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Dualization in algebraic K-theory and the invariant e^1 of quadratic forms over schemes

by

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Abstract. In the classical Witt theory over a field F, the study of quadratic forms begins with two simple invariants: the dimension of a form modulo 2, called the dimension index and denoted $e^0 : W(F) \to \mathbb{Z}/2$, and the discriminant e^1 with values in $k_1(F) = F^*/F^{*2}$, which behaves well on the fundamental ideal $I(F) = \ker(e^0)$.

Here a more sophisticated situation is considered, of quadratic forms over a scheme and, more generally, over an exact category with duality. Our purposes are:

- to establish a theory of the invariant e^1 in this generality;
- to provide computations involving this invariant and show its usefulness.

We define a relative version of e^1 for pairs of quadratic forms with the same value of e^0. This is first done in terms of loops in some bisimplicial set whose fundamental group is the K_1 of the underlying exact category, and next translated into the language of 4-term double exact sequences, which allows us to carry out actual computations. An unexpected difficulty is that the value of the relative e^1 need not vanish even if both forms are metabolic. To make the invariant well defined on the Witt classes, we study the subgroup H generated by the values of e^1 on the pairs of metabolic forms and define the codomain for e^1 by factoring out this subgroup from some obvious subquotient of K_1. This proves to be a correct definition of the small k_1 for categories; it agrees with Milnor's usual k_1 in the case of fields.

Next we provide applications of this new invariant by computing it for some pairs of forms over the projective line and for some forms over conics.

1. Introduction. To obtain a proper generalization of the classical notion of the discriminant of a quadratic form

$$e^1 : W(F) \to k_1(F),$$
$$e^1(\langle a_1, \ldots, a_n\rangle) = (-1)^{n(n-1)/2} a_1 \cdot \cdots \cdot a_n \bmod F^{*2}$$

to symmetric bilinear forms over schemes, the framework of exact categories with duality seems to be the best one, as it involves K-theory.

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Section 2 contains a short review of the required results from K-theory; the details are partially published [11]–[13] or will be published subsequently [14]. The group $K_1(\mathcal{M}) = \pi_1(\Omega BQ(\mathcal{M}), 0)$ of an exact category $\mathcal{M}$ is generated by loops corresponding to short double exact sequences, and the presentation of $K_1(\mathcal{M})$ by these generators and relations, due to Nenashev, is known (see Section 2.5 below). In general a double exact sequence of arbitrary length defines a loop in $\Omega BQ(\mathcal{M})$ (or in another K-theory space of $\mathcal{M}$). As K-theory spaces we use the G-construction $G(\mathcal{M})$ and its self-dual version, denoted $T(\mathcal{M})$ here and in [14, App. B], or $W(\mathcal{M})$ in [13]. We are interested in exact categories with a duality functor D (Section 2.1 below); in that case an action of the two-element group $\{1, D\}$ on $K_1(\mathcal{M})$ arises.

The main example of an exact category with a duality functor is the category of locally free sheaves of $\mathcal{O}_X$-modules of finite rank on a scheme X with the duality functor $D : V \mapsto V\hat{}\otimes L$ (L a line bundle) with either the canonical isomorphism of V with its double dual (for symmetric L-valued forms) or the negative of this isomorphism (for skew symmetric L-valued forms).

For each of the Witt groups $W^+(X, L) = W(X, L)$, $W^-(X, L)$ of a scheme X of forms with values in a line bundle L there is a natural homomorphism

$$e^0 : W^{\pm}(X, L) \to E^0(X, L)$$

where $E^0(X, L)$ is a certain subfactor of $K_0(X)$, a member of the family

$$E^n(X, L) = E^n_+(X, L), E^n_-(X, L)$$

of subfactors of $K_n(X)$, namely the Tate cohomology groups of $\{1, D\}$ with values in the group $K_1(X)$ (see Definition 3.2 below). The homomorphism e^0 (depending on L) is induced by the forgetful functor and reduces to the usual dimension index e^0 in the classical case of $X = \mathbf{Spec}(F)$, F a field of characteristic different from 2. The pull-back

$$\begin{tikzcd} W_2(X, L) \arrow[r] \arrow[d] & W(X, L) \arrow[d, "e^0"] \\ W(X, L) \arrow[r, "e^0"] & E^0(X, L) \end{tikzcd}$$

i.e. the set of pairs $([(V, \alpha)], [(W, \beta)])$ of Witt classes with equal e^0 values, may be parametrized by a set $\mathcal{W}(X, L)$ of pairs of exact sequences

$$\begin{cases} 0 \leftarrow V \overset{b}{\leftarrow} B \overset{a}{\leftarrow} A \leftarrow 0, \ \alpha \\ 0 \leftarrow W \underset{b'}{\leftarrow} B \underset{a'}{\leftarrow} A \leftarrow 0, \ \beta \end{cases}$$

with the same objects A, B (we call such a pair a *common resolution* of V, W) and (skew-)self-dual isomorphisms $\alpha : V \to V\hat{}\otimes L$, $\beta : W \to W\hat{}\otimes L$ representing given Witt classes:

$$\begin{cases} 0 \leftarrow V \leftarrow B \leftarrow A \leftarrow 0, \ \alpha \\ 0 \leftarrow W \leftarrow B \leftarrow A \leftarrow 0, \ \beta \end{cases} \mapsto ([(V,\alpha)],[(W,\beta)])$$

(Corollary 3.2). In this case we define the relative discriminant $\varepsilon^1(\alpha \div \beta) \in E^1(X,L)$ (Definition 3.4) by the formula

$$\varepsilon^1(\alpha \div \beta) = \text{ class of d.e.s. } DA \underset{Da'}{\overset{Da}{\leftleftarrows}} DB \underset{Db'\circ\beta\circ b'}{\overset{Db\circ\alpha\circ b}{\leftleftarrows}} B \underset{a'}{\overset{a}{\leftleftarrows}} A$$

(we say that the double exact sequence is obtained by gluing the common resolution with its L-dual along α and β). This relative discriminant map $\varepsilon^1 : \mathcal{W}(X,L) \to E^1(X,L)$ is additive:

$$\varepsilon^1((\alpha \oplus \alpha') \div (\beta \oplus \beta')) = \varepsilon^1(\alpha \div \beta) + \varepsilon^1(\alpha' \div \beta'),$$

it vanishes on pairs of equal forms:

$$\varepsilon^1(\alpha \div \alpha) = 0,$$

and its value does not depend on the choice of a common resolution (Theorem 3.3), but need not be constant on Witt equivalence classes. In fact, there exist pairs of hyperbolic forms with the same e^0 values and nontrivial relative discriminant (Example 3.1).

Let $\mathcal{H}(X,L)$ be the subgroup of $E^1(X,L)$ consisting of the relative discriminants of pairs of hyperbolic forms with equal e^0 values. Consider the set of common resolutions of pairs of hyperbolic forms, and the natural map of this set into $E^0_-(X,L)$. The relative discriminant map is constant on each fibre of this map (Prop. 3.5): if a pair of hyperbolic forms with equal e^0 values defines the trivial element of the group $E^0_-(X,L)$, then the relative discriminant of the pair is trivial. It follows that $E^0_-(X,L)$ maps onto $\mathcal{H}(X,L)$. In particular $\mathcal{H}(X,L)$ is trivial provided $E^0_-(X,L)$ is trivial.

Note that even in the case of a projective line over a field the group $\mathcal{H}(X,\mathcal{O}_X)$ is nontrivial (Remark 3.1).

We define (Definition 3.5) the first k-group of X (with respect to the dualization $D : V \mapsto \mathcal{H}om_{\mathcal{O}_X}(V,L)$) as

$$k_1(X,L) = E^1(X,L)/\mathcal{H}(X,L)$$

and the discriminant map (depending on L)

$$e^1 : I(X,L) \to k_1(X,L), \quad e^1(\varphi) = \varepsilon^1(\varphi \div 0) \bmod \mathcal{H}(X,L),$$

where $I(X,L) = \operatorname{Ker} e^0$ (Definition 3.6). In the classical case of $X = \operatorname{Spec}(F)$, F a field of characteristic different from 2, there is no nontrivial line bundle L, $I(X)$ is the fundamental ideal of the Witt ring $W(X)$, $E^0_-(X) = 0$, $k_1(X) = E^1(X) = \dot{F}/\dot{F}^2$ and

$$e^1(\langle a_1,\dots,a_{2n}\rangle) = \varepsilon^1(\langle a_1,\dots,a_{2n}\rangle \div 0) = (-1)^{2n(2n-1)/2} a_1 \cdots a_n$$

is the usual discriminant of a quadratic form (Example 3.2).

The discriminant map is clearly functorial: given a morphism $f : X \to Y$ of schemes, the functor f^* induces homomorphisms of all groups involved, and

$$e^1 \circ f^* = f^* \circ e^1.$$

In the particular case of a variety X over a field F such that $K_1(X) = K_1(F) \otimes_{\mathbb{Z}} K_0(X)$ the E^1-groups of X are

$$E^1(X, L) = k_1(F) \otimes_{\mathbb{Z}} E^0(X, L) \oplus \mu_2(F) \otimes_{\mathbb{Z}} E^0_-(X, L),$$

$$E^1_-(X, L) = k_1(F) \otimes_{\mathbb{Z}} E^0_-(X, L) \oplus \mu_2(F) \otimes_{\mathbb{Z}} E^0(X, L),$$

where $\mu_2(F)$ is the group of square roots of 1 in F.

If X is a variety over F (so all Witt groups of X are $W(F)$-modules), the map e^1 satisfies

$$I(F)I(X, L) \subset \operatorname{Ker} e^1$$

(Theorem 3.9). The framework of exact categories with duality provides uniform notation for the cases of symmetric and skew-symmetric forms, and different line bundles L.

There are some immediate applications of the relative discriminant to classes of metabolic spaces in a Grothendieck group. The discriminant map is applied to the Witt group of symmetric bilinear forms over an anisotropic conic with values in a line bundle.

The paper depends on bisimplicial computations done by Sasha Nenashev.

2. Witt groups and K-theory

2.1. Dualization and forms. Exact categories and their higher algebraic K-theory were defined in [18] by D. Quillen as follows. An *exact category* $\mathfrak{M}$ is an additive category $\mathfrak{M}$ (with the isomorphism class of each object forming a set), embedded as a full subcategory of an abelian category $\mathfrak{A}$, and closed under extensions in $\mathfrak{A}$. Quillen also gave an axiomatic definition of an exact category:

DEFINITION 2.1. An *exact category* $\mathfrak{M} = (\mathfrak{M}, \mathfrak{E})$ is an additive category $\mathfrak{M}$ (with the isomorphism class of each object forming a set), with a family $\mathfrak{E}$ of exact (in $\mathfrak{A}$) sequences

$$0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0 \tag{2.1}$$

(called *admissible exact sequences*) satisfying the following conditions:

(a) all split exact sequences of objects of $\mathfrak{M}$ are in $\mathfrak{E}$; if (2.1) is in $\mathfrak{E}$, then α is a kernel of β in $\mathfrak{M}$ and β is a cokernel of α in $\mathfrak{M}$;

(b) a composition of admissible epimorphisms (monomorphisms) is an admissible epimorphism (monomorphism);

$$\begin{tikzcd} A \arrow[r, tail, "\alpha"] \arrow[dr, tail, dashed, "\beta\circ\alpha"'] & B \arrow[d, tail, "\beta"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} A & B \arrow[l, two heads, "\alpha"'] \\ & C \arrow[u, two heads, "\beta"'] \arrow[ul, two heads, dashed, "\alpha\circ\beta"] \end{tikzcd}$$

a (co)base change of an admissible epimorphism (monomorphism) is an admissible epimorphism (monomorphism);

$$\begin{tikzcd} C \arrow[r, tail, dashed, "\alpha'"] & C \amalg_A B \\ A \arrow[u, "\varphi"] \arrow[r, tail, "\alpha"] & B \arrow[u, "\varphi'"'] \end{tikzcd} \qquad \begin{tikzcd} A \times_B C \arrow[r, two heads, dashed, "\alpha'"] \arrow[d, "\varphi'"'] & C \arrow[d, "\varphi"] \\ A \arrow[r, two heads, "\alpha"] & B \end{tikzcd}$$

(c) if $M \to M''$ possesses a kernel in $\mathfrak{M}$ and the composition $N \to M \to M''$ is an admissible epimorphism, then $M \to M''$ is an admissible epimorphism; the dual statement for monomorphisms holds true.

$$\begin{tikzcd} \operatorname{Ker}\alpha \arrow[r] & M \arrow[r, two heads, dashed, "\alpha"] & M'' \\ & N \arrow[u] \arrow[ur, two heads] & \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Coker}\alpha & M \arrow[l] \arrow[d] & M' \arrow[l, tail, dashed, "\alpha"'] \arrow[dl, tail] \\ & N & \end{tikzcd}$$

For example, given an admissible exact sequence (2.1) and a map $N \xrightarrow{\varphi} M''$ the sequence

$$0 \to M' \xrightarrow{\overline{\alpha}} M \times_{M''} N \xrightarrow{p_2} N \to 0$$

is an admissible exact sequence.

It is known now that condition (c) is a consequence of (a)–(b) (Keller, [5, App. A]).

We will use $\rightarrowtail$ and $\twoheadrightarrow$ to denote admissible monomorphisms and admissible epimorphisms respectively.

An *admissible subobject* is a kernel of an admissible epimorphism.

Recall that an *exact functor* between exact categories is an additive functor which takes admissible exact sequences to admissible exact sequences.

2.2. Q-construction. Given an exact category $(\mathfrak{M}, \mathfrak{A}, \mathfrak{E})$, the category $Q\mathfrak{M}$ has the same objects as $\mathfrak{M}$, and a morphism from M to M' in $Q\mathfrak{M}$ is a class of diagrams

$$M \overset{\beta}{\twoheadleftarrow} N \overset{\alpha}{\rightarrowtail} M'$$

up to an isomorphism which induces the identity on M and M'.

The composition of morphisms $M \overset{\beta}{\twoheadleftarrow} N \overset{\alpha}{\rightarrowtail} M'$ and $M' \overset{\delta}{\twoheadleftarrow} N' \overset{\gamma}{\rightarrowtail} M''$ in $Q\mathfrak{M}$ is defined by fiber product:

$$M \xleftarrow{\beta\circ p_1} N \times_{M'} N' \xrightarrow{\gamma\circ p_2} M'.$$

It is clear that composition is well-defined and associative. If we assume that the isomorphism classes of diagrams $M \overset{\beta}{\twoheadleftarrow} N \overset{\alpha}{\rightarrowtail} M'$ form a set for each M, M', then $Q\mathfrak{M}$ is a well defined category.

Each admissible monomorphism $N \overset{\alpha}{\rightarrowtail} M$ gives rise to a morphism $\alpha_! : N \to M$ in $Q\mathfrak{M}$ represented by the diagram

$$\alpha_! : N \overset{1_N}{\twoheadleftarrow} N \overset{\alpha}{\rightarrowtail} M,$$

and morphisms of this type are called *injective*.

Dually, each admissible epimorphism $M \overset{\beta}{\twoheadleftarrow} N$ defines a morphism $\beta^! : M \to N$ in $Q\mathfrak{M}$ represented by the diagram

$$\beta^! : M \overset{\beta}{\twoheadleftarrow} N \overset{1_N}{\rightarrowtail} N,$$

and such morphisms are called *surjective*.

Note that $(M \overset{\beta}{\twoheadleftarrow} N \overset{\alpha}{\rightarrowtail} M') = \alpha_! \circ \beta^!$. There is a dual decomposition: given a map $M \overset{\beta}{\twoheadleftarrow} N \overset{\alpha}{\rightarrowtail} M'$ the fiber product

$$\begin{tikzcd} N \arrow[r, tail, "\alpha"] \arrow[d, "\beta"'] & M' \arrow[d, "\delta"] \\ M \arrow[r, tail, "\gamma"] & M \times_N M' \end{tikzcd}$$

defines a decomposition

$$M \overset{\beta}{\twoheadleftarrow} N \overset{\alpha}{\rightarrowtail} M' = \delta^! \circ \gamma_!.$$

Conversely, a diagram $M \overset{\gamma}{\rightarrowtail} N' \overset{\delta}{\twoheadleftarrow} M'$ by means of a fiber sum $M \amalg_{N'} M'$ defines a morphism $M \overset{\beta}{\twoheadleftarrow} N \overset{\alpha}{\rightarrowtail} M' = \alpha_! \circ \beta^!$ with the decomposition. In fact, this is an alternative way to define morphisms in $Q\mathfrak{M}$. It may be convenient to regard a morphism in the category $Q\mathfrak{M}$ as a bicartesian square

$$\begin{tikzcd} N \arrow[r, tail, "\alpha"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\square"] & M' \arrow[d, "\delta"] \\ M \arrow[r, tail, "\gamma"] & N' \end{tikzcd}$$

and agree that usually we omit one corner of the square for short. The $\square$ sign indicates that the square is bicartesian.

PROPOSITION 2.1.

(a) *If α and α' are composable monomorphisms in $\mathfrak{M}$, then $(\alpha \circ \alpha')_! = \alpha_! \circ \alpha'_!$ in $Q\mathfrak{M}$.*
(b) *If β and β' are composable epimorphisms in $\mathfrak{M}$, then $(\beta' \circ \beta)^! = \beta^! \circ \beta'^!$ in $Q\mathfrak{M}$.*
(c) $(1_M)_! = (1_M)^! = 1_M$.
(d) *For a bicartesian square*

$$\begin{tikzcd} N \arrow[r, tail, "\alpha"] \arrow[d, two heads, "\beta"'] \arrow[dr, phantom, "\square"] & M' \arrow[d, two heads, "\delta"] \\ M \arrow[r, tail, "\gamma"'] & N' \end{tikzcd}$$

with admissible arrows in $\mathfrak{M}$ we have $\alpha_! \circ \beta^! = \delta^! \circ \gamma_!$ in $Q\mathfrak{M}$. ■

2.2.1. *The universal property of the Q-construction.* Suppose we are given a category $\mathfrak{C}$ and for each object M in $\mathfrak{M}$ an object hM of $\mathfrak{C}$, and for each $N \overset{\alpha}{\rightarrowtail} M'$ (resp. $M \overset{\beta}{\twoheadleftarrow} N$) in $\mathfrak{M}$ a map $\alpha_! : hN \to hM'$ (resp. $\beta^! : hM \to hN$) such that the properties (a)–(d) of Proposition 2.1 hold. Then it is clear that this data induces a unique functor $F : Q\mathfrak{M} \to \mathfrak{C}$, $F(M) = hM$, compatible with the operations $\alpha \mapsto \alpha_!$, $\beta \mapsto \beta^!$.

This universal property of the Q-construction shows that an exact functor $F : \mathfrak{M} \to \mathfrak{M}'$ between exact categories induces a functor $Q\mathfrak{M} \to Q\mathfrak{M}'$, $M \mapsto FM$, $\alpha_! \mapsto (F\alpha)_!$, $\beta^! \mapsto (F\beta)^!$. Also for the dual category $\mathfrak{M}^o$ of an exact category there is an isomorphism of categories

$$Q\mathfrak{M}^o = Q\mathfrak{M}$$

such that the injective arrows in the former correspond to surjective arrows in the latter and conversely.

2.2.2. *Isomorphisms.* For an isomorphism $N \overset{\alpha}{\rightarrowtail} M$ the maps $\alpha_! : N \to M$ and $\alpha^{-1!} : N \to M$ are equal, since there is a commutative diagram

$$\begin{tikzcd} N \arrow[d, equal] & N \arrow[l, two heads, "1_N"'] \arrow[r, tail, "\alpha"] \arrow[d, "\alpha"] & M \arrow[d, equal] \\ N & M \arrow[l, two heads, "\alpha^{-1}"'] \arrow[r, tail, "1_M"] & M \end{tikzcd}$$

Conversely, a map in $Q\mathfrak{M}$ which is both injective and surjective is an isomorphism and it is of the form $\alpha_! = \alpha^{-1!}$ for a unique isomorphism α in $\mathfrak{M}$.

2.2.3. *Zero maps.* Let $0^M : 0 \rightarrowtail M$ and $0_M : M \twoheadrightarrow 0$ denote unique maps in the additive category $\mathfrak{M}$. The set $\mathrm{Mor}_{Q\mathfrak{M}}(0, M)$ is in 1-1 correspondence with the set of admissible subobjects of M (i.e. admissible monomorphisms $N \overset{\alpha}{\rightarrowtail} M$ up to automorphism of N over M) since each such morphism by the definition is given by the diagram

$$0_\alpha^M : 0 \overset{0_N}{\twoheadleftarrow} N \overset{\alpha}{\rightarrowtail} M.$$

Thus the set $\mathrm{Mor}_{Q\mathfrak{M}}(0, M)$ is partially ordered, with the smallest element $0_!^M$ and the greatest element $0^!_M$. There are decompositions $0_!^M = \alpha_! \circ 0_!^N$ and $0_\alpha^M = \alpha_! \circ 0^!_N$ in view of commutativity of

$$\begin{tikzcd}
0\times_N N = 0 \arrow[r, tail] \arrow[d, equal] \arrow[dr, phantom, "\square"] & N \arrow[r, tail, "\alpha"] \arrow[d] & M \\
0 \arrow[r, tail] \arrow[d] & N \arrow[ur, "\alpha_!"'] & \\
0 \arrow[ur, "0_!^N"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
N\times_N N = N \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\square"] & N \arrow[r, tail, "\alpha"] \arrow[d, equal] & M \\
N \arrow[r, equal] \arrow[d] & N \arrow[ur, "\alpha_!"'] & \\
0 \arrow[ur, "0^!_N"'] & &
\end{tikzcd}$$

Dually, given an admissible epimorphism $M \overset{\beta}{\twoheadrightarrow} M''$ with kernel $N \overset{\alpha}{\rightarrowtail} M$, there are decompositions $0^!_M = \beta^! \circ 0^!_{M''}$ and $0_\alpha^M = \beta^! \circ 0_!^{M''}$ in view of commutativity of

$$\begin{tikzcd}
N = \operatorname{Ker}\beta = 0 \times_{M''} M \arrow[r, tail, "\alpha"] \arrow[d] & M \arrow[d, "\beta"] \\
0 \arrow[r, tail] & M''
\end{tikzcd}$$

Proposition 2.2. *An admissible exact sequence* (2.1) *produces the following commutative diagram in $Q\mathfrak{M}$:*

$$\begin{tikzcd}
& & 0 \arrow[dll, "0_!^{M''}"'] \arrow[dd, "0_\alpha^M"] \arrow[drr, "0^!_{M'}"] & & \\
M'' \arrow[drr, "\beta^!"] & & & & M' \arrow[dll, "\alpha_!"'] \\
& & M & & \\
& 0 \arrow[uul, "0^!_{M''}"] \arrow[ur, "0^!_M"'] & & 0 \arrow[ul, "0_!^M"] \arrow[uur, "0_!^{M'}"'] &
\end{tikzcd}$$

Proof. We have the decompositions $0_!^M = \alpha_! \circ 0_!^{M'}$ and $0_\alpha^M = \alpha_! \circ 0^!_{M'}$. ∎

Quillen [18, I.2, Theorem 1 p. 18] proved

THEOREM 2.3. *The fundamental group $\pi_1(B(Q\mathfrak{M}), 0)$ is canonically isomorphic to the Grothendieck group $K_0(\mathfrak{M})$.* ∎

It is obvious from the proof that the class $[M] \in K_0(\mathfrak{M})$ corresponds under this isomorphism to the loop $|0^!_M| - |0^M_!|$ formed by the path $|0^!_M|$ corresponding to the 1-simplex $0^!_M : 0 \to M$ and $|0^M_!|$ corresponding to the 1-simplex $0^M_! : 0 \to M$.

This theorem motivates the following definition:

DEFINITION 2.2. The *higher algebraic K-theory groups* of an exact category $\mathfrak{M}$ are the homotopy groups of the classifying space of the category $Q\mathfrak{M}$:

$$K_i(\mathfrak{M}) = \pi_{i+1}(B(Q\mathfrak{M}), 0).$$

A *K-theory space* is a topological space $X(\mathfrak{M})$ homotopy equivalent to $\Omega B(Q\mathfrak{M})$, i.e. such that

$$K_i(\mathfrak{M}) = \pi_i(X(\mathfrak{M}), *)$$

(see e.g. Nenashev [13]).

2.3. Duality

DEFINITION 2.3. A *duality* or a structure of a *Hermitian category* on $\mathfrak{M}$ (cf. [19, p. 241]) is a pair (D, δ), where the *dualization functor* $D : \mathfrak{M} \to \mathfrak{M}$ is an exact contravariant functor and $\delta : 1_{\mathfrak{M}} \to D^2$ is a natural isomorphism such that $(D\delta) \circ \delta_D = 1_D$, i.e. the diagram

$$\begin{tikzcd} & D^3M \arrow[dr, "D(\delta_M)"] & \\ DM \arrow[ur, "\delta_{DM}"] \arrow[rr, "1"] & & DM \end{tikzcd}$$

commutes for every object M of $\mathfrak{M}$.

Note that by definition the dual of an admissible exact sequence is an admissible exact sequence.

We focus our attention on the following three examples:

1. Let $\mathfrak{M}$ be the category of finite-dimensional vector spaces over some fixed field F, $DV = V^* = \mathrm{Hom}_F(V, F)$,

$$D(f)(\phi) = \phi \circ f$$

and $\delta_V : V \to D^2V = V^{**}$ the canonical isomorphism of V and its double dual:

$$\delta_V(v)(\phi) = \phi(v)$$

for $\phi : V \to F$. The condition $(D\delta) \circ \delta_D = 1_D$ means that the map defined by

$$\psi \mapsto \delta_{DV}(\psi), \qquad \delta_{DV}(\psi)(\phi) = \phi(\psi),$$

composed with $D(\delta_V)$,

$$D(\delta_V)(\chi) = \chi \circ \delta_V,$$

produces the map $\psi \mapsto (v \mapsto \psi(v))$, i.e. the map $\psi \mapsto \psi$. If $f : V \times V \to F$ is a symmetric bilinear form, then it is customary to factor out the kernel and deal exclusively with nonsingular symmetric bilinear forms. For such a form there is an isomorphism—the adjoint linear map $\phi : V \to DV$—defined by $\phi(v)(w) = f(v, w)$. The symmetry $f(v, w) = f(w, v)$ of f is equivalent to the condition $(D\phi) \circ \delta_V = \phi$.

2. Let $\mathfrak{M}$ be the category of vector bundles on a scheme X. We set

$$\begin{aligned} D\mathcal{V} &= \mathcal{V}\hat{} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{V}, \mathcal{O}_X), \\ D\varphi &= \varphi\hat{} = - \circ \varphi : \mathcal{W}\hat{} \to \mathcal{V}\hat{} \text{ for } \varphi : \mathcal{V} \to \mathcal{W} \end{aligned}$$

and $\delta_{\mathcal{V}} : \mathcal{V} \to D^2\mathcal{V} = \mathcal{V}\hat{}\hat{}$ is the canonical isomorphism of a $\mathcal{V}$ with its double dual. In this case we define a symmetric bilinear form as its adjoint, an isomorphism $\varphi : \mathcal{V} \to \mathcal{V}\hat{}$ such that $\varphi\hat{} \circ \delta_{\mathcal{V}} = D\varphi \circ \delta_{\mathcal{V}} = \varphi$. Note that on the fibers a family of usual symmetric bilinear forms arises, parametrized by points of X.

3. Let $\mathfrak{M}$ be the category of vector bundles on a scheme X, and $\mathcal{L}$ a line bundle on X. We set

$$D\mathcal{V} = \mathcal{V}\hat{} \otimes_{\mathcal{O}_X} \mathcal{L} = \mathcal{H}om_{\mathcal{O}_X}(V, \mathcal{L}), \qquad D\varphi = \varphi\hat{} \otimes_{\mathcal{O}_X} 1_{\mathcal{L}},$$

and let

$$\delta_{\mathcal{V}} : \mathcal{V} \to D^2\mathcal{V} = (\mathcal{V}\hat{} \otimes_{\mathcal{O}_X} \mathcal{L})\hat{} \otimes_{\mathcal{O}_X} \mathcal{L}$$

be a composition of canonical isomorphisms (note that $D^2\mathcal{V} \cong \mathcal{V}\hat{}\hat{} \otimes (\mathcal{L}\hat{} \otimes_{\mathcal{O}_X} \mathcal{L}) \cong \mathcal{V}\hat{}\hat{}$). An isomorphism $\varphi : \mathcal{V} \to \mathcal{V}\hat{} \otimes_{\mathcal{O}_X} \mathcal{L}$ is an *$\mathcal{L}$-valued symmetric bilinear* form if $(\varphi\hat{} \otimes 1_{\mathcal{L}}) \circ \delta_{\mathcal{V}} = D\varphi \circ \delta_{\mathcal{V}} = \varphi$.

Changing $\delta_{\mathcal{V}}$ to the negative of the canonical isomorphism

$$\mathcal{V} \xrightarrow{\sim} (\mathcal{V}\hat{} \otimes_{\mathcal{O}_X} \mathcal{L})\hat{} \otimes_{\mathcal{O}_X} \mathcal{L}$$

gives a formalism for $\mathcal{L}$-valued skew-symmetric bilinear forms.

One may easily generalize concepts known for example 1 to the other two.

Definition 2.4. For a category $\mathfrak{M}$ with a duality (D, δ) a morphism $\varphi : V \to DV$ is a *self-dual morphism* if $D\varphi \circ \delta_V = \varphi$, or the diagram

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\delta_V"'] & DV \arrow[d, "1"] \\ D^2V \arrow[r, "D\varphi"] & DV \end{tikzcd}$$

commutes. A *symmetric bilinear form* (or a *symmetric bilinear space*) is a self-dual isomorphism $\varphi : V \to DV$.

Compare [2, Def. 1.1.2] and the following remark.

Example 2.1. $(0,0)$ is a symmetric bilinear form for arbitrary (D,δ).

There are obvious notions of isomorphism of symmetric bilinear forms and of direct sum of symmetric bilinear forms. In all three examples there is a well defined bi-exact tensor product of symmetric bilinear forms.

Definition 2.5. Let $\varphi : V \to DV$ be a symmetric bilinear form. For an admissible monomorphism $\alpha : U \rightarrowtail V$ an *orthogonal complement* $U^\perp$ of $U \overset{\alpha}{\rightarrowtail} V$ is a kernel of the composition $V \xrightarrow{\varphi} DV \xrightarrow{D\alpha} DU$, so there is an exact sequence

$$0 \to U^\perp \rightarrowtail V \xrightarrow{(D\alpha)\circ\varphi} DU.$$

The kernel $U^\perp$ exists, since α is admissible.

A direct generalization of the above notions to self-dual morphisms which are not isomorphisms is impossible.

Example 2.2. Let $\mathfrak{M}$ be a full subcategory of the category of finitely generated abelian groups, having free abelian groups as objects, and exact sequences of free abelian groups as admissible exact sequences, with the dualization functor

$$DA = \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z}).$$

Multiplication by 2 is a self-dual map, and even a bimorphism (a monomorphism and an epimorphism) in $\mathfrak{M}$. It is a symmetric bilinear map which is not a form. It has a nontrivial cokernel $\mathbb{Z}/2\mathbb{Z}$ in the ambient abelian category, but the functor D is not defined on $\mathbb{Z}/2\mathbb{Z}$.

Example 2.3. Let $\varphi : V \to DV$ be a self-dual morphism. The morphism φ has a kernel in the ambient abelian category, but it need not have a kernel in the category $\mathfrak{M}$; even if it does have a kernel in $\mathfrak{M}$, it may differ from the kernel in the ambient abelian category (e.g. consider the opposite to the category $\mathfrak{M}$ of the last example, and the same morphism), unless the kernel in the ambient abelian category is an object of $\mathfrak{M}$.

To avoid the situation described in the last example, we define admissible symmetric bilinear maps as follows:

Definition 2.6. An *admissible symmetric bilinear map* (briefly: an a.s.b.m.) $\varphi : V \to DV$ is a self-dual morphism which has a decomposition $\varphi = \mu \circ \eta$ with μ an admissible monomorphism and η an admissible epimorphism.

Remark 2.1. This is a terminological novelty: a symmetric bilinear form is nonsingular by definition; a symmetric bilinear map may be singular.

An a.s.b.m. φ has both a kernel and a cokernel, and $\operatorname{Coker}\varphi \cong D\operatorname{Ker}\varphi$. An a.s.b.m. is a symmetric bilinear form iff $\operatorname{Ker}\varphi = 0$. If $\varphi : V \to DV$ is an a.s.b.m. and $\varphi = \mu \circ \eta$, then there is a commutative diagram

$$\begin{tikzcd}
V \arrow[r, two heads, "\eta"] \arrow[d, "\delta_V"', "\sim"] & W \arrow[r, tail, "\mu"] \arrow[d, "\psi"] & DV \arrow[d, equal] \\
D^2V \arrow[r, two heads, "D\mu"] & DW \arrow[r, tail, "D\eta"] & DV
\end{tikzcd}$$

with an isomorphism ψ which must be self-dual. Thus every a.s.b.m. is of the form

$$\varphi = D\eta \circ \psi \circ \eta$$

for a symmetric bilinear form ψ and an admissible epimorphism η.

If $\varphi : V \to DV$ is an a.s.b.m. with a decomposition as above, $\alpha : U \to V$ is a morphism, $\operatorname{Ker}\varphi = (R \rightarrowtail V)$ factors through α,

$$(R \rightarrowtail V) = (R \xrightarrow{\beta} U \xrightarrow{\alpha} V)$$

and $\beta : R \to U$ has a cokernel in $\mathfrak{M}$, then β is an admissible monomorphism. If, in addition, $\alpha : U \to V$ is an admissible monomorphism and $\operatorname{Coker}\beta = U \twoheadrightarrow S$, then $S \to W$ is an admissible monomorphism. Thus it has an orthogonal complement $S^{\perp}$ in the symmetric bilinear space (W, ψ). Then $V \times_W S^{\perp}$ is an object of $\mathfrak{M}$, $V \times_W S^{\perp} \to S^{\perp}$ is an admissible epimorphism, and $V \times_W S^{\perp} \to V$ is an admissible monomorphism. Moreover, $V \times_W S^{\perp} \rightarrowtail V$ is a kernel of $D\alpha \circ \varphi$ and $R \to V \times_W S^{\perp}$ is an admissible monomorphism.

Definition 2.7. Let $\varphi : V \to DV$ be an a.s.b.m. If $\alpha : U \rightarrowtail V$ is an admissible subobject such that $\operatorname{Ker}\varphi$ factors through α with cokernel in $\mathfrak{M}$, then the *orthogonal complement* $U^{\perp} \rightarrowtail V$ of $\alpha : U \rightarrowtail V$ is the kernel of $D\alpha \circ \varphi$,

$$(U^{\perp} \rightarrowtail V) = \operatorname{Ker}(D\alpha \circ \varphi).$$

An orthogonal complement of an admissible subobject is itself an admissible subobject, since the sequence

$$U^{\perp} \rightarrowtail V \twoheadrightarrow DS$$

is admissible. Moreover, $\operatorname{Ker}\varphi$ factors through $U^{\perp} \rightarrowtail V$ with cokernel in $\mathfrak{M}$.

If $\varphi : V \to DV$ is a symmetric bilinear form and $\kappa : V \twoheadrightarrow V/U$ is a cokernel of an admissible subobject $j : U \rightarrowtail V$, then φ induces an isomorphism $U^{\perp} \cong D(V/U)$:

$$\begin{tikzcd}
0 \arrow[r] & D(V/U) \arrow[r, "D\kappa"] & DV \arrow[r, "Di"] & DU \arrow[r] & 0 \\
0 \arrow[r] & U^{\perp} \arrow[r, "j"] \arrow[u, dashed, "\overline{\varphi}"] & V \arrow[r, "Di\circ\varphi"] \arrow[u, "\varphi"'] & DU \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

LEMMA 2.4. *If $i : U \to V$ is an admissible subobject such that* $\operatorname{Ker}\varphi$ *factors through U with cokernel in $\mathfrak{M}$, then $U^{\perp\perp} = U$.*

Proof. Dualization of the exact sequence

$$0 \to U^{\perp} \xrightarrow{j} V \xrightarrow{Di\circ\varphi} DU \to 0$$

yields an exact sequence

$$0 \to D^2U \xrightarrow{D\varphi\circ D^2 i} DV \xrightarrow{Dj} D(U^{\perp}) \to 0.$$

Since $\operatorname{Ker}\varphi$ factors through U, it follows from the next lemma that the sequences

$$\begin{aligned} &0 \to D^2U \xrightarrow{D^2 i} D^2V \xrightarrow{Dj\circ D\varphi} D(U^{\perp}),\\ &0 \to U \xrightarrow{D^2 i\circ\delta_U} D^2V \xrightarrow{Dj\circ D\varphi} D(U^{\perp}),\\ &0 \to U \xrightarrow{\delta_V\circ i} D^2V \xrightarrow{Dj\circ D\varphi} D(U^{\perp}),\\ &0 \to U \xrightarrow{i} V \xrightarrow{Dj\circ\varphi} D(U^{\perp}) \end{aligned}$$

are exact in the ambient abelian category. Hence $(U \xrightarrow{i} V) = \operatorname{Ker}(Dj\circ\varphi)$, i.e. $U = U^{\perp\perp}$. ∎

LEMMA 2.5. *Assume there are morphisms*

$$\alpha : A \to B, \quad \beta : B \to C, \quad \gamma : C \to D$$

in an abelian category. If $\operatorname{Ker}\beta$ *factors through α, then the exactness of*

$$0 \to A \xrightarrow{\beta\circ\alpha} C \xrightarrow{\gamma} D$$

implies the exactness of

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\gamma\circ\beta} D. \ ∎$$

The subobjects $U, U^{\perp}$ of V may have various mutual positions; two particular cases are important.

DEFINITION 2.8. Let $\varphi : V \to DV$ be an a.s.b.m., and let $i : U \to V$ be a morphism. Then $i : U \to V$ is *nonsingular* if $Di\circ\varphi\circ i : U \to DU$ is an isomorphism.

Note that every nonsingular $i : U \to V$ is a direct summand with complementary summand $U^{\perp} \to V$.

DEFINITION 2.9. Let $\varphi : V \to DV$ be an a.s.b.m., and let $i : U \to V$ be an admissible monomorphism such that $\operatorname{Ker}\varphi$ factors through i. Then $i : U \to V$ is *totally isotropic* (or *sublagrangian*) if it factors through $j : U^{\perp} \to V$ (i.e. there exists an $\alpha' : U \to U^{\perp}$ such that $i = j\circ\alpha'$) and the cokernel $U^{\perp}/U$ is in $\mathfrak{M}$.

Of course, for an a.s.b.m. $\varphi : V \to DV$ all nontrivial subobjects are nonsingular only if it is a symmetric bilinear form. Nevertheless, there are symmetric forms with singular nontrivial subobjects. If $i : U \to V$ is nonsingular, then $Di \circ \varphi \circ i : U \to DU$ is an isomorphism, the induced symmetric bilinear form $\varphi|_U$.

If $\varphi : V \to DV$ is a symmetric bilinear form and $i : U \to V$ is totally isotropic, then $Di \circ \varphi$ induces an isomorphism $V/U^\perp \to DU$ and a morphism $V/U \to DU$. If $U \rightarrowtail V$ is totally isotropic, then $U \to U^\perp$ is an admissible monomorphism.

Definition 2.10. Let $\varphi : V \to DV$ be an a.s.b.m. An admissible monomorphism $\alpha : U \to V$ is a *Lagrangian* (or a *metabolizer*) if it is the kernel of $D\alpha \circ \varphi$ ($D\alpha \circ \varphi \circ \alpha = 0$ and given $\beta : T \to V$ such that $D\alpha \circ \varphi \circ \beta = 0$, there exists a unique $\overline{\beta} : T \to U$ such that $\alpha \circ \overline{\beta} = \beta$), i.e. the diagram

$$\begin{tikzcd}
U \arrow[r, tail, "\alpha"] \arrow[d, "\delta_U"'] & V \arrow[r, "D\alpha\circ\varphi"] \arrow[d, "\varphi"] & DU \arrow[d, "1"] \\
D^2U \arrow[r, "D\varphi\circ D^2\alpha"] & DV \arrow[r, two heads, "D\alpha"] & DU
\end{tikzcd}$$

with exact rows commutes. An a.s.b.m. $\varphi : V \to DV$ is *metabolic* if it possesses a Lagrangian.

If (V, φ) is a metabolic space ("space" means that φ is an isomorphism), then in addition the map $D\alpha \circ \varphi$ is an epimorphism and $D\varphi \circ D^2\alpha$ is a monomorphism. In such a case $Di \circ \varphi$ induces an isomorphism $V/U \to DU$.

Example 2.4. A *hyperbolic form*

$$\begin{bmatrix} 0 & 1_{DV} \\ \delta_V & 0 \end{bmatrix} : V \oplus DV \to DV \oplus D^2V \cong V \oplus DV$$

is a metabolic space with the Lagrangian $V \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} V \oplus DV$.

Example 2.5. For an arbitrary symmetric bilinear form (V, φ), the form

$$\begin{bmatrix} \varphi & 0 \\ 0 & -\varphi \end{bmatrix} : V \oplus V \to DV \oplus DV$$

has the Lagrangian $V \xrightarrow{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} V \oplus V$, so it is metabolic.

Example 2.6. For an arbitrary self-dual map $\alpha : U \to DU$ (i.e. with $D\alpha \circ \delta_U = \alpha$) the *split metabolic space*

$$H(U, \alpha) = \left(U \oplus DU, \begin{bmatrix} \alpha & 1_{DU} \\ \delta_U & 0 \end{bmatrix} \right)$$

is metabolic, since the subobject

$$\begin{bmatrix} 0 \\ 1_{DU} \end{bmatrix} : DU \to U \oplus DU$$

is a Lagrangian:

$$\begin{aligned} \operatorname{Ker}\left(D \begin{bmatrix} 0 \\ 1_{DU} \end{bmatrix} \begin{bmatrix} \alpha & 1_{DU} \\ \delta_U & 0 \end{bmatrix} \right) &= \operatorname{Ker}\left([0, 1_{D^2U}] \begin{bmatrix} \alpha & 1_{DU} \\ \delta_U & 0 \end{bmatrix} \right) \\ &= \operatorname{Ker}\,[\delta_U, 0] = \begin{bmatrix} 0 \\ 1_{DU} \end{bmatrix}. \end{aligned}$$

EXAMPLE 2.7. For a split metabolic space $H(U,\alpha) = \left(U \oplus DU, \left[\begin{smallmatrix} \alpha & 1_{DU} \\ \delta_U & 0 \end{smallmatrix}\right]\right)$ and an arbitrary map $\gamma : U \to DU$, there is an isomorphism $H(U,\alpha) \cong H(U, \alpha + \gamma + D\gamma \circ \delta_U)$. The isomorphism is given by

$$\begin{bmatrix} 1_U & 0 \\ \gamma & 1_{DU} \end{bmatrix} : U \oplus DU \to U \oplus DU$$

since

$$\begin{aligned} &\left(D \begin{bmatrix} 1_U & 0 \\ \gamma & 1_{DU} \end{bmatrix} \right) \begin{bmatrix} \alpha & 1_{DU} \\ \delta_U & 0 \end{bmatrix} \begin{bmatrix} 1_U & 0 \\ \gamma & 1_{DU} \end{bmatrix} \\ &\quad = \begin{bmatrix} 1_{DU} & D\gamma \\ 0 & 1_{D^2U} \end{bmatrix} \begin{bmatrix} \alpha & 1_{DU} \\ \delta_U & 0 \end{bmatrix} \begin{bmatrix} 1_U & 0 \\ \gamma & 1_{DU} \end{bmatrix} \\ &\quad = \begin{bmatrix} 1_{DU} & D\gamma \\ 0 & 1_{D^2U} \end{bmatrix} \begin{bmatrix} \alpha + \gamma & 1_{DU} \\ \delta_U & 0 \end{bmatrix} = \begin{bmatrix} \alpha + \gamma + D\gamma \circ \delta_U & 1_{DU} \\ \delta_U & 0 \end{bmatrix}. \end{aligned}$$

It follows that a split metabolic space need not be hyperbolic; it is so for symmetric bilinear forms over a field of characteristic 2.

EXAMPLE 2.8. If

$$U \overset{i}{\rightarrowtail} V \overset{j}{\twoheadrightarrow} W$$

is an admissible exact sequence, then $\left[\begin{smallmatrix} i & 0 \\ 0 & Dj \end{smallmatrix}\right] : U \oplus DW \to V \oplus DV$ is a Lagrangian for the hyperbolic space $\left(V \oplus DV, \left[\begin{smallmatrix} 0 & 1 \\ \delta_V & 0 \end{smallmatrix}\right]\right)$. In fact,

$$\begin{aligned} D \begin{bmatrix} i & 0 \\ 0 & Dj \end{bmatrix} \circ \begin{bmatrix} 0 & 1 \\ \delta_V & 0 \end{bmatrix} &= \begin{bmatrix} Di & 0 \\ 0 & D^2 j \end{bmatrix} \circ \begin{bmatrix} 0 & 1 \\ \delta_V & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & Di \\ D^2 j \circ \delta_V & 0 \end{bmatrix} = \begin{bmatrix} 0 & Di \\ \delta_W \circ j & 0 \end{bmatrix} \end{aligned}$$

and the sequence

$$U \oplus DW \xrightarrow{\left[\begin{smallmatrix} i & 0 \\ 0 & Dj \end{smallmatrix}\right]} V \oplus DV \xrightarrow{\left[\begin{smallmatrix} 0 & Di \\ \delta_W \circ j & 0 \end{smallmatrix}\right]} DU \oplus D^2W$$

is exact, since if for any $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] : K \to V \oplus DV$ the equality

$$\begin{bmatrix} 0 & Di \\ \delta_W \circ j & 0 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} Di \circ g \\ \delta_W \circ j \circ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

holds, then g factors through $\operatorname{Ker} Di$, $g = Dj \circ \overline{g}$. Therefore f factors through $\operatorname{Ker}(\delta_W \circ j) = \operatorname{Ker} j$, $f = i \circ \bar{f}$, so

$$\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & Dj \end{bmatrix} \begin{bmatrix} \bar{f} \\ \overline{g} \end{bmatrix}.$$

2.4. Witt groups

PROPOSITION 2.6. *Let $\varphi : V \to DV$ be an a.s.b.m. If $i : U \rightarrowtail V$ is totally isotropic and*

$$(U \xrightarrow{i} V) = (U \xrightarrow{\bar{i}} U^{\perp} \xrightarrow{j} V),$$

and if $\kappa : U^{\perp} \to U^{\perp}/U$ is the natural map, then φ induces a unique symmetric bilinear form $\widetilde{\varphi} : U^{\perp}/U \to D(U^{\perp}/U)$ such that

$$D\kappa \circ \widetilde{\varphi} \circ \kappa = Dj \circ \varphi \circ j.$$

Proof. By assumption there is an exact sequence $U^{\perp} \overset{j}{\rightarrowtail} V \xrightarrow{(Di)\circ\varphi} DU$. Hence

$$0 = D(Di \circ \varphi \circ j) = Dj \circ D\varphi \circ D^2 i,$$
$$0 = 0 \circ \delta_U = Dj \circ D\varphi \circ D^2 i \circ \delta_U = Dj \circ D\varphi \circ \delta_V \circ i = Dj \circ \varphi \circ i,$$

and i factors through $\operatorname{Ker}(Dj \circ \varphi)$. Moreover

$$0 = Dj \circ \varphi \circ i = Dj \circ \varphi \circ j \circ \bar{i},$$

so $\bar{i}$ factors through $\operatorname{Ker}(Dj \circ \varphi \circ j)$ and there is an induced map $\overline{\varphi} : U^{\perp}/U \to D(U^{\perp})$ such that

$$\overline{\varphi} \circ \kappa = Dj \circ \varphi \circ j.$$

The commutative diagram

$$\begin{tikzcd}
U \arrow[r, tail, "i"] \arrow[d, tail, "\bar{i}"] & V \arrow[r, "Di\circ\varphi"] \arrow[d, "1"] & DU \\
U^{\perp} \arrow[r, tail, "j"] \arrow[d, two heads, "\kappa"'] & V \arrow[r, two heads, "Dj\circ\varphi"] & D(U^{\perp}) \arrow[u, two heads, "D\bar{i}"'] \\
U^{\perp}/U \arrow[urr, "\overline{\varphi}"'] & & D(U^{\perp}/U) \arrow[u, tail, "D\kappa"']
\end{tikzcd}$$

has exact columns, and the induced map $\overline{\varphi} : U^{\perp}/U \to D(U^{\perp})$ has the property

$$D\bar{i} \circ \overline{\varphi} \circ \kappa = D\bar{i} \circ Dj \circ \varphi \circ j = D(j \circ \bar{i}) \circ \varphi \circ j = Di \circ \varphi \circ j = 0,$$

which yields

$$D\bar{i} \circ \overline{\varphi} = 0,$$

since κ is an epimorphism. It follows that $\overline{\varphi}$ factors through $D\kappa : D(U^{\perp}/U) \to D(U^{\perp})$; let $\widetilde{\varphi} : U^{\perp}/U \to D(U^{\perp}/U)$ be such that $\overline{\varphi} = D\kappa \circ \widetilde{\varphi}$. This induced map $\widetilde{\varphi}$ is the unique map which has the property that

$$D\kappa \circ \widetilde{\varphi} \circ \kappa = Dj \circ \varphi \circ j.$$

since $D\kappa$ is a monomorphism and κ is an epimorphism. It follows that $\widetilde{\varphi}$ is symmetric, since $D\widetilde{\varphi} \circ \delta_{U^{\perp}/U}$ has the same property:

$$\begin{aligned}
D(D\kappa \circ \widetilde{\varphi} \circ \kappa) \circ \delta_{U^{\perp}} &= D(Dj \circ \varphi \circ j) \circ \delta_{U^{\perp}},\\
D\kappa \circ D\widetilde{\varphi} \circ D^2\kappa \circ \delta_{U^{\perp}} &= Dj \circ D\varphi \circ D^2 j \circ \delta_{U^{\perp}},\\
D\kappa \circ D\widetilde{\varphi} \circ \delta_{U^{\perp}/U} \circ \kappa &= Dj \circ D\varphi \circ \delta_V \circ j,\\
D\kappa \circ (D\widetilde{\varphi} \circ \delta_{U^{\perp}/U}) \circ \kappa &= Dj \circ \varphi \circ j,\\
D\widetilde{\varphi} \circ \delta_{U^{\perp}/U} &= \widetilde{\varphi}.
\end{aligned}$$

It is easy to check that $\widetilde{\varphi}$ is an isomorphism. ∎

In the above case (V, φ) and $(U^{\perp}/U, \widetilde{\varphi})$ are said to be *directly Witt equivalent.*

EXAMPLE 2.9. A metabolic form is directly Witt equivalent to $0 = (0, 0)$.

EXAMPLE 2.10. Let $\varphi : V \to DV$ be an a.s.b.m. with a kernel $k : R \rightarrowtail V$. Then $R^{\perp} = V$:

$$\begin{aligned}
\delta_V(R^{\perp}) &= \delta_V(\operatorname{Ker}(Dk \circ \varphi)) = \delta_V(\operatorname{Ker}(Dk \circ D\varphi \circ \delta_V))\\
&= \operatorname{Ker}(Dk \circ D\varphi) = \operatorname{Ker}(D(\varphi \circ k)) = \operatorname{Ker} D(R \xrightarrow{0} DV)\\
&= \operatorname{Ker}(D^2V \xrightarrow{0} DR) = D^2 V
\end{aligned}$$

so $R^{\perp} = V$ and (V, φ) is directly Witt equivalent to $(V/R, \widetilde{\varphi})$.

In general Witt equivalence is the transitive completion of direct Witt equivalence.

DEFINITION 2.11. Two symmetric bilinear forms $\varphi : V \to DV$ and $\psi : U \to DU$ are *Witt equivalent*, $(V, \varphi) \approx (U, \psi)$, if there exist metabolic forms $\chi_1 : M_1 \to DM_1$ and $\chi_2 : M_2 \to DM_2$ such that

$$(V, \varphi) \oplus (M_1, \chi_1) \cong (U, \psi) \oplus (M_2, \chi_2).$$

REMARK 2.2. In the classical algebraic theory of quadratic forms, $(V, \varphi) \cong (W, \psi)$ or $\varphi \cong \psi$ denotes isomorphism, $(V, \varphi) = (W, \psi)$ or $\varphi = \psi$ denotes Witt equivalence, and $\varphi \approx a$ signifies that φ represents a. There is no notion of representing elements in the present categorical context, so we may reserve $=$ for the identity relation and $\approx$ for Witt equivalence.

PROPOSITION 2.7. *Two a.s.b.m.* $\varphi : V \to DV$ *and* $\psi : U \to DU$ *are Witt equivalent,* $(V, \varphi) \approx (U, \psi)$*, iff the space* $(V \oplus U, \varphi \oplus (-\psi))$ *is Witt equivalent to* 0.

Proof. It is obvious that if for some metabolic (W, μ) the form $(V \oplus U \oplus W, \varphi \oplus (-\psi) \oplus \mu)$ is metabolic, then

$$(V, \varphi) \oplus ((U, \psi) \oplus (U, -\psi) \oplus (W, \mu)) \cong (U, \psi) \oplus (V \oplus U \oplus W, \varphi \oplus (-\psi) \oplus \mu)$$

so (V, φ) and (U, ψ) are Witt equivalent.

Conversely, if $(V, \varphi) \approx (U, \psi)$ and $(V, \varphi) \oplus (W, \mu) \cong (U, \psi) \oplus (W', \mu')$ for some metabolic (W, μ), (W', μ'), then the graph of this isomorphism is a Lagrangian in

$$(V, \varphi) \oplus (W, \mu) \oplus (U, -\psi) \oplus (W', -\mu'),$$

so $(V \oplus U, \varphi \oplus (-\psi))$ is Witt equivalent to 0. ■

Even for projective modules over a ring, Witt equivalence of spaces of equal rank is weaker than isomorphism.

EXAMPLE 2.11. For $X = \mathbf{Spec}\,\mathbb{Z}$, the well known lattice Γ_8 generated by all $e_i + e_j$, and $\frac{1}{2}\sum_{i=1}^{8} e_i$ in the Euclidean space $\mathbb{R}^8$ with the usual scalar product defines a rank 8 free abelian group together with a self-dual isomorphism $\beta : \Gamma_8 \to \mathrm{Hom}(\Gamma_8, \mathbb{Z})$ which is not isomorphic to $8 \cdot \langle 1 \rangle = \langle 1, 1, 1, 1, 1, 1, 1, 1 \rangle$ since the integer $\beta(u)(u)$ is even for arbitrary $u \in \Gamma_8$ (see [10, Chapt. 2]). Nevertheless, Γ_8 and $8 \cdot \langle 1 \rangle$ are Witt equivalent, since (Theorem 4.3 of [10, Chapt. 2]) the space $\Gamma_8 \oplus 8 \cdot \langle -1 \rangle$ is hyperbolic.

EXAMPLE 2.12. For $X = \mathbf{Spec}\,\mathbb{R}[x, y]$ it is known that $W(\mathbb{R}) \to W(\mathbb{R}[x, y])$ is an isomorphism (the Karoubi theorem). Parimala [15] produced a sequence of invertible symmetric matrices

$$S_n = \begin{bmatrix} 4 + y^{2n}(1 + x^2) & xy^n(1 + y^{2n}) & 0 & y^n(1 + x^2y^{2n}) \\ xy^n(1 + y^{2n}) & 1 + x^2y^{4n} & -y^n(1 + x^2y^{2n}) & 0 \\ 0 & -y^n(1 + x^2y^{2n}) & 4 + y^{2n}(1 + x^2) & xy^n(1 + y^{2n}) \\ y^n(1 + x^2y^{2n}) & 0 & xy^n(1 + y^{2n}) & 1 + x^2y^{4n} \end{bmatrix}$$

such that over $\mathbb{R}[x, y]$:

- S_0 is congruent to the identity matrix,
- if $m \neq n$, then S_m is not congruent to S_n,
- for $n > 0$ the bilinear space $P_n = (\mathbb{R}[x, y]^4, S_n)$ is not extended from $\mathbb{R}$,
- for $n > 0$ the bilinear space $P_n = (\mathbb{R}[x, y]^4, S_n)$ is indecomposable (so it has no orthogonal base).

Thus for $n > 0$ the bilinear space P_n is not isomorphic to any symmetric bilinear space of dimension four extended from $\mathbb{R}$. On the other hand, over the field $\mathbb{R}(x,y)$ the space P_n is isomorphic to a totally positive space

$$\left\langle 1+x^2y^{4n}, \frac{4+2x^2y^{4n}+y^{2n}+y^{6n}x^4}{1+x^2y^{4n}}, \frac{4+4x^2y^{4n}}{4+2x^2y^{4n}+y^{2n}+y^{6n}x^4}, \frac{16}{4+4x^2y^{4n}}\right\rangle$$

and the map $W(\mathbb{R}) \cong W(\mathbb{R}[x,y]) \to W(\mathbb{R}(x,y))$ is an injection, so over $\mathbb{R}[x,y]$ this form is Witt equivalent to $4 \cdot \langle 1 \rangle$.

The symmetric bilinear space P_n extends from $\mathbf{Spec}\,\mathbb{R}[x,y] = \mathbb{A}^2_{\mathbb{R}}$ to the whole projective plane $\mathbb{P}^2_{\mathbb{R}}$ (Theorem 4.1 of [7]).

Note that a space Witt equivalent to 0 need not be metabolic, even in the exact category of finitely generated projective modules over a commutative ring.

It is easy to see that Witt equivalence is transitive.

Proposition 2.8. *Let $\varphi : V \to DV$ be an a.s.b.m. If $i : U \to V$ is totally isotropic, has a decomposition*

$$(U \xrightarrow{i} V) = (U \xrightarrow{\bar{i}} U^{\perp} \xrightarrow{j} V)$$

and $\kappa : U^{\perp} \twoheadrightarrow U^{\perp}/U$, then (V,φ) and $(U^{\perp}/U, \widetilde{\varphi})$ are Witt equivalent.

Proof. The a.s.b.m. $(V,\varphi) \oplus (U^{\perp}/U, -\widetilde{\varphi})$ is metabolic, since the map

$$\begin{bmatrix} j \\ \kappa \end{bmatrix} : U^{\perp} \to V \oplus U^{\perp}/U$$

coincides with its orthogonal complement: first

$$[Dj \circ \varphi \quad -D\kappa \circ \widetilde{\varphi}] \begin{bmatrix} j \\ \kappa \end{bmatrix} = 0,$$

and secondly $U^{\perp\perp} = U$, so there are exact sequences

$$U^{\perp} \overset{j}{\rightarrowtail} V \xrightarrow{Di\circ\varphi} \!\!\!\!\rightarrow DU, \qquad U \overset{i}{\rightarrowtail} V \xrightarrow{Dj\circ\varphi} \!\!\!\!\rightarrow DU^{\perp}.$$

If $[Dj \circ \varphi \quad -D\kappa \circ \widetilde{\varphi}]\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] = 0$ for a map $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] : K \to V \oplus U^{\perp}/U$, then

$$\begin{aligned} Dj \circ \varphi \circ f - D\kappa \circ \widetilde{\varphi} \circ g &= 0, \\ D\bar{i} \circ Dj \circ \varphi \circ f - D\bar{i} \circ D\kappa \circ \widetilde{\varphi} \circ g &= 0, \\ D(j \circ \bar{i}) \circ \varphi \circ f - D(\kappa \circ \bar{i}) \circ \widetilde{\varphi} \circ g &= 0, \\ Di \circ \varphi \circ f &= 0. \end{aligned}$$

Now, f factors through $(U^{\perp} \xrightarrow{j} V) = \operatorname{Ker}(Di \circ \varphi)$, i.e. $f = j \circ \bar{f}$ and

$$\begin{aligned} Dj \circ \varphi \circ f - D\kappa \circ \widetilde{\varphi} \circ g &= 0, \\ Dj \circ \varphi \circ j \circ \bar{f} - D\kappa \circ \widetilde{\varphi} \circ g &= 0, \\ D\kappa \circ \widetilde{\varphi} \circ \kappa \circ \bar{f} - D\kappa \circ \widetilde{\varphi} \circ g &= 0, \\ D\kappa \circ \widetilde{\varphi} \circ (\kappa \circ \bar{f} - g) &= 0, \\ \widetilde{\varphi} \circ (\kappa \circ \bar{f} - g) &= 0, \\ \kappa \circ \bar{f} - g &= 0, \\ g &= \kappa \circ \bar{f}, \end{aligned}$$

and $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] = \left[\begin{smallmatrix} j\circ\bar{f} \\ \kappa\circ\bar{f} \end{smallmatrix}\right] = \left[\begin{smallmatrix} j \\ \kappa \end{smallmatrix}\right] \circ \bar{f}$. Thus $\left[\begin{smallmatrix} j \\ \kappa \end{smallmatrix}\right] : U^{\perp} \to V \oplus U^{\perp}/U$ is the kernel of $[Dj \circ \varphi \ - D\kappa \circ \widetilde{\varphi}]$, which is an orthogonal complement of $\left[\begin{smallmatrix} j \\ \kappa \end{smallmatrix}\right] : U^{\perp} \to V \oplus U^{\perp}/U$. ∎

Definition 2.12. The *Witt group* $W(\mathfrak{M}, D, \delta)$ of a duality (D, δ) consists of all Witt equivalence classes of symmetric bilinear spaces with the operation $\oplus$ induced by direct sum.

In the case of vector spaces over a field F of characteristic different from 2 every metabolic form is hyperbolic and the Witt ring of the duality of example **1** is the usual Witt ring $W(F)$ of the field F. In the case of example **2**, of dualization $DV = V\hat{}$ of vector bundles, we obtain the usual Witt ring $W(X)$ of a scheme X, introduced by Knebusch [6]. The third example is less known (but not new, see e.g. [3]); it yields the Witt group $W(X, \mathcal{L})$ of (Witt classes of) $\mathcal{L}$-valued symmetric bilinear forms of X. Note that the usual Witt rings of Severi–Brauer varieties are known [16], as also are the Witt groups of $\mathcal{L}$-valued symmetric bilinear forms [17].

2.5. Description of $K_1(\mathfrak{M})$. A *double short exact sequence*, briefly d.s.e.s., is a pair of admissible short exact sequences with the same objects:

$$\begin{pmatrix} C \overset{\beta}{\twoheadleftarrow} B \overset{\alpha}{\hookleftarrow} A \\ C \overset{\delta}{\twoheadleftarrow} B \overset{\gamma}{\hookleftarrow} A \end{pmatrix}.$$

We will abbreviate the above notation to

$$(*) \qquad C \underset{\delta}{\overset{\beta}{\twoheadleftarrow}} B \underset{\gamma}{\overset{\alpha}{\hookleftarrow}} A$$

and refer to $C \overset{\beta}{\twoheadleftarrow} B \overset{\alpha}{\hookleftarrow} A$ as the upper short exact sequence, and to $C \overset{\delta}{\twoheadleftarrow} B \overset{\gamma}{\hookleftarrow} A$ as the lower short exact sequence of the d.s.e.s. $(*)$. We use a quite unusual convention: if the sequence is depicted vertically, with arrows going from top to bottom, then the upper exact sequence is to the left of the arrow, as if the arrow were rotated like a rigid body together with its upper and lower sides.

The d.s.e.s. $(*)$ defines a path in the K-theory space of the category $\mathfrak{M}$, e.g. in the G-construction of $\mathfrak{M}$ from (A, A) to (B, B) and, together with the d.s.e.s.'s

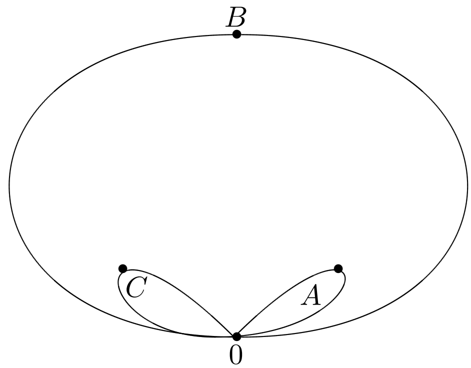
$$A \overset{1_A}{\underset{1_A}{\twoheadleftarrow}} A \leftarrowtail 0, \qquad B \overset{1_B}{\underset{1_B}{\twoheadleftarrow}} B \leftarrowtail 0$$

a loop $\ell = \ell(\alpha, \gamma; \beta, \delta)$,

$$\ell : \begin{tikzcd} (A,A) \arrow[rr, "(*)"] & & (B,B) \\ & (0,0) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

It is easy to realize an element of $K_1(\mathfrak{M}) = \pi_2(B(Q\mathfrak{M}), 0)$ produced by a double exact sequence: each of the exact sequences in the pair produces a complex of dimension 2 of parachute form, which appears when one glues together three 0 vertices of the pentagonal diagram of Proposition 2.2. So one may put one parachute inside the other and glue together equal paths on the border. The result for the double exact sequence $A \overset{\alpha}{\underset{\alpha'}{\rightrightarrows}} B \overset{\beta}{\underset{\beta'}{\rightrightarrows}} C$ is topologically equivalent to the "pretzel" surface $\left|A \overset{\alpha}{\underset{\alpha'}{\rightrightarrows}} B \overset{\beta}{\underset{\beta'}{\rightrightarrows}} C\right|$ with two holes as in the figure:

(2.2)



One obtains an element of $\pi_2(B(Q\mathfrak{M}), 0)$ putting a baloon inside this surface and inflating it. More formally this surface results from an octagon with sides oriented as the border of the diagram

(2.3)
$$\begin{tikzcd}
0 \arrow[r, "0^!_A"] \arrow[d, "0^C_!"'] \arrow[dr, "0^B_\alpha"'] & A \arrow[d, "\alpha_!"] & 0 \arrow[l, "0^A_!"'] \arrow[d, "0^A_!"] \arrow[dl, "0^B_!"] \\
C \arrow[r, "\beta^!"] & B & A \arrow[l, "\alpha'_!"'] \\
0 \arrow[r, "0^!_C"'] \arrow[u, "0^!_C"] \arrow[ur, "0^!_B"] & C \arrow[u, "\beta'^!"] & 0 \arrow[l, "0^C_!"] \arrow[u, "0^!_A"'] \arrow[ul, "0^B_{\alpha'}"']
\end{tikzcd}$$

THEOREM 2.9. *$K_1(\mathfrak{M})$ may be described as follows:*

(a) *Every element of $K_1(\mathfrak{M})$ is represented by the loop $\ell = \ell(\alpha, \gamma; \beta, \delta)$ of a d.s.e.s.*

(b) *$K_1(\mathfrak{M})$ is an abelian group generated by all d.s.e.s. in $\mathfrak{M}$, subject to the following relations:*

  (i) *the class of (the loop of) the d.s.e.s. with equal upper and lower short exact sequences is zero;*

  (ii) (3 × 3 lemma) *for any diagram of six d.s.e.s.'s*

$$\begin{tikzcd}
A'' & A \arrow[l, two heads, "b", "b'"'] & A' \arrow[l, tail, "a", "a'"'] \\
B'' \arrow[from=u, tail, "i"', "i'"] & B \arrow[l, two heads, "d", "d'"'] \arrow[from=u, tail, "h"', "h'"] & B' \arrow[l, tail, "c", "c'"'] \arrow[from=u, tail, "g"', "g'"] \\
C'' \arrow[from=u, two heads, "l"', "l'"] & C \arrow[l, two heads, "f", "f'"'] \arrow[from=u, two heads, "k"', "k'"] & C' \arrow[l, tail, "e", "e'"'] \arrow[from=u, two heads, "j"', "j'"]
\end{tikzcd}$$

  such that the diagram of the upper short exact sequences commutes, and the diagram of the lower exact sequences commutes, the alternate sums of rows and columns coincide:

$$\begin{aligned}
&\ell(a, a'; b, b') - \ell(c, c'; d, d') + \ell(e, e'; f, f') \\
&\qquad\qquad = \ell(g, g'; j, j') - \ell(h, h'; k, k') + \ell(i, i'; l, l').
\end{aligned}$$

Proof. (a): [11, Theorem 2.1]; (b): [12, Theorem]. ∎

Given an object A of $\mathfrak{M}$ and $\alpha \in \mathrm{Aut}(A)$ we put

$$\ell(\alpha) = \ell(0 \to A, 0 \to A; A \xrightarrow{1} A, A \xrightarrow{\alpha} A).$$

REMARK 2.3. In fact, a pair $\alpha, \alpha' \in \mathrm{Aut}(A)$ gives rise to two double short exact sequences:

(2.4)
$$A \underset{\alpha'}{\overset{\alpha}{\twoheadleftarrow}} A \leftarrowtail 0$$

and

$$0 \twoheadleftarrow A \overset{\alpha}{\underset{\alpha'}{\leftarrowtail}} A \tag{2.5}$$

whose classes are opposite to each other in $K_1(\mathfrak{M})$. There is a choice of signs: we choose the class of $(\alpha')^{-1}\alpha$ in $K_1(\mathfrak{M})$ to be given by (2.5).

LEMMA 2.10 ([12, Lemma 3.2]). *Given an object A of $\mathfrak{M}$,*

(i) *the class $\ell(\alpha)$ of the automorphism $\alpha = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] \in \mathrm{Aut}(A \oplus A)$ in $K_1(\mathfrak{M})$ vanishes;*
(ii) *the class of the d.s.e.s. of the form*

$$A \overset{[0,1]}{\underset{[-1,0]}{\twoheadleftarrow}} A \oplus A \overset{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]}{\underset{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]}{\leftarrowtail}} A$$

vanishes in $K_1(\mathfrak{M})$. ∎

We define the action of the dualization functor D on $K_1(\mathfrak{M})$ by

$$\left|A \overset{\alpha}{\underset{\alpha'}{\rightarrowtail}} B \overset{\beta}{\underset{\beta'}{\twoheadrightarrow}} C\right| = \left|DC \overset{D\beta'}{\underset{D\beta}{\rightarrowtail}} DB \overset{D\alpha'}{\underset{D\alpha}{\twoheadrightarrow}} DA\right|.$$

2.6. Nenashev's K-theory space $T(\mathfrak{M})$. It is difficult to do computations with spheroids, so we follow the idea of Nenashev: we use a bisimplicial space, homotopy equivalent to $\Omega B(Q\mathfrak{M})$. Since it is vital to this investigation, we restate here the notions and results of [13, Sect. 2.4], and an unpublished result of Nenashev, the 3×4 lemma.

The definition of the loop corresponding to a four-term double exact sequence may also be stated in terms of a self-dual K-theory space $T(\mathfrak{M})$ introduced by A. Nenashev as a bisimplicial set, a mixture of $G(\mathfrak{M})$ and $G(\mathfrak{M}^{\mathrm{op}})$: a $(p,0)$-simplex of $T(\mathfrak{M})$ is a p-simplex of $G(\mathfrak{M})$ and a $(0,q)$-simplex of $T(\mathfrak{M})$ is a q-simplex of $G(\mathfrak{M}^{\mathrm{op}})$. Both the embeddings $G(\mathfrak{M}) \to T(\mathfrak{M})$ and $G(\mathfrak{M}^{\mathrm{op}}) \to T(\mathfrak{M})$ are homotopy equivalences. More precisely $T(\mathfrak{M})$ is the result of the Nenashev mapping cone construction $C^{(1,-1)}$ applied to the square

$$\begin{tikzcd} \mathcal{M} & \mathcal{M} \arrow[l, "\mathrm{Id}"'] \arrow[d, "\mathrm{Id}"] \\ \mathcal{M} \times \mathcal{M} \arrow[from=1-1, "\mathrm{diag}"'] & \mathcal{M} \arrow[l, "\mathrm{diag}"'] \end{tikzcd}$$

(see [13, Sect. 2.4]).

Let $\mathfrak{M}$ be an exact category. Then $T(\mathfrak{M})$ is the following bisimplicial set.

A (p,q)-simplex is given by five families of objects:

$$A_{i,k},\ A'_{i,k},\ A_{j/i,k},\ A_{i,l/k},\ A_{j/i,l/k}$$

for $i, j = 0, 1, \ldots, p$, $i < j$, $k, l = 0, 1, \ldots, q$, $k < l$, and eight families of admissible short exact sequences:

$$
\begin{array}{rr}
A_{i,k} \rightarrowtail A_{j,k} \twoheadrightarrow A_{j/i,k}, & A_{i,l/k} \rightarrowtail A_{i,k} \twoheadrightarrow A_{i,l}, \\
A'_{i,k} \rightarrowtail A'_{j,k} \twoheadrightarrow A_{j/i,k}, & A_{i,l/k} \rightarrowtail A'_{i,k} \twoheadrightarrow A'_{i,l}, \\
A_{i,l/k} \rightarrowtail A_{j,k} \twoheadrightarrow A_{i,l}, & A_{i,l/k} \rightarrowtail A'_{j,k} \twoheadrightarrow A'_{i,l}, \\
A_{i,l/k} \rightarrowtail A_{j,l/k} \twoheadrightarrow A_{j/i,l/k}, & A_{j/i,l/k} \rightarrowtail A_{j/i,k} \twoheadrightarrow A_{j/i,l},
\end{array}
$$

such that all diagrams

$$
\begin{tikzcd}
A_{i,l/k} \arrow[r, tail] \arrow[d, tail] & A_{j,l/k} \arrow[r, two heads] \arrow[d, tail] & A_{j/i,l/k} \arrow[d, tail] \\
A_{i,k} \arrow[r, tail] \arrow[d, two heads] & A_{j,k} \arrow[r, two heads] \arrow[d, two heads] & A_{j/i,k} \arrow[d, two heads] \\
A_{i,l} \arrow[r, tail] & A_{j,l} \arrow[r, two heads] & A_{j/i,l}
\end{tikzcd}
\qquad
\begin{tikzcd}
A_{i,l/k} \arrow[r, tail] \arrow[d, tail] & A_{j,l/k} \arrow[r, two heads] \arrow[d, tail] & A_{j/i,l/k} \arrow[d, tail] \\
A'_{i,k} \arrow[r, tail] \arrow[d, two heads] & A'_{j,k} \arrow[r, two heads] \arrow[d, two heads] & A_{j/i,k} \arrow[d, two heads] \\
A'_{i,l} \arrow[r, tail] & A'_{j,l} \arrow[r, two heads] & A_{j/i,l}
\end{tikzcd}
$$

commute.

In other words a (p, q)-simplex is a pair of admissible filtrations-cofiltrations

$$
\left(
\begin{tikzcd}
A_{0,0} \arrow[r, tail] \arrow[d, two heads] & A_{1,0} \arrow[r, tail] \arrow[d, two heads] & A_{2,0} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & A_{p,0} \arrow[d, two heads] \\
A_{0,1} \arrow[r, tail] \arrow[d, two heads] & A_{1,1} \arrow[r, tail] \arrow[d, two heads] & A_{2,1} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & A_{p,1} \arrow[d, two heads] \\
\vdots \arrow[d, two heads] & \vdots \arrow[d, two heads] & \vdots \arrow[d, two heads] & & \vdots \arrow[d, two heads] \\
A_{0,q} \arrow[r, tail] & A_{1,q} \arrow[r, tail] & A_{2,q} \arrow[r, tail] & \cdots \arrow[r, tail] & A_{p,q}
\end{tikzcd}
\quad
\begin{tikzcd}
A'_{0,0} \arrow[r, tail] \arrow[d, two heads] & A'_{1,0} \arrow[r, tail] \arrow[d, two heads] & A'_{2,0} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & A'_{p,0} \arrow[d, two heads] \\
A'_{0,1} \arrow[r, tail] \arrow[d, two heads] & A'_{1,1} \arrow[r, tail] \arrow[d, two heads] & A'_{2,1} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & A'_{p,1} \arrow[d, two heads] \\
\vdots \arrow[d, two heads] & \vdots \arrow[d, two heads] & \vdots \arrow[d, two heads] & & \vdots \arrow[d, two heads] \\
A'_{0,q} \arrow[r, tail] & A'_{1,q} \arrow[r, tail] & A'_{2,q} \arrow[r, tail] & \cdots \arrow[r, tail] & A'_{p,q}
\end{tikzcd}
\right)
\tag{2.6}
$$

with fixed common: subquotients $A_{j/i,k}$, subkernels $A_{i,l/k}$, induced cofiltrations of subquotients and induced filtrations of subkernels.

The degeneracy maps are defined by duplicating a row or a column in both diagrams (2.6) and reindexing. The boundaries are defined by deleting a row or a column in both diagrams of (2.6) and reindexing.

Obviously all $(p, 0)$-simplexes for $p = 0, 1, 2, \ldots$ form a simplicial subset isomorphic to $G(\mathfrak{M})$ and all $(0, p)$-simplexes for $p = 0, 1, 2, \ldots$ form a simplicial subset isomorphic to $G(\mathfrak{M}^{\mathrm{op}})$. Moreover, Nenashev proved the following theorem.

THEOREM 2.11 ([13, Theorem 2.5]). *The embeddings*

$$G(\mathfrak{M}) = T_{\cdot,0}(\mathfrak{M}) \hookrightarrow T(\mathfrak{M})$$

and

$$G(\mathfrak{M}^{\mathrm{op}}) = T_{0,\cdot}(\mathfrak{M}) \hookrightarrow T(\mathfrak{M})$$

are homotopy equivalences. ■

Thus the geometric realization of $T(\mathfrak{M})$ is a K-theory space:

$$K_n(\mathfrak{M}) = \pi_n(T(\mathfrak{M})).$$

The main advantage of this particular K-theory space is its self-duality. Namely, for any bisimplicial set X let $\overline{X}$ be the bisimplicial set

$$\overline{X}_{m,n} = X_{n,m}.$$

There is a canonical homeomorphism of the geometric realizations

$$\phi_X : |\overline{X}| \to |X|$$

which takes $x \times \Delta^m \times \Delta^n$ to $x \times \Delta^n \times \Delta^m$ for any $x \in X_{n,m}$. The set $T(\mathfrak{M})$ is self-dual in the sense that $T(\mathfrak{M}^{\mathrm{op}}) = \overline{T(\mathfrak{M})}$, which yields a homeomorphism

$$|T(\mathfrak{M}^{\mathrm{op}})| = |\overline{T(\mathfrak{M})}| \to |T(\mathfrak{M})|.$$

A duality functor $D : \mathfrak{M} \to \mathfrak{M}^{\mathrm{op}}$ induces a bisimplicial map

$$T(\mathfrak{M}) \to T(\mathfrak{M}^{\mathrm{op}}) = \overline{T(\mathfrak{M})}$$

which maps a (p, q)-simplex (2.6) onto the (q, p)-simplex

$$
\left(
\begin{tikzcd}
DA_{0,0} \arrow[r, tail] \arrow[d, two heads] & DA_{0,1} \arrow[r, tail] \arrow[d, two heads] & DA_{0,2} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & DA_{0,q} \arrow[d, two heads] \\
DA_{1,0} \arrow[r, tail] \arrow[d, two heads] & DA_{1,1} \arrow[r, tail] \arrow[d, two heads] & DA_{1,2} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & DA_{1,q} \arrow[d, two heads] \\
\vdots \arrow[d, two heads] & \vdots \arrow[d, two heads] & \vdots \arrow[d, two heads] & & \vdots \arrow[d, two heads] \\
DA_{p,0} \arrow[r, tail] & DA_{p,1} \arrow[r, tail] & DA_{p,2} \arrow[r, tail] & \cdots \arrow[r, tail] & DA_{p,q}
\end{tikzcd}
\quad
\begin{tikzcd}
DA'_{0,0} \arrow[r, tail] \arrow[d, two heads] & DA'_{0,1} \arrow[r, tail] \arrow[d, two heads] & DA'_{0,2} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & DA'_{0,q} \arrow[d, two heads] \\
DA'_{1,0} \arrow[r, tail] \arrow[d, two heads] & DA'_{1,1} \arrow[r, tail] \arrow[d, two heads] & DA'_{1,2} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & DA'_{1,q} \arrow[d, two heads] \\
\vdots \arrow[d, two heads] & \vdots \arrow[d, two heads] & \vdots \arrow[d, two heads] & & \vdots \arrow[d, two heads] \\
DA'_{p,0} \arrow[r, tail] & DA'_{p,1} \arrow[r, tail] & DA'_{p,2} \arrow[r, tail] & \cdots \arrow[r, tail] & DA'_{p,q}
\end{tikzcd}
\right)
$$

with subquotients $DA_{i,l/k}$ and subkernels $DA_{j/i,k}$. Then $G(\mathfrak{M}) = T_{\cdot,0}(\mathfrak{M})$ is mapped onto $G(\mathfrak{M}^{\mathrm{op}}) = T_{0,\cdot}(\mathfrak{M})$.

There is a commutative diagram of spaces

$$
\begin{tikzcd}
{|T(\mathfrak{M})|} \arrow[r, "D"] & {|\overline{T(\mathfrak{M})}|} \arrow[r, "\phi_{T(\mathfrak{M})}"] & {|T(\mathfrak{M})|} \\
{|G(\mathfrak{M})|} \arrow[u] \arrow[rr, "D"] & & {|G(\mathfrak{M}^{\mathrm{op}})|} \arrow[u]
\end{tikzcd}
$$

A vertex (i.e. a $(0,0)$-simplex) of $T(\mathfrak{M})$ is a pair of objects (P, P') of $\mathfrak{M}$. Thus, given a pair of objects, think of it as a vertex in the triangulation of the geometric realization $|T(\mathfrak{M})|$. We indicate this by writing $\overset{(P,P')}{\bullet}$.

Given two vertices (P_0, P_0') and (P_1, P_1'), there may generally be edges of two types connecting (P_0, P_0') to (P_1, P_1'), namely $(1,0)$- and $(0,1)$-simplices; imagine something like

$$
\overset{(P_0,P_0')}{\bullet} \longrightarrow \overset{(P_1,P_1')}{\bullet}
$$

in both cases. The $(1,0)$-edges are in one-to-one correspondence with the pairs of short exact sequences of the form

$$(P_0 \rightarrowtail P_1 \twoheadrightarrow P_{1/0},\ P'_0 \rightarrowtail P'_1 \twoheadrightarrow P_{1/0}), \tag{2.7}$$

and it is essential that the cokernel objects in them are identical. The $(1,0)$-simplices in $T(\mathfrak{M})$ are therefore the same as the edges in the G-construction. In fact, the $(-,0)$-part of $T(\mathfrak{M})$ is isomorphic to $G(\mathfrak{M})$, which gives an embedding $G(\mathfrak{M}) \subset T(\mathfrak{M})$.

The $(0,1)$-simplices connecting (P_0, P'_0) to (P_1, P'_1) are given by pairs of short exact sequences

$$(P_0 \twoheadleftarrow P_1 \leftarrowtail P_{0\backslash 1},\ P'_0 \twoheadleftarrow P'_1 \leftarrowtail P_{0\backslash 1}) \tag{2.8}$$

in $\mathfrak{M}$ with identical kernel objects. This is the same as the edges in $G(\mathfrak{M}^{\mathrm{op}})$, and in fact, $G(\mathfrak{M}^{\mathrm{op}})$ embeds into $T(\mathfrak{M})$ as its $(0,-)$-part.

A $(1,1)$-simplex (a cell of square shape in the geometric realization) is defined by a pair of diagrams

$$\begin{tikzcd}
A_{0,1/0} \arrow[r, tail] \arrow[d, tail] & A_{1,1/0} \arrow[r, two heads] \arrow[d, tail] & A_{1/0,1/0} \arrow[d, tail] \\
A_{0,1} \arrow[r, tail] \arrow[d, two heads] & A_{1,1} \arrow[r, two heads] \arrow[d, two heads] & A_{1/0,1} \arrow[d, two heads] \\
A_{0,0} \arrow[r, tail] & A_{1,0} \arrow[r, two heads] & A_{1/0,0}
\end{tikzcd}
\qquad
\begin{tikzcd}
A_{0,1/0} \arrow[r, tail] \arrow[d, tail] & A_{1,1/0} \arrow[r, two heads] \arrow[d, tail] & A_{1/0,1/0} \arrow[d, tail] \\
A'_{0,1} \arrow[r, tail] \arrow[d, two heads] & A'_{1,1} \arrow[r, two heads] \arrow[d, two heads] & A_{1/0,1} \arrow[d, two heads] \\
A'_{0,0} \arrow[r, tail] & A'_{1,0} \arrow[r, two heads] & A_{1/0,0}
\end{tikzcd} \tag{2.9}$$

with identical upper horizontal and rightmost vertical short exact sequences. The vertices of this "square" are the pairs $(A_{i,j}, A'_{i,j})$ with $i, j \in \{0,1\}$. Its four edges are given by the corresponding pairs of short exact sequences in these diagrams.

The map $D : |T(\mathfrak{M})| \to |T(\mathfrak{M})|$ takes a vertex (P, P') to (DP, DP'). If a $(1,0)$-edge e from (P_0, P'_0) to (P_1, P'_1) is given by (2.7), then

$$De = (DP_0 \twoheadleftarrow DP_1 \leftarrowtail DP_{1/0},\ DP'_0 \twoheadleftarrow DP'_1 \leftarrowtail DP_{1/0}),$$

which is a $(0,1)$-simplex.

Recall that by Theorem 2.9 given a double short exact sequence $u = \big(C \underset{\delta}{\overset{\beta}{\twoheadleftarrow}} B \underset{\gamma}{\overset{\alpha}{\leftarrowtail}} A\big)$, we may associate to it a 3-edge loop

$$\ell(u) = \Big(\begin{tikzcd} \underset{(0,0)}{\bullet} \arrow[r, tail, "e(A)"] & \underset{(A,A)}{\bullet} \arrow[r, tail, "e(u)"] & \underset{(B,B)}{\bullet} & \underset{(0,0)}{\bullet} \arrow[l, tail, "e(B)"'] \end{tikzcd} \Big)$$

in $G(\mathfrak{M}) = T_{\cdot,0}(\mathfrak{M})$, where $e(A) = \big(0 \to A \xrightarrow{1} A,\ 0 \to A \xrightarrow{1} A\big)$ is the canonical edge from the base point $(0,0)$ to (A,A) (similarly for B), and $e(u)$ is the $(1,0)$-simplex given by u.

The map $D : |T(\mathfrak{M})| \to |T(\mathfrak{M})|$ takes the loop $\ell(u)$ to the loop

$$D\ell(u) = \left(\begin{tikzcd} \overset{(0,0)}{\bullet} \arrow[r, dashed, "De(A)"] & \overset{(DA,DA)}{\bullet} \arrow[r, dashed, "De(u)"] & \overset{(DB,DB)}{\bullet} & \overset{(0,0)}{\bullet} \arrow[l, dashed, "De(B)"'] \end{tikzcd} \right)$$

in $G(\mathfrak{M}^{\mathrm{op}}) = T_{0,\cdot}(\mathfrak{M})$, where $De(A) = \big(0 \overset{0}{\leftarrow} DA \overset{1}{\leftarrow} DA,\ 0 \overset{0}{\leftarrow} DA \overset{1}{\leftarrow} DA\big)$ (the same for B), and

$$De(u) = \big(DA \xleftarrow{D\alpha} DB \xleftarrow{D\beta} DC,\ DA \xleftarrow{D\gamma} DB \xleftarrow{D\delta} DC\big).$$

There is also the dual d.s.e.s.

$$Du = \big(DA \underset{D\gamma}{\overset{D\alpha}{\longleftarrow}} DB \underset{D\delta}{\overset{D\beta}{\longleftarrow}} DC\big)$$

and its loop

$$\ell(Du) = \left(\begin{tikzcd} \overset{(0,0)}{\bullet} \arrow[r, "e(DC)"] & \overset{(DC,DC)}{\bullet} \arrow[r, "e(Du)"] & \overset{(DB,DB)}{\bullet} & \overset{(0,0)}{\bullet} \arrow[l, "e(DB)"'] \end{tikzcd} \right)$$

We restate here Proposition 3.1, Corollary 3.2 and Lemma 3.3 of [13]:

PROPOSITION 2.12. *$D\ell(u)$ is homotopic to $\ell(Du)$ in $T(\mathfrak{M})$.*

COROLLARY 2.13. *If $m(u)$ is the class of $\ell(u)$ in $K_1(\mathfrak{M})$, then $Dm(u) = m(Du)$.*

LEMMA 2.14. *For any object X in $\mathfrak{M}$ put $e^{\mathrm{op}}(X) = \big(0 \overset{0}{\leftarrow} X \overset{1}{\leftarrow} X,\ 0 \overset{0}{\leftarrow} X \overset{1}{\leftarrow} X\big) \in T_{0,1}(\mathfrak{M})$. Then the loop*

$$\left(\begin{tikzcd} \overset{(0,0)}{\bullet} \arrow[r, "e(X)"] & \overset{(X,X)}{\bullet} & \overset{(0,0)}{\bullet} \arrow[l, "e^{\mathrm{op}}(X)"] \end{tikzcd} \right)$$

in $T(\mathfrak{M})$ is contractible.

Proof. A contracting 2-cell is given by the $(1,1)$-simplex

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[d, tail] & X \arrow[r, two heads, "1"] \arrow[d, tail, "1"'] & X \arrow[d, tail, "1"] \\ 0 \arrow[r, tail] \arrow[d, two heads] & X \arrow[r, two heads, "1"] \arrow[d, two heads] & X \arrow[d, two heads] \\ 0 \arrow[r, tail] & 0 \arrow[r, two heads] & 0 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, tail] \arrow[d, tail] & X \arrow[r, two heads, "1"] \arrow[d, tail, "1"'] & X \arrow[d, tail, "1"] \\ 0 \arrow[r, tail] \arrow[d, two heads] & X \arrow[r, two heads, "1"] \arrow[d, two heads] & X \arrow[d, two heads] \\ 0 \arrow[r, tail] & 0 \arrow[r, two heads] & 0 \end{tikzcd} \quad \blacksquare$$

Proof of Proposition 2.12. By Lemma 2.14, we can replace the edges $De(A) = e^{\mathrm{op}}(DA)$ and $De(B) = e^{\mathrm{op}}(DB)$ in $D\ell(u)$ by $e(DA)$ and $e(DB)$ respectively, and the edge $e(DC)$ in $\ell(Du)$ by $e^{\mathrm{op}}(DC)$. It then follows that

$D\ell(u)\ell(Du)^{-1}$ is homotopic to the contour of the $(1,1)$-simplex

$$\begin{tikzcd}
DC \arrow[r, tail, "1"] \arrow[d, tail, "1"'] & DC \arrow[r, two heads] \arrow[d, tail, "D\beta"'] & 0 \arrow[d, tail] \\
DC \arrow[r, tail, "D\beta"] \arrow[d, two heads] & DB \arrow[r, tail, "D\alpha"] \arrow[d, two heads, "D\alpha"'] & DA \arrow[d, two heads, "1"'] \\
0 \arrow[r, tail] & DA \arrow[r, two heads, "1"'] & DA
\end{tikzcd}
\qquad
\begin{tikzcd}
DC \arrow[r, tail, "1"] \arrow[d, tail, "1"'] & DC \arrow[r, two heads] \arrow[d, tail, "D\delta"'] & 0 \arrow[d, tail] \\
DC \arrow[r, tail, "D\delta"] \arrow[d, two heads] & DB \arrow[r, tail, "D\gamma"] \arrow[d, two heads, "D\gamma"'] & DA \arrow[d, two heads, "1"'] \\
0 \arrow[r, tail] & DA \arrow[r, two heads, "1"'] & DA
\end{tikzcd}
\qquad \blacksquare$$

Definition 2.13. Given a four-term double exact sequence $D \overset{c_1}{\underset{c_2}{\twoheadleftarrow}} C \overset{b_1}{\underset{b_2}{\longleftarrow}} B \overset{a_1}{\underset{a_2}{\leftarrowtail}} A$, let $B \overset{u_1}{\twoheadrightarrow} X_1$ be $\operatorname{Coker} a_1$, $B \overset{u_2}{\twoheadrightarrow} X_2$ be $\operatorname{Coker} a_2$, $X_1 \overset{v_1}{\rightarrowtail} C$ be $\operatorname{Ker} c_1$ and $X_2 \overset{v_2}{\rightarrowtail} C$ be $\operatorname{Ker} c_2$. The corresponding element $\ell\big(D \overset{c_1}{\underset{c_2}{\twoheadleftarrow}} C \overset{b_1}{\underset{b_2}{\longleftarrow}} B \overset{a_1}{\underset{a_2}{\leftarrowtail}} A\big)$ of $K_1(\mathfrak{M})$ is the class of the loop

$$\begin{tikzcd}
 & (B,B) & \\
(0,0) \arrow[ur] \arrow[dr] & & (X_1,X_2) \arrow[ul, dashed] \\
 & (C,C) \arrow[ur] &
\end{tikzcd}$$

consisting of the paths:

- $e(B)$ from $(0,0)$ to (B,B) given by the $(1,0)$-simplex $B \overset{1}{\underset{1}{\twoheadleftarrow}} B \leftarrowtail 0$;
- from (X_1,X_2) to (B,B) given by the $(0,1)$-simplex
$$\begin{pmatrix} X_1 \overset{u_1}{\twoheadleftarrow} B \overset{a_1}{\leftarrowtail} A \\ X_2 \overset{u_2}{\twoheadleftarrow} B \overset{a_2}{\leftarrowtail} A \end{pmatrix};$$
- from (C,C) to (X_1,X_2) given by the $(1,0)$-simplex
$$\begin{pmatrix} D \overset{c_1}{\twoheadleftarrow} C \overset{v_1}{\leftarrowtail} X_1 \\ D \overset{c_2}{\twoheadleftarrow} C \overset{v_2}{\leftarrowtail} X_2 \end{pmatrix};$$
- $e(C)$ from $(0,0)$ to (C,C) given by the $(1,0)$-simplex $C \overset{1}{\underset{1}{\twoheadleftarrow}} C \leftarrowtail 0$.

Remark 2.4. There is another choice of signs: given a double short exact sequence $(*)$,

$$\ell(\alpha,\gamma;\beta,\delta) = \ell\Big(C \overset{\beta}{\underset{\delta}{\twoheadleftarrow}} B \overset{\alpha}{\underset{\gamma}{\leftarrowtail}} A \leftarrowtail 0\Big)$$

and $\ell(\alpha,\gamma;\beta,\delta)$ is the negative of $\ell\big(0 \twoheadleftarrow C \overset{\beta}{\underset{\delta}{\twoheadleftarrow}} B \overset{\alpha}{\underset{\gamma}{\leftarrowtail}} A\big)$.

Here are some standard ways to replace a pair of adjoint edges in a combinatorial path by another pair of edges and get a homotopic path as a result.

- Suppose we are given a pair of $(1,0)$-simplices sharing the source:

$$\begin{tikzcd}(Q,Q') & (P,P') \arrow[l, "e_1"'] \arrow[r, "e_2"] & (R,R')\end{tikzcd} \tag{2.10}$$

Then e_1 and e_2 are given by data of the form

$$e_1 = (P \rightarrowtail Q \twoheadrightarrow M,\ P' \rightarrowtail Q' \twoheadrightarrow M),$$
$$e_2 = (P \rightarrowtail R \twoheadrightarrow N,\ P' \rightarrowtail R' \twoheadrightarrow N).$$

Following [4] we choose push out objects $S = Q \amalg_P R$, $S' = Q' \amalg_{P'} R'$ and consider the two $(2,0)$-simplices

$$\begin{tikzcd}(Q,Q') \arrow[r, "\tilde{e}_2"] \arrow[dr, phantom, "t_1"] & (S,S') \\ (P,P') \arrow[u, "e_1"] \arrow[r, "e_2"'] \arrow[ur] \arrow[ur, phantom, "t_2"'] & (R,R') \arrow[u, "\tilde{e}_1"']\end{tikzcd}$$

given by the diagrams

$$t_1 = \left(\begin{tikzcd}[ampersand replacement=\&] \& \& N \\ \& M \arrow[r, tail] \& M \oplus N \arrow[u, two heads] \\ P \arrow[r, tail] \& Q \arrow[u, two heads] \arrow[r, tail] \& S \arrow[u, two heads]\end{tikzcd} \quad \begin{tikzcd}[ampersand replacement=\&] \& \& N \\ \& M \arrow[r, tail] \& M \oplus N \arrow[u, two heads] \\ P' \arrow[r, tail] \& Q' \arrow[u, two heads] \arrow[r, tail] \& S' \arrow[u, two heads]\end{tikzcd}\right),$$

$$t_2 = \left(\begin{tikzcd}[ampersand replacement=\&] \& \& N \\ \& M \arrow[r, tail] \& M \oplus N \arrow[u, two heads] \\ P \arrow[r, tail] \& R \arrow[u, two heads] \arrow[r, tail] \& S \arrow[u, two heads]\end{tikzcd} \quad \begin{tikzcd}[ampersand replacement=\&] \& \& N \\ \& M \arrow[r, tail] \& M \oplus N \arrow[u, two heads] \\ P' \arrow[r, tail] \& R' \arrow[u, two heads] \arrow[r, tail] \& S' \arrow[u, two heads]\end{tikzcd}\right).$$

This enables us to replace the 2-edge path (2.10) by the homotopic path

$$\begin{tikzcd}(Q,Q') \arrow[r, "\tilde{e}_2"] & (S,S') & (R,R') \arrow[l, "\tilde{e}_1"']\end{tikzcd}$$

Note that everything here is happening in the $G(\mathfrak{M})$-part of $T(\mathcal{M})$.

• Given a pair of $(0,1)$-simplices with a common source

(2.11)
$$\begin{tikzcd}[column sep=large]
\underset{(Q,Q')}{\bullet} & \underset{(P,P')}{\bullet} \arrow[l, dashed, two heads, "e_1"'] \arrow[r, dashed, two heads, "e_2"] & \underset{(R,R')}{\bullet}
\end{tikzcd}$$

where

$$e_1 = (P \twoheadleftarrow Q \leftarrowtail M,\ P' \twoheadleftarrow Q' \leftarrowtail M'),$$
$$e_2 = (P \twoheadleftarrow Q \leftarrowtail N,\ P' \twoheadleftarrow Q' \leftarrowtail N'),$$

take the pullbacks $S = Q \times_P R,\ S' = Q' \times_{P'} R'$ and form the two $(0,2)$-simplices

$$t_1 = \left(\begin{tikzcd}
& & N \arrow[d, tail] \\
& M \arrow[d, tail] & M \oplus N \arrow[l, two heads] \arrow[d, tail] \\
P & Q \arrow[l, two heads] & S \arrow[l, two heads]
\end{tikzcd}\qquad
\begin{tikzcd}
& & N \arrow[d, tail] \\
& M \arrow[d, tail] & M \oplus N \arrow[l, two heads] \arrow[d, tail] \\
P' & Q' \arrow[l, two heads] & S' \arrow[l, two heads]
\end{tikzcd}\right),$$

$$t_2 = \left(\begin{tikzcd}
& & N \arrow[d, tail] \\
& M \arrow[d, tail] & M \oplus N \arrow[l, two heads] \arrow[d, tail] \\
P & R \arrow[l, two heads] & S \arrow[l, two heads]
\end{tikzcd}\qquad
\begin{tikzcd}
& & N \arrow[d, tail] \\
& M \arrow[d, tail] & M \oplus N \arrow[l, two heads] \arrow[d, tail] \\
P' & R' \arrow[l, two heads] & S' \arrow[l, two heads]
\end{tikzcd}\right).$$

The whole picture has the form

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\overset{(Q,Q')}{\bullet} \arrow[r, dashed, two heads, "\tilde{e}_2"] \arrow[dr, phantom, "t_1" near start] & \overset{(S,S')}{\bullet} \arrow[dl, dashed, two heads] \\
\underset{(P,P')}{\bullet} \arrow[u, dashed, two heads, "e_1"] \arrow[r, dashed, two heads, "e_2"'] \arrow[ur, phantom, "t_2" near end] & \underset{(R,R')}{\bullet} \arrow[u, dashed, two heads, "\tilde{e}_1"']
\end{tikzcd}$$

It follows that the path

$$\begin{tikzcd}[column sep=large]
\underset{(Q,Q')}{\bullet} \arrow[r, dashed, two heads, "\tilde{e}_2"] & \underset{(S,S')}{\bullet} & \underset{(Q,Q')}{\bullet} \arrow[l, dashed, two heads, "\tilde{e}_1"']
\end{tikzcd}$$

is homotopic to the given one. Everything here lies in the $G(\mathfrak{M}^{\mathrm{op}})$-part of $T(\mathcal{M})$.

• Suppose we are given a $(0,1)$-simplex followed by a $(1,0)$-simplex:

(2.12)
$$\begin{tikzcd}[column sep=large]
\underset{(P,P')}{\bullet} \arrow[r, dashed, two heads, "e_2"] & \underset{(Q,Q')}{\bullet} \arrow[r, "e_2"] & \underset{(R,R')}{\bullet}
\end{tikzcd}$$

where

$$\begin{aligned} e_1 &= (P \twoheadleftarrow Q \leftarrowtail M,\ P' \twoheadleftarrow Q' \leftarrowtail M),\\ e_2 &= (Q \rightarrowtail R \twoheadrightarrow N,\ Q' \rightarrowtail R' \twoheadrightarrow N). \end{aligned}$$

Choose pushout objects $S = P \amalg_Q R$, $S' = P' \amalg_{Q'} R'$ and consider the $(1,1)$-simplex given by the diagrams

$$\begin{tikzcd} M \arrow[r, tail] \arrow[d, tail] & M \arrow[r, two heads] \arrow[d, tail] & 0 \arrow[d, tail] \\ Q \arrow[r, tail] \arrow[d, two heads] & R \arrow[r, two heads] \arrow[d, two heads] & N \arrow[d, two heads] \\ P \arrow[r, tail] & S \arrow[r, two heads] & N \end{tikzcd} \qquad \begin{tikzcd} M' \arrow[r, tail] \arrow[d, tail] & M' \arrow[r, two heads] \arrow[d, tail] & 0 \arrow[d, tail] \\ Q' \arrow[r, tail] \arrow[d, two heads] & R' \arrow[r, two heads] \arrow[d, two heads] & N' \arrow[d, two heads] \\ P' \arrow[r, tail] & S' \arrow[r, two heads] & N' \end{tikzcd}$$

It looks like

$$\begin{tikzcd} (Q,Q') \arrow[r, "e_2"] & (R,R') \\ (P,P') \arrow[u, dashed, two heads, "e_1"] \arrow[r, "\tilde{e}_2"'] & (S,S') \arrow[u, dashed, two heads, "\tilde{e}_1"'] \end{tikzcd}$$

which enables us to replace the path (2.12) by the homotopic path

$$\begin{tikzcd} (P,P') \arrow[r, "\tilde{e}_2"] & (S,S') \arrow[r, dashed, "\tilde{e}_1"] & (R,R') \end{tikzcd}$$

- Given a path of the form

(2.13) $$\begin{tikzcd} (P,P') \arrow[r, "e_1"] & (Q,Q') \arrow[r, dashed, "e_2"] & (R,R') \end{tikzcd}$$

where

$$\begin{aligned} e_1 &= (P \rightarrowtail Q \twoheadrightarrow M,\ P' \rightarrowtail Q' \twoheadrightarrow M),\\ e_2 &= (Q \twoheadleftarrow R \quad N,\ Q' \twoheadleftarrow R' \quad N), \end{aligned}$$

choose pullbacks $S = P \times_Q R$ and $S' = P' \times_{Q'} R'$ and consider the $(1,1)$-simplex

$$\begin{tikzcd} N \arrow[r, tail] \arrow[d, tail] & N \arrow[r, two heads] \arrow[d, tail] & 0 \arrow[d, tail] \\ S \arrow[r, tail] \arrow[d, two heads] & R \arrow[r, two heads] \arrow[d, two heads] & M \arrow[d, two heads] \\ P \arrow[r, tail] & Q \arrow[r, two heads] & M \end{tikzcd} \qquad \begin{tikzcd} N' \arrow[r, tail] \arrow[d, tail] & N' \arrow[r, two heads] \arrow[d, tail] & 0 \arrow[d, tail] \\ S' \arrow[r, tail] \arrow[d, two heads] & R' \arrow[r, two heads] \arrow[d, two heads] & M' \arrow[d, two heads] \\ P' \arrow[r, tail] & Q' \arrow[r, two heads] & M' \end{tikzcd}$$

Thus we can replace (2.13) by

$$\begin{tikzcd}[column sep=large] \underset{(P,P')}{\bullet} \arrow[r, dashed, "\tilde{e}_2"] & \underset{(S,S')}{\bullet} \arrow[r, "\tilde{e}_1"] & \underset{(R,R')}{\bullet} \end{tikzcd}$$

where

$$\tilde{e}_1 = (S \rightarrowtail R \twoheadrightarrow M,\, S' \rightarrowtail R' \twoheadrightarrow M),$$

$$\tilde{e}_2 = (Q \twoheadleftarrow R \leftarrowtail N,\, Q' \twoheadleftarrow R' \leftarrowtail N).$$

The main technical tool for computations with four-term double exact sequences is the following proposition proved by A. Nenashev:

PROPOSITION 2.15 (3 × 4 lemma). *Suppose we are given a diagram*

$$\begin{tikzcd} D' \arrow[d, tail, "d_1'"', "d_2'"] & C' \arrow[l, two heads, "h_1'"', "h_2'"] \arrow[d, tail, "c_1'"', "c_2'"] & B' \arrow[l, "g_1'"', "g_2'"] \arrow[d, tail, "b_1'"', "b_2'"] & A' \arrow[l, tail, "f_1'"', "f_2'"] \arrow[d, tail, "a_1'"', "a_2'"] \\ D \arrow[d, two heads, "d_1"', "d_2"] & C \arrow[l, two heads, "h_1"', "h_2"] \arrow[d, two heads, "c_1"', "c_2"] & B \arrow[l, "g_1"', "g_2"] \arrow[d, two heads, "b_1"', "b_2"] & A \arrow[l, tail, "f_1"', "f_2"] \arrow[d, two heads, "a_1"', "a_2"] \\ D'' & C'' \arrow[l, two heads, "h_1''"', "h_2''"] & B'' \arrow[l, "g_1''"', "g_2''"] & A'' \arrow[l, tail, "f_1''"', "f_2''"] \end{tikzcd} \tag{2.14}$$

which consists of three double four-term exact sequences and four d.s.e.s., such that each of the two diagrams with indices $i = 1$ or 2 commutes. Let q', q, and q'' denote the horizontal double four-term exact sequences and l_A, l_B, l_C, l_D the vertical double short exact sequences. Then

$$\ell(q') - \ell(q) + \ell(q'') = -\ell(l_A) + \ell(l_B) - \ell(l_C) + \ell(l_D)$$

in $K_1(\mathfrak{M})$.

Proof. We simplify things and do not display the unnecessary parts of loops, in the following sense. Let $\widetilde{T}(\mathcal{M})$ denote the bisimplicial subset of $T(\mathcal{M})$ which consists of the diagonal bisimplices, i.e. the bisimplices given by pairs of identical diagrams. It is easily seen to be contractible and we can therefore compute $K_1(\mathfrak{M}) = \pi_1(T(\mathcal{M}))$ as the relative fundamental group: $\pi_1(T(\mathcal{M})) = \pi_1(T(\mathcal{M}), \widetilde{T}(\mathcal{M}))$. In this context, we can omit the side edges of the type $e(X)$ in the definition of $\ell(q)$ and $\ell(l)$ and work with relative loops; $\ell(l)$ (resp. $\ell(q)$) amounts then to the single edge $e(l)$ (resp. the pair of edges $(e_{0,1}(q), e_{1,0}(q))$).

With the notation of Definition 2.13, the given 3×4 diagram may be cut into two 3×3 diagrams:

$$
\begin{tikzcd}
D' \arrow[d, tail, "d'_i"'] & C' \arrow[l, two heads, "h'_i"'] \arrow[d, tail, "c'_i"] & X'_i \arrow[l, tail, "v'_i"'] \arrow[d, tail, "x'_i"] \\
D \arrow[d, two heads, "d_i"'] & C \arrow[l, two heads, "h_i"'] \arrow[d, two heads, "c_i"] & X_i \arrow[l, tail, "v_i"'] \arrow[d, two heads, "x_i"] \\
D'' & C'' \arrow[l, two heads, "h''_i"'] & X''_i \arrow[l, tail, "v''_i"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X'_i \arrow[d, tail, "x'_i"'] & B' \arrow[l, two heads, "u'_i"'] \arrow[d, tail, "b'_i"'] & A' \arrow[l, tail, "f'_i"'] \arrow[d, tail, "a'_i"] \\
X_i \arrow[d, two heads, "x_i"'] & B \arrow[l, two heads, "u_i"'] \arrow[d, two heads, "b_i"'] & A \arrow[l, tail, "f_i"'] \arrow[d, two heads, "a_i"] \\
X''_i & B'' \arrow[l, two heads, "u''_i"'] & A'' \arrow[l, tail, "f''_i"']
\end{tikzcd}
\qquad (i = 0, 1) \tag{2.15}
$$

The same argument as in the proof of Proposition 2.12 lets us replace the $(1,0)$-edge $e(l_C)$ from (C', C') to (C, C) by a homotopic (rel. $\widetilde{T}(\mathcal{M})$) $(0,1)$-edge from (C'', C'') to (C, C). The element

$$\ell(q'') + \ell(l_C) - \ell(q) - \ell(l_B) + \ell(q')$$

can thus be represented by the following path:

$$
\begin{tikzcd}
(B',B') \arrow[d, "e_{1,0}(l_B)"] & (X'_1,X'_2) \arrow[l, dashed, "e_{0,1}(q')"'] \arrow[r, "e_{1,0}(q')"'] & (C',C') \\
(B,B) & & \\
& (X_1,X_2) \arrow[ul, dashed, "e_{0,1}(q)"] \arrow[dr, "e_{1,0}(q)"] & \\
& & (C,C) \\
(B'',B'') & (X''_1,X''_2) \arrow[l, dashed, "e_{0,1}(q'')"] \arrow[r, "e_{1,0}(q'')"] & (C'',C'') \arrow[u, dashed, two heads, "e_{0,1}(l_C)"']
\end{tikzcd}
\tag{2.16}
$$

We will now construct several homotopies in order to show that the above expression equals $-\ell(l_A) + \ell(l_D)$.

1. Denote by P_i a pullback object for

$$X''_i \overset{v''_i}{\rightarrowtail} C'' \overset{c_i}{\twoheadleftarrow} C, \qquad i = 1, 2,$$

and apply (2.13) to the edges

$$
\begin{tikzcd}
\bullet \arrow[r, "e_{1,0}(q'')"] & \bullet \arrow[r, dashed, "e_{0,1}(l_C)"] & \bullet
\end{tikzcd}
$$

in (2.16). We get two other edges

$$
\begin{tikzcd}
(X''_1,X''_2) \arrow[r, dashed, "\tilde{e}_{0,1}(l_C)"] & (P_1,P_2) \arrow[r, "\tilde{e}_{1,0}(q'')"] & (C,C)
\end{tikzcd}
$$

the homotopy being provided by the $(1,1)$-simplex

(2.17)
$$\begin{tikzcd}
C' \arrow[r, tail, "1"] \arrow[d, tail, "\tilde{c}'_i"'] & C' \arrow[r, two heads] \arrow[d, tail, "c'_i"'] & 0 \arrow[d, tail] \\
P_i \arrow[r, tail, "\tilde{v}''_i"] \arrow[d, two heads, "\tilde{c}_i"'] & C \arrow[r, two heads, "\tilde{h}''_i"] \arrow[d, two heads, "c_i"'] & D'' \arrow[d, two heads, "1"] \\
X''_i \arrow[r, tail, "v''_i"] & C'' \arrow[r, two heads, "h''_i"] & D''
\end{tikzcd} \qquad (i=1,2).$$

The arrows $\tilde{c}'_i$, $\tilde{c}_i$, $\tilde{v}''_i$, and $\tilde{h}''_i$ in this diagram are induced by the arrows c'_i, c_i, v''_i, and h''_i of (2.15).

2. As $c_i v_i = v''_i x_i$, there is a unique arrow $X_i \xrightarrow{p_i} P_i$ with $\tilde{c}_i p_i = \tilde{x}_i$ and $\tilde{v}''_i p_i = v_i$ $(i=1,2)$. We leave it to the reader to deduce from Quillen's third axiom of an exact category that this arrow is an admissible monomorphism (see the example after Definition 2.1) and we have the following pair of diagrams:

(2.18)
$$\begin{tikzcd}
X_i \arrow[r, tail, "p_i"] \arrow[d, tail, "1"'] & P_i \arrow[r, two heads, "r_i"] \arrow[d, tail, "\tilde{v}''_i"] & D' \arrow[d, tail, "d'_i"] \\
X_i \arrow[r, tail, "v_i"] \arrow[d, two heads] & C \arrow[r, two heads, "h_i"] \arrow[d, two heads, "d_i h_i"] & D \arrow[d, two heads, "d_i"] \\
0 \arrow[r, tail] & D'' \arrow[r, two heads, "1"] & D''
\end{tikzcd} \qquad (i=1,2).$$

The arrows r_i here are induced by the other arrows in the diagrams.

3. Denote by U_i a pushout object for $B \overset{b'_i}{\twoheadleftarrow} B' \overset{u'_i}{\rightarrowtail} X'_i$, $i=1,2$, and apply (2.12) to the edges $\bullet \xleftarrow{e_{1,0}(l_B)} \bullet \overset{e_{0,1}(q')}{\dashleftarrow} \bullet$ in (2.16). We then get two other edges

$$\begin{tikzcd}
\underset{(B,B)}{\bullet} & \underset{(U_1,U_2)}{\bullet} \arrow[l, dashed, "\tilde{e}_{0,1}(q')"'] & \underset{(X'_1,X'_2)}{\bullet} \arrow[l, "\tilde{e}_{1,0}(l_B)"']
\end{tikzcd}$$

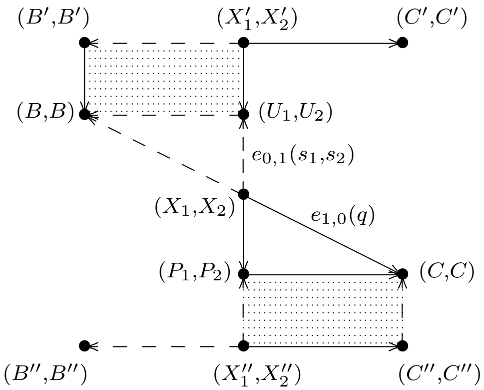
the homotopy being provided by the $(1,1)$-simplex

(2.19)
$$\begin{tikzcd}
A' \arrow[r, tail, "1"] \arrow[d, tail, "f'_i"'] & A' \arrow[r, two heads] \arrow[d, tail, "\tilde{f}'_i"] & 0 \arrow[d, tail] \\
B' \arrow[r, tail, "b'_i"] \arrow[d, "u'_i"'] & B \arrow[r, two heads, "b''_i"] \arrow[d, "\tilde{u}'_i"] & B'' \arrow[d, two heads, "1"] \\
X'_i \arrow[r, tail, "\tilde{b}'_i"] & U_i \arrow[r, two heads, "\tilde{b}''_i"] & B''
\end{tikzcd} \qquad (i=1,2).$$

4. Dualizing 2, we get admissible epimorphisms $U_i \overset{s_i}{\twoheadrightarrow} X_i$ with $s_i\tilde{u}_i' = u_i$ and $s_i\tilde{b}_i' = x_i'$ $(i = 1, 2)$ and the diagrams

$$\begin{tikzcd} A' \arrow[r, tail, "1"] \arrow[d, tail, "a_i'"'] & A' \arrow[r, two heads] \arrow[d, tail, "f_i a_i'"] & 0 \arrow[d, tail] \\ A \arrow[r, tail, "f_i"] \arrow[d, two heads, "a_i''"'] & B \arrow[r, two heads, "u_i"] \arrow[d, two heads, "\tilde{u}_i'"] & X_i \arrow[d, two heads, "1"] \\ A'' \arrow[r, tail, "t_i"] & U_i \arrow[r, two heads, "s_i"] & X_i \end{tikzcd} \qquad (i = 1, 2). \tag{2.20}$$

The arrows t_i here are induced by the other arrows in these diagrams. This results in the following figure:



in which:

- the horizontal arrow from (P_1, P_2) to (C, C) is $\tilde{e}_{1,0}(q'')$;
- the horizontal arrow from (X_1'', X_2'') to (C'', C'') is $e_{1,0}(q'')$;
- the vertical arrow from (X_1, X_2) to (P_1, P_2) is

$$e_{1,0}(p_1, p_2) = (X_1 \overset{p_1}{\rightarrowtail} P_1 \twoheadrightarrow r_1D',\ X_2 \overset{p_2}{\rightarrowtail} P_2 \twoheadrightarrow r_2D'),$$

and the shadowed areas are $(1, 1)$-simplices. The triangular contour with vertices $(X_1, X_2), (P_1, P_2), (C, C)$ is an admissible triple of edges in the G-part of $T(\mathcal{M})$, in the sense of [12, Definition, p. 207]. By [12, Prop. 4.5], the diagrams (2.18) show that the associated d.s.e.s. is l_D. Dually, the triangular contour with vertices $(X_1, X_2), (U_1, U_2), (B, B)$ is a (co)admissible triple in the G^{op}-part (we leave it to the reader to dualize the definition and related arguments), the associated d.s.e.s. being l_A. Thus by [12, Prop. 4.5] and its dual version, the path

$$(B'',B'') \xleftarrow{e_{0,1}(q'')} (X_1'',X_2'') \xrightarrow{\tilde{e}_{1,0}(l_C)} (P_1,P_2) \xleftarrow{e_{1,0}(p_1,p_2)} (X_1,X_2) \xrightarrow{e_{0,1}(s_1,s_2)} (U_1,U_2) \xleftarrow{\tilde{e}_{1,0}(l_B)} (X_1',X_2') \xrightarrow{e_{1,0}(q')} (C',C') \tag{2.21}$$

represents the element $\ell(v') - \ell(v) + \ell(v'') + \ell(l_C) - \ell(l_B) + \ell(l_A) - \ell(l_D)$ in $\pi_1(T(\mathcal{M}), \widetilde{T}(\mathcal{M}))$, and it remains to show that this path is contractible.

5. We leave it to the reader to check the following elementary assertion:

LEMMA 2.16. *In an abelian category, given a commutative diagram*

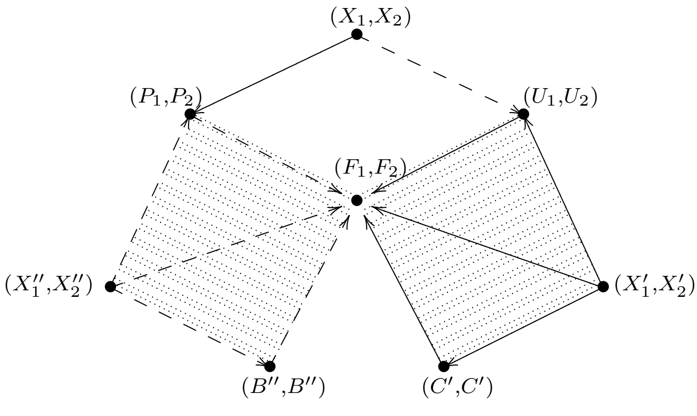
$$\begin{tikzcd} B' \arrow[r, tail] \arrow[d] & B \arrow[r, two heads] \arrow[d] & B'' \arrow[d] \\ C' \arrow[r, tail] & C \arrow[r, two heads] & C'' \end{tikzcd}$$

whose rows are s.e.s's, there is a natural map $C' \amalg_{B'} B \to C \times_{C''} B''$ and this map is an isomorphism. ∎

Let F_i, $i = 1, 2$, denote a choice for the pullback/pushout object of the lemma applied to the i-th arrows in the B-C-part of diagram (2.14). Then

$$C' \amalg_{X'_i} U_i \cong C' \amalg_{X'_i} (X'_i \amalg_{B'} B) \cong C' \amalg_{B'} B \cong F_i,$$
$$P_i \times_{X''_i} B'' \cong (C \times_{C''} X''_i) \times_{X''_i} B'' \cong C \times_{C''} B'' \cong F_i$$

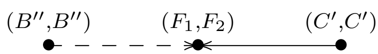
and we get the following picture:



where the shadowed rectangles are as in (2.11) and (2.10) respectively. The remaining rectangular contour can be filled by the $(1,1)$-simplex

$$\begin{tikzcd} A'' \arrow[r, tail, "1"] \arrow[d, tail, "t_i"'] & A'' \arrow[r, two heads] \arrow[d, tail] & 0 \arrow[d, tail] \\ U_i \arrow[r, tail] \arrow[d, two heads, "s_i"'] & F_i \arrow[r, two heads] \arrow[d, two heads] & D' \arrow[d, two heads, "1"] \\ X_i \arrow[r, tail, "p_i"] & P_i \arrow[r, two heads, "r_i"] & D' \end{tikzcd} \qquad (i = 1, 2).$$

Thus the path (2.21) is homotopic to the two-edge path



The latter can be contracted to $\widetilde{T}(\mathcal{M})$ by means of the $(1,1)$-simplex

$$\begin{tikzcd}
C' \arrow[r, tail, "1"] \arrow[d, tail, "1"'] & C' \arrow[r, two heads] \arrow[d, tail] & 0 \arrow[d, tail] \\
C' \arrow[r, tail] \arrow[d, two heads] & F_i \arrow[r, two heads] \arrow[d, two heads] & B'' \arrow[d, two heads, "1"] \\
0 \arrow[r, tail] & B'' \arrow[r, two heads, "1"] & B''
\end{tikzcd}
\qquad (i = 1, 2). \ \blacksquare$$

COROLLARY 2.17.

$$\ell\Big(D \underset{c'}{\overset{c}{\twoheadleftarrow}} C \underset{b'}{\overset{b}{\leftarrow}} B \underset{a'}{\overset{a}{\leftarrowtail}} A\Big) + \ell\Big(H \underset{h'}{\overset{h}{\twoheadleftarrow}} G \underset{g'}{\overset{g}{\leftarrow}} F \underset{f'}{\overset{f}{\leftarrowtail}} E\Big)$$

$$= \ell\left(D \oplus H \underset{\left[\begin{smallmatrix} c' & 0 \\ 0 & h' \end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix} c & 0 \\ 0 & h \end{smallmatrix}\right]}{\twoheadleftarrow}} C \oplus G \underset{\left[\begin{smallmatrix} b' & 0 \\ 0 & g' \end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix} b & 0 \\ 0 & g \end{smallmatrix}\right]}{\longleftarrow}} B \oplus F \underset{\left[\begin{smallmatrix} a' & 0 \\ 0 & f' \end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix} a & 0 \\ 0 & f \end{smallmatrix}\right]}{\leftarrowtail}} A \oplus E\right). \ \blacksquare$$

COROLLARY 2.18.

$$\ell\Big(D \underset{c'}{\overset{c}{\twoheadleftarrow}} C \underset{b'}{\overset{b}{\leftarrow}} B \underset{a'}{\overset{a}{\leftarrowtail}} A\Big) + \ell\Big(D \underset{c}{\overset{c'}{\twoheadleftarrow}} C \underset{b}{\overset{b'}{\leftarrow}} B \underset{a}{\overset{a'}{\leftarrowtail}} A\Big) = 0. \ \blacksquare$$

LEMMA 2.19. *For arbitrary automorphisms $\alpha, \alpha' \in \operatorname{Aut}(A)$ and arbitrary exact sequences $A \overset{b}{\twoheadleftarrow} B \overset{c}{\leftarrowtail} C$, $F \overset{f}{\twoheadleftarrow} E \overset{e}{\leftarrowtail} A$ we have*

$$\ell\Big(0 \twoheadleftarrow A \underset{\alpha'}{\overset{\alpha}{\leftarrow}} A \leftarrowtail 0\Big) = \ell\Big(F \underset{f}{\overset{f}{\twoheadleftarrow}} E \underset{e\circ\alpha'\circ b}{\overset{e\circ\alpha\circ b}{\longleftarrow}} B \underset{c}{\overset{c}{\leftarrowtail}} C\Big)$$

in $K_1(\mathfrak{M})$.

Proof. Apply the 3×3 lemma and the 3×4 lemma to the diagrams

$$\begin{tikzcd}
0 \arrow[d, tail] & A \arrow[l, two heads] \arrow[d, tail, "e"', "e"] & A \arrow[l, tail, "\alpha", "\alpha'"'] \arrow[d, tail, "1"', "1"] \\
F \arrow[d, two heads, "1"', "1"] & E \arrow[l, two heads, "f", "f"'] \arrow[d, two heads, "f"', "f"] & A \arrow[l, tail, "e\circ\alpha", "e\circ\alpha'"'] \arrow[d, two heads] \\
F & F \arrow[l, two heads, "1", "1"'] & 0 \arrow[l, tail]
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[d, tail] & 0 \arrow[l, two heads] \arrow[d, tail] & C \arrow[l] \arrow[d, tail, "a"', "a"] & C \arrow[l, tail, "1", "1"'] \arrow[d, tail, "1"', "1"] \\
F \arrow[d, two heads, "1"', "1"] & E \arrow[l, two heads, "f", "f"'] \arrow[d, two heads, "1"', "1"] & B \arrow[l, "e\circ\alpha\circ b", "e\circ\alpha'\circ b"'] \arrow[d, two heads, "b"', "b"] & C \arrow[l, tail, "c", "c"'] \arrow[d, two heads] \\
F & E \arrow[l, two heads, "f", "f"'] & A \arrow[l, "e\circ\alpha", "e\circ\alpha'"'] & 0 \arrow[l, tail]
\end{tikzcd}
\ \blacksquare$$

LEMMA 2.20.

$$\ell\Big(D \underset{c'}{\overset{c}{\twoheadleftarrow}} C \underset{b'}{\overset{b}{\leftarrow}} B \underset{a'}{\overset{a}{\leftarrowtail}} A\Big) + \ell\Big(D \underset{c''}{\overset{c'}{\twoheadleftarrow}} C \underset{b''}{\overset{b'}{\leftarrow}} B \underset{a''}{\overset{a'}{\leftarrowtail}} A\Big)$$

$$= \ell\Big(D \underset{c''}{\overset{c}{\twoheadleftarrow}} C \underset{b''}{\overset{b}{\leftarrow}} B \underset{a''}{\overset{a}{\leftarrowtail}} A\Big).$$

Proof. Apply the 3×4 lemma to the diagram

$$\begin{tikzcd}[column sep=huge, row sep=large]
D \arrow[d, tail, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"', "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"] & C \arrow[l, two heads, "c", "c''"'] \arrow[d, tail, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"', "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"] & B \arrow[l, "b", "b''"'] \arrow[d, tail, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"', "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"] & A \arrow[l, tail, "a", "a''"'] \arrow[d, tail, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"', "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"] \\
D \oplus D \arrow[d, two heads, "{[0\,1]}"', "{[-1\,0]}"] & C \oplus C \arrow[l, two heads, "{\left[\begin{smallmatrix}c & 0\\0 & c'\end{smallmatrix}\right]}", "{\left[\begin{smallmatrix}c' & 0\\0 & c''\end{smallmatrix}\right]}"'] \arrow[d, two heads, "{[0\,1]}"', "{[-1\,0]}"] & B \oplus B \arrow[l, two heads, "{\left[\begin{smallmatrix}b & 0\\0 & b'\end{smallmatrix}\right]}", "{\left[\begin{smallmatrix}b' & 0\\0 & b''\end{smallmatrix}\right]}"'] \arrow[d, two heads, "{[0\,1]}"', "{[-1\,0]}"] & A \oplus A \arrow[l, two heads, "{\left[\begin{smallmatrix}a & 0\\0 & a'\end{smallmatrix}\right]}", "{\left[\begin{smallmatrix}a' & 0\\0 & a''\end{smallmatrix}\right]}"'] \arrow[d, two heads, "{[0\,1]}"', "{[-1\,0]}"] \\
D & C \arrow[l, two heads, "c'", "c'"'] & B \arrow[l, "b'", "b'"'] & A \arrow[l, tail, "a'", "a'"']
\end{tikzcd}$$ ■

3. The group $E^1(X)$ and the invariant e^1. For the sake of completeness we state the following obvious fact:

LEMMA 3.1. *Let $\mathfrak{M}$ be a small exact category. For $A, B \in \mathrm{Ob}(\mathfrak{M})$ the following conditions are equivalent:*

(i) *the equality $[A] = [B]$ in $K_0(\mathfrak{M})$ holds;*

(ii$_1$) *there exist $P, Q, R \in \mathrm{Ob}(\mathfrak{M})$ and $\alpha, \beta, \alpha', \beta' \in \mathrm{Mor}(\mathfrak{M})$ such that the sequences*

$$0 \leftarrow R \overset{\beta}{\leftarrow} Q \overset{\alpha}{\leftarrow} A \oplus P \leftarrow 0,$$
$$0 \leftarrow R \overset{\beta'}{\longleftarrow} Q \overset{\alpha'}{\longleftarrow} B \oplus P \leftarrow 0$$

are exact admissible;

(ii$_2$) *there exist $P, Q, R \in \mathrm{Ob}(\mathfrak{M})$ and $\gamma, \mu, \gamma', \mu' \in \mathrm{Mor}(\mathfrak{M})$ such that the sequences*

$$\begin{gathered} 0 \leftarrow A \oplus P \overset{\mu}{\leftarrow} Q \overset{\gamma}{\leftarrow} R \leftarrow 0, \\ 0 \leftarrow B \oplus P \overset{\mu'}{\longleftarrow} Q \overset{\gamma'}{\longleftarrow} R \leftarrow 0 \end{gathered} \tag{3.1}$$

are exact admissible.

Moreover, if $(\mathfrak{M}, D, \delta)$ is an exact category with duality, then one may assume that in each case P carries a hyperbolic form $\chi : P \to DP$.

Proof. It is obvious that each of conditions (ii$_1$), (ii$_2$) implies $[A] = [B]$ in $K_0(\mathfrak{M})$. Assume that (i) holds. By [18, Theorem 1], $K_0(\mathfrak{M}) = \mathcal{A}/\mathcal{B}$ is a factor group of the free abelian group $\mathcal{A}$ generated by the classes of isomorphic objects in $\mathrm{Ob}(\mathfrak{M})$ modulo the subgroup $\mathcal{B}$ generated by the expressions

$$[Y] - [X] - [Z]$$

for all admissible exact sequences $Z \twoheadleftarrow Y \hookleftarrow X$. Thus if $[A] = [B]$, then there exist two admissible exact sequences

$$Z \twoheadleftarrow Y \overset{i}{\hookleftarrow} X, \quad W \twoheadleftarrow V \overset{j}{\hookleftarrow} U$$

and an isomorphism

$$\rho : B \oplus X \oplus Z \oplus V \xrightarrow{\sim} A \oplus Y \oplus U \oplus W.$$

Thus for

$$\begin{aligned}
&P = X \oplus U,\\
&Q = A \oplus Y \oplus U \oplus W \cong B \oplus X \oplus Z \oplus V,\\
&R = Z \oplus W,\\
&\alpha = \begin{bmatrix} 1_A & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1_U \\ 0 & 0 & 0 \end{bmatrix} : A \oplus X \oplus U \to A \oplus Y \oplus U \oplus W,\\
&\alpha' = \rho \circ \begin{bmatrix} 1_B & 0 & 0 \\ 0 & 1_X & 0 \\ 0 & 0 & 0 \\ 0 & 0 & j \end{bmatrix} : B \oplus X \oplus U \to A \oplus Y \oplus U \oplus W,
\end{aligned}$$

(ii_1) holds. Analogously (i) implies (ii_2) for the same Q,

$$P = Z \oplus W, \quad R = X \oplus U$$

and suitable morphisms.

If there is a duality in $\mathfrak{M}$, then substitution of $P \oplus DP$ for P, $Q \oplus DP$ for Q, and $\alpha \oplus 1_{DP}$ for α, etc., yields an analogous exact sequence with a hyperbolic form χ defined on P. ■

We will refer to the pair of exact sequences $A \oplus P \twoheadleftarrow Q \hookleftarrow R$, $B \oplus P \twoheadleftarrow Q \hookleftarrow R$ with hyperbolic P as a *common resolution* of $A \oplus P$, $B \oplus P$. Let us rephrase the last result in a more convenient form:

COROLLARY 3.2. *Given two a.s.b.m. $\varphi : A \to DA$, $\psi : B \to DB$ such that $[A] = [B]$ in $K_0(\mathfrak{M})$, there exist:*

- *a hyperbolic form (P, χ),*
- *an object Q,*
- *admissible epimorphisms $\pi_A : Q \to A \oplus P$, and $\pi_B : Q \to B \oplus P$ with kernels $R \rightrightarrows Q$ having a common source R,*
- *a.s.b.m. $D\pi_A \circ (\varphi \oplus \chi) \circ \pi_A : Q \to DQ$ and $D\pi_B \circ (\psi \oplus \chi) \circ \pi_B : Q \to DQ$ Witt equivalent to φ and ψ respectively.*

For Witt equivalence compare Example 2.10.

Definition 3.1. Let $\mathfrak{M}$ be an exact category with a duality (D,δ). With a.s.b.m. $\alpha : Q \to DQ$, $\beta : Q \to DQ$ with respective kernels $\gamma, \gamma' : R \rightarrowtail Q$, we associate a self-dual double exact sequence

$$DR \underset{D\gamma'}{\overset{D\gamma}{\twoheadleftarrow}} DQ \underset{\beta}{\overset{\alpha}{\leftarrow}} Q \underset{\gamma'}{\overset{\gamma}{\leftarrowtail}} R. \tag{3.2}$$

In particular, with symmetric bilinear forms $\varphi : A \xrightarrow{\sim} DA$, $\psi : B \xrightarrow{\sim} DB$ such that $[A]=[B]$ in $K_0(\mathfrak{M})$, and a common resolution $A \oplus P \twoheadleftarrow Q \leftarrowtail R$, $B \oplus P \twoheadleftarrow Q \leftarrowtail R$ with hyperbolic (P,χ) we associate a self-dual double exact sequence

$$DR \underset{D\gamma'}{\overset{D\gamma}{\twoheadleftarrow}} DQ \underset{D\mu'\circ(\psi\oplus\chi)\circ\mu'}{\overset{D\mu\circ(\varphi\oplus\chi)\circ\mu}{\longleftarrow}} Q \underset{\gamma'}{\overset{\gamma}{\leftarrowtail}} R \tag{3.3}$$

with admissible γ, γ'. We will refer to this double exact sequence as *gluing a common resolution with its dual.*

We want to refine this to a map defined on Witt equivalence classes, so we shall define an appropriate target group.

3.1. E-groups. An exact dualization functor $D : \mathfrak{M} \to \mathfrak{M}$ defines an action of the two-element group $\{1, D\}$ on the K-groups of $\mathfrak{M}$ (by action on d.s.e.s.) One of the objectives of this paper is the study of the following groups for $n=1$:

Definition 3.2. For $D : K_n(\mathfrak{M}) \to K_n(\mathfrak{M})$ the *E^n-groups* of the exact category $\mathfrak{M}$ with duality D are

$$E^n(\mathfrak{M};D) = E^n_+(\mathfrak{M};D) = \mathrm{Ker}(1-D)/\mathrm{Im}(1+D),$$
$$E^n_-(\mathfrak{M};D) = \mathrm{Ker}(1+D)/\mathrm{Im}(1-D).$$

Equivalently one may define the E^n-groups as the homology groups of the complex

$$\cdots \xrightarrow{1+D} K_n(\mathfrak{M}) \xrightarrow{1-D} K_n(\mathfrak{M}) \xrightarrow{1+D} K_n(\mathfrak{M}) \xrightarrow{1-D} \cdots \tag{3.4}$$

or the Tate cohomology groups

$$E^n(\mathfrak{M};D) = \widehat{H}^{2p}(\{1,D\}, K_n(\mathfrak{M})), \qquad E^n_-(\mathfrak{M};D) = \widehat{H}^{2p-1}(\{1,D\}, K_n(\mathfrak{M})).$$

Note that the case $n=0$ was used extensively in [21]–[23]. Let us recall that in those papers we used the map

$$e^0 : W(\mathfrak{M}) \to E^0(\mathfrak{M})$$

induced by the forgetful functor, and more general maps $e^0 : W^{\pm}(\mathfrak{M}, D, \delta) \to E^0(\mathfrak{M}; D)$. The group

$$E^0_-(\mathfrak{M};D) = \{x \in K_0(\mathfrak{M}) : Dx = -x\}/\{y - Dy : y \in K_0(\mathfrak{M})\}$$

also occurred. Each pair of hyperbolic spaces $(M \oplus DM, \left[\begin{smallmatrix} 0 & 1 \\ \delta_M & 0 \end{smallmatrix}\right])$, $(N \oplus DN, \left[\begin{smallmatrix} 0 & 1 \\ \delta_N & 0 \end{smallmatrix}\right])$ such that

$$[M \oplus DM] = [N \oplus DN] \quad \text{in } K_0(\mathfrak{M})$$

defines an element $[M] - [N]$ of $E^0_-(\mathfrak{M}; D)$.

Now we are interested in the case when for two symmetric bilinear forms (A, φ), (B, ψ) the equality

$$e^0(A, \varphi) = e^0(B, \psi)$$

holds. By Corollary 3.2, $e^0(A, \varphi) = e^0(B, \psi)$ iff there exist $M, N \in \mathrm{Ob}(\mathfrak{M})$ such that

$$[A \oplus M \oplus DM] = [B \oplus N \oplus DN] \quad \text{in } K_0(\mathfrak{M}).$$

Definition 3.3. Given two symmetric bilinear forms (A, φ), (B, ψ) and admissible exact sequences

$$A \overset{\beta}{\twoheadleftarrow} P \overset{\alpha}{\leftarrowtail} Q, \qquad B \overset{\nu}{\twoheadleftarrow} P \overset{\mu}{\leftarrowtail} Q$$

denote

$$\varepsilon^1 \begin{pmatrix} A \overset{\beta}{\twoheadleftarrow} P \overset{\alpha}{\leftarrowtail} Q,\ \varphi \\ B \overset{\nu}{\twoheadleftarrow} P \overset{\mu}{\leftarrowtail} Q,\ \psi \end{pmatrix} = \ell\Big(DQ \underset{D\mu}{\overset{D\alpha}{\twoheadleftarrow}} DP \underset{D\nu\circ\psi\circ\nu}{\overset{D\beta\circ\varphi\circ\beta}{\longleftarrow}} P \underset{\mu}{\overset{\alpha}{\leftarrowtail}} Q\Big).$$

Theorem 3.3. *Given two a.s.b.m. (A, φ), (B, ψ), and two common resolutions*

$$A \overset{\beta}{\twoheadleftarrow} P \overset{\alpha}{\leftarrowtail} Q, \qquad A \overset{b}{\twoheadleftarrow} R \overset{a}{\leftarrowtail} S,$$
$$B \overset{\nu}{\twoheadleftarrow} P \overset{\mu}{\leftarrowtail} Q, \qquad B \overset{d}{\twoheadleftarrow} R \overset{c}{\leftarrowtail} S,$$

we get

$$\varepsilon^1 \begin{pmatrix} A \overset{\beta}{\twoheadleftarrow} P \overset{\alpha}{\leftarrowtail} Q,\ \varphi \\ B \overset{\nu}{\twoheadleftarrow} P \overset{\mu}{\leftarrowtail} Q,\ \psi \end{pmatrix} - \varepsilon^1 \begin{pmatrix} A \overset{b}{\twoheadleftarrow} R \overset{a}{\leftarrowtail} S,\ \varphi \\ B \overset{d}{\twoheadleftarrow} R \overset{c}{\leftarrowtail} S,\ \psi \end{pmatrix} \in (1 + D)K_1(\mathfrak{M}).$$

Proof. The class

$$\begin{aligned} &\varepsilon^1 \begin{pmatrix} A \overset{\beta}{\twoheadleftarrow} P \overset{\alpha}{\leftarrowtail} Q,\ \varphi \\ B \overset{\nu}{\twoheadleftarrow} P \overset{\mu}{\leftarrowtail} Q,\ \psi \end{pmatrix} - \varepsilon^1 \begin{pmatrix} A \overset{b}{\twoheadleftarrow} R \overset{a}{\leftarrowtail} S,\ \varphi \\ B \overset{d}{\twoheadleftarrow} R \overset{c}{\leftarrowtail} S,\ \psi \end{pmatrix} \\ &\qquad = \varepsilon^1 \begin{pmatrix} A \overset{\beta}{\twoheadleftarrow} P \overset{\alpha}{\leftarrowtail} Q,\ \varphi \\ B \overset{\nu}{\twoheadleftarrow} P \overset{\mu}{\leftarrowtail} Q,\ \psi \end{pmatrix} + \varepsilon^1 \begin{pmatrix} B \overset{d}{\twoheadleftarrow} R \overset{c}{\leftarrowtail} S,\ \psi \\ A \overset{b}{\twoheadleftarrow} R \overset{a}{\leftarrowtail} S,\ \varphi \end{pmatrix} \end{aligned}$$

corresponds to the double long exact sequence:

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
 & DA \oplus DB \arrow[d, tail, "\left[\begin{smallmatrix} D\beta & 0 \\ 0 & Dd \end{smallmatrix}\right]"', "\left[\begin{smallmatrix} 0 & D\nu \\ Db & 0 \end{smallmatrix}\right]"] & P \oplus R \arrow[l, two heads, "\left[\begin{smallmatrix} \varphi\circ\beta & 0 \\ 0 & \psi\circ d \end{smallmatrix}\right]"', "\left[\begin{smallmatrix} 0 & \varphi\circ b \\ \psi\circ\nu & 0 \end{smallmatrix}\right]"] \arrow[d] & Q \oplus S \arrow[l, tail, "\left[\begin{smallmatrix} \alpha & 0 \\ 0 & c \end{smallmatrix}\right]"', "\left[\begin{smallmatrix} \mu & 0 \\ 0 & a \end{smallmatrix}\right]"] \arrow[d] \\
DQ \oplus DS \arrow[d] & DP \oplus DR \arrow[l, two heads, "\left[\begin{smallmatrix} D\alpha & 0 \\ 0 & Dc \end{smallmatrix}\right]"', "\left[\begin{smallmatrix} D\mu & 0 \\ 0 & Da \end{smallmatrix}\right]"] \arrow[d, "\left[\begin{smallmatrix} D\alpha & 0 \\ 0 & Dc \end{smallmatrix}\right]"', "\left[\begin{smallmatrix} D\mu & 0 \\ 0 & Da \end{smallmatrix}\right]"] & P \oplus R \arrow[l, "\left[\begin{smallmatrix} \overline{\varphi} & 0 \\ 0 & \overline{\psi} \end{smallmatrix}\right]"', "\left[\begin{smallmatrix} \underline{\psi} & 0 \\ 0 & \underline{\varphi} \end{smallmatrix}\right]"] & Q \oplus S \arrow[l, tail, "\left[\begin{smallmatrix} \alpha & 0 \\ 0 & c \end{smallmatrix}\right]"', "\left[\begin{smallmatrix} \mu & 0 \\ 0 & a \end{smallmatrix}\right]"] \\
DQ \oplus DS & DQ \oplus DS \arrow[l]
\end{tikzcd}
$$

where

$$\overline{\varphi} = D\beta \circ \varphi \circ \beta, \qquad \underline{\varphi} = Db \circ \varphi \circ b, \qquad \overline{\psi} = Dd \circ \psi \circ d, \qquad \underline{\psi} = D\nu \circ \psi \circ \nu.$$

It follows that if

$$
\begin{aligned}
t &= \ell\Big(DQ \oplus DS \xleftarrow[\left[\begin{smallmatrix} D\mu & 0 \\ 0 & Da \end{smallmatrix}\right]]{\left[\begin{smallmatrix} D\alpha & 0 \\ 0 & Dc \end{smallmatrix}\right]} DP \oplus DR \xleftarrow[\left[\begin{smallmatrix} \underline{\psi} & 0 \\ 0 & \underline{\varphi} \end{smallmatrix}\right]]{\left[\begin{smallmatrix} \overline{\varphi} & 0 \\ 0 & \overline{\psi} \end{smallmatrix}\right]} P \oplus R \xleftarrow[\left[\begin{smallmatrix} \mu & 0 \\ 0 & a \end{smallmatrix}\right]]{\left[\begin{smallmatrix} \alpha & 0 \\ 0 & c \end{smallmatrix}\right]} Q \oplus S\Big), \\
u &= \ell\Big(DQ \oplus DS \xleftarrow[\left[\begin{smallmatrix} D\mu & 0 \\ 0 & Da \end{smallmatrix}\right]]{\left[\begin{smallmatrix} D\alpha & 0 \\ 0 & Dc \end{smallmatrix}\right]} DP \oplus DR \xleftarrow[\left[\begin{smallmatrix} 0 & D\nu \\ Db & 0 \end{smallmatrix}\right]]{\left[\begin{smallmatrix} D\beta & 0 \\ 0 & Dd \end{smallmatrix}\right]} DA \oplus DB\Big), \\
v &= \ell\Big(DQ \oplus DS \xleftarrow[\left[\begin{smallmatrix} D\mu & 0 \\ 0 & Da \end{smallmatrix}\right]]{\left[\begin{smallmatrix} D\alpha & 0 \\ 0 & Dc \end{smallmatrix}\right]} DP \oplus DR \xleftarrow[\left[\begin{smallmatrix} 0 & D\nu\circ\psi \\ Db\circ\varphi & 0 \end{smallmatrix}\right]]{\left[\begin{smallmatrix} D\beta\circ\varphi & 0 \\ 0 & Dd\circ\psi \end{smallmatrix}\right]} A \oplus B\Big),
\end{aligned}
$$

then $u = v$ (since these d.s.e.s.'s are isomorphic) and $t = u + Dv = u + Du$ by the 3×4 lemma, since there is a commutative diagram

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
 & DA \oplus DB \arrow[d, tail, "\left[\begin{smallmatrix} D\beta & 0 \\ 0 & Dd \end{smallmatrix}\right]"', "\left[\begin{smallmatrix} 0 & D\nu \\ Db & 0 \end{smallmatrix}\right]"] & P \oplus R \arrow[l, two heads, "\left[\begin{smallmatrix} \varphi\circ\beta & 0 \\ 0 & \psi\circ d \end{smallmatrix}\right]"', "\left[\begin{smallmatrix} 0 & \varphi\circ b \\ \psi\circ\nu & 0 \end{smallmatrix}\right]"] \arrow[d] & Q \oplus S \arrow[l, tail, "\left[\begin{smallmatrix} \alpha & 0 \\ 0 & c \end{smallmatrix}\right]"', "\left[\begin{smallmatrix} \mu & 0 \\ 0 & a \end{smallmatrix}\right]"] \arrow[d] \\
DQ \oplus DS \arrow[d] & DP \oplus DR \arrow[l, two heads, "\left[\begin{smallmatrix} D\alpha & 0 \\ 0 & Dc \end{smallmatrix}\right]"', "\left[\begin{smallmatrix} D\mu & 0 \\ 0 & Da \end{smallmatrix}\right]"] \arrow[d, "\left[\begin{smallmatrix} D\alpha & 0 \\ 0 & Dc \end{smallmatrix}\right]"', "\left[\begin{smallmatrix} D\mu & 0 \\ 0 & Da \end{smallmatrix}\right]"] & P \oplus R \arrow[l, "\left[\begin{smallmatrix} \overline{\varphi} & 0 \\ 0 & \overline{\psi} \end{smallmatrix}\right]"', "\left[\begin{smallmatrix} \underline{\varphi} & 0 \\ 0 & \underline{\psi} \end{smallmatrix}\right]"] & Q \oplus S \arrow[l, tail, "\left[\begin{smallmatrix} \alpha & 0 \\ 0 & c \end{smallmatrix}\right]"', "\left[\begin{smallmatrix} \mu & 0 \\ 0 & a \end{smallmatrix}\right]"] \\
DQ \oplus DS & DQ \oplus DS \arrow[l]
\end{tikzcd}
$$

with exact rows and columns. ∎

Definition 3.4. The *relative discriminant* $\varepsilon^1(\varphi \div \psi)$ of a pair (A, φ), (B, ψ) of a.s.b.m.'s with common resolution is

$$\varepsilon^1(\varphi \div \psi) = \varepsilon^1\begin{pmatrix} A \overset{\beta}{\twoheadleftarrow} P \overset{\alpha}{\leftarrowtail} Q, \ \varphi \\ B \overset{\nu}{\twoheadleftarrow} P \overset{\mu}{\leftarrowtail} Q, \ \psi \end{pmatrix} \bmod (1 + D)K_1(\mathfrak{M}).$$

This is well defined by Theorem 3.3. Clearly

$$\varepsilon^1(\varphi \div \psi) \in \operatorname{Ker}(1 - D) \tag{3.5}$$

by the very construction.

Lemma 3.4. *If (A, φ) is metabolic and $\iota : L \rightarrowtail A$ is a Lagrangian, then*

$$\varepsilon^1\left(\varphi \div \begin{bmatrix} 0 & 1 \\ \delta_L & 0 \end{bmatrix}\right) = 0.$$

Proof. If $L \xrightarrow{\iota} A$ is a Lagrangian, then the sequence

$$DL \xleftarrow{D\iota \circ \varphi} A \overset{\iota}{\leftarrowtail} L$$

is exact. Thus one may form exact sequences

$$\begin{aligned} A \xleftarrow{[1,0]} A \oplus L \xleftarrow{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} L, \\ L \oplus DL \xleftarrow[\left[\begin{smallmatrix} 1 & 0 \\ D\iota\circ\varphi & 0 \end{smallmatrix}\right]]{} A \oplus L \xleftarrow[\left[\begin{smallmatrix} \iota \\ 0 \end{smallmatrix}\right]]{} L. \end{aligned}$$

Since

$$D\varphi \circ D^2\iota \circ \delta_L = D\varphi \circ \delta_A \circ \iota = \varphi \circ \iota,$$

gluing them with their duals yields a double exact sequence

$$DL \underset{[0,1]}{\overset{[D\iota,0]}{\twoheadleftarrow}} DA \oplus DL \underset{\left[\begin{smallmatrix} 0 & \varphi\circ\iota \\ D\iota\circ\varphi & 0 \end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix} \varphi & 0 \\ 0 & 0 \end{smallmatrix}\right]}{\longleftarrow}} A \oplus L \underset{\left[\begin{smallmatrix} \iota \\ 0 \end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]}{\leftarrowtail}} L,$$

which may be "resolved" as follows: apply the 3×4 lemma to the commutative diagram

$$\begin{tikzcd}
 & L \arrow[d, tail, "\left[\begin{smallmatrix}\varphi\circ\iota\\ 0\end{smallmatrix}\right]"', "\left[\begin{smallmatrix}\varphi\circ\iota\\ 0\end{smallmatrix}\right]"] & L \oplus L \arrow[l, two heads, "{[-1,0]}"', "{[0,1]}"] \arrow[d, tail, "\left[\begin{smallmatrix}-\iota & 0\\ 0 & 1\end{smallmatrix}\right]"', "\left[\begin{smallmatrix}-\iota & 0\\ 0 & 1\end{smallmatrix}\right]"] & L \arrow[l, tail, "\left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right]"', "\left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right]"] \arrow[d, "1"', "-1"] \\
DL \arrow[d, "1"', "-1"] & DA \oplus DL \arrow[l, two heads] \arrow[d, two heads, "\left[\begin{smallmatrix}D\iota & 0\\ 0 & 1\end{smallmatrix}\right]"', "\left[\begin{smallmatrix}D\iota & 0\\ 0 & 1\end{smallmatrix}\right]"] & A \oplus L \arrow[l] \arrow[d, two heads, "{[D\iota\circ\varphi,0]}"', "{[D\iota\circ\varphi,0]}"] & L \arrow[l, tail] \\
DL & DL \oplus DL \arrow[l, two heads, "{[0,1]}"', "{[-1,0]}"] & DL \arrow[l, tail, "\left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right]"', "\left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right]"] &
\end{tikzcd}$$

Then Lemma 2.10(b) applied to the upper and the lower d.s.e.s. shows that

$$\varepsilon^1\left(\varphi \div \begin{bmatrix} 0 & 1 \\ \delta_L & 0 \end{bmatrix}\right) = \{-1\}([L] + [DL]) = (1 + D)(\{-1\}[L])$$

(here $\{-1\} \in K_1(\mathfrak{M})$ corresponds to $-1 \in \operatorname{Aut}(L)$), so a metabolic form and its hyperbolic form produce exactly 0 in $E^1(\mathfrak{M}, D)$. ∎

Proposition 3.5. *If $\big(K \oplus DK, \left[\begin{smallmatrix} 0 & 1 \\ \delta_K & 0 \end{smallmatrix}\right]\big)$ and $\big(L \oplus DL, \left[\begin{smallmatrix} 0 & 1 \\ \delta_L & 0 \end{smallmatrix}\right]\big)$ are hyperbolic spaces such that $[K \oplus DK] = [L \oplus DL]$ in $K_0(\mathfrak{M})$, and*

$$[K] - [L] \equiv 0 \bmod (1 - D)K_0(\mathfrak{M}),$$

then

$$\varepsilon^1\big(\left[\begin{smallmatrix} 0 & 1 \\ \delta_K & 0 \end{smallmatrix}\right] \div \left[\begin{smallmatrix} 0 & 1 \\ \delta_L & 0 \end{smallmatrix}\right]\big) \equiv 0 \bmod (1 + D)K_1(\mathfrak{M}).$$

Proof. Denote $H(X) = X \oplus DX$ for an object X. The second assumption implies the existence of objects X, Y of $\mathfrak{M}$ such that $[K] - [L] = ([X] - [DX]) - ([Y] - [DY])$, i.e.

$$[K \oplus DX \oplus Y] = [L \oplus X \oplus DY].$$

The common resolution may be chosen in a special way: as a direct sum

$$H(K \oplus X \oplus Y \oplus R \oplus S) \twoheadleftarrow P \oplus P' \oplus DR \oplus DS \leftarrowtail Q \oplus Q',$$
$$H(L \oplus X \oplus Y \oplus R \oplus S) \twoheadleftarrow P \oplus P' \oplus DR \oplus DS \leftarrowtail Q \oplus Q'$$

of exact sequences

$$(K \oplus DX \oplus Y) \oplus R \xleftarrow{\left[\begin{smallmatrix}\kappa\\ \rho\end{smallmatrix}\right]} P \xleftarrow{\alpha} Q,$$

$$(L \oplus X \oplus DY) \oplus R \xleftarrow{\left[\begin{smallmatrix}l\\ r\end{smallmatrix}\right]} P \xleftarrow{a} Q,$$

and

$$(DK \oplus X \oplus DY) \oplus S \xtwoheadleftarrow{\left[\begin{smallmatrix}\kappa' \\ \sigma\end{smallmatrix}\right]} P' \xleftarrowtail{\alpha'} Q',$$

$$(DL \oplus DX \oplus Y) \oplus S \xtwoheadleftarrow{\left[\begin{smallmatrix}l' \\ s\end{smallmatrix}\right]} P' \xleftarrowtail{a'} Q',$$

with added $DR \overset{1}{\twoheadleftarrow} DR \leftarrowtail 0$ and $DS \overset{1}{\twoheadleftarrow} DS \leftarrowtail 0$ respectively.

This common resolution glued with its dual yields the double exact sequence

$$DQ \oplus DQ' \xtwoheadleftarrow[\left[\begin{smallmatrix}D\alpha & 0 & 0 & 0 \\ 0 & D\alpha' & 0 & 0\end{smallmatrix}\right]]{\left[\begin{smallmatrix}Da & 0 & 0 & 0 \\ 0 & Da' & 0 & 0\end{smallmatrix}\right]} DP \oplus DP' \oplus R \oplus S$$

$$\xleftarrow[\left[\begin{smallmatrix}0 & v & D\rho & 0 \\ Dv & 0 & 0 & Ds \\ \delta\circ r & 0 & 0 & 0 \\ 0 & \delta\circ s & 0 & 0\end{smallmatrix}\right]]{\left[\begin{smallmatrix}0 & u & D\rho & 0 \\ Du & 0 & 0 & D\sigma \\ \delta\circ\rho & 0 & 0 & 0 \\ 0 & \delta\circ\sigma & 0 & 0\end{smallmatrix}\right]} P \oplus P' \oplus DR \oplus DS \xleftarrowtail[\left[\begin{smallmatrix}a & 0 \\ 0 & a' \\ 0 & 0 \\ 0 & 0\end{smallmatrix}\right]]{\left[\begin{smallmatrix}\alpha & 0 \\ 0 & \alpha' \\ 0 & 0 \\ 0 & 0\end{smallmatrix}\right]} Q \oplus Q',$$

which is a direct sum of the double exact sequence

$$DQ' \xtwoheadleftarrow[[Da', 0]]{[D\alpha', 0]} DP' \oplus R \xleftarrow[\left[\begin{smallmatrix}Dv & Ds \\ \delta\circ r & 0\end{smallmatrix}\right]]{\left[\begin{smallmatrix}Du & D\sigma \\ \delta\circ\rho & 0\end{smallmatrix}\right]} P \oplus DS \xleftarrowtail[\left[\begin{smallmatrix}a \\ 0\end{smallmatrix}\right]]{\left[\begin{smallmatrix}\alpha \\ 0\end{smallmatrix}\right]} Q$$

with one isomorphic to its dual. ■

The assumption on the class in the group $E_{-}^{0}(\mathfrak{M}; D)$ is necessary:

Example 3.1. Let X be a projective line $X = \mathbf{Proj}\, F[x, y]$ over a field F, where x, y are homogeneous coordinates. Consider vector bundles on X with the dualization

$$DA = A\hat{} = \mathcal{H}\mathrm{om}(A, \mathcal{O}_X)$$

and the canonical isomorphism $A \to A\hat{}\,\hat{}$ as δ_A. The equality

$$[\mathcal{O}_X(-1)] + [\mathcal{O}_X(1)] = 2[\mathcal{O}_X]$$

in $K_0(X)$ follows from the exactness of the sequence

$$\mathcal{O}_X(1) \xtwoheadleftarrow{[x, y]} \mathcal{O}_X \oplus \mathcal{O}_X \xleftarrowtail{\left[\begin{smallmatrix}-y \\ x\end{smallmatrix}\right]} \mathcal{O}_X(-1).$$

Consider the hyperbolic forms

$$\varphi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : \mathcal{O}_X(-1) \oplus \mathcal{O}_X(1) \to \mathcal{O}_X(1) \oplus \mathcal{O}_X(-1),$$

$$\psi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : \mathcal{O}_X \oplus \mathcal{O}_X \to \mathcal{O}_X \oplus \mathcal{O}_X.$$

The common resolution

$$\mathcal{O}_X \oplus \mathcal{O}_X \xleftarrow{\left[\begin{smallmatrix}0&1&0\\0&0&1\end{smallmatrix}\right]} \mathcal{O}_X(-1) \oplus \mathcal{O}_X \oplus \mathcal{O}_X \xleftarrow{\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]} \mathcal{O}_X(-1),$$

$$\mathcal{O}_X(-1) \oplus \mathcal{O}_X(1) \xleftarrow{\left[\begin{smallmatrix}1&0&0\\0&x&y\end{smallmatrix}\right]} \mathcal{O}_X(-1) \oplus \mathcal{O}_X \oplus \mathcal{O}_X \xleftarrow{\left[\begin{smallmatrix}0\\-y\\x\end{smallmatrix}\right]} \mathcal{O}_X(-1)$$

glued with its dual yields the double exact sequence

$$\mathcal{O}_X(1) \xleftarrow[{[0,-y,x]}]{[1,0,0]} \mathcal{O}_X(1) \oplus \mathcal{O}_X^2 \xleftarrow[\left[\begin{smallmatrix}0&x&y\\x&0&0\\y&0&0\end{smallmatrix}\right]]{\left[\begin{smallmatrix}0&0&0\\0&0&1\\0&1&0\end{smallmatrix}\right]} \mathcal{O}_X(-1) \oplus \mathcal{O}_X^2 \xleftarrow[\left[\begin{smallmatrix}0\\-y\\x\end{smallmatrix}\right]]{\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]} \mathcal{O}_X(-1). \tag{3.6}$$

Since the short double exact sequence

$$\mathcal{O}_X(-1) \oplus \mathcal{O}_X^2 \xleftarrow[1]{\left[\begin{smallmatrix}1&0&0\\y&1&0\\x&0&1\end{smallmatrix}\right]} \mathcal{O}_X(-1) \oplus \mathcal{O}_X^2 \leftarrowtail 0$$

produces 0 in $K_1(X)$, the class of (3.1) in $K_1(X)$ is, by the 3×4 lemma, the same as the class of the double exact sequence

$$\mathcal{O}_X(1) \xleftarrow[{[0,-y,x]}]{[1,0,0]} \mathcal{O}_X(1) \oplus \mathcal{O}_X^2 \xleftarrow[\left[\begin{smallmatrix}0&x&y\\x&0&0\\y&0&0\end{smallmatrix}\right]]{\left[\begin{smallmatrix}0&0&0\\x&0&1\\y&1&0\end{smallmatrix}\right]} \mathcal{O}_X(-1) \oplus \mathcal{O}_X^2 \xleftarrow[\left[\begin{smallmatrix}0\\-y\\x\end{smallmatrix}\right]]{\left[\begin{smallmatrix}1\\-y\\-x\end{smallmatrix}\right]} \mathcal{O}_X(-1),$$

which in turn is the middle row of the commutative diagram

$$
\begin{tikzcd}[column sep=large, row sep=huge]
\mathcal{O}_X(1) \arrow[d] & \mathcal{O}_X^2 \arrow[l, two heads, "{[y,x]}", "{[y,x]}"'] \arrow[d, tail, "{\left[\begin{smallmatrix} y & x \\ -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]}"', "{\left[\begin{smallmatrix} y & x \\ -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]}"] & \mathcal{O}_X(-1) \arrow[l, tail, "{\left[\begin{smallmatrix} -x \\ y \end{smallmatrix}\right]}", "{\left[\begin{smallmatrix} -x \\ y \end{smallmatrix}\right]}"'] \arrow[d, tail, "{\left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]}"', "{\left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]}"] & \\
\mathcal{O}_X(1) & \mathcal{O}_X(1) \oplus \mathcal{O}_X^2 \arrow[l, two heads] \arrow[d, two heads, "{[1,y,-x]}"', "{[1,y,-x]}"] & \mathcal{O}_X(-1) \oplus \mathcal{O}_X^2 \arrow[l] \arrow[d, two heads, "{\left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]}"', "{\left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]}"] & \mathcal{O}_X(-1) \arrow[l, tail] \arrow[d] \\
& \mathcal{O}_X(1) & \mathcal{O}_X^2 \arrow[l, two heads, "{[-x,y]}", "{[x,y]}"'] & \mathcal{O}_X(-1). \arrow[l, tail, "{\left[\begin{smallmatrix} -y \\ -x \end{smallmatrix}\right]}", "{\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]}"']
\end{tikzcd}
$$

It follows that the relative discriminant is the class of the short double exact sequence

$$
\begin{tikzcd}[column sep=large]
\mathcal{O}_X(1) & \mathcal{O}_X^2 \arrow[l, two heads, "{[-x,y]}", "{[x,y]}"'] & \mathcal{O}_X(-1), \arrow[l, tail, "{\left[\begin{smallmatrix} -y \\ -x \end{smallmatrix}\right]}", "{\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]}"']
\end{tikzcd}
$$

which occurs in the commutative diagram

$$
\begin{tikzcd}[column sep=huge, row sep=large]
0 \arrow[d, tail] & 0 \arrow[l] \arrow[d, tail] & 0 \arrow[l] \arrow[d, tail] \\
\mathcal{O}_X(1) \arrow[d, two heads, "1"', "1"] & \mathcal{O}_X^2 \arrow[l, two heads, "{[-x,y]}", "{[x,y]}"'] \arrow[d, two heads, "{\left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]}"', "1"] & \mathcal{O}_X(-1) \arrow[l, tail, "{\left[\begin{smallmatrix} -y \\ -x \end{smallmatrix}\right]}", "{\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]}"'] \arrow[d, two heads, "1"', "-1"] \\
\mathcal{O}_X(1) & \mathcal{O}_X^2 \arrow[l, two heads, "{[x,y]}", "{[x,y]}"'] & \mathcal{O}_X(-1) \arrow[l, tail, "{\left[\begin{smallmatrix} y \\ -x \end{smallmatrix}\right]}", "{\left[\begin{smallmatrix} y \\ -x \end{smallmatrix}\right]}"']
\end{tikzcd}
$$

It follows from Theorem 2.9 that the relative discriminant equals

$$
-\{-1\}[\mathcal{O}_X] - \{-1\}[\mathcal{O}_X(-1)] = \{-1\}([\mathcal{O}_X] - [\mathcal{O}_X(-1)])
$$

($\{-1\}$ has order 2 in $K_1(F)$), which is not 0.

Remark 3.1. It is easy to see that if the classes of symmetric bilinear forms φ, ψ are equal in the Grothendieck group of symmetric bilinear spaces, then $\varepsilon^1(\varphi \div \psi) = 0$. It follows that the Grothendieck ring of symmetric bilinear spaces of the projective line over a field F differs from the Grothendieck ring of symmetric bilinear spaces of the field F, in contrast to the case of Witt rings.

As an example shows, to define an invariant of Witt equivalence one must factor out the relative discriminants of pairs of hyperbolic forms.

DEFINITION 3.5. Let $\mathcal{H}(\mathfrak{M}; D, \delta)$ be the subgroup of $E^1(\mathfrak{M}; D)$ generated by the classes of all the relative discriminants $\varepsilon^1(\mu \div \nu)$ of pairs of hyperbolic spaces (the class of hyperbolic spaces depends on δ). The *k_1-group* of $(\mathfrak{M}; D, \delta)$ is the factor group

$$k_1(\mathfrak{M}; D) = E^1(\mathfrak{M}; D)/\mathcal{H}(\mathfrak{M}; D, \delta).$$

For applications it is important (by Proposition 3.5) that to compute the group $\mathcal{H}(\mathfrak{M}; D, \delta)$ it is enough to compute the relative discriminants of pairs of hyperbolic spaces corresponding to nonzero elements of $E^0_-(\mathfrak{M}; D)$.

COROLLARY 3.6. *There is a natural surjective homomorphism $E^0_-(\mathfrak{M}; D) \to \mathcal{H}(\mathfrak{M}; D, \delta)$. In particular $\mathcal{H}(\mathfrak{M}; D, \delta) = 0$ whenever $E^0_-(\mathfrak{M}; D) = 0$.*

Proof. Consider the pullback $\mathfrak{X} = \mathfrak{X}(\mathfrak{M}; D, \delta)$,

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & K_0(\mathcal{M}) \\ \downarrow & & \downarrow{\scriptstyle 1+D} \\ K_0(\mathcal{M}) & \xrightarrow{1+D} & K_0(\mathcal{M}) \end{array}$$

There are two homomorphisms defined on $\mathfrak{X}$:

- $([K], [L]) \mapsto [K] - [L] \bmod (1 - D)K_0(\mathfrak{M}) \in E^0_-(\mathfrak{M}; D)$,
- $([K], [L]) \mapsto \varepsilon^1(H(K), H(L)) \in E^1(\mathfrak{M}; D)$.

Proposition 3.5 states that the kernel of the first is contained in the kernel of the second, so there is a homomorphism $E^0_-(\mathfrak{M}; D) \to E^1(\mathfrak{M}; D)$.

Whenever two hyperbolic forms $H(A)$, $H(B)$ such that $[H(A)] = [H(B)]$ in $K_0(\mathcal{M})$ are given, we have

$$\begin{aligned} [A] + D[A] &= [B] + D[B], \\ [A] - [B] &= D[B] - D[A] = -D([A] - [B]) \end{aligned}$$

so $[A] - [B]$ defines an element of $E^0_-(\mathfrak{M}; D)$ which maps onto the class $\varepsilon^1(H(A) \div H(B))$. ∎

For instance, for the projective line $X = \mathbb{P}^1_F$ from Example 3.1, the group

$$\mathcal{H}(X) = \mathcal{H}(\mathfrak{M}; D, \delta) = \mu_2(F) \cdot E^0_-(X) \tag{3.7}$$

has two elements: 0 and $\{-1\}([\mathcal{O}_X] - [\mathcal{O}_X(-1)])$.

COROLLARY 3.7. *If for symmetric bilinear spaces (A, φ) and (B, ψ) there are metabolic forms (M, μ), (M', μ'), (N, ν) and (N', ν') with common res-*

olutions

$$A \oplus M \twoheadleftarrow P \leftarrowtail Q, \quad B \oplus N \twoheadleftarrow P \leftarrowtail Q,$$
$$A \oplus M' \twoheadleftarrow R \leftarrowtail S, \quad B \oplus N' \twoheadleftarrow R \leftarrowtail S,$$

then the difference

$$\varepsilon^1(\varphi \oplus \mu \div \psi \oplus \nu) - \varepsilon^1(\varphi \oplus \mu' \div \psi \oplus \nu')$$

belongs to $\mathcal{H}(\mathfrak{M}; D, \delta)$.

Proof. This is a formal consequence of Theorem 3.3 and the fact that the difference of the double exact sequences

$$DQ \twoheadleftarrow DP \leftarrow P \leftarrowtail Q, \quad DS \twoheadleftarrow DR \leftarrow R \leftarrowtail S$$

is the class of the direct sum of the first one and of the second one turned upside-down:

$$DQ \oplus DS \twoheadleftarrow DP \oplus DR \leftarrow P \oplus R \leftarrowtail Q \oplus S.$$

Denote by u its class in $K_1(\mathfrak{M})$. Next choose a common resolution

$$M' \oplus N \oplus W \twoheadleftarrow U \leftarrowtail T, \quad M \oplus N' \oplus W \twoheadleftarrow U \leftarrowtail T$$

for some metabolic (W, ω). Since

$$\varepsilon^1(\mu' \oplus \nu \oplus \omega \div \mu \oplus \nu' \oplus \omega) \equiv 0 \bmod \mathcal{H}(\mathfrak{M}; D, \delta),$$

the class v in $K_1(\mathfrak{M})$ defined by the direct sum

$$DQ \oplus DS \oplus DT \twoheadleftarrow DP \oplus DR \oplus DU \leftarrow P \oplus R \oplus U \leftarrowtail Q \oplus S \oplus T$$

is congruent to u modulo $\mathcal{H}(\mathfrak{M}; D, \delta)$. But this class corresponds to the common resolution of the isomorphic forms

$$(X, \xi) = (A \oplus M \oplus B \oplus N' \oplus M' \oplus N \oplus W, \varphi \oplus \mu \oplus \psi \oplus \nu' \oplus \mu' \oplus \nu \oplus \omega),$$
$$(Y, \upsilon) = (B \oplus N \oplus A \oplus M' \oplus M \oplus N' \oplus W, \psi \oplus \nu \oplus \varphi \oplus \mu' \oplus \mu \oplus \nu' \oplus \omega).$$

By Theorem 3.3 this class is congruent modulo $\mathcal{H}(\mathfrak{M}; D, \delta)$ to the class of

$$0 \twoheadleftarrow DX \underset{\xi}{\overset{\xi}{\leftarrow}} X \leftarrowtail 0.$$

It follows that $u \equiv 0$ modulo $\mathcal{H}(\mathfrak{M}; D, \delta)$. ∎

The goal of this paper is to define the discriminant map:

DEFINITION 3.6. Let $I(\mathfrak{M}; D, \delta) = \operatorname{Ker} e^0$. The *discriminant map*

$$e^1 : I(\mathfrak{M}; D, \delta) \to k_1(\mathfrak{M}; D)$$

is defined as follows:

$$\begin{array}{ll} \text{if} & [A \oplus M \oplus DM] = [N \oplus DN] \text{ in } K_0(\mathfrak{M}), \\ \text{then} & e^1(A, \varphi) = \varepsilon^1(\varphi \oplus \mu \div \nu) \bmod \mathcal{H}(\mathfrak{M}; D, \delta), \end{array}$$

where $\mu : M \oplus DM \to DM \oplus M$ and $\nu : N \oplus DN \to DN \oplus N$ are hyperbolic forms.

Let us state several elementary properties of these notions.

PROPOSITION 3.8. *For arbitrary exact categories with duality* $(\mathfrak{M}, (D, \delta))$, $(\mathfrak{N}, (D', \delta'))$,

(i) *the* E^n*-groups are elementary* 2*-groups* (*groups of exponent* 2).
(ii) *if* $f : \mathfrak{M} \to \mathfrak{N}$ *is an exact functor which commutes with the duality, then* f *induces homomorphisms*

$$f : W(\mathfrak{M}; D, \delta) \to W(\mathfrak{N}; D', \delta'),\ \ f : I(\mathfrak{M}; D, \delta) \to I(\mathfrak{N}; D', \delta'),$$
$$f : E^n(\mathfrak{M}; D, \delta) \to E^n(\mathfrak{N}; D', \delta'),\ \ f : E^n_-(\mathfrak{M}; D, \delta) \to E^n_-(\mathfrak{N}; D', \delta'),$$
$$f : k_1(\mathfrak{M}; D, \delta) \to k_1(\mathfrak{N}; D, \delta)$$

such that the diagrams

$$\begin{tikzcd} W(\mathfrak{M}; D, \delta) \arrow[r, "e^0"] \arrow[d, "f"'] & E^0(\mathfrak{M}; D) \arrow[d, "f"] \\ W(\mathfrak{N}; D', \delta') \arrow[r, "e^0"] & E^0(\mathfrak{N}; D') \end{tikzcd} \qquad \begin{tikzcd} I(\mathfrak{M}; D, \delta) \arrow[r, "e^1"] \arrow[d, "f"'] & k_1(\mathfrak{M}; D) \arrow[d, "f"] \\ I(\mathfrak{N}; D', \delta') \arrow[r, "e^1"] & k_1(\mathfrak{N}; D') \end{tikzcd}$$

commute.

Proof. (i) If $(1 \pm D)a = 0$, then $2a = (1 \mp D)a \equiv 0 \bmod \mathrm{Im}(1 \mp D)$. ∎

The following example provides a motivation for the notation e^1 and $I(\mathfrak{M}; D, \delta)$, and shows that the above defined notion generalizes the usual notion of the discriminant of a quadratic form.

EXAMPLE 3.2. Let $\mathfrak{M}$ be the category of vector spaces over a field F with the usual dualization. In this case it is obvious that e^0 coincides with the usual dimension index, so $I(\mathfrak{M}) = I(F)$. Moreover, $E^0_-(F) = 0$, so $k_1(\mathfrak{M}) = E^1(\mathfrak{M})$. Note that D acts trivially on $K_1(F)$, so

$$E^1(F) \overset{\text{df}}{=} E^1(\mathfrak{M}) = K_1(F)/2K_1(F) = k_1(F),$$

hence the new notation $k_1(F)$ is consistent with the one introduced in [9]. Moreover,

$$E^1_-(F) \overset{\text{df}}{=} E^1_-(\mathfrak{M}) = \mu_2(F) = \{1, -1\}.$$

The group $E^1(F) \cong F^*/F^{*2}$ is usually denoted by $g(F)$ (the square classes group) in the theory of quadratic forms. If $e^0(A, \varphi) = 0$, then A is an even-dimensional vector space, $\dim(A) = 2k$, so there exists an isomorphism $\rho : B \oplus B^* \to A$ of a space $B \oplus B^*$ supporting a hyperbolic form χ, with A. Thus one may choose the exact sequences (3.1) in a special way:

$$A \xleftarrow{\rho} B \oplus B^* \leftarrow 0 \leftarrow 0, \qquad B \oplus B^* \xleftarrow{1} B \oplus B^* \leftarrow 0 \leftarrow 0.$$

The double exact sequence (3.3)

$$0 \twoheadleftarrow B \oplus B^* \overset{\chi}{\underset{\rho^*\circ\varphi\circ\rho}{\leftleftarrows}} B \oplus B^* \leftarrowtail 0$$

defines the element

$$\det(\varphi)\det(\chi) = (-1)^k \det(\varphi) = (-1)^{2k(2k-1)/2} \det(\varphi) \bmod 2K_1(F),$$

which is exactly the discriminant of (A, φ).

The additional E^1-group $E^1_-(F) = \mu_2(F) = \{1, -1\}$ has no direct interpretation yet.

Example 3.3. If $\mathfrak{M}$ is the category of vector bundles on a scheme X with the usual dualization ($D = \hat{\ } = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(-, \mathcal{O}_X)$), then we write

$$W(X) = W(\mathfrak{M}), \quad I(X) = I(\mathfrak{M}), \quad E^n(X) = E^n(\mathfrak{M}), \quad k_1(X) = k_1(\mathfrak{M}).$$

Let $f : Y \to X$ be a morphism of schemes, and $\mathfrak{N}$ be the category of vector bundles on Y. The exact functor $f^* : \mathfrak{M} \to \mathfrak{N}$ commutes with the dualization, so it induces homomorphisms on E-groups and Witt groups, and homomorphisms of E^0 and W commute with e^0. Thus the homomorphism of Witt groups maps $I(X)$ into $I(Y)$, and commutes with e^1. This means that if, in particular, Y is a point, then the above defined discriminant reduces to the usual discriminant on fibers.

It is known that there exist symmetric bilinear forms of even rank (and hence with e^0 trivial on fibers) which have nontrivial global e^0, say on split projective quadrics of even dimension. So there is no reason to expect in general that local discriminants determine the value of e^1.

The following property of e^1 provides a motivation for adopting the classical notation $I(X)$ for the notion defined above:

Theorem 3.9. *Consider the category $\mathcal{P}_X$ of locally free sheaves of finite rank on a variety X over a field F with a line bundle L (possibly trivial), the usual duality functor $DA = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(A, L)$ and δ either the canonical isomorphism, or its negative. Let moreover $c \in \dot{F}$ be a nonzero constant. Then for each form (A, φ) such that $e^0(\varphi) = 0$ we have*

$$e^1(c \cdot \varphi) = e^1(\varphi) \quad \text{in } k_1(X).$$

Proof. Choose metabolic (M, μ), (N, ν) such that $A \oplus M$ and N have a common resolution:

$$A \oplus M \overset{\pi}{\twoheadleftarrow} P \overset{\rho}{\leftarrowtail} Q, \qquad N \overset{\pi'}{\twoheadleftarrow} P \overset{\rho'}{\leftarrowtail} Q.$$

Note that $(M, c \cdot \mu)$ and $(N, c \cdot \nu)$ are metabolic forms, and the four-term double exact sequences needed to compute discriminants are the rows of the commutative diagram

$$\begin{tikzcd}[column sep=large]
DQ \arrow[d, "1"', "1"] & DP \arrow[l, two heads, shift right, "D\rho"'] \arrow[l, two heads, shift left, "D\rho'"] \arrow[d, "1"', "1"] & & P \arrow[ll, shift right, "cD\pi\circ(\varphi\oplus\mu)\circ\pi"'] \arrow[ll, shift left, "cD\pi'\circ(\varphi\oplus\mu)\circ\pi'"] \arrow[d, "c"', "c"] & Q \arrow[l, tail, shift right, "D\rho"'] \arrow[l, tail, shift left, "D\rho'"] \arrow[d, "c"', "c"] \\
DQ & DP \arrow[l, two heads, shift right, "\rho"'] \arrow[l, two heads, shift left, "\rho'"] & & P \arrow[ll, shift right, "D\pi\circ(\varphi\oplus\mu)\circ\pi"'] \arrow[ll, shift left, "D\pi'\circ(\varphi\oplus\mu)\circ\pi'"] & Q \arrow[l, tail, shift right, "\rho"'] \arrow[l, tail, shift left, "\rho'"]
\end{tikzcd}$$

with isomorphic vertical arrows. It follows from the 3×4 lemma that both rows define the same class in $K_1(X)$, so $e^1(c \cdot \varphi) = e^1(\varphi)$. ∎

Corollary 3.10. *Under the assumptions of the theorem,*

$$I(F) \cdot I(X) \subset \operatorname{Ker}(e^1). \ ∎$$

4. The map $e^1 : I(X) \to k_1(X)$ for a certain projective variety X. Here we assume that the dualization (D, δ) is fixed, and suppress it in the notation. It is easy to compute the E^1-groups in the following particular case:

Theorem 4.1. *Assume that X is a quasiprojective variety over a field F such that $K_1(X) = K_0(X) \otimes_{\mathbb{Z}} K_1(F)$. Then*

$$E^1(X) \cong E^1(F) \otimes E^0(X) \oplus E^1_-(F) \otimes E^0_-(X),$$
$$E^1_-(X) \cong E^1(F) \otimes E^0_-(X) \oplus E^1_-(F) \otimes E^0(X).$$

Proof. The E-groups are the Tate cohomology groups of the group $\{1, D\}$:

$$E^1(X) = \hat{H}^0(\{1, D\}, K_1(F) \otimes K_0(X)),$$
$$E^1_-(X) = \hat{H}^1(\{1, D\}, K_1(F) \otimes K_0(X)).$$

By the universal coefficients formula there are exact sequences

$$0 \to K_1(F) \otimes \hat{H}^i(\{1, D\}, K_0(X)) \to E^1_{\pm}(X) \\ \to \operatorname{Tor}_1(K_1(F), \hat{H}^{i+1}(\{1, D\}, K_0(X))) \to 0,$$

i.e. exact sequences

$$0 \to K_1(F) \otimes E^0(X) \to E^1(X) \to \operatorname{Tor}_1(K_1(F), E^0_-(X)) \to 0,$$
$$0 \to K_1(F) \otimes E^0_-(X) \to E^1_-(X) \to \operatorname{Tor}_1(K_1(F), E^0(X)) \to 0.$$

$E^0(X)$ and $E^0_-(X)$ are elementary 2-groups (groups of exponent 2), so

$$K_1(F) \otimes E^0_{\pm}(X) = g(F) \otimes E^0_{\pm}(X),$$
$$\operatorname{Tor}_1(K_1(F), E^0_{\pm}(X)) = \mu_2(F) \otimes E^0_{\pm}(X),$$

and the assertion is proved. ∎

It is obvious that $\mathcal{H}(X) = \mathcal{H}(\mathfrak{M}) \subset \mu_2(F) \otimes E^0_-(X)$.

Conjecture 4.2. *Under the assumptions of Theorem* 4.1,

$$\mathcal{H}(X) = E^1_-(F) \otimes E^0_-(X), \qquad k_1(X) = E^1(F) \otimes E^0(X).$$

Example 4.1. The projective space $X = \mathbb{P}^n_F = \mathbf{Proj}\, F[x_0, x_1, \ldots, x_n]$ with the usual dualization functor ($D = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(-, \mathcal{O}_X)$) satisfies the assumption of Theorem 4.1. In this case the E^0-groups are known:

$$E^0(X) = \mathbb{Z}/2\mathbb{Z}[\mathcal{O}_X] \quad \text{and} \quad E^0_-(X) = \begin{cases} 0 & \text{for even } n, \\ \mathbb{Z}/2\mathbb{Z}H^n & \text{for odd } n, \end{cases}$$

where $H = 1 - [\mathcal{O}_X(-1)]$ is the class of a hyperplane section ([21, Prop. 2.1.3], [23, Prop. 5.1], [22, Cor. 3.4]). Thus:

$$E^1(X) = \begin{cases} g(F) \cdot [\mathcal{O}_X] & \text{for even } n, \\ g(F) \cdot [\mathcal{O}_X] \oplus \mu_2(F) \cdot H^n & \text{for odd } n, \end{cases}$$
$$\mathcal{H}(X) = \begin{cases} 0 & \text{for even } n, \\ \mu_2(F) \cdot H^n & \text{for odd } n, \end{cases}$$
$$E^1_-(X) = \begin{cases} \mu_2(F) \cdot [\mathcal{O}_X] & \text{for even } n, \\ \mu_2(F) \cdot [\mathcal{O}_X] \oplus g(F)H^n & \text{for odd } n. \end{cases}$$

It is known that the canonical map $W(F) \to W(X)$ induced by the inverse image functor $V \mapsto V \otimes_F \mathcal{O}_X$ for the structure map $X \to \mathbf{Spec}\,(F)$ is an isomorphism ([1, Satz]). Therefore $I(X) = I(F) \otimes_F \mathcal{O}_X$ and the map $e^1 : I(X) \to k_1(X)$ is surjective for even n.

For a field F of characteristic different from 2, the classes of hyperbolic spaces form a cyclic direct summand of even order in the Grothendieck group of symmetric bilinear spaces generated by hyperbolic planes ([24, Th. 1.1]). This is not so for odd-dimensional projective spaces. First of all, there is an infinite sequence of hyperbolic planes

$$\mathfrak{h}_k = \mathcal{O}_X(k) \oplus \mathcal{O}_X(-k) \quad \text{for } k = 0, 1, 2, \ldots.$$

Corollary 4.3. *On the projective space $X = \mathbb{P}^n_F$ of odd dimension n there exists a hyperbolic space whose class in the Grothendieck group of symmetric bilinear spaces is not a multiple of the standard hyperbolic plane $\mathfrak{h}_0 = \left(\mathcal{O}^2_X, \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]\right)$.*

Proof. For the case $n = 1$ see Example 3.1. For $n > 1$ there exists a pair of hyperbolic spaces with relative discriminant $\{-1\}H^n \neq 0$, since $\mathcal{H}$ is nontrivial.

Let $n = 2k - 1$ and

$$A = \sum_{i=0}^{k-1} \binom{2k-1}{2i} \mathfrak{h}_i, \tag{4.1}$$

$$B = \sum_{i=0}^{k-1} \binom{2k-1}{2i+1} \mathfrak{h}_i. \tag{4.2}$$

The identity $H^{n+1} = 0$ implies that A and B define equal classes in $K_0(X)$. The proof of Corollary 3.6 shows that $\varepsilon^1(A \div B)$ is nonzero. Thus A, B cannot both be multiples of the standard hyperbolic plane $\mathfrak{h}_0$. ∎

4.1. A conic. We compute the E_0-groups and E_1-groups of a projective conic X:

$$q = z_0^2 - az_1^2 - bz_2^2$$

for the anisotropic quadratic form $\langle 1, -a, -b\rangle$ over a field F defined on the space V over F with base v_0, v_1, v_2 (z_0, z_1, z_2 is the dual base of the dual space V^*), so that the projective conic

$$X = \mathbf{Proj}\, S(V^*)/(q)$$

has no rational points.

The Clifford algebra $C(q)$ has generators

$$1, v_0, v_1, v_2, v_0v_1, v_0v_2, v_1v_2, v_0v_1v_2,$$

and relations

$$v_iv_j = -v_jv_i \quad \text{for } i \neq j,$$
$$v_0^2 = 1, \quad v_1^2 = -a, \quad v_2^2 = -b.$$

The elements $1, v_0v_1, v_0v_2, v_2v_1$ form a base of the even Clifford algebra C_0, which is isomorphic to the quaternion algebra $\left(\frac{a,b}{F}\right)$:

$$\begin{aligned}
(v_0v_1)^2 &= a, & (v_0v_2)(v_2v_1) &= -(v_2v_1)(v_0v_2),\\
(v_0v_2)^2 &= b, & (v_0v_1)(v_2v_1) &= -(v_1v_2)(v_0v_1),\\
(v_2v_1)^2 &= -ab, & (v_0v_1)(v_0v_2) &= -(v_0v_2)(v_0v_1) = v_2v_1.
\end{aligned}$$

We denote as usual

$$i = v_0v_1, \quad j = v_0v_2, \quad k = v_2v_1.$$

Let

$$\varphi = z_0v_0 + z_1v_1 + z_2v_2 \in \Gamma(X, \mathcal{O}_X(1) \otimes C_1)$$

be the "generic zero vector" of q,

$$\varphi^2 = (z_0v_0 + z_1v_1 + z_2v_2)^2 = z_0^2 - az_1^2 - bz_2^2 = 0.$$

The complex

$$\cdots \xrightarrow{\varphi\cdot} \mathcal{O}_X(-n) \otimes_F C_{n+2} \xrightarrow{\varphi\cdot} \mathcal{O}_X(1-n) \otimes_F C_{n+1}$$
$$\xrightarrow{\varphi\cdot} \mathcal{O}_X(2-n) \otimes_F C_n \xrightarrow{\varphi\cdot} \cdots$$

(subscripts in C_n are taken modulo 2) is exact and splits locally ([20, Prop. 8.2(a)]). For the bases $1, i, j, k$ of C_0 and $v_0, v_1, v_2, v_2v_1v_0$ of C_1 the maps $\varphi\cdot -$ have the matrices

$$\begin{bmatrix} z_0 & -az_1 & -bz_2 & 0 \\ -z_1 & z_0 & 0 & bz_2 \\ -z_2 & 0 & z_0 & -az_1 \\ 0 & z_2 & -z_1 & z_0 \end{bmatrix} \quad \text{for } \mathcal{O}_X(2s-1)\otimes_F C_1 \xrightarrow{\varphi\cdot} \mathcal{O}_X(2s)\otimes C_0,$$

$$\begin{bmatrix} z_0 & az_1 & bz_2 & 0 \\ z_1 & z_0 & 0 & -bz_2 \\ z_2 & 0 & z_0 & az_1 \\ 0 & -z_2 & z_1 & z_0 \end{bmatrix} \quad \text{for } \mathcal{O}_X(2s)\otimes_F C_0 \xrightarrow{\varphi\cdot} \mathcal{O}_X(2s+1)\otimes C_1.$$

Definition 4.1. The *Swan sheaf* $\mathcal{U}$ is defined as the cokernel

$$\mathcal{U} = \mathcal{U}_0 = \operatorname{Coker}(\mathcal{O}_X(-2)\otimes_F C_0 \xrightarrow{\varphi\cdot} \mathcal{O}_X(-1)\otimes_F C_1).$$

Let $E_0^{\pm}(X)$, $E_0^{\pm}(X,L)$ be the E-groups of the dualization

$$A\hat{} = \mathcal{H}\mathrm{om}(A, \mathcal{O}_X), \qquad A^{\hat{}L} = \mathcal{H}\mathrm{om}(A, \mathcal{O}_X(-1)),$$

with canonical δ, δ_L respectively.

Proposition 4.4. *The E-groups of X are:*

$$E_0(X) = (\mathbb{Z}/2)\cdot[\mathcal{O}_X], \qquad E_0^-(X) = (\mathbb{Z}/2)\cdot(2-[\mathcal{U}]),$$
$$E_0(X,L) = 0, \qquad E_0^-(X,L) = 0.$$

Proof. There is an exact sequence

$$0 \leftarrow \mathcal{O}_X(1) \leftarrow \mathcal{O}_X^3 \leftarrow \mathcal{O}_X(-1)^3 \leftarrow \mathcal{O}_X(-2) \leftarrow 0$$

inherited from the projective plane. Now to use the results of [20], note that the sheaf $\mathcal{O}_X(1)$ has the truncated canonical resolution

$$0 \leftarrow \mathcal{O}_X(1) \leftarrow \mathcal{O}_X^3 \leftarrow \mathcal{U} \leftarrow 0. \tag{4.3}$$

The Swan sheaf $\mathcal{U}$ is a right module over

$$\operatorname{End}_X(\mathcal{U}) = C_0(\langle 1, -a, -b\rangle) \cong \left(\frac{a,b}{F}\right).$$

The quaternion algebra $\mathcal{D} = \left(\frac{a,b}{F}\right)$ is a skew field, so $\mathcal{U}$ is an indecomposable vector bundle. The sheaf $\mathcal{U}$ has rank 2, so there is a nonsingular skew symmetric pairing $\mathcal{U}\otimes_{\mathcal{O}_X}\mathcal{U} \to \bigwedge^2\mathcal{U}$ given by exterior multiplication. Thus

$$\mathcal{U} \cong \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{U}, \textstyle\bigwedge^2\mathcal{U}).$$

Taking the highest exterior powers in the truncated canonical resolution yields

$$\mathcal{O}_X(1)\otimes \textstyle\bigwedge^2\mathcal{U} \cong \mathcal{O}_X, \qquad \bigwedge^2\mathcal{U} \cong \mathcal{O}_X(-1).$$

It follows that

$$\mathcal{U} \cong \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{U}, \mathcal{O}_X(-1)) \cong \mathcal{U}\hat{\ } \otimes \mathcal{O}_X(-1).$$

The formula for the determinant of the tensor product shows that in general

$$\textstyle\bigwedge^2 \mathcal{U}(n) = \bigwedge^2 (\mathcal{U} \otimes \mathcal{O}_X(n)) \cong (\bigwedge^2 \mathcal{U}) \otimes (\mathcal{O}_X(n))^{\otimes 2} \cong \mathcal{O}_X(2n-1)$$

and

$$\mathcal{U}(n) \cong \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{U}(n), \mathcal{O}_X(2n-1)) \cong \mathcal{U}(n)\hat{\ } \otimes \mathcal{O}_X(2n-1),$$

so $\mathcal{U}(n)$ carries a skew symmetric $\mathcal{O}_X(2n-1)$-valued bilinear form.

It follows that $K_0(X)$ is a free abelian group of rank 2 with a free base $1 = [\mathcal{O}_X]$, $[\mathcal{U}]$ and identities

$$H^3 = 0,$$
$$[\mathcal{O}_X(1)] - 3 + [\mathcal{U}] = 0 \quad \text{or} \quad [\mathcal{O}_X(1)]H = 2 - [\mathcal{U}]$$

where $H = 1 - [\mathcal{O}_X(-1)]$ is the class of the hyperplane section, as above. By [21, Corollary 3.4.3] there is an additional identity

$$[\mathcal{U}] + [\mathcal{U}\hat{\ }] = 4 \quad \text{or} \quad [\mathcal{U}] + [\mathcal{U}(1)] = 4. \tag{4.4}$$

Thus

$$[\mathcal{U}] + [\mathcal{U}(1)] = 3 - [\mathcal{O}_X(1)] + 3[\mathcal{O}_X(1)] - [\mathcal{O}_X(2)] = 4,$$
$$1 - 2[\mathcal{O}_X(1)] + [\mathcal{O}_X(2)] = 0 \quad \text{or} \quad H^2 = 0.$$

The same result (that $H^2 = 0$) may be obtained easily by checking locally that the sequence

$$0 \leftarrow \mathcal{O}_X(1) \xleftarrow{\alpha} \mathcal{O}_X^2 \xleftarrow{\beta} \mathcal{O}_X(-1) \leftarrow 0, \qquad \alpha = [z_0, z_1], \qquad \beta = \begin{bmatrix} z_1 \\ -z_0 \end{bmatrix},$$

is exact. Here $z_i \in \Gamma(X, \mathcal{O}_X(1)) = \mathrm{Hom}_X(\mathcal{O}_X(k), \mathcal{O}_X(k+1))$ are global morphisms of $\mathcal{O}_X$-modules under consideration.

The condition $H^2 = 0$ implies $2 - [\mathcal{U}] = [\mathcal{O}_X(1)]H = H$, and

$$0 = (2 - [\mathcal{U}])^2.$$

It follows that

$$[\mathcal{U}]^2 = 4[\mathcal{U}] - 4,$$
$$[\mathcal{O}_X(1)] = 3 - [\mathcal{U}] = 1 + (2 - [\mathcal{U}]),$$
$$[\mathcal{O}_X(-1)] = [\mathcal{U}] - 1 = 1 - (2 - [\mathcal{U}]).$$

There is a ring isomorphism $K_0(X) \cong \mathbb{Z}[u]/(u^2)$ which maps $2 - [\mathcal{U}]$ onto the coset of u and, by (4.4),

$$1\hat{\ } = 1, \qquad u\hat{\ } = -u.$$

Denote $L = \mathcal{O}_X(-1)$. With respect to the base $1, 2-[\mathcal{U}]$ the involutions $\hat{}$ and $\hat{}^L$ have matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

respectively.

Now the assertion follows immediately. ■

Note that the discriminant map e^1 is defined on the whole $W(X, L)$.

The argument used in the above proof can be reformulated as follows:

COROLLARY 4.5. *In $K_0(X)$ the action of the involutions induced by the duality functors $\hat{}$, $\hat{}^L$ respectively, is given by*

$$(x + y[\mathcal{U}])\hat{} = (x + 4y) - y[\mathcal{U}], \qquad (x + y[\mathcal{U}])\hat{}^L = -x + (x + y)[\mathcal{U}].$$

More precisely: the map $\hat{}$ on $K_0(X)$ is induced by the functors:

- the identity from F-modules to F-modules (the summand x),
- the forgetful functor from $\mathcal{D}$-modules to F-modules (the summand $4y$),
- the identity functor from $\mathcal{D}$-modules to $\mathcal{D}$-modules (the summand $y[\mathcal{U}]$),

while the map $\hat{}^L$ is induced by the functors:

- the identity from F-modules to F-modules (the summand x),
- the natural functor from F-modules to $\mathcal{D}$-modules (the summand $x[\mathcal{U}]$),
- the identity functor from $\mathcal{D}$-modules to $\mathcal{D}$-modules (the summand $y[\mathcal{U}]$).

Proposition 4.4 states that $E_0^-(X, L) = 0$. Hence the group $\mathcal{H}$ is trivial and $k_1(X, L) = E_1(X, L)$.

To compute E^1-groups let us recall some facts from the K-theory of quaternion algebras.

The reduced norm is an injective homomorphism

$$\mathrm{Nrd} : K_1(\mathcal{D}) \to K_1(F)$$

such that for any splitting field E ($E \otimes_F \mathcal{D} \cong M_2(E)$ is the matrix algebra over E), if E/F is finite, then the diagram

$$\text{(4.5)} \qquad \begin{tikzcd} K_1(M_2(E)) \arrow[r, equal] & K_1(E \otimes_F \mathcal{D}) \arrow[d, "N"] \\ K_1(E) \arrow[u, "\sim"] \arrow[d, "N_{E/F}"'] & K_1(\mathcal{D}) \arrow[dl, "\mathrm{Nrd}"] \\ K_1(F) & \end{tikzcd}$$

commutes [8]. It follows that $K_1(\mathcal{D}) = \dot{\mathcal{D}}/[\dot{\mathcal{D}}, \dot{\mathcal{D}}]$ is isomorphic to the group B of values of the quadratic form $\langle 1, -a, -b, ab\rangle$ in $K_1F = \dot{F}$, and the involution $\hat{}$ acts trivially on $K_1(\mathcal{D})$.

Note that the composition $K_1(F) \to K_1(\mathcal{D}) \xrightarrow{\mathrm{Nrd}} K_1(F)$ is the map $x \mapsto x^2$. We will identify $K_1(\mathcal{D})$ with B via the map Nrd.

For example:

• If the involution $\hat{}\ : K_0(X) \to K_0(X)$ maps $x + y[\mathcal{U}] \in K_0(X)$ to $x + 4y - y[\mathcal{U}]$, then $\hat{}\ : K_1(X) \to K_1(X)$ maps $x + y[\mathcal{U}] \in K_1(X)$ to $x + N_{\mathcal{D}/F}(y) - y[\mathcal{U}]$; if $x = \{\xi\} \in K_1(F)$, $y = \{\zeta\} \in K_1(\mathcal{D})$, then

$$x + N_{\mathcal{D}/F}(y) = \{\xi N_{\mathcal{D}/F}(\zeta)\} = \{\xi \,\mathrm{Nrd}(\zeta)^2\} \in K_1(F),$$
$$-y = \{\zeta^{-1}\} \in K_1(\mathcal{D}),$$

and we write $\mathrm{Nrd}(\zeta^{-1}) \in B$ instead of $\{\zeta^{-1}\} \in K_1(\mathcal{D})$.

• If the involution $\hat{}^{L} : K_0(X) \to K_0(X)$ maps $x + y[\mathcal{U}] \in K_0(X)$ to $-x + (x+y)[\mathcal{U}]$, then $\hat{}^{L} : K_1(X) \to K_1(X)$ maps $x + y[\mathcal{U}] \in K_1(X)$ to

$$-x + (r(x) + y)[\mathcal{U}] = \{\xi^{-1}\} + \{\xi\zeta\}[\mathcal{U}],$$

where $r : K_1(F) \to K_1(D)$ is the canonical map; after identification of $K_1(\mathcal{D})$ with B we write $\mathrm{Nrd}(\xi\zeta) = \xi^2\,\mathrm{Nrd}(\zeta) \in B$ instead of $\{\xi\zeta\} \in K_1(\mathcal{D})$.

It follows from Corollary 4.5 that in $K_1(X) = K_1(F) \oplus K_1(\mathcal{D})[\mathcal{U}] = \dot{F} \oplus B \cdot [\mathcal{U}]$ for $t \in \dot{F}$, $u \in B$ we have

$$(t, u[\mathcal{U}])\hat{} = (tu^2, u^{-1}[\mathcal{U}]), \qquad (t, u[\mathcal{U}])^{\hat{}L} = (t^{-1}, t^2u[\mathcal{U}]).$$

Thus the following result is obvious.

Theorem 4.6. *For a projective conic X: $x_0^2 - ax_1^2 - bx_2^2 = 0$ given by the anisotropic quadratic form $\langle 1, -a, -b\rangle$ over a field F and the line bundle $L = \mathcal{O}_X(-1)$ let $E^1_\pm(X) = E^1_\pm(\mathcal{P}_X; \hat{}, \delta)$ and $E^1_\pm(X, L) = E^1_\pm(\mathcal{P}_X; \hat{}^{L}, \delta_L)$. Then*

$$E^1(X) = g(F)[\mathcal{O}_X] \oplus (\mu_2(F) \cap B)[\mathcal{U}], \qquad E^-_1(X) = \mu_2(F)[\mathcal{O}_X] \oplus B/B^2,$$
$$E^1(X, L) \cong \mu_2(F) \oplus B/\dot{F}^2[\mathcal{U}], \qquad E^-_1(X, L) \cong \mu_2(F) \oplus B/\dot{F}^2. \ \blacksquare$$

The following computation looks strange, but will be used below, in the proof of Proposition 4.13.

Consider two $\mathcal{O}_X(-1)$-valued bilinear forms: the first is a *skew symmetric* form

$$\pi : \mathcal{U} \to \mathcal{U}\hat{} \otimes \mathcal{O}_X(-1) = \mathcal{U}^{\hat{}L}$$

given by exterior multiplication $\mathcal{U} \times \mathcal{U} \to \bigwedge^2 \mathcal{U} = \mathcal{O}_X(-1)$, and the second is the *symmetric* hyperbolic form

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : \mathcal{O}_X \oplus \mathcal{O}_X(-1) \to \mathcal{O}_X(-1) \oplus \mathcal{O}_X = (\mathcal{O}_X \oplus \mathcal{O}_X(-1))^{\hat{}L}.$$

Then use the common resolution

$$0 \leftarrow \mathcal{O}_X(1) \xleftarrow{[z_0,-z_1,z_2]} \mathcal{O}_X^3 \xleftarrow{f} \mathcal{U} \leftarrow 0,$$

$$0 \leftarrow \mathcal{O}_X(1) \xleftarrow{[z_0,-z_1,0]} \mathcal{O}_X^3 \xleftarrow{\left[\begin{smallmatrix} 0 & z_1 \\ 0 & z_0 \\ 1 & 0 \end{smallmatrix}\right]} \mathcal{O}_X \oplus \mathcal{O}_X(-1) \leftarrow 0$$

to produce a four-term double exact sequence.

PROPOSITION 4.7. *The class of the double exact sequence*

$$\mathcal{O}_X(1) \xleftarrow[{[z_0,-z_1,0]}]{[z_0,-z_1,z_2]} \mathcal{O}_X^3 \xleftarrow[\left[\begin{smallmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_0 \\ z_1 & z_0 & 0 \end{smallmatrix}\right]]{\left[\begin{smallmatrix} 0 & -z_2 & -z_1 \\ z_2 & 0 & -z_0 \\ z_1 & z_0 & 0 \end{smallmatrix}\right]} \mathcal{O}_X(-1)^3 \xleftarrow[\left[\begin{smallmatrix} z_0 \\ -z_1 \\ 0 \end{smallmatrix}\right]]{\left[\begin{smallmatrix} z_0 \\ -z_1 \\ z_2 \end{smallmatrix}\right]} \mathcal{O}_X(-2) \tag{4.6}$$

in $K_1(X) = K_1(F)[\mathcal{O}_X] \oplus K_1(\mathcal{D})[\mathcal{U}]$ *equals* $\{-1\}[\mathcal{O}_X] + \{1\}[\mathcal{U}]$.

Proof. Two resolutions

$$0 \leftarrow \mathcal{O}_X(1) \xleftarrow{[z_0,-z_1,z_2]} \mathcal{O}_X^3 \xleftarrow{f} \mathcal{U} \leftarrow 0,$$

$$0 \leftarrow \mathcal{O}_X(1) \xleftarrow{[z_0,-z_1,0]} \mathcal{O}_X^3 \xleftarrow{\left[\begin{smallmatrix} 0 & z_1 \\ 0 & z_0 \\ 1 & 0 \end{smallmatrix}\right]} \mathcal{O}_X \oplus \mathcal{O}_X(-1) \leftarrow 0$$

glued with their $\hat{\ }^L$-duals along π and $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ give the double exact sequence

$$\mathcal{O}_X(1) \xleftarrow[{[z_0,-z_1,0]}]{[z_0,-z_1,z_2]} \mathcal{O}_X^3 \xleftarrow[\left[\begin{smallmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_0 \\ z_1 & z_0 & 0 \end{smallmatrix}\right]]{\left[\begin{smallmatrix} 0 & -z_2 & -z_1 \\ z_2 & 0 & -z_0 \\ z_1 & z_0 & 0 \end{smallmatrix}\right]} \mathcal{O}_X(-1)^3 \xleftarrow[\left[\begin{smallmatrix} z_0 \\ -z_1 \\ 0 \end{smallmatrix}\right]]{\left[\begin{smallmatrix} z_0 \\ -z_1 \\ z_2 \end{smallmatrix}\right]} \mathcal{O}_X(-2). \tag{4.7}$$

It may be resolved as follows: in the commutative diagram

$$\begin{tikzcd}[column sep=large, row sep=huge]
\mathcal{O}_X(1) \arrow[d, tail, "1"', "1"] & \mathcal{O}_X^2 \arrow[l, two heads, "{[z_0,-z_1]}", "{[z_0,-z_1]}"'] \arrow[d, tail, "\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]"', "\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]"] & \mathcal{O}_X(-1) \arrow[l, tail, "\left[\begin{smallmatrix} z_1 \\ z_0 \end{smallmatrix}\right]", "\left[\begin{smallmatrix} z_1 \\ z_0 \end{smallmatrix}\right]"'] \arrow[d, tail, "\left[\begin{smallmatrix} 0 \\ 0 \\ -1 \end{smallmatrix}\right]"', "\left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]"] & 0 \arrow[l] \arrow[d] \\
\mathcal{O}_X(1) \arrow[d] & \mathcal{O}_X^3 \arrow[l, two heads] \arrow[d, two heads, "{[0,0,1]}"', "{[0,0,1]}"] & \mathcal{O}_X(-1)^3 \arrow[l] \arrow[d, two heads, "\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right]"', "\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right]"] & \mathcal{O}_X(-2) \arrow[l, tail] \arrow[d, two heads, "1"', "1"] \\
0 & \mathcal{O}_X \arrow[l] & \mathcal{O}_X(-1)^2 \arrow[l, two heads, "{[z_1,z_0]}", "{[z_1,z_0]}"'] & \mathcal{O}_X(-2) \arrow[l, tail, "\left[\begin{smallmatrix} z_0 \\ -z_1 \end{smallmatrix}\right]", "\left[\begin{smallmatrix} z_0 \\ -z_1 \end{smallmatrix}\right]"']
\end{tikzcd}$$

all displayed d.s.e.s.'s but one have equal upper and lower part. Thus the double exact sequence (4.7) defines the same element of $K_1(X)$ as the one defined by the split d.s.e.s.

$$\mathcal{O}_X(-1)^2 \xleftarrow[\left[\begin{smallmatrix}1&0&0\\0&1&0\end{smallmatrix}\right]]{\left[\begin{smallmatrix}1&0&0\\0&1&0\end{smallmatrix}\right]} \mathcal{O}_X(-1)^3 \xleftarrow[\left[\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right]]{\left[\begin{smallmatrix}0\\0\\-1\end{smallmatrix}\right]} \mathcal{O}_X(-1).$$

The same element of $K_1(X)$ is defined by the automorphism of taking the negative on $\mathcal{O}_X(-1)$, since there is a commutative diagram of d.s.e.s.'s

$$\begin{tikzcd}
0 \arrow[d, tail] & \mathcal{O}_X(-1) \arrow[l, two heads] \arrow[d, tail] & \mathcal{O}_X(-1) \arrow[l, two heads, "-1"', "1"] \arrow[d, tail, "1"', "1"] \\
\mathcal{O}_X(-1)^2 \arrow[d, two heads, "1"', "1"] & \mathcal{O}_X(-1)^3 \arrow[l, two heads, "\left[\begin{smallmatrix}1&0&0\\0&1&0\end{smallmatrix}\right]"', "\left[\begin{smallmatrix}1&0&0\\0&1&0\end{smallmatrix}\right]"] \arrow[d, two heads, "\left[\begin{smallmatrix}1&0&0\\0&1&0\end{smallmatrix}\right]"', "\left[\begin{smallmatrix}1&0&0\\0&1&0\end{smallmatrix}\right]"] & \mathcal{O}_X(-1) \arrow[l, tail, "\left[\begin{smallmatrix}0\\0\\-1\end{smallmatrix}\right]"', "\left[\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right]"] \arrow[d, two heads] \\
\mathcal{O}_X(-1)^2 & \mathcal{O}_X(-1)^2 \arrow[l, two heads, "1"', "1"] & 0 \arrow[l, tail]
\end{tikzcd}$$

Since

$$\{-1\}[\mathcal{O}_X(-1)] = \{-1\}([\mathcal{U}] - 1) = \{\mathrm{Nrd}(-1)\}[\mathcal{U}] + \{-1\}[\mathcal{O}_X]$$

in $K_1(X)$, the assertion follows. ∎

This computation is important for determining discriminants of generators of the Witt group $W(X, L)$ found by Susane Pumplün [17, Cor. 4.4].

A (skew) symmetric isomorphism $\phi : \mathcal{U} \to D\mathcal{U}(-1)$ yields an involution $* = *^\phi$ on $\mathrm{End}_X(\mathcal{U})$: if $\psi \in \mathrm{End}_X(\mathcal{U})$ then

$$\psi^* = \phi^{-1} \circ (D\psi \otimes 1_{\mathcal{O}_X(-1)}) \circ \phi.$$

PROPOSITION 4.8. *The involution $*^\pi$ coincides with the conjugation $\alpha \mapsto \overline{\alpha}$ on the quaternion algebra* $\mathrm{End}_X(\mathcal{U}) = C_0 = \left(\frac{a,b}{F}\right)$*:*

$$(q_0 + q_1 i + q_2 j + q_3 k)^{*^\pi} = q_0 - q_1 i - q_2 j - q_3 k.$$

Proof. Replace $\mathcal{U}$ with isomorphic $\varphi\mathcal{U} \subset \mathcal{O}_X \otimes C_0$. The exact sequence

$$\varphi \cdot \mathcal{U} \hookrightarrow \mathcal{O}_X \otimes C_0 \twoheadrightarrow \mathcal{U}(1)$$

splits locally, so it yields an exact sequence on each of the principal affine open subsets

$$D(z_m) = \{(z_0 : z_1 : z_2) : z_m \neq 0\} \quad \text{for } m = 0, 1, 2.$$

Locally, in $D(z_m)$, the module $\varphi\cdot\mathcal{U}(D(z_m))$ as a submodule of $\mathcal{O}(D(z_m))\otimes_F C_0$ is generated by products of φ times $v_0, v_1, v_2, v_2v_1v_0$:

$$\begin{array}{ll}\frac{1}{z_m}(z_0 - z_1 i - z_2 j), & \frac{1}{z_m}(-az_1 + z_0 i + z_2 k),\\ \frac{1}{z_m}(-bz_2 + z_0 j - z_1 k), & \frac{1}{z_m}(bz_2 i - az_1 j + z_0 k).\end{array} \tag{4.8}$$

The module $\mathcal{U}(1)|_{D(x_m)} = \mathcal{U}|_{D(z_m)}$ is generated by the cosets of $1, i, j, k$ modulo $\varphi\cdot\mathcal{U}|_{D(z_m)}$. Thus the module $\bigwedge^2\mathcal{U}|_{D(z_m)}$ is generated by the exterior products

$$1\wedge i,\quad 1\wedge j,\quad 1\wedge k,\quad i\wedge j,\quad i\wedge k,\quad j\wedge k$$

subject to twelve identities of the form

$$(\text{generator (4.8)})\wedge(\text{one of } 1, i, j, k) = 0,$$

e.g.

$$\begin{array}{ll}\frac{1}{z_m}(z_1 1\wedge i + z_2 1\wedge j) = 0, & \frac{1}{z_m}(z_0 1\wedge i + z_2 1\wedge k) = 0,\\ \frac{1}{z_m}(z_0 1\wedge i + z_2 i\wedge j) = 0, & \frac{1}{z_m}(az_1 i\wedge j + z_0 i\wedge k) = 0,\\ \frac{1}{z_m}(bz_2 1\wedge j - z_1 j\wedge k) = 0. & \end{array}$$

Thus exterior multiplication

$$\wedge : \mathcal{U}\otimes\mathcal{U}\to\textstyle\bigwedge^2\mathcal{U} = \mathcal{O}_X(-1)$$

has in $D(z_m)$ the "Gram matrix"

$$\frac{1}{z_m}\begin{bmatrix} 0 & z_2 & -z_1 & -z_0\\ -z_2 & 0 & -z_0 & -az_1\\ z_1 & z_0 & 0 & -bz_2\\ z_0 & az_1 & bz_2 & 0\end{bmatrix}.$$

Now it is easy to see that

$$\alpha i\wedge\beta = -\alpha\wedge\beta i,\qquad \alpha j\wedge\beta = -\alpha\wedge\beta j,\qquad \alpha k\wedge\beta = -\alpha\wedge\beta k$$

for all $\alpha,\beta\in\mathcal{U}(D(z_m))$, so

$$(q_0 + q_1 i + q_2 j + q_3 k)^{*\pi} = q_0 - q_1 i - q_2 j - q_3 k$$

holds in every $D(z_m)$, $m = 0, 1, 2$, and hence holds globally. ∎

Corollary 4.9. *For every pure quaternion $\gamma\in\left(\frac{a,b}{F}\right)$ the map*

$$\pi\circ(\cdot\gamma) : \mathcal{U}\to\mathcal{U}\hat{\ }(-1)$$

defines a nonsingular symmetric bilinear form with values in $\mathcal{O}_X(-1)$.

Proof. Locally

$$(\pi \circ (\cdot\gamma))(\alpha)(\beta) = \pi(\alpha\cdot\gamma)(\beta) = (\alpha\cdot\gamma)\wedge\beta = -\alpha\wedge(\beta\cdot\gamma)$$
$$= (\beta\cdot\gamma)\wedge\alpha = (\pi\circ(\cdot\gamma))(\beta)(\alpha). \ ■$$

Susane Pumplün proved the following fact:

PROPOSITION 4.10. *The Witt group $W(X,L)$ is generated by the classes of the forms*

$$p = \big(\mathrm{tr}_{F'/F}(\mathcal{O}_{X'}(-1)), \mathrm{tr}_{F'/F}(c\,\mathrm{id})\big), \qquad c\in F'. \tag{4.9}$$

Here F'/F is a quadratic extension which splits $\langle 1,-a,-b\rangle$ (splits $\big(\frac{a,b}{F}\big)$).

Proof. [17, Cor. 4.4]; the canonical bundle $\mathcal{I}$ is our $\mathcal{U}$. ■

We will refer to forms of the kind (4.9) as *Pumplün generators.*

COROLLARY 4.11. *For every pure quaternion $\gamma\in\big(\frac{a,b}{F}\big)$ there exists a constant $c\in\dot F$ such that*

$$\pi\circ(\cdot\gamma) = c\cdot p$$

for a suitable Pumplün generator p.

Proof. The bundle $\mathcal{U}$ is indecomposable, so the assertion follows immediately from [17, Prop. 4.1]. ■

Note that if $\pi\circ(\cdot\gamma) = c\cdot p$, then $\pi\circ(\cdot c^{-1}\gamma) = p$. Moreover the quadratic extension F' needed to define the Pumplün generator p is $F' = F(\gamma)\cong F[\sqrt{-\mathrm{Nrd}(\gamma)}]$.

PROPOSITION 4.12. *For every Pumplün generator p there exist a pure quaternion $\eta\in\big(\frac{a,b}{F}\big)$ and a constant $c\in F^{\cdot}$ such that*

$$p\circ(\cdot\eta) = c\pi.$$

Proof. By the Noether–Skolem theorem every involution of the algebra $\big(\frac{a,b}{F}\big)$ is the composition of the canonical involution with an inner automorphism. Let $\gamma\in\big(\frac{a,b}{F}\big)$ be a quaternion such that

$$(\cdot\,\xi)^{*^p} = \overline{\gamma^{-1}\xi\gamma}$$

for all $\xi\in\big(\frac{a,b}{F}\big)$. A priori there are two cases. If $\overline{\gamma}=\gamma$, then $*^p$ is conjugation, and η is any pure quaternion. If $\overline{\gamma}\neq\gamma$, then $\eta=\overline{\gamma}-\gamma$ is a pure quaternion and

$$\eta^{*^p} = \overline{\gamma^{-1}\delta\gamma} = -\eta$$

and in any case $p\circ(\cdot\eta)$ is skew symmetric, since locally

$$(p\circ(\cdot\eta))(\alpha)(\beta) = p(\alpha\eta)(\beta) = p(\alpha)(-\beta\eta) = -p(\beta\eta)(\alpha) = -(p\circ(\cdot\eta))(\beta)(\alpha).$$

Hence $p\circ(\cdot\eta)$ must be a scalar multiple of π. ■

REMARK 4.1. It follows that the first case (when $*^p$ is conjugation) never occurs: if $p = \pi \circ (\cdot\, \eta^{-1})$ for a pure quaternion η, then

$$\begin{aligned} p(\alpha\gamma)(\beta) &= \pi(\alpha\gamma\eta^{-1})(\beta) = \pi(\alpha)(\beta\overline{\gamma\eta^{-1}}) = -\pi(\alpha)(\beta\eta^{-1}\overline{\gamma}) \\ &= -\pi(\alpha)(\beta\eta^{-1}\overline{\gamma}\eta\eta^{-1}) = -\pi(\alpha\overline{\eta^{-1}})(\beta\eta^{-1}\overline{\gamma}\eta) \\ &= \pi(\alpha\eta^{-1})(\beta\eta^{-1}\overline{\gamma}\eta) = p(\alpha)(\beta\eta^{-1}\overline{\gamma}\eta), \end{aligned} \tag{4.10}$$

where

$$\gamma^{*^p} = \eta^{-1}\overline{\gamma}\eta = \overline{\eta\gamma\eta^{-1}},$$

so

$$\pi^{-1} \circ \alpha^{\wedge L} \circ \pi = -\alpha,$$

while the composition $\pi \circ \alpha : \mathcal{U} \to \mathcal{U}^{\wedge L}$ is a symmetric bilinear form.

PROPOSITION 4.13. *If α is an invertible anti-selfadjoint endomorphism of the sheaf $\mathcal{U}$, then*

$$e^1(\pi \circ \alpha) = (-1, \mathrm{Nrd}(\alpha)\dot{F}^2) \in \mu_2(F) \oplus B/\dot{F}^2.$$

Proof. The commutative diagram

$$\begin{tikzcd} & 0 \arrow[d, tail] & 0 \arrow[d, tail] & \\ 0 & \mathcal{U}^{\wedge L} \arrow[l, two heads] \arrow[d, "1"', "1"] & \mathcal{U} \arrow[l, tail, "\pi\circ\alpha", "\pi"'] \arrow[d, "\alpha"', "1"] & 0 \arrow[l, tail] \\ 0 & \mathcal{U}^{\wedge L} \arrow[l, two heads] & \mathcal{U} \arrow[l, tail, "\pi", "\pi"'] & 0 \arrow[l, tail] \end{tikzcd} \tag{4.11}$$

implies that $\ell\big(0 \twoheadleftarrow \mathcal{U} \xleftarrow[\pi]{\pi\circ\alpha} \mathcal{U} \leftarrowtail 0\big)$ is the element of $K_1(D)$ corresponding to $\alpha \in \mathrm{Aut}(\mathcal{U})$, which is obviously the reduced norm $\mathrm{Nrd}(\alpha)$. By Lemma 2.19 the element $\ell\big(0 \twoheadleftarrow \mathcal{U} \xleftarrow[\pi]{\pi\circ\alpha} \mathcal{U} \leftarrowtail 0\big)$ coincides with the element defined by

$$\mathcal{O}_X(1) \xleftarrow[{[z_0,-z_1,0]}]{[z_0,-z_1,z_2]} \mathcal{O}_X^3 \xleftarrow[f\circ\pi\circ Df]{f\circ\pi\circ\alpha\circ Df} \mathcal{O}_X(-1)^3 \xleftarrow[{\left[\begin{smallmatrix} z_0 \\ -z_1 \\ 0 \end{smallmatrix}\right]}]{\left[\begin{smallmatrix} z_0 \\ -z_1 \\ z_2 \end{smallmatrix}\right]} \mathcal{O}_X(-2). \tag{4.12}$$

Moreover by Lemma 2.20 the sum of the element of $K_1(X)$ defined by (4.12) and the element defined by (4.7) coincides with the element defined by the double exact sequence

$$\mathcal{O}_X(1) \xleftarrow[{[z_0,-z_1,0]}]{[z_0,-z_1,z_2]} \mathcal{O}_X^3 \xleftarrow[{\left[\begin{smallmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_0 \\ z_1 & z_0 & 0 \end{smallmatrix}\right]}]{f\circ\pi\circ\alpha\circ Df} \mathcal{O}_X(-1)^3 \xleftarrow[{\left[\begin{smallmatrix} z_0 \\ -z_1 \\ 0 \end{smallmatrix}\right]}]{\left[\begin{smallmatrix} z_0 \\ -z_1 \\ z_2 \end{smallmatrix}\right]} \mathcal{O}_X(-2)$$

which represents $e^1(\pi\circ\alpha)$. Thus

$$e^1(\pi\circ\alpha)=\varepsilon^1\left(\pi\circ\alpha\div\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)=(-1,\mathrm{Nrd}(\alpha)).\ \blacksquare$$

Remark 4.2. The group B is generated by the reduced norms of pure quaternions; in fact, every nonzero quaternion is either a pure quaternion or a product of two pure quaternions.

Corollary 4.14. *If* $\mathrm{Nrd}(\alpha\beta)$ *is not a square, then the forms* $\pi\circ\alpha$ *and* $\pi\circ\beta$ *are not Witt equivalent.*

This corollary may be partially reversed:

Proposition 4.15. *If* γ, δ *are pure quaternions such that* $\mathrm{Nrd}(\gamma)=\mathrm{Nrd}(\delta)$, *then for some* $c\in\dot F$ *the spaces* $(\mathcal{U},\pi\circ(\cdot\,\gamma))$ *and* $(\mathcal{U},c\cdot\pi\circ(\cdot\,\delta))$ *are isomorphic.*

Proof. If $\gamma+\delta=0$ then $c=-1$. The condition $\mathrm{Nrd}(\gamma)=\mathrm{Nrd}(\delta)$ yields

$$\gamma^2=-\mathrm{Nrd}(\gamma)=-\mathrm{Nrd}(\delta)=\delta^2.$$

If $\gamma+\delta\neq 0$ then for $\lambda=\gamma+\delta$,

$$\gamma\lambda=\lambda\delta,\quad\text{i.e.}\quad\gamma=\lambda\delta\lambda^{-1}.$$

Thus locally

$$\begin{aligned}(\pi\circ(\cdot\,\gamma))(\alpha)(\beta)&=\pi(\alpha\gamma)(\beta)=\pi(\alpha\lambda\delta\lambda^{-1})(\beta)\\&=\frac{1}{\mathrm{Nrd}(\lambda)}\pi(\alpha\lambda\delta\overline{\lambda})(\beta)=\frac{1}{\mathrm{Nrd}(\lambda)}\pi(\alpha\lambda\delta)(\beta\lambda)\\&=\frac{1}{\mathrm{Nrd}(\lambda)}(\pi\circ(\cdot\,\delta))(\alpha\lambda)(\beta\lambda),\end{aligned}$$

which means that $\cdot\,\lambda\in\mathrm{End}_X(\mathcal{U})$ is an isomorphism of $\mathrm{Nrd}(\lambda)\pi\circ(\cdot\,\gamma)$ and $\pi\circ(\cdot\,\delta)$. $\blacksquare$

Corollary 4.16. *The discriminant map* $e^1: W(X,L)\to\mu_2(F)\oplus B/\dot F^2$ *is given by the formula*

$$e^1\Big(\sum_{m=1}^{n}\pi\circ(\cdot\,\alpha_m)\Big)=((-1)^n,\mathrm{Nrd}(\alpha_1\cdots\alpha_n)).$$

Corollary 4.17. *If* n *is odd or* $\mathrm{Nrd}(\alpha_1\cdots\alpha_n)$ *is not a square, then the element* $\sum_{m=1}^{n}\pi\circ(\cdot\,\alpha_m)$ *is not zero in* $W(X,L)$.

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the construction of e^1 given here is designed especially for explicit computations, as is shown in Section 4. Moreover, there are some hopes to define higher maps e^n in a similar way.

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