2] in the realm of normal, countably paracompact, strongly zero-dimensional spaces:

Let X be a countably paracompact normal space. The following are equivalent:

(a) $\dim X = 0$;

(b) the closure of $C(X)$ in $\mathrm{CL}(X \times \mathbb{R})$ with the Vietoris topology consists of all $F \in \mathrm{CL}(X \times \mathbb{R})$ such that $F(x) \neq \emptyset$ for every $x \in X$ and F maps isolated points into singletons.

In the present paper we give the solution of the above approximation problem for locally connected locally compact paracompact spaces.

Main Theorem. *Let X be a locally connected locally compact paracompact space, and let F be a closed subset of $X \times \mathbb{R}$. Then F is in the closure of $C(X)$ in* $\mathrm{CL}(X \times \mathbb{R})$ *with the Vietoris topology if and only if*:

(1) *for every $x \in X$, $F(x)$ is nonempty*;

(2) *for every $x \in X$, $F(x)$ is connected*;

(3) *for every isolated $x \in X$, $F(x)$ is a singleton set*;

(4) *F is upper semicontinuous*;

(5) *F forces local semiboundedness.*

There are other theorems like this in the literature; see [Be2] about relations that are approximated in the Hausdorff distance by Baire class one functions, and [Mc1] about the closure of densely continuous forms in the Vietoris topology. Moreover there is a rich literature concerning approximation of a multifunction from above by a decreasing sequence of "continuous" multifunctions (see [Hu], [DB], [DBM]).

2. Preliminaries. We refer to Beer [Be1] and Engelking [En] for basic notions. If X and Y are nonempty sets, a *set-valued mapping* or *multifunction* from X to Y is a mapping that assigns to each element of X a (possibly empty) subset of Y. If T is a set-valued mapping from X to Y, then its *graph* is $\{(x,y) : y \in T(x)\}$.

If F is a subset of $X \times Y$ and $x \in X$, define $F(x) = \{y : (x,y) \in F\}$. Thus we can assign to each subset F of $X \times Y$ a set-valued mapping which takes the value $F(x)$ at each point $x \in X$. Then F is the graph of this set-valued mapping. In our paper we will identify mappings with their graphs.

To describe the hypertopologies we will work with, we introduce the following notation. Let (X,τ) be a topological space and $\mathrm{CL}(X)$ be the hyperspace of all nonempty closed subsets of X. For $U \subset X$, define

$$U^+ = \{A \in \mathrm{CL}(X) : A \subset U\}, \qquad U^- = \{A \in \mathrm{CL}(X) : A \cap U \neq \emptyset\}.$$

If $\mathcal{U}$ is a family of open sets in X, define $\mathcal{U}^- = \bigcap\{U^- : U \in \mathcal{U}\}$.

A subbase for the Vietoris (resp., locally finite) topology on $\mathrm{CL}(X)$ (see [Be1]) consists of the sets of the form U^+ with $U \in \tau$ and the form $\mathcal{U}^-$ with $\mathcal{U} \subset \tau$ finite (resp., locally finite).

Throughout the paper X will be a Hausdorff topological space. We use $\mathrm{CL}_{\mathrm{V}}(X \times \mathbb{R})$ to denote $\mathrm{CL}(X \times \mathbb{R})$ with the Vietoris topology.

3. Necessary conditions for relations approximated by continuous functions in the Vietoris topology. To prove the Main Theorem we start with some necessary conditions for an $F \in \mathrm{CL}(X \times \mathbb{R})$ to be in the closure of $C(X)$ in $\mathrm{CL}_{\mathrm{V}}(X \times \mathbb{R})$.

The following three results are known but we include them for the reader's convenience.

REMARK 3.1 (see [HM]). It is easy to verify that if F is in the closure of $C(X)$ in $\mathrm{CL}_{\mathrm{V}}(X \times \mathbb{R})$ then $F(x) \neq \emptyset$ for every $x \in X$, and F maps isolated points of X to singletons.

LEMMA 3.2 (see [HM]). *Let X be a locally connected regular space. If F is in the closure of $C(X)$ in $\mathrm{CL}_{\mathrm{V}}(X \times \mathbb{R})$ then for every $x \in X$, $F(x)$ is connected.*

Proof. Suppose, by way of contradiction, that F is in the closure of $C(X)$ in $\mathrm{CL}_{\mathrm{V}}(X \times \mathbb{R})$ but $F(x)$ is not connected for some $x \in X$. Then there exist $r < s < t$ in $\mathbb{R}$ such that (x,r) and (x,t) are in F while (x,s) is not. Let U be a connected open neighbourhood of x in X such that the closed set $\overline{U} \times \{s\}$ in $X \times \mathbb{R}$ is disjoint from F. Define W to be the complement of this closed set in $X \times \mathbb{R}$. Also define $W_1 = U \times (-\infty, s)$ and $W_2 = U \times (s, \infty)$. Then $W^+ \cap W_1^- \cap W_2^-$ is a neighbourhood of F in $\mathrm{CL}_{\mathrm{V}}(X \times \mathbb{R})$ and must therefore contain some $f \in C(X)$. It follows that $(-\infty, s)$ and (s, ∞) separate the set $f(U)$, which contradicts the fact that $f(U)$ is connected because f is continuous. ∎

LEMMA 3.3 (see [HM]). *Let X be a locally connected regular space. If F is in the closure of $C(X)$ in $\mathrm{CL}_{\mathrm{V}}(X \times \mathbb{R})$, then F is the graph of an upper semicontinuous multifunction.*

Proof. Suppose, by way of contradiction, that F is in the closure of $C(X)$ in $\mathrm{CL}_{\mathrm{V}}(X \times \mathbb{R})$, but F is not the graph of an upper semicontinuous multifunction. Then there is an $x \in X$ and a neighbourhood V of $F(x)$ in $\mathbb{R}$ such that for every neighbourhood U of x, $F(x')$ is not contained in V for some $x' \in U$. Since $F(x)$ is connected by Lemma 3.2, there is an open interval V' containing $F(x)$ such that $\overline{V'} \subset V$, say $V' = (a, b)$ (the case of an infinite interval is similar). Let U be a connected neighbourhood of x such that $\overline{U} \times \{a, b\}$ is disjoint from F. So there exists an $x' \in U$ such that $F(x')$ is not contained in V. Define $W = X \times \mathbb{R} \setminus \overline{U} \times \{a, b\}$, $W_1 = U \times V'$, and $W_2 = U \times \mathbb{R} \setminus \overline{V'}$. Then $F \in W^+ \cap W_1^- \cap W_2^-$, so there is an $f \in W^+ \cap W_1^- \cap W_2^- \cap C(X)$. But then $f(U)$ is contained in $\mathbb{R} \setminus \{a, b\}$ and intersects both (a, b) and $\mathbb{R} \setminus [a, b]$, which contradicts the fact that $f(U)$ is connected.

We say that $F \in \mathrm{CL}(X \times \mathbb{R})$ *forces local semiboundedness* if every closed subset of $X \times \mathbb{R}$ which is disjoint from F is locally semibounded. Here a closed subset G of $X \times \mathbb{R}$ is *locally semibounded* if for every $x \in X$ there exists a neighbourhood U of x and $n \in \mathbb{N}$ such that for every component C of G, $C \cap U \times \mathbb{R} \subset U \times (-\infty, n]$ or $C \cap U \times \mathbb{R} \subset U \times [-n, \infty)$.

LEMMA 3.4. *If F is in the closure of $C(X)$ in $\mathrm{CL}_{\mathrm{V}}(X \times \mathbb{R})$ then F forces local semiboundedness.*

Proof. Let F be in the closure of $C(X)$ in $\mathrm{CL}_{\mathrm{V}}(X \times \mathbb{R})$ and suppose it does not force local semiboundedness. Then there exists $G \in \mathrm{CL}(X \times \mathbb{R})$ with $F \cap G = \emptyset$ such that G is not locally semibounded at some $x \in X$.

Let $W = X \times \mathbb{R} \setminus G$. Then $F \in W^+$, so there is an $f \in W^+ \cap C(X)$. Note that for every $z \in X$, $(z, f(z)) \notin G$. Now there is a neighbourhood U of x and a bounded open interval V containing $f(x)$ such that $U \times V \subset W$ and $f(U) \subset V$. Let $n \in \mathbb{N}$ with $V \subset (-n, n)$.

Since G is not locally semibounded at x, there exists a component C of G containing elements (x_1, t_1) and (x_2, t_2) such that $x_1, x_2 \in U$, $t_1 \in (-\infty, -n)$, and $t_2 \in (n, \infty)$. Then $f(x_1), f(x_2) \in (t_1, t_2)$.

Define $L = \{(x, t) \in X \times \mathbb{R} : t < f(x)\}$ and $M = \{(x, t) \in X \times \mathbb{R} : t > f(x)\}$. Then L and M are disjoint open subsets of $X \times \mathbb{R}$ containing (x_1, t_1) and (x_2, t_2), respectively.

Since C is connected, there exists $(x, t) \in C \setminus (L \cup M)$. Then $t = f(x)$, so that $(x, f(x)) \in C \subset G$, which is a contradiction. ∎

4. The proof of the Main Theorem. The Main Theorem follows from the theorem below. In it, $Z(F)$ is the complement of the union of open sets U such that for some finite interval (a,b), all $F(x)$ with $x \in U$ intersect (a,b).

THEOREM. *Let X be a locally connected locally compact paracompact space, and let F be a closed subset of $X \times \mathbb{R}$ such that*:

(1) *for every $x \in X$, $F(x)$ is nonempty*;
(2) *for every $x \in X$, $F(x)$ is connected*;
(3) *F is upper semicontinuous*;
(4) *F forces local semiboundedness.*

If W is an open subset of $X \times \mathbb{R}$ containing F and U is an open subset of X containing $Z(F)$, then there exists an open subset W_0 of $X \times \mathbb{R}$ such that:

(a) $W_0 \subset W$;
(b) *for every $x \in X \setminus U$, $F(x) \subset W_0(x)$*;
(c) *for every $x \in X$, $W_0(x)$ is nonempty*;
(d) *for every $x \in X$, $W_0(x)$ is connected.*

LEMMA 4.1. *Let X be a Baire space and $F \in \mathrm{CL}(X \times \mathbb{R})$ be the graph of an upper semicontinuous multifunction with nonempty values. Then $Z(F)$ is a nowhere dense closed subset of X.*

Proof. It is evident that $Z(F)$ is closed. Let $\{[a_n, b_n] : n \in \mathbb{N}\}$ be an enumeration of all intervals with rational ends. For each $n \in \mathbb{N}$, put

$$F_n = \{x \in X : F(x) \cap [a_n, b_n] \neq \emptyset\}.$$

The upper semicontinuity of F implies that F_n is closed for every $n \in \mathbb{N}$. It is easy to verify that the sets F_n cover X. The union of the interiors of F_n is dense in X and disjoint from $Z(F)$. ■

For the proof of the Theorem, we begin with some notation. For every subset S of $X \times \mathbb{R}$, define

$$\begin{aligned} S\updownarrow &= \{(x,-t) : (x,t) \in S\}, \\ S\downarrow &= \{(x,s) \in X \times \mathbb{R} : \exists t \in \mathbb{R} \text{ with } s \le t \text{ and } (x,t) \in S\}, \\ S\uparrow &= \{(x,s) \in X \times \mathbb{R} : \exists t \in \mathbb{R} \text{ with } t \le s \text{ and } (x,t) \in S\}. \end{aligned}$$

Also define

$$\begin{aligned} F_* &= \{(x,t) \in X \times \mathbb{R} : t < \inf F(x)\}, \\ F^* &= \{(x,t) \in X \times \mathbb{R} : \sup F(x) < t\}. \end{aligned}$$

LEMMA 4.2. *The sets F_* and F^* are disjoint open subsets of $X \times \mathbb{R}$, and every component of $X \times \mathbb{R} \setminus W$ is contained in one of them.*

Proof. To show that F_* is open in $X \times \mathbb{R}$, let $(x,t) \in F_*$ and let $\varepsilon = \min F(x) - t$. Since F is upper semicontinuous, x has a neighbourhood U' such that for every $x' \in U'$, $F(x') \subset (\min F(x) - \varepsilon/2, \infty)$. Then $U' \times (-\infty, t + \varepsilon/2)$ is a neighbourhood of (x,t) contained in F_*. A similar argument shows that F^* is open in $X \times \mathbb{R}$.

Clearly F, F_*, F^* are pairwise disjoint. Note that if $(x,t) \in X \times \mathbb{R} \setminus (F_* \cup F^*)$, then $\inf F(x) \le t \le \sup F(x)$. Because $F(x)$ is connected, $t \in F(x)$, so that $(x,t) \in F$. Therefore $\{F, F_*, F^*\}$ partitions $X \times \mathbb{R}$. Since $F \subset W$, it follows that every component of $X \times \mathbb{R} \setminus W$ is contained in either F_* or F^*. ∎

Let $\mathcal{C}_*$ and $\mathcal{C}^*$ be the families of components of $X \times \mathbb{R} \setminus W$ contained in F_* and F^*, respectively.

The next three lemmas develop a general tool for locally connected locally compact paracompact spaces that we need to apply to $\mathcal{C}_*$ and $\mathcal{C}^*$.

Lemma 4.3. *Let Y be a locally connected regular space, let U be a connected open subset of Y, and let A be a compact subset of U. Then there exists a connected open subset V of Y such that $A \subset V \subset \overline{V} \subset U$.*

Proof. There exists a finite family $\mathcal{V}$ of connected open subsets of X such that $A \subset \bigcup \mathcal{V}$ and $\overline{V} \subset U$ for each $V \in \mathcal{V}$. For any two distinct $V_1, V_2 \in \mathcal{V}$, there exists a finite chain of connected open sets with closure in U, beginning with V_1 and ending with V_2. Let V be the (finite) union of all the sets in all these chains. Then $A \subset V$ and $\overline{V} \subset U$. Also V is connected because the union of each chain of sets is connected, and the chains connect every pair of members of $\mathcal{V}$. ∎

Lemma 4.4. *Let Y be a locally connected space, and let $\mathcal{C}$ be a locally finite family of connected subsets of Y. Then the family of components of $\bigcup\{\overline{C} : C \in \mathcal{C}\}$ is discrete in Y.*

Proof. The set $\bigcup\{\overline{C} : C \in \mathcal{C}\}$ is a closed subset of Y, so its components are closed in Y; let $\mathcal{D}$ be the family of those components. To show that $\mathcal{D}$ is discrete in Y, let $y \in Y$. Then there exists a neighbourhood U of y such that the family $\mathcal{C}'$ of members of $\mathcal{C}$ that intersect U is finite. For every $C \in \mathcal{C}'$, there exists a $D_C \in \mathcal{D}$ that contains C. Then the family $\mathcal{D}' = \{D_C : C \in \mathcal{C}'\}$ only contains the members of $\mathcal{D}$ that intersect U. Now y can be in at most one member of $\mathcal{D}'$, so that $U \setminus \bigcup\{D \in \mathcal{D}' : y \notin D\}$ is a neighbourhood of y that intersects at most one member of $\mathcal{D}$. ∎

Lemma 4.5. *Let Y be a locally connected locally compact paracompact space, let U be an open subset of Y, and let A be a closed subset of Y contained in U. Then there exists a closed subset B of Y such that $A \subset B \subset U$ and the family of components of B is discrete in Y.*

Proof. First suppose that Y is σ-compact. Then we can write $Y = \bigcup\{Y_n : n \in \mathbb{N}\}$ where each Y_n is compact and contained in the interior of Y_{n+1}. Let $Y_0 = Y_{-1} = \emptyset$. For each $n \in \mathbb{N}$, define $A_n = A \cap Y_n \setminus Y_{n-1}^\circ$ and $U_n = U \cap Y_{n+1}^\circ \setminus Y_{n-2}$ (where S° denotes the interior of S). Then each A_n is compact and contained in the open set U_n. Also $\bigcup\{A_n : n \in \mathbb{N}\} = A$ and $\bigcup\{U_n : n \in \mathbb{N}\} = U$.

For each $n \in \mathbb{N}$, let $\mathcal{U}_n$ be the family of components of U_n that intersect the set A_n, which is a finite family since A_n is compact. Then for each $W \in \mathcal{U}_n$, $A_n \cap W$ is compact; so by Lemma 4.3, there exists a connected open subset V_W of Y such that $A_n \cap W \subset V_W$ and $\overline{V}_W \subset W$. Define $\mathcal{V}_n = \{V_W : W \in \mathcal{U}_n\}$, and observe that $\bigcup \mathcal{V}_n \subset U_n$.

Now letting n vary, define $\mathcal{V} = \bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$. The family $\{U_n : n \in \mathbb{N}\}$ is locally finite because for each n, $U_n \cap U_{n+j} = \emptyset$ for $j \geq 3$. Therefore, the family $\{\bigcup \mathcal{V}_n : n \in \mathbb{N}\}$ is locally finite. Define $B = \bigcup\{\overline{V} : V \in \mathcal{V}\}$, which is closed because $\mathcal{V}$ is locally finite. By definition, $A \subset B \subset U$; and by Lemma 4.4, the family of components of B is discrete in Y.

Now we consider the general case of a locally compact paracompact space Y. In this case, we can write Y as the topological sum $\bigoplus\{Y_\lambda : \lambda \in \Lambda\}$ of σ-compact spaces Y_λ. Then for each $\lambda \in \Lambda$, let $U_\lambda = U \cap Y_\lambda$ and $A_\lambda = A \cap Y_\lambda$. By the argument above, for each $\lambda \in \Lambda$, there exists a closed subset B_λ of Y_λ such that $A_\lambda \subset B_\lambda \subset U_\lambda$ and the family of components of B_λ is discrete in Y_λ. Define $B = \bigoplus\{B_\lambda : \lambda \in \Lambda\}$. Then B is closed in Y, $A \subset B \subset U$, and the family of components of B is discrete in Y. ∎

By Lemma 4.5, there exist families $\mathcal{D}_*$ and $\mathcal{D}^*$ of closed connected subsets of $X \times \mathbb{R}$ contained in F_* and F^*, respectively, such that $\bigcup \mathcal{C}_* \subset \bigcup \mathcal{D}_*$, $\bigcup \mathcal{C}^* \subset \bigcup \mathcal{D}^*$, and $\mathcal{D}_*$ and $\mathcal{D}^*$ are discrete in $X \times \mathbb{R}$.

Define

$$\begin{aligned}
H_* &= \{(x,t) \in U \times \mathbb{R} : 0 < t < \inf F(x)\},\\
H^* &= \{(x,t) \in U \times \mathbb{R} : \sup F(x) < t < 0\},\\
\mathcal{H}_* &= \{D \in \mathcal{D}_* : D \subset H_*\},\\
\mathcal{H}^* &= \{D \in \mathcal{D}^* : D \subset H^*\}.
\end{aligned}$$

Let $\mathcal{K}_*$ be the set of $D \in \mathcal{H}_*$ such that for some $n \in \mathbb{N}$, there exist $D_1, \ldots, D_n \in \mathcal{H}_*$ and $D_0 \in \mathcal{D}_* \setminus \mathcal{H}_*$ that satisfy $D_n = D$ and for every $j = 1, \ldots, n$, $D_j{\uparrow} \cap D_{j-1}{\downarrow} \neq \emptyset$.

Similarly, let $\mathcal{K}^*$ be the set of $D \in \mathcal{H}^*$ such that for some $n \in \mathbb{N}$, there exist $D_1, \ldots, D_n \in \mathcal{H}^*$ and $D_0 \in \mathcal{D}^* \setminus \mathcal{H}^*$ that satisfy $D_n = D$ and for every $j = 1, \ldots, n$, $D_j{\downarrow} \cap D_{j-1}{\uparrow} \neq \emptyset$.

Define

$$\begin{aligned}
\mathcal{L}_* &= \mathcal{H}_* \setminus \mathcal{K}_*, &\quad \mathcal{M}_* &= \mathcal{D}_* \setminus \mathcal{L}_*,\\
\mathcal{L}^* &= \mathcal{H}^* \setminus \mathcal{K}^*, &\quad \mathcal{M}^* &= \mathcal{D}^* \setminus \mathcal{L}^*.
\end{aligned}$$

Also define

$$L_* = \bigcup\{D\uparrow : D \in \mathcal{L}_*\}, \quad M_* = \bigcup\{D\downarrow : D \in \mathcal{M}_*\},$$
$$L^* = \bigcup\{D\downarrow : D \in \mathcal{L}^*\}, \quad M^* = \bigcup\{D\uparrow : D \in \mathcal{M}^*\}.$$

Lemma 4.6. *The intersections $L_* \cap M_*$ and $L^* \cap M^*$ are empty.*

Proof. Suppose that $L_* \cap M_* \neq \emptyset$. Then there exist $D \in \mathcal{L}_*$ and $E \in \mathcal{M}_*$ such that $D\uparrow \cap E\downarrow \neq \emptyset$. If E were not in $\mathcal{H}_*$, then since $D\uparrow \cap E\downarrow \neq \emptyset$, D would be in $\mathcal{K}_*$. Since it is not, it follows that E is in $\mathcal{H}_*$. Therefore, E is in $\mathcal{K}_*$, so there exist $D_1, \ldots, D_n \in \mathcal{H}_*$ and $D_0 \in \mathcal{D}_* \setminus \mathcal{H}_*$ that satisfy $D_n = E$ and for every $j = 1, \ldots, n$, $D_i\uparrow \cap D_{i-1}\downarrow \neq \emptyset$. Let $D_{n+1} = D$. Then $D_{n+1}\uparrow \cap D_n\downarrow = D\uparrow \cap E\downarrow \neq \emptyset$. Therefore, D is in $\mathcal{K}_*$, which is a contradiction. Hence $L_* \cap M_* = \emptyset$. A similar argument shows that $L^* \cap M^* = \emptyset$. ∎

Lemma 4.7. *The sets L_* and L^* are closed in $X \times R$.*

Proof. Let (x_i, s_i) be a net in L_* that converges to some $(x, s) \in X \times \mathbb{R}$. Then for each i, there exists a $D_i \in \mathcal{L}_*$ such that $(x_i, s_i) \in D_i\uparrow$, and so, there exists an $r_i \in R$ such that $0 < r_i \leq s_i$ and $(x_i, r_i) \in D_i$. The net (x_i, r_i) has a cluster point (x, r) with some $r \in [0, s]$. Since $\mathcal{L}_*$ is a subfamily of $\mathcal{D}_*$, it is discrete in $X \times \mathbb{R}$, so $(x, r) \in D$ for some $D \in \mathcal{L}_*$. Thus $(x, s) \in D\uparrow \subset L_*$, so that L_* is closed in $X \times \mathbb{R}$. A similar argument shows that L^* is closed in $X \times \mathbb{R}$. ∎

In order to show that M_* is closed in $X \times \mathbb{R}$, we introduce the following sets and lemmas, which culminate with Lemma 4.11. We will leave out the details for the similar argument that M^* is closed in $X \times \mathbb{R}$.

For every $D \subset F_*$, define

$$\widehat{D} = D \cup (D \cap X \times [0, \infty))\updownarrow \cup (D \setminus U \times \mathbb{R})\downarrow.$$

Lemma 4.8. *For every $D \subset F_*$, the following are true*:

(1) $\widehat{D} \subset D\downarrow \subset F_*$;

(2) *if D is closed in $X \times \mathbb{R}$, then $\widehat{D}$ is closed in $X \times \mathbb{R}$;*

(3) *if D is connected and not contained in H_*, then $\widehat{D}$ is connected.*

Proof. To show that $\widehat{D} \subset D\downarrow$, first observe that $D \subset D\downarrow$ and $(D \setminus U \times \mathbb{R})\downarrow \subset D\downarrow$. Also for each $(x, -t) \in (D \cap X \times [0, \infty))\updownarrow$, we have $(x, t) \in D \cap X \times [0, \infty)$, so that $-t \leq 0 \leq t$. This means that $(D \cap X \times [0, \infty))\updownarrow \subset D\downarrow$, and hence $\widehat{D} \subset D\downarrow$.

Now suppose that D is closed in $X \times \mathbb{R}$. Then $D \cap X \times [0, \infty)$ is closed in $X \times \mathbb{R}$. Since $(D \cap X \times [0, \infty))\updownarrow$ is homeomorphic to $D \cap X \times [0, \infty)$, it is also closed in $X \times \mathbb{R}$.

It remains to show that $(D \setminus U \times \mathbb{R})\downarrow$ is closed in $X \times \mathbb{R}$. Let (x_i, s_i) be a net in $(D \setminus U \times R)\downarrow$ that converges to some (x, s) in $X \times \mathbb{R}$. Then $x \notin U$, so $x \notin Z(F)$. Thus x has a neighbourhood U_0 such that $\{\inf F(y) : y \in U_0\}$ is bounded above by some $b \in \mathbb{R}$. We may assume that each $x_i \in U_0$. Also for each i, there exists a $t_i \in \mathbb{R}$ such that $s_i \le t_i$ and $(x_i, t_i) \in D \setminus U \times \mathbb{R}$. Now $D \subset F_*$, so each $t_i \le b$. Then there exists a $t \in [s, b]$ such that (x, t) is a cluster point of the net (x_i, t_i). Since $D \setminus U \times \mathbb{R}$ is closed in $X \times \mathbb{R}$, $(x, t) \in D \setminus U \times \mathbb{R}$. Therefore $(x, s) \in (D \setminus U \times \mathbb{R})\downarrow$, as required.

Finally, suppose that D is connected and not contained in H_*. First note that either $D \setminus U \times \mathbb{R} = \emptyset$ or $(D \setminus U \times \mathbb{R})\downarrow$ is the union of connected sets each of which intersects the connected set D. Therefore, $D \cup (D \setminus U \times \mathbb{R})\downarrow$ is connected. So we may assume that $D \cap X \times [0, \infty) \ne \emptyset$.

Suppose first that $D \cap X \times \{0\} = \emptyset$. Since D is connected, it follows that $D \subset X \times (0, \infty)$. In this case, $(D \cap X \times [0, \infty))\updownarrow = D\updownarrow$, which is homeomorphic to D and hence connected. Also D is contained in F_* but not in H_*, so $D \setminus U \times \mathbb{R} \ne \emptyset$. Then $(D \setminus U \times \mathbb{R})\downarrow$ intersects both D and $D\updownarrow$, and it follows that $\widehat{D}$ is connected in this case.

Now suppose that $D \cap X \times \{0\} \ne \emptyset$. If $D \cap (X \times (-\infty, 0]) = D \cap (X \times \{0\})$, then of course $D \cup (D \cap (X \times [0, \infty))\updownarrow = D \cup D\updownarrow$ and thus $\widehat{D}$ is connected. If $D \cap (X \times [0, \infty)) = D \cap (X \times \{0\})$, then $D \cup (D \cap (X \times [0, \infty))\updownarrow = D$ and so $\widehat{D}$ is connected.

Suppose now that both $D \cap (X \times (-\infty, 0]) \ne \emptyset$ and $D \cap (X \times [0, \infty)) \ne \emptyset$. To show that $\widehat{D}$ is connected, it suffices to show that $\widetilde{D}$ is connected, where $\widetilde{D} = (D \cap X \times (-\infty, 0]) \cup (D \cap X \times [0, \infty))\updownarrow$. If $\widetilde{D}$ were not connected, then there would be disjoint open subsets H and K in $\widetilde{D} \subset X \times (-\infty, 0]$ such that $\widetilde{D} = H \cup K$. Then $(H \cup H\updownarrow) \cap D$ and $(K \cup K\updownarrow) \cap D$ would be two nonempty open disjoint sets of D with $(H \cup H\updownarrow) \cap D \cup (K \cup K\updownarrow) \cap D = D$, contradicting the connectedness of D. ∎

It follows from Lemma 4.8 that for every $D \in \mathcal{D}_* \setminus \mathcal{H}_*$, $\widehat{D}$ is a closed connected subset of $X \times \mathbb{R}$.

Lemma 4.9. *For every subfamily $\mathcal{D}$ of $\mathcal{D}_* \setminus \mathcal{H}_*$, the set $\bigcup\{\widehat{D} : D \in \mathcal{D}\}$ is closed in $X \times \mathbb{R}$.*

Proof. Put $L = \bigcup\{D : D \in \mathcal{D}\}$. Then $\widehat{L} = \bigcup\{\widehat{D} : D \in \mathcal{D}\}$, so by Lemma 4.8 we are done since $\mathcal{D}$ is discrete in $X \times \mathbb{R}$ (being a subfamily of $\mathcal{D}_*$). ∎

Let $\mathcal{E}_*$ be the family of sets E of the form

$$E = D_0 \cup D_1 \cup \cdots \cup D_n \cup (D_1\uparrow \cap D_0\downarrow) \cup \cdots \cup (D_n\uparrow \cap D_{n-1}\downarrow)$$

such that $n \in \mathbb{N}$, $D_1, \ldots, D_n \in \mathcal{K}_*$, $D_0 \in \mathcal{D}_* \setminus \mathcal{H}_*$, and for every $j = 1, \ldots, n$, $D_j\uparrow \cap D_{j-1}\downarrow \ne \emptyset$.

Then every member of $\mathcal{K}_*$ is contained in some member of $\mathcal{E}_*$. Also, for every $E \in \mathcal{E}_*$, $E \subset \bigcup\{D\downarrow : D \in \mathcal{M}_*\}$, so that $\widehat{E} \subset \bigcup\{D\downarrow : D \in \mathcal{M}_*\}$.

Lemma 4.10. *For every $E \in \mathcal{E}_*$, $\widehat{E}$ is a closed connected subset of $X \times \mathbb{R}$.*

Proof. Since no $E \in \mathcal{E}_*$ is contained in H_*, it suffices, by Lemma 4.8, to show that each E in $\mathcal{E}_*$ is a closed and connected subset of $X \times \mathbb{R}$. So let E be as in the definition of $\mathcal{E}_*$. Since $D_0 \cup D_1 \cup \cdots \cup D_n$ is closed in $X \times \mathbb{R}$, to show that E is closed in $X \times \mathbb{R}$, we need to show that $D_j{\uparrow} \cap D_{j-1}{\downarrow}$ is closed in $X \times \mathbb{R}$ for each $j = 1, \ldots, n$; so let such a j be fixed.

Let (x_i, s_i) be a net in $D_j{\uparrow} \cap D_{j-1}{\downarrow}$ that converges to some (x, s) in $X \times \mathbb{R}$. Then for each i, there exist $r_i, t_i \in \mathbb{R}$ such that $0 < r_i \le s_i \le t_i$, $(x_i, r_i) \in D_j$, and $(x_i, t_i) \in D_{j-1}$.

It is easy to verify that there is a cluster point (x, r) of (x_i, r_i) and of course $(x, r) \in D_j$, since D_j is closed.

We claim that also the net (x_i, t_i) has a cluster point. Suppose, otherwise. Then for each neighbourhood V of x and $p \in \mathbb{N}$, there exists an n such that $x_n \in V$ and $t_n > p$.

If $j - 1 = 0$ put $G = \widehat{D}_0$, and if $j - 1 > 0$ put

$$G = D_{j-1} \cup D_{j-1}{\updownarrow} \cup \{z\} \times [-q, q],$$

where (z, q) is any point in D_{j-1}.

Then G is a closed connected subset of $X \times \mathbb{R}$ disjoint from F. Since for every neighbourhood V of x and $p \in \mathbb{N}$, $G \cap V \times \mathbb{R}$ is contained neither in $V \times (-\infty, p]$ nor in $V \times [-p, \infty)$, this contradicts the fact that F forces local semiboundedness.

Thus (x_i, t_i) has a cluster point (x, t), which is in D_{j-1} since D_{j-1} is closed. Now $0 \le r \le s \le t$, so $(x, s) \in D_j{\uparrow} \cap D_{j-1}{\downarrow}$. This finishes the proof that E is closed in $X \times \mathbb{R}$.

To show that E is connected, first note that $D_0, D_1, \ldots, D_n$ are all connected. Thus it suffices to show that for each $j = 1, \ldots, n$, $D_{j-1} \cup D_j \cup (D_j{\uparrow} \cap D_{j-1}{\downarrow})$ is connected; so let such a j be fixed. Since there exist some $(x, s) \in D_j{\uparrow} \cap D_{j-1}{\downarrow}$, there exist $r, t \in \mathbb{R}$ such that $r \le s \le t$, $(x, r) \in D_j$, and $(x, t) \in D_{j-1}$. Then the nonempty connected set $\{x\} \times [r, t]$ is contained in $D_j{\uparrow} \cap D_{j-1}{\downarrow}$. Now $\{x\} \times [r, t]$ intersects both D_{j-1} and D_j, so that the union $D_{j-1} \cup D_j \cup \{x\} \times [r, t]$ is connected.

Finally, note that each $(x, s) \in D_j{\uparrow} \cap D_{j-1}{\downarrow}$ is in some $\{x\} \times [r, t]$ that intersects both D_{j-1} and D_j. Therefore, $D_{j-1} \cup D_j \cup (D_j{\uparrow} \cap D_{j-1}{\downarrow})$ can be written as the union of connected subsets of $D_{j-1} \cup D_j \cup (D_j{\uparrow} \cap D_{j-1}{\downarrow})$ each of which intersects the connected set D_{j-1}, so that $D_{j-1} \cup D_j \cup (D_j{\uparrow} \cap D_{j-1}{\downarrow})$ must be connected. ∎

Lemma 4.11. *The set $\bigcup\{\widehat{E} : E \in \mathcal{E}_*\}$ is closed in $X \times \mathbb{R}$.*

Proof. Let $E_* = \bigcup\{E : E \in \mathcal{E}_*\}$. Now $\widehat{E}_* = \bigcup\{\widehat{E} : E \in \mathcal{E}_*\}$, so that by Lemma 4.8 it suffices to show that E_* is closed in $X \times \mathbb{R}$. To this end, let (x_i, s_i) be a net in E_* that converges to some (x, s) in $X \times \mathbb{R}$. Now for each i, $(x_i, s_i) \in E_i$ for some $E_i \in \mathcal{E}_*$. So for each i, we can write

$$E_i = D_0^i \cup D_1^i \cup \cdots \cup D_{n_i}^i \cup (D_1^i{\uparrow} \cap D_0^i{\downarrow}) \cup \cdots \cup (D_{n_i}^i{\uparrow} \cap D_{n_i-1}^i{\downarrow}),$$

where $n_i \in \mathbb{N}$, $D_0^i \in \mathcal{D}_* \setminus \mathcal{H}_*$, and $D_1^i, \ldots, D_{n_i}^i \in \mathcal{H}_*$ with $D_j^i{\uparrow} \cap D_{j-1}^i{\downarrow} \neq \emptyset$ for $j = 1, \ldots, n_i$.

Since $\mathcal{D}_*$ is discrete in $X \times \mathbb{R}$, we may suppose that each

$$(x_i, s_i) \in (D_1^i{\uparrow} \cap D_0^i{\downarrow}) \cup \cdots \cup (D_{n_i}^i{\uparrow} \cap D_{n_i-1}^i{\downarrow}).$$

Then for each i, there exist $j_i \in \{1, \ldots, n_i\}$ and $r_i, t_i \in \mathbb{R}$ such that $r_i \leq s_i \leq t_i$, $(x_i, r_i) \in D_{j_i}^i$, and $(x_i, t_i) \in D_{j_i-1}^i$. Because each $0 < r_i \leq s_i$, and the net (s_i) converges to s, we may assume that the net (x_i, r_i) converges to (x, r) for some $r \in \mathbb{R}$ with $0 \leq r \leq s$. By the discreteness of $\mathcal{D}_*$, there is some i_0 such that $(x, r) \in D_{j_{i_0}}^{i_0}$. Then we may assume that for each i, $(x_i, r_i) \in D_{j_{i_0}}^{i_0}$.

Then of course for each i, $(x_i, t_i) \in D_{j_{i_0}-1}^{i_0}$. Since all (x_i, t_i) belong to the same element $D_{j_{i_0}-1}^{i_0}$, to prove that the net (x_i, t_i) has a cluster point we use the same argument as in the proof of Lemma 4.10.

Let (x, t) be a cluster point of the net (x_i, t_i). Then of course $(x, t) \in D_{j_{i_0}-1}^{i_0}$ since $D_{j_{i_0}-1}^{i_0}$ is closed. Now $0 \leq r \leq s \leq t$, so $(x, s) \in D_{j_{i_0}}^{i_0}{\uparrow} \cap D_{j_{i_0}-1}^{i_0}{\downarrow}$. If $j_{i_0} - 1 = 0$ define

$$E = D_0^{i_0} \cup D_1^{i_0} \cup (D_1^{i_0}{\uparrow} \cap D_0^{i_0}{\downarrow}).$$

If $j_{i_0} - 1 > 0$, define

$$\begin{aligned} E = D_0^{i_0} \cup D_1^{i_0} \cup \cdots \cup D_{j_{i_0}-1}^{i_0} \cup D_{j_{i_0}}^{i_0} \\ \cup (D_1^{i_0}{\uparrow} \cap D_0^{i_0}{\downarrow}) \cup \cdots \cup (D_{j_{i_0}}^{i_0}{\uparrow} \cap D_{j_{i_0}-1}^{i_0}{\downarrow}). \end{aligned}$$

Then $E \in \mathcal{E}_*$ and $(x, s) \in E \subset E_*$, so that E_* is indeed closed in $X \times \mathbb{R}$. ■

LEMMA 4.12. *The sets M_* and M^* are closed in $X \times \mathbb{R}$.*

Proof. To show that M_* is closed in $X \times \mathbb{R}$, let (x_i, s_i) be a net in M_* that converges to some (x, s) in $X \times \mathbb{R}$. Let

$$G = \bigcup\{\widehat{D} : D \in \mathcal{D}_* \setminus \mathcal{H}_*\} \cup \bigcup\{\widehat{E} : E \in \mathcal{E}_*\}$$

which, by Lemmas 4.9 and 4.11, is closed in $X \times \mathbb{R}$. Since $G \cap F = \emptyset$, G is locally semibounded. So there exists a neighbourhood U_0 of x and an $n \in \mathbb{N}$ such that for every component C of G, $C \cap U_0 \times \mathbb{R} \subset U_0 \times (-\infty, n]$ or $C \cap U_0 \times \mathbb{R} \subset U_0 \times [-n, \infty)$.

Let D be an arbitrary member of $\mathcal{M}_*$. Suppose, by way of contradiction, that $D \cap U_0 \times (n, \infty) \neq \emptyset$. Then $D{\updownarrow} \cap U_0 \times (-\infty, -n) \neq \emptyset$. Let C be the component of G that contains D. If $D \in \mathcal{D}_* \setminus \mathcal{H}_*$, then $D{\updownarrow} \subset \widehat{D} \subset C$ by Lemma 4.8. If $D \in \mathcal{H}_*$, then $D \in \mathcal{K}_*$. In this case, $D \subset E$ for some $E \in \mathcal{E}_*$, so that $D{\updownarrow} \subset \widehat{D} \subset \widehat{E} \subset C$ by Lemma 4.8. In either case, $C \cap U_0 \times \mathbb{R}$ is contained neither in $U_0 \times (-\infty, n]$ nor in $U_0 \times [-n, \infty)$, which is a contradiction. This shows that for every $D \in \mathcal{M}_*$, $D \cap U_0 \times (n, \infty) = \emptyset$.

We may assume that each $x_i \in U_0$. For each i, $(x_i, s_i) \in D_i{\downarrow}$ for some $D_i \in \mathcal{M}_*$. Since $D_i \cap U_0 \times (n, \infty) = \emptyset$, there exists a $t_i \in [s_i, n]$ such that $(x_i, t_i) \in D_i$. Thus (x, t) is a cluster point of the net (x_i, t_i) for some $t \in [s, n]$. Because D_* is discrete in $X \times \mathbb{R}$, $(x, t) \in D$ for some $D \in \mathcal{M}_*$. But then $(x, s) \in D{\downarrow}$, which shows that M_* is closed. ■

Now define

$$P = L_* \cup L^* \cup M_* \cup M^*, \qquad W_0 = W \setminus P.$$

Lemma 4.13. *The set W_0 is an open subset of $X \times \mathbb{R}$ that satisfies conditions* (a)–(d) *of the Theorem.*

Proof. The set W_0 is open in $X \times \mathbb{R}$ by Lemmas 4.7 and 4.12, and it clearly satisfies condition (a).

To show that it satisfies (b), let $x \in X \setminus U$. Then $H_*(x) = \emptyset$, so for each $D \in \mathcal{L}_*$, $D(x) = \emptyset$ because $D \subset H_*$. Therefore $L_*(x) = \emptyset$, and hence $F(x) \cap L_*(x) = \emptyset$. Also, it is clear that $F(x) \cap M_*(x) = \emptyset$. Similarly, $F(x) \cap L^*(x) = \emptyset$ and $F(x) \cap M^*(x) = \emptyset$, so that $F(x) \subset W_0(x)$.

To show that W_0 satisfies (c) and (d), let $x \in X$. Suppose first that $0 < \inf F(x)$. Then $H^*(x) = \emptyset$, so that $L^*(x) = \emptyset$. If $L_*(x) \neq \emptyset$, then $M^*(x) \subset L_*(x)$; also $L_*(x) \cap M_*(x) = \emptyset$, so that $\mathbb{R} \setminus P(x) = \mathbb{R} \setminus (L_* \cup M_*)(x)$ is a nonempty open interval. If $L_*(x) = \emptyset$, then $M_*(x) \cap M^*(x) = \emptyset$, so that $\mathbb{R} \setminus P(x) = \mathbb{R} \setminus (M_* \cup M^*)(x)$ is again a nonempty open interval.

For the case that $\sup F(x) < 0$, we argue in a similar manner to show that $\mathbb{R} \setminus P(x)$ is a nonempty open interval. Finally, if $0 \in F(x)$, then $L_*(x) = \emptyset$ and $L^*(x) = \emptyset$, so that again $\mathbb{R} \setminus P(x)$ is a nonempty open interval.

Now $X \times \mathbb{R} \setminus W \subset P$, so $\mathbb{R} \setminus P(x) \subset W(x)$. Then $W_0(x) = W(x) \setminus P(x) = \mathbb{R} \setminus P(x)$, which is a nonempty open interval. ■

Now we prove the Main Theorem of our paper:

Main Theorem. *Let X be a locally connected locally compact paracompact space, and let F be a closed subset of $X \times \mathbb{R}$. Then F is in the closure of $C(X)$ in* $\mathrm{CL}_{\mathrm{V}}(X \times \mathbb{R})$ *if and only if*:

(1) *for every $x \in X$, $F(x)$ is nonempty*;

(2) *for every $x \in X$, $F(x)$ is connected*;

(3) *for every isolated $x \in X$, $F(x)$ is a singleton set*;

(4) *F is upper semicontinuous*;

(5) *F forces local semiboundedness.*

Proof. First, let $F \in \mathrm{CL}(X\times\mathbb{R})$ be in the closure of $C(X)$ in $\mathrm{CL}_{\mathrm{V}}(X\times\mathbb{R})$. Then Remark 3.1 and Lemmas 3.2–3.4 imply that F satisfies conditions (1)–(5).

Now, let $F \in \mathrm{CL}(X \times \mathbb{R})$ satisfy (1)–(5) and let

$$G^{+} \cap \bigcap_{i\in I} W_i^{-}, \qquad I \text{ finite},$$

be a basic open subset of $\mathrm{CL}_{\mathrm{V}}(X \times \mathbb{R})$ that contains F.

We may assume that each W_i equals $U_i \times V_i$, where $\overline{U}_i \times \overline{V_i} \subset G$ and V_i is a bounded open interval.

We can suppose that there is a finite set $J \subset I$ and points x_j, y_j^l, $l \in \{1,\ldots,n_j\}$, $j \in J$, such that $x_j \neq x_i$ for $j \neq i$, j, $i \in J$, $y_j^l \in F(x_j)$ for all $l \in \{1,\ldots,n_j\}$, $(x_j, y_j^l) \in U_i \times V_i$ for some $i \in I$ and every $i \in I$, and $(x_j, y_j^l) \in U_i \times V_i$ for some $j \in J$ and $l \in \{1,\ldots,n_j\}$.

Put $L = \{x_j : x_j \notin Z(F)\}$. There is an open set $U \subset X \times \mathbb{R}$ such that $Z(F) \subset U$ and $\overline{U} \cap L = \emptyset$. Put further $H = \{x_j : x_j \in Z(F)\}$ and $H' = \{x_j \in H : F(x_j) \neq \mathbb{R}\}$.

Let $\{O(x_j) : x_j \in H\}$ be a family of pairwise disjoint neighbourhoods of elements of H contained in U with $O(x_j) \subset U_i$ for some U_i.

Now let $x_j \in H'$. Then $F(x_j) = [t_j, \infty)$ or $F(x_j) = (-\infty, t_j]$. There are an open set $V(x_j)$ and $\varepsilon > 0$ such that $x_j \in V(x_j)$, $\overline{V(x_j)} \subset O(x_j)$, $\overline{V(x_j)} \times (t_j - \varepsilon, t_j + \varepsilon) \subset G$ and $F(z) \subset (t_j - \varepsilon, \infty)$ or $F(z) \subset (-\infty, t_j + \varepsilon)$ for every $z \in \overline{V(x_j)}$. Put $B_j = (-\infty, t_j - \varepsilon]$ or $B_j = [t_j + \varepsilon, \infty)$ respectively.

Set

$$W = G \setminus \bigcup \{\overline{V(x_j)} \times B_j : x_j \in H'\}.$$

It is easy to verify that W is an open set in $X \times \mathbb{R}$ which contains F. Now by the Theorem there exists an open subset W_0 of $X \times \mathbb{R}$ such that: $W_0 \subset W$; for every $x \in X \setminus U$, $F(x) \subset W_0(x)$; and for every $x \in X$, $W_0(x)$ is nonempty and connected.

By Lemma 4.1 in [HM] there is an $f \in W_0^{+} \cap C(X)$. We will modify f to g as follows. Let $I(X)$ denote the set of all isolated points of X. For every $x_j \in L \cap I(X)$ we put $g(x_j) = F(x_j)$. For every $x_j \in L \setminus I(X)$ we have $F(x_j) \subset W_0(x_j)$; i.e. there is an open interval $I(x_j)$ containing $f(x_j)$ and y_j^l, $l \in \{1,\ldots,n_j\}$, with $\overline{I(x_j)} \subset W_0(x_j)$. Let $\{G(x_j) : x_j \in L \setminus I(X)\}$ be a pairwise disjoint family of open sets such that for every $x_j \in L \setminus I(X)$, $x_j \in G(x_j)$, $G(x_j) \cap \overline{U} = \emptyset$, $G(x_j) \times \overline{I(x_j)} \subset W_0$, $f(G(x_j)) \subset I(x_j)$ and $G(x_j) \subset U_i$ for some $i \in I$.

Now as in the proof of Lemma 4.2 in [HM], using the Tietze extension theorem we modify f to g on every $G(x_j)$ in such a way that g takes all the values y_j^l, $l \in \{1,\ldots,n_j\}$, on $G(x_j)$.

For every $x_j \in H$ we have $\{f(x_j), y_j^l : l \in \{1,\dots,n_j\}\} \subset W(x_j)$ and since $W(x_j)$ is connected, there is an open interval $J(x_j)$ containing $f(x_j)$ and y_j^l, $l \in \{1,\dots,n_j\}$, with $\overline{J(x_j)} \subset W(x_j)$.

We can suppose that for every $x_j \in H$, $O(x_j)\times\overline{J(x_j)} \subset W$ and $f(O(x_j)) \subset J(x_j)$. Again as above we modify f to g on every $O(x_j)$ in such a way that g takes all the values y_j^l, $l \in \{1,\dots,n_j\}$, on every $O(x_j)$.

It is easy to verify that the constructed function g belongs to $G^+ \cap \bigcap_{i\in I} W_i^-$. ■

5. Examples. We end with five examples, starting with an example of a closed subset F_1 of $\mathbb{R}^3$ that satisfies conditions (1) through (4) of the Main Theorem but not condition (5). Then we modify F_1 in two different ways to obtain closed subsets F_2 and F_3 of $\mathbb{R}^3$ that satisfy all the conditions (1) through (5). We then modify F_2 to obtain another closed subset F_4 of $\mathbb{R}^3$ that satisfies conditions (1) through (4) but not (5); and finally we modify F_4 to obtain a closed subset F_5 that again satisfies all the conditions (1) through (5). So the relations F_2, F_3, and F_5 have Vietoris approximations by continuous real-valued functions on $\mathbb{R}^2$, while the relations F_1 and F_4 do not. These examples illustrate the subtlety of approximating a relation by a continuous function in the Vietoris topology, caused primarily by condition (5).

Example 5.1. Let $A = \{(x,y,z) \in \mathbb{R}^3 : x = 0\}$, $B = \{(x,y,z) \in \mathbb{R}^3 : x \le 0,\ z = 0\}$, $C = \{(x,y,z) \in \mathbb{R}^3 : x > 0,\ z = 1/x\}$, and for every $n \in \mathbb{N}$, let $A_n = \{(x,y,z) \in \mathbb{R}^3 : x = 1/n,\ z \le n\}$. Then the relation $F_1 = A\cup B\cup C\cup\{A_n : n \in \mathbb{N}\}$ is a closed subset of $\mathbb{R}^3$ that satisfies conditions (1) through (4) but not (5). To see why condition (5) is not satisfied, for each $n \in \mathbb{N}$, let $x_n \in (1/(n+1), 1/n)$ and let $S_n = \{(x,y,z) \in \mathbb{R}^3 : x = x_n,\ -n \le y \le n,\ z \in \{-n,n\}\} \cup \{(x,y,z) \in \mathbb{R}^3 : x = x_n,\ y \in \{-n,n\},\ -n \le z \le n\}$. Then $G = \bigcup\{S_n : n \in \mathbb{N}\}$ is a closed subset of $\mathbb{R}^3$ that is disjoint from F_1 but is not locally semibounded at $(0,y)$ for any $y \in \mathbb{R}$. Note that the S_n are the components of G, so that G is not connected.

Example 5.2. Let $P = \{(x,y,z) \in \mathbb{R}^3 : z = 0\}$. Then the relation $F_2 = P\cup A\cup\bigcup\{A_n : n \in \mathbb{N}\}$ is a closed subset of $\mathbb{R}^3$ that satisfies conditions (1) through (5).

Example 5.3. Let $\mathbb{Z}$ be the set of integers, and let $D = \{(x,y,z) \in \mathbb{R}^3 : x > 0,\ y \in \mathbb{Z},\ z \le 1/x\}$. Then the relation $F_3 = F_1 \cup D$ is a closed subset of $\mathbb{R}^3$ that satisfies conditions (1) through (5).

Example 5.4. For every $n \in \mathbb{N}$, let $H_n = \{(x,y,z) \in \mathbb{R}^3 : x = 1/n,\ n-1 < y < n,\ z < n\}$. Then the relation $F_4 = F_2 \setminus \bigcup\{H_n : n \in \mathbb{N}\}$ is closed in $\mathbb{R}^3$ and satisfies conditions (1) through (4) but not condition (5). In this case, a non-locally semibounded closed subset G of $\mathbb{R}^3$ can be found that is disjoint from F_4 and, unlike the G in Example 5.1, it is connected.

Example 5.5. For every $n \in \mathbb{N}$, let $D_n = \{(x, y, z) \in \mathbb{R}^3 : 1/(n+1) \leq x \leq 1/n,\ y = n-1,\ z \leq 1/x\}$. Then the relation $F_5 = F_4 \cup \bigcup\{D_n : n \in \mathbb{N}\}$ is a closed subset of $\mathbb{R}^3$ that satisfies conditions (1) through (5).

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