FUNDAMENTA
MATHEMATICAE
169 (2001)

Forcing relation on minimal interval patterns

by

Jozef Bobok (Praha)

Abstract. Let $\mathcal{M}$ be the set of pairs (T,g) such that $T\subset\mathbb{R}$ is compact, $g:T\to T$ is continuous, g is minimal on T and has a piecewise monotone extension to $\operatorname{conv}T$. Two pairs $(T,g),(S,f)$ from $\mathcal{M}$ are equivalent if the map $h:\operatorname{orb}(\min T,g)\to\operatorname{orb}(\min S,f)$ defined for each $m\in\mathbb{N}_0$ by $h(g^m(\min T))=f^m(\min S)$ is increasing on $\operatorname{orb}(\min T,g)$. An equivalence class of this relation—a minimal (oriented) pattern A—is exhibited by a continuous interval map $f:I\to I$ if there is a set $T\subset I$ such that $(T,f|T)=(T,f)\in A$. We define the forcing relation on minimal patterns: A forces B if all continuous interval maps exhibiting A also exhibit B. In Theorem 3.1 we show that for each minimal pattern A there are maps exhibiting only patterns forced by A. Using this result we prove that the forcing relation on minimal patterns is a partial ordering. Our Theorem 3.2 extends the result of [B], where pairs (T,g) with T finite are considered.

0. Introduction. The question of coexistence of different types of closed invariant sets arises in the theory of discrete dynamical systems. In dimension one for interval maps different types of such sets have been investigated. Using the equivalence relation on cycles (finite invariant sets), the notion of a pattern (generalized pattern) has been defined and a law of coexistence of different patterns, now usually called the forcing relation, has been studied [B], [ALM]. Furthermore, recent results [Bl1], [Bl2], [Y] show that the essential parts of the theory of forcing of finite invariant sets could be extended to the more general case of infinite minimal sets exhibited by interval maps. The aim of this paper is to make a few steps in this direction.

In order to achieve our goal we define an equivalence relation on the set of all minimal pairs exhibited by interval maps and consider a minimal (oriented) pattern as an equivalence class of this relation. Our main results generalizing Theorems 2.6.13 and 2.5.1 from [ALM] are the following.

2000 *Mathematics Subject Classification*: 26A18, 37B05, 37E05, 37E99.
Key words and phrases: interval map, minimal pattern, forcing relation.
The author was supported by GA of Czech Republic, contract 201/00/0859.

THEOREM 3.1. *Let A, B be minimal patterns. Then the following conditions are equivalent.*

(i) *A forces B.*
(ii) *For some $(T, g) \in A$, g_T exhibits the pattern B.*

THEOREM 3.2. *The forcing relation on minimal patterns is a partial ordering.*

The paper is organized as follows: In Section 1 we give some basic notation and definitions. Section 2 is devoted to the lemmas used throughout the paper. The main result of this section is Lemma 2.6. In Section 3 we prove Theorems 3.1 and 3.2.

Acknowledgements. The author thanks Milan Kuchta for useful discussions and remarks.

1. Notation and definitions. By $\mathbb{R}, \mathbb{N}, \mathbb{N}_0$ we denote the sets of real, positive and nonnegative integer numbers respectively. Let I be a closed finite subinterval of $\mathbb{R}$. We consider the space $C(I)$ of all continuous maps f which are defined on I and map it into itself. For $f \in C(I)$ and an interval (maybe degenerate) $J \subset I$ the set $\operatorname{orb}(J, f) = \{f^i(J) : i \in \mathbb{N}_0\}$ is called the *orbit* of J. We will write $\operatorname{orb}(x, f)$ if $J = \{x\}$. A point $x \in I$ is called *periodic* if $f^n(x) = x$ for some $n \in \mathbb{N}$. The minimal such n is called the *period* of x and the set $\operatorname{orb}(x, f)$ is called a *cycle*. The union of all cycles of f is denoted by $\operatorname{Per}(f)$. For $T \subset \mathbb{R}$, we say that $g : T \to T$ is *minimal* on T if for each $x \in T$, $\operatorname{orb}(x, g)$ is dense in T. We denote by $\operatorname{conv} X$ the convex hull of a set $X \subset \mathbb{R}$.

(T, g)-monotone maps. For a pair (T, g), where $T \subset \mathbb{R}$ is compact and $g : T \to T$ is continuous, a map $\widetilde{g} \in C(\operatorname{conv} T)$ is said to be (T, g)-*monotone* if $\widetilde{g}|T = g$ and $\widetilde{g}|J$ is strictly monotone or constant for any interval $J \subset \operatorname{conv} T$ such that $J \cap T = \emptyset$. In particular, the (T, g)-monotone map which is affine on each component of $\operatorname{conv} T \setminus T$ is denoted by g_T. We use the notation $C(T, g)$ for the set of all (T, g)-monotone maps. A pair (T, g) is said to be *piecewise monotone* if there are $k \in \mathbb{N}$ and points $\min T = c_0 < c_1 < \ldots < c_k < c_{k+1} = \max T$ such that g_T is monotone on each $[c_i, c_{i+1}]$, $i = 0, \ldots, k$. The least k with this property is called the *modality* of (T, g).

The set $\mathcal{M}$ of minimal pairs. We define $\mathcal{M}$ as the set of all piecewise monotone pairs (T, g) such that $T \subset \mathbb{R}$ is compact, $g : T \to T$ is continuous and g is minimal on T. It is well known that for $(T, g) \in \mathcal{M}$ exactly one of the following two possibilities is satisfied [BCp]: (i) T is finite and so a cycle; (ii) T is a Cantor set. We denote the sets of pairs corresponding to (i), (ii) by $\mathcal{M}_{\mathrm{p}}$, $\mathcal{M}_\infty$ respectively. Thus, $\mathcal{M} = \mathcal{M}_{\mathrm{p}} \cup \mathcal{M}_\infty$.

Let $f \in C(I = [a,b])$. Following [P], we consider open sets $\mathcal{Z}(f)$, $\mathcal{C}(f)$, $\mathcal{R}(f)$, where

$$\mathcal{Z}(f) = \{x \in (a,b) : \text{there is } \varepsilon > 0 \text{ such that } f^n \text{ is strictly monotone on } (x-\varepsilon, x+\varepsilon) \text{ for all } n \in \mathbb{N}_0\},$$

$$\mathcal{C}(f) = \{x \in (a,b) : \text{there is } \varepsilon > 0 \text{ such that } f^n \text{ is constant on } (x-\varepsilon, x+\varepsilon) \text{ for some } n \in \mathbb{N}_0\}$$

and $\mathcal{R}(f) = \mathcal{Z}(f) \cup \mathcal{C}(f)$. Clearly, $\mathcal{Z}(f) \cap \mathcal{C}(f) = \emptyset$.

Canonical pairs. A pair $(T,g) \in \mathcal{M}$ is said to be *canonical* if there is a (T,g)-monotone map $\widetilde{g} \in C(\operatorname{conv} T)$ such that $\mathcal{R}(\widetilde{g}) = \emptyset$.

Let $\mathcal{I}$ be the set of all closed finite subintervals of $\mathbb{R}$. In what follows we use the notation $C(\mathcal{I}) = \bigcup_{I \in \mathcal{I}} C(I)$. For two closed sets $K, L \subset \mathbb{R}$ we write $K < L$ if $\max K < \min L$.

Sequences of the same order. Assume there are sequences $\{K_i^1\}_{i \in \mathbb{N}_0}$, $\{K_i^2\}_{i \in \mathbb{N}_0}$ such that

(i) K_i^j is a point or a closed interval,
(ii) either $K_{i(1)}^j \cap K_{i(2)}^j = \emptyset$ or $K_{i(1)}^j = K_{i(2)}^j$ for $i(1) \neq i(2)$.

We say that the sequences $\{K_i^1\}_{i \in \mathbb{N}_0}, \{K_i^2\}_{i \in \mathbb{N}_0}$ *have the same order* if

$$K_{i(1)}^1 < K_{i(2)}^1 \;\Leftrightarrow\; K_{i(1)}^2 < K_{i(2)}^2, \qquad i(1), i(2) \in \mathbb{N}_0.$$

In particular, for $f_1, f_2 \in C(\mathcal{I})$ and closed (degenerate) intervals J, K, the orbits $\operatorname{orb}(J, f_1)$, $\operatorname{orb}(K, f_2)$ have the same order if it is true for the sequences $\{f_1^i(J)\}_{i \in \mathbb{N}_0}$, $\{f_2^i(K)\}_{i \in \mathbb{N}_0}$.

Minimal patterns. Pairs $(T,g), (S,f) \in \mathcal{M}$ are said to be *equivalent* if the orbits $\operatorname{orb}(\min T, g), \operatorname{orb}(\min S, f)$ have the same order. An equivalence class A of this relation will be called a *minimal* (*oriented*) *pattern* or briefly a pattern. If A is a pattern and $(T,g) \in A$ we say that the pair (T,g) has pattern A and we use the symbol $[(T,g)]$ to denote the pattern A. If (T,g) is a cycle then $[(T,g)]$ is called a *periodic pattern.* Note that all pairs of a pattern A have the same modality.

A function $f \in C(I)$ has a pair $(T,g) \in \mathcal{M}$ if $f|T = g$. In this case we say that f *exhibits* the pattern $A = [(T,g)]$ and we often write $(T,f) \in A$.

Now we define the forcing relation on minimal patterns.

Forcing relation. A pattern A *forces* a pattern B if all maps in $C(\mathcal{I})$ exhibiting A also exhibit B. Sometimes we use the symbol $A \hookrightarrow B$.

A relation which is reflexive, transitive and weakly antisymmetric ($A \hookrightarrow B$ and $B \hookrightarrow A$ implies $A = B$) is called a partial ordering.

Concerning the forcing relation the following result is known.

Theorem 1.1 ([B], [ALM]). *The forcing relation on periodic patterns is a partial ordering.*

2. Lemmas. In the first lemma we recall known properties of minimal dynamical systems that will be useful when proving our results. These assertions are valid for any minimal dynamical system (X,f), where X is a compact metric space and $f:X\to X$ is continuous.

Lemma 2.1. (i) *If $(T,g)\in\mathcal{M}$ and $R\subset T$ is open in T then there is a positive integer k with the property $\bigcup_{l=0}^{k} g^{l}(R)=T$.*

(ii) ([BCp]) *A pair (T,g) is minimal if and only if $T=\overline{\mathrm{orb}}(t=\min T,g)$, where t is strongly recurrent, i.e. for any open neighborhood $[t,t+\varepsilon)$ of t in T there is a positive integer n_0 such that $\{g^i(t)\}_{i=j-1}^{j-1+n_0}\cap[t,t+\varepsilon)\neq\emptyset$ for each $j\in\mathbb{N}$.*

Proof. (i) It follows directly that there is $k\in\mathbb{N}$ for which $\bigcup_{l=0}^{k} g^{-l}(R)=T$. Now $T=g^{k}(\bigcup_{l=0}^{k} g^{-l}(R))=\bigcup_{l=0}^{k} g^{k-l}(g^{l}(g^{-l}(R)))$. ∎

In order to study the forcing relation on minimal patterns we need some method that will help us to recognize that a fixed map $f\in C(I)$ exhibits a minimal pattern A. The following lemma satisfies this requirement.

Lemma 2.2. *Let $f\in C(I)$ and $(T,g)\in\mathcal{M}$. Assume there is a sequence $\{\mathcal{K}_i\}_{i\in\mathbb{N}_0}$ such that*

(i) *$\mathcal{K}_i\subset I$ is a point or a closed interval,*

(ii) *either $\mathcal{K}_{i(1)}\cap\mathcal{K}_{i(2)}=\emptyset$ or $\mathcal{K}_{i(1)}=\mathcal{K}_{i(2)}$ for $i(1)\neq i(2)$,*

(iii) *$f^i(\mathcal{K}_0)=\mathcal{K}_i$, and for some $t\in T$ the orbits $\mathrm{orb}(\mathcal{K}_0,f)$, $\mathrm{orb}(t,g)$ have the same order.*

Then there is $T^*\subset I$ such that $\max\mathcal{K}_0\le\min T^*$ and $(T^*,f)\in[(T,g)]$.

Proof. The conclusion is well known when $(T,g)\in\mathcal{M}_{\mathrm{p}}$ [ALM]. So suppose that $(T,g)\in\mathcal{M}_\infty$.

We start our proof by choosing a point t^* which will be useful when defining T^*. Without loss of generality we can assume that $t\in T$ is a right-side limit point of T. Consider a sequence $\{m_i\}_{i\in\mathbb{N}}$ of positive integers for which the sequence $\{g^{m_i}(t)\}$ is decreasing and $\lim_i g^{m_i}(t)=t$, and put $t^*=\inf\bigcup_{i\in\mathbb{N}} f^{m_i}(\mathcal{K}_0)$. Clearly $\max\mathcal{K}_0\le t^*$. We show that

(iv) the map $h:\mathrm{orb}(t^*,f)\to\mathrm{orb}(t,g)$ defined by $h(f^m(t^*))=g^m(t)$, $m\in\mathbb{N}_0$, is increasing on $\mathrm{orb}(t^*,f)$.

First we prove that $\mathrm{orb}(t^*,f)$ is infinite and t^* is its limit point. Notice that by (i)–(iii), $\lim_i\mathrm{diam}(\mathcal{K}_{m_i})=0$, hence the continuity of f gives for each $j\in\mathbb{N}_0$,

$$\lim_i \mathcal{K}_{m_i+j}=f^j(t^*). \tag{1}$$

Since $(T,g)\in\mathcal{M}$, we can consider a sequence $\{k_n\}_{n\in\mathbb{N}}$ for which $\lim_n g^{k_n}(t) = t$,

$$\ldots < g^{k_{n+1}}(t) < g^{k_n}(t) < \ldots < g^{k_2}(t) < g^{k_1}(t)$$

and each intersection $(g^{k_{n+1}}(t), g^{k_n}(t))\cap T$ is infinite. Fix $m\in\mathbb{N}$. Then $g^j(t)\in(g^{k_{m+1}}(t), g^{k_m}(t))$ for some $j\in\mathbb{N}$ and since $\lim_i g^{m_i}(t)=t$, for each $i\ge i_0$ we also have $g^{m_i+j}(t)\in(g^{k_{m+1}}(t), g^{k_m}(t))$. By (i)–(iii) again the situation is similar for $\operatorname{orb}(\mathcal{K}_0,f)$. We have $\lim_n \mathcal{K}_{k_n}=t^*$,

$$\ldots < \mathcal{K}_{k_{n+1}} < \mathcal{K}_{k_n} < \ldots < \mathcal{K}_{k_2} < \mathcal{K}_{k_1}$$

and for each $i\ge i_0$,

$$\mathcal{K}_{k_{m+1}} < \mathcal{K}_{m_i+j} < \mathcal{K}_{k_m}. \tag{2}$$

Using (1), (2) we can see that $f^j(t^*)\in[\max\mathcal{K}_{k_{m+1}}, \min\mathcal{K}_{k_n}]$. Since m was arbitrary, $\operatorname{orb}(t^*,f)$ is infinite and t^* is a limit point of $\operatorname{orb}(t^*,f)$. Let us show (iv). If for some $k,l\in\mathbb{N}_0$ we have $g^k(t)<g^l(t)$ then for each $i\ge i_0$, $g^{m_i+k}(t)<g^{m_i+l}(t)$ and from (i)–(iii) also $\mathcal{K}_{m_i+k}<\mathcal{K}_{m_i+l}$. It follows from (1) that $f^k(t^*)\le f^l(t^*)$. But we already know that $\operatorname{orb}(t^*,f)$ is infinite. This implies $f^k(t^*)<f^l(t^*)$. Summarizing, the map $h:\operatorname{orb}(t^*,f)\to\operatorname{orb}(t,g)$ defined by $h(f^m(t^*))=g^m(t)$, $m\in\mathbb{N}_0$, is increasing on $\operatorname{orb}(t^*,f)$, which proves (iv).

Put $T^*=\overline{\operatorname{orb}}(t^*,f)$. By (iv), the orbits $\operatorname{orb}(t^*,f),\operatorname{orb}(t,g)$ have the same order.

Let us show that $(T^*,f)\in\mathcal{M}$. By Lemma 2.1(ii) it is sufficient to show that $t^*\in T^*$ is a strongly recurrent point in the system (T^*,f). Consider a neighborhood $U_\varepsilon=[t^*,t^*+\varepsilon)$ of t^* in T^*. Then $f^k(t^*)\in U_\varepsilon$ for some k. Since t is strongly recurrent in (T,g), for an open neighborhood $[t,g^k(t))$ of t in T there is a positive integer n_0 such that $\{g^i(t)\}_{i=j-1}^{j-1+n_0}\cap[t,g^k(t))\ne\emptyset$ for each $j\in\mathbb{N}$. By (iv) we know that the orbits $\operatorname{orb}(t,g)$ and $\operatorname{orb}(t^*,f)$ have the same order. It follows immediately that for the same value n_0 we have $\{f^i(t^*)\}_{i=j-1}^{j-1+n_0}\cap[t^*,f^k(t^*))\ne\emptyset$ for each $j\in\mathbb{N}$. Thus $(T^*,f)\in\mathcal{M}$.

Similarly we can prove that $\operatorname{orb}(\min T,g)$ and $\operatorname{orb}(\min T^*,f)$ have the same order. Thus $(T^*,f)\in[(T,g)]$.

The proof of Lemma 2.2 is finished. ■

Let A be a minimal pattern. In what follows we outline the procedure that allows us to show that A contains canonical pairs. The reason for being brief is that in order to develop the whole procedure in a systematic way, we would have to repeat (often with small modifications) the proofs from [P] and the amount of space necessary for doing that would be too large compared to the advantages.

Let J,K be two compact subintervals of $\mathbb{R}$; we denote by $\mathcal{H}(J,K)$, resp. $\mathcal{H}(J)$ the set of all continuous nondecreasing maps mapping J onto K, resp. J. For $h\in\mathcal{H}(J,K)$ we put

$$\operatorname{supp}(h) = \{x \in J : h(L) \text{ is nondegenerate for each open interval } L \subset J \text{ with } x \in L\}.$$

Let $(T, g) \in \mathcal{M}_\infty$ and $\widetilde{g} \in C(T, g)$. Then if $\operatorname{conv} T \setminus \mathcal{R}(\widetilde{g})$ is perfect, one can find maps $h \in \mathcal{H}(\operatorname{conv} T)$ with $\operatorname{supp}(h) = \operatorname{conv} T \setminus \mathcal{R}(\widetilde{g})$ and $f \in C(\operatorname{conv} T)$ with $\mathcal{R}(f) = \emptyset$, $f \in C(h(T), f)$ and such that

$$f \circ h = h \circ \widetilde{g} \quad \text{on } \operatorname{conv} T. \tag{$*$}$$

This known statement can be proved using the fact that for each $J \subset \operatorname{conv} T$, $h(J)$ is nondegenerate if and only if $h(\widetilde{g}(J))$ is nondegenerate (see, for example, [ALM], [P]).

Our strategy will be to show that the set $\operatorname{conv} T \setminus \mathcal{R}(\widetilde{g})$ is really perfect and then we prove the needed common properties of the maps $f, \widetilde{g}$ given by $(*)$.

Lemma 2.3. *Let $(T, g) \in \mathcal{M}_\infty$ and $\widetilde{g} \in C(T, g)$. Then:*

(i) *$T \cap \mathcal{R}(\widetilde{g}) = \emptyset$ and $\operatorname{conv} T \setminus \mathcal{R}(\widetilde{g})$ is a perfect set.*
(ii) *The maps $\widetilde{g} = g_T$ and f exhibit the same patterns.*
(iii) *$(h(T), f)$ is a canonical pair and $[(T, g)] = [(h(T), f)]$.*

Proof. The conclusions are true if $\mathcal{R}(\widetilde{g}) = \emptyset$. In this case $h = \mathrm{id}$ and $f = \widetilde{g}$. Thus, in the following we suppose that $\mathcal{R}(\widetilde{g}) \neq \emptyset$.

Recall that the sets $\mathcal{Z}(\widetilde{g})$, $\mathcal{C}(\widetilde{g}), \mathcal{R}(\widetilde{g})$ are open. It follows from the definition of $\mathcal{Z}(\widetilde{g})$ that $\widetilde{g}(\mathcal{Z}(\widetilde{g})) \subset \mathcal{Z}(\widetilde{g})$. Notice that if J is a component of $\mathcal{Z}(\widetilde{g})$, then there is a component K of $\mathcal{Z}(\widetilde{g})$ such that $\widetilde{g}(J) \subset K$. The fact that $\widetilde{g}$ is continuous implies that $\widetilde{g}(\overline{J}) \subset \overline{K}$.

(i) Clearly, $T \cap \mathcal{C}(\widetilde{g}) = \emptyset$. Let $J \subset \operatorname{conv} T$ be open. If $T \cap J \neq \emptyset$, then by Lemma 2.1(i), J contains a point $x \in T$ such that for $j \in \mathbb{N}$ we have $\widetilde{g}^j(x) \in T$ and the map $\widetilde{g}$ is not strictly monotone on $(\widetilde{g}^j(x) - \varepsilon, \widetilde{g}^j(x) + \varepsilon)$ for any positive ε. This implies $T \cap \mathcal{Z}(\widetilde{g}) = T \cap \mathcal{R}(\widetilde{g}) = \emptyset$.

Assume that $x \in \operatorname{conv} T$ is an isolated point of $\operatorname{conv} T \setminus \mathcal{R}(\widetilde{g})$. Then for a sufficiently small positive ε', $(x - \varepsilon', x) \cup (x, x + \varepsilon') \subset \mathcal{R}(\widetilde{g})$ and by the definition of $\mathcal{C}(\widetilde{g})$, $(x - \varepsilon', x) \cup (x, x + \varepsilon') \not\subseteq \mathcal{C}(\widetilde{g})$. If $j \in \mathbb{N}_0$ is the least such that $\widetilde{g}^{j+1}$ is not strictly monotone on $(x - \varepsilon, x + \varepsilon)$ then $\widetilde{g}^j(x) \in T$ and at least one of the two intersections $\widetilde{g}^{-j}(T) \cap (x - \varepsilon, x) \cap \mathcal{Z}(\widetilde{g})$, $\widetilde{g}^{-j}(T) \cap (x, x + \varepsilon) \cap \mathcal{Z}(\widetilde{g})$ has to be nonempty for each positive ε. This is impossible since $T \cap \mathcal{R}(\widetilde{g}) = \emptyset$ and $\widetilde{g}(\mathcal{Z}(\widetilde{g})) \subset \mathcal{Z}(\widetilde{g})$.

Thus we can consider the maps $h \in \mathcal{H}$ and $f \in C(\operatorname{conv} T)$ satisfying $(*)$.

(ii) First we prove that if g_T exhibits A then so does f. We distinguish two cases.

Case 1. Let $S \subset \operatorname{conv} T$ and $(S, g_T) \in \mathcal{M}_\infty$. Put $A = [(S, g_T)]$. By our definition the map g_T exhibits the pattern A. We now show that also $(h(S), f) \in A$.

Note that the open set $\mathcal{R}(g_T)$ has countably many components. Since T is a Cantor set and (T,g) is piecewise monotone one can find $s \in S$ such that for each component J of $\mathcal{R}(g_T)$ we have $\operatorname{orb}(s,g) \cap \overline{J} = \emptyset$. This means that the h introduced in $(*)$ is increasing on $\operatorname{orb}(s, g_T)$. Now, using Lemma 2.2 for $\mathcal{K}_i = h(g_T^i(s))$, $i \in \mathbb{N}_0$, we can verify that $t^* = h(s)$, $T^* = h(S)$, $h(\min S) = \min h(S)$ and the orbits $\operatorname{orb}(\min S, g_T)$ and $\operatorname{orb}(\min h(S), f)$ have the same order. We conclude that $(h(S), f) \in A$.

Case 2. Similarly, let $S \subset \operatorname{conv} T$, $(S, g_T) \in \mathcal{M}_{\mathrm{p}}$ and $A = [(S, g_T)]$. If for each component J of $\mathcal{Z}(g_T)$ we have $\#(S \cap \overline{J}) \le 1$, the conclusion follows directly from $(*)$. Now we show that in fact the opposite case cannot hold.

Assume to the contrary that there is a component J such that $m = \#(S \cap \overline{J}) \ge 2$. Then $s_{\min} = \min(S \cap \overline{J}) < s_{\max} = \max(S \cap \overline{J})$ and there are components $J_1 = J, \dots, J_n$ of $\mathcal{Z}(g_T)$ such that $S \subset \bigcup_{i=1}^n \overline{J}_i$, $g_T(\overline{J}_i) \subset \overline{J}_{i+1}$ and $g_T(\overline{J}_n) \subset \overline{J}_1$, $m = \#(S \cap \overline{J}_i)$. Since g_T is affine on each J_i and $g_T^{2n}(\overline{J}_1) \subset \overline{J}_1$, the map g_T^{2n} has slope one on $\overline{J}_1$ and $g_T^{2n}(s_{\min}) = s_{\min}$, $g_T^{2n}(s_{\max}) = s_{\max}$. In particular, this implies that $m = 2$. Since by (i) we have $T \cap \mathcal{R}(g_T) = \emptyset$, we can consider the components $K_1, \dots, K_n$ of $\operatorname{conv} T \setminus T$ for which $J_i \subset K_i$. Clearly $g_T(K_i) \supset K_{i+1}$, hence there is an interval $K \subset K_1$ such that $g_T^{2n}(K) = K_1$. We know that g_T^{2n} has slope one on K and hence $K = K_1$. But this contradicts our choice of the infinite pair $(T,g) \in \mathcal{M}_\infty$.

In order to finish the proof of (ii) we have to show that any pattern exhibited by f is also exhibited by g_T. Take $S \subset \operatorname{conv} T$ for which $(S,f) \in \mathcal{M}$, and put $s = \min S$. If we define $s_0 = \max h^{-1}(s)$, by $(*)$ we see that $f^m(s) = h(g_T^m(s_0))$ for each $m \in \mathbb{N}_0$, hence the map $h|\operatorname{orb}(s_0, g_T)$ is increasing on $\operatorname{orb}(s_0, g_T)$ and we can use Lemma 2.2 again putting $\mathcal{K}_i = g_T^i(s_0)$, $i \in \mathbb{N}_0$. We conclude that $(\overline{\operatorname{orb}}(s_0, g_T), g_T) \in [(S,f)]$.

(iii) We know that $\mathcal{R}(f) = \emptyset$ and $f \in C(h(T), f)$. Now, put $S = T$ in the proof of (ii).

The proof of the lemma is finished. ∎

The following lemma can be considered to belong to folklore knowledge. For the sake of completeness we present its proof (cf. [BCv, Th. 2.1]).

Lemma 2.4. *Let $(T,g), (S,f) \in \mathcal{M}_\infty$ be canonical pairs. The following conditions are equivalent.*

(i) $[(T,g)] = [(S,f)]$.

(ii) *For each $\widetilde{g} \in C(T,g)$ and $\widetilde{f} \in C(S,f)$ with $\mathcal{R}(\widetilde{g}) = \mathcal{R}(\widetilde{f}) = \emptyset$ there is an increasing map $H \in \mathcal{H}(\operatorname{conv} T, \operatorname{conv} S)$ such that $\widetilde{f} \circ H = H \circ \widetilde{g}$ on* $\operatorname{conv} T$, *i.e. the maps $\widetilde{g}, \widetilde{f}$ are topologically conjugate.*

Proof. The implication (ii)⇒(i) is clear.

Take $\widetilde{g} \in C(T, g)$ and $\widetilde{f} \in C(S, f)$ such that $\mathcal{R}(\widetilde{g}) = \mathcal{R}(\widetilde{f}) = \emptyset$. Let

$$h : \operatorname{orb}(t = \min T, g) \to \operatorname{orb}(s = \min S, f)$$

be the map ensured by the equivalence of $(T, g), (S, f)$.

First we show that h extends to a strictly monotone continuous map $\widetilde{h}$ on T such that $\widetilde{h}(T) = S$ and $f \circ \widetilde{h} = \widetilde{h} \circ g$ on T.

Because of the monotonicity of h on $\operatorname{orb}(t, g)$ it is sufficient to prove that whenever $x \in T$ and $\lim_i g^{m_i}(t) = x$, then $\lim_i f^{m_i}(s) = y$; in such a case we put $\widetilde{h}(x) = y$. The claim is true if $x \in T$ is a one-sided limit point of T. Suppose that for suitable sequences $\{m_i\}, \{n_i\}$ of positive integers we have

$$g^{m_i}(t) < g^{m_{i+1}}(t) < \ldots < x < \ldots < g^{n_{i+1}}(t) < g^{n_i}(t),$$

$\lim_i g^{m_i}(t) = \lim_i g^{n_i}(t) = x$ and at the same time

$$\lim_i f^{m_i}(h(t)) = \lim_i f^{m_i}(s) = u < v = \lim_i f^{n_i}(h(t)) = \lim_i f^{n_i}(s).$$

In particular this means that $(u, v) \cap \operatorname{orb}(s, f) = \emptyset$. Notice that for each $j \in \mathbb{N}_0$, $\widetilde{f}^j((u, v))$ is nondegenerate, otherwise we would have $J \subset \mathcal{C}(\widetilde{f})$ for some nondegenerate interval $J \subset (u, v)$. Moreover, if $\operatorname{int}(\widetilde{f}^j((u, v))) \cap \operatorname{orb}(s, f) = \emptyset$ for each $j \in \mathbb{N}_0$, we get $(u, v) \subset \mathcal{R}(\widetilde{f})$, which is impossible again. Thus we can consider the least positive integer j for which there is $k \in \mathbb{N}$ such that $f^k(s) \in \operatorname{int}(\widetilde{f}^j((u, v)))$. By our choice of j, $\operatorname{int}(\widetilde{f}^j((u, v))) = \operatorname{int}(\operatorname{conv}\{\widetilde{f}^j(u), \widetilde{f}^j(v)\})$ and if we take a sequence $\{k_i\}$ of positive integers for which $\lim_i f^{k_i}(s) = f^k(s)$, then for each $i \geq i_0$ and $l \geq l_0$ ($i_0, l_0 \in \mathbb{N}$ are sufficiently large) we get

$$f^{k_i}(s) \in \operatorname{int}(\operatorname{conv}\{f^{m_l + j}(s), f^{n_l + j}(s)\}),$$

hence from the equivalence of $(T, g), (S, f)$ also

$$g^{k_i}(t) \in \operatorname{int}(\operatorname{conv}\{g^{m_l + j}(t), g^{n_l + j}(t)\}).$$

This implies $g^{k_i}(t) = g^j(x)$ for each $i \in \mathbb{N}$—a contradiction.

From what we proved above, $\widetilde{h}$ has the following properties: $\widetilde{h} : T \to S$ is a continuous extension of $h : \operatorname{orb}(t, g) \to \operatorname{orb}(s, f)$, it is nondecreasing and $f \circ \widetilde{h} = \widetilde{h} \circ g$ on T. Repeating our proof for $h^{-1} : \operatorname{orb}(s, f) \to \operatorname{orb}(t, g)$ we see that $\widetilde{h}$ is even increasing on T, which finishes the first part of the proof.

In the second part we need to show that there is an increasing map $H \in \mathcal{H}(\operatorname{conv} T, \operatorname{conv} S)$ such that

$$H|T = \widetilde{h}, \qquad \widetilde{f} \circ H = H \circ \widetilde{g} \quad \text{on } \operatorname{conv} T.$$

For $k \in \mathbb{N}_0$ we define a sequence $\{T_k\}$ by $T_0 = T$, $T_k = T_{k-1} \cup (\widetilde{g}^{-1}(T_{k-1}) \cap \operatorname{conv} T)$ and similarly $\{S_k\}$ from S and $\widetilde{f}$. Notice that $T_0 \subset T_1 \subset \ldots$, $\widetilde{g}(T_{k+1}) \subset T_k$, $S_0 \subset S_1 \subset \ldots$ and $\widetilde{f}(S_{k+1}) \subset S_k$. Put $H_0 = \widetilde{h}$. Suppose that we have already defined a map $H_k : T_k \to S_k$ which is increasing and $\widetilde{f} \circ H_k = H_k \circ \widetilde{g}$ on T_k. By our assumption for $x \in T_k$ and $y = H_k(x) \in S_k$, if $\widetilde{g}^{-1}(x) = \{t_1(x) < \ldots < t_m(x)\}$ and $\widetilde{f}^{-1}(y) = \{s_1(y) < \ldots < s_n(y)\}$ then $m = n$ and the map $H_{k+1} : T_{k+1} \to S_{k+1}$ defined by $H_{k+1}(t_i(x)) = s_i(y)$ for all $t_i(x) \in$

T_{k+1} extends H_k, it is increasing and $\widetilde{f}\circ H_{k+1} = H_{k+1}\circ\widetilde{g}$ on T_{k+1}. Now, using the maps H_k we can define an increasing map $\widetilde{H} : \bigcup T_k \to \bigcup S_k$ by $\widetilde{H}(x) = H_k(x)$ for $x \in T_k$. Note that $\widetilde{H}^{-1} : \bigcup S_k \to \bigcup T_k$ is also increasing and since $\mathcal{R}(\widetilde{g}) = \mathcal{R}(\widetilde{f}) = \emptyset$, the set $\bigcup T_k$, resp. $\bigcup S_k$ is dense in $\operatorname{conv} T$, resp. $\operatorname{conv} S$. Now, it follows immediately that $\widetilde{H}$ extends to a continuous increasing H defined on $\operatorname{conv} T$ such that $\widetilde{f}\circ H = H\circ\widetilde{g}$ on $\operatorname{conv} T$. This proves the lemma. ∎

Definition. Let $f \in C(I)$ and $[x, y] \subset I$. We define

$$\operatorname{sign}_f([x,y]) = \begin{cases} +1, & f(x) < f(y), \\ -1, & f(x) > f(y). \end{cases}$$

Lemma 2.5. *Let $f \in C(I)$, $[a, b] \subset I$, $[c, d] \subset I$, $f(a) \neq f(b)$ and*

$$\operatorname{conv}\{f(a), f(b)\} \supset [c, d].$$

Then there are $a^*, b^* \in [a, b]$ such that $f([a^*, b^*]) = [c, d]$, $f(\{a^*, b^*\}) = \{c, d\}$ and $\operatorname{sign}_f([a^*, b^*]) = \operatorname{sign}_f([a, b])$.

Proof. If $f(a) > f(b)$ then $a^* = \sup\{x \in [a, b] : f(x) = d\}$ and $b^* = \inf\{x \in [a^*, b] : f(x) = c\}$. The second case is similar. ∎

The key lemma follows. Its "periodic part" was proved in [BK].

Lemma 2.6. *Let $f \in C(I)$, and assume there is a compact set $S \subset I$ with $f(S) \subset S$. Then for $f_S \in C(S, f)$ and $T \subset \operatorname{conv} S$ such that $(T, f_S) \in \mathcal{M}$ there is $T^* \subset \operatorname{conv} S$ for which $(T^*, f) \in [(T, f_S)]$.*

Proof. The case when $(T, f_S) \in \mathcal{M}_{\mathrm{p}}$ was proved in [BK, Th. 3.12]. Therefore we suppose that $(T, f_S) \in \mathcal{M}_\infty$.

If $T \cap S \neq \emptyset$, put $T^* = T$. So, we can assume that $T \cap S = \emptyset$. Define $t = \min T$.

Let $f_S^i(t) \in J_i$ for $i \in \mathbb{N}_0$ where each J_i is the closure of a component of $\operatorname{conv} S \setminus S$. Obviously f_S is strictly monotone on each J_i and $f(J_i) \supset J_{i+1}$. Moreover, since $T \cap S = \emptyset$ we can consider the least finite set $\{I_1, \ldots, I_k\}$ of components of $\operatorname{conv} S \setminus S$ such that every J_i is from $\{I_1, \ldots, I_k\}$. Define the map $p : \mathbb{N}_0 \to \{1, \ldots, k\}$ by

$$p : i \mapsto p_i \iff J_i = I_{p_i}.$$

The map p is periodic if there is a positive integer n such that $p_i = p_{i+n}$ for each $i \in \mathbb{N}_0$. Let us show that such an n does not exist. We know that $f_S^i(t) \in J_i = I_{p_i}$. If such an n exists, then $f_S^n(T \cap I_{p_0}) = T \cap I_{p_0}$ and since f_S is affine on each I_{p_i}, f_S^n or f_S^{2n} is increasing on I_{p_0}. But then $f_S^n(t)$ or $f_S^{2n}(t)$ has to be equal to t—a contradiction with our assumption $(T, f_S) \in \mathcal{M}_\infty$. So p is not periodic. Notice that this is equivalent to the fact that for any different $i(1), i(2) \in \mathbb{N}$ there exists $i \in \mathbb{N}_0$ for which $f_S^i(f_S^{i(1)}(t)) \in I_{p_{i+i(1)}}$, $f_S^i(f_S^{i(2)}(t)) \in I_{p_{i+i(2)}}$ and $\operatorname{int}(I_{p_{i+i(1)}}) \cap \operatorname{int}(I_{p_{i+i(2)}}) =$

$\emptyset$, i.e. the points $f_S^{i(1)}(t), f_S^{i(2)}(t) \in \operatorname{orb}(t, f_S)$ have different trajectories with respect to $\{I_1, \ldots, I_k\}$.

Set $I_i^1 = I_{p_i}$ for $i \in \mathbb{N}_0$. We define closed intervals I_i^j, $(i,j) \in \mathbb{N}_0 \times \mathbb{N}$, by the conditions $I_i^j \subset I_i^{j-1}$ and $f_S(I_i^j) = I_{i+1}^{j-1}$ (clearly $f_S(I_i^{j-1}) \supset I_{i+1}^{j-1}$). Put $\mathcal{I}_i = \bigcap_{j\in\mathbb{N}} I_i^j$. We have $f_S^i(t) \in \mathcal{I}_i$ for each $i \in \mathbb{N}_0$; by our definition of the intervals I_i^j we even get $f_S^i(\mathcal{I}_0) = \mathcal{I}_i$. Clearly $\mathcal{I}_i$ is a point or a closed interval.

Now we show that $\mathcal{I}_{i(1)} \cap \mathcal{I}_{i(2)} = \emptyset$ for $i(1) \neq i(2)$. Define n as the least positive integer for which the trajectories of the points $f_S^{i(1)}(t), f_S^{i(2)}(t)$ differ. If there is an $x \in \mathcal{I}_{i(1)} \cap \mathcal{I}_{i(2)}$, then $\{x\} = \mathcal{I}_{i(1)} \cap \mathcal{I}_{i(2)}$ and $f_S^n(x) \in S$, since $I_{i(1)}^j = I_{i(2)}^j$ for $j \in \{1, \ldots, n\}$ and $f_S^n(I_{i(1)}^{n+1}), f_S^n(I_{i(2)}^{n+1})$ belong to the different intervals $I_{p_{n+i(1)}}, I_{p_{n+i(2)}}$ with $\{f_S^n(x)\} = I_{p_{n+i(1)}} \cap I_{p_{n+i(2)}}$. So we have already shown that the intersection of two $\mathcal{I}$'s can be at most one-point. Since $f_S^i(t) \in \mathcal{I}_i$, this immediately shows that $\operatorname{orb}(t, f_S)$ and $\operatorname{orb}(\mathcal{I}_0, f_S)$ have the same order. In particular, the minimality of (T, f_S) implies that both orbits have infinitely many elements in every interval from $\{I_1, \ldots, I_k\}$. On the other hand, by assumption, $f(S) \subset S$, hence also $f_S(S) \subset S$. Now the reader can see that, supposing $\{x\} = \mathcal{I}_{i(1)} \cap \mathcal{I}_{i(2)}$ there have to be positive integers $n_2 > n_1 > n$ for which $\mathcal{I}_{n_2} \supset \mathcal{I}_{n_1}$. Summarizing, $\mathcal{I}_{i(1)} \cap \mathcal{I}_{i(2)} = \emptyset$ for $i(1) \neq i(2)$ and $\mathcal{I}_i \cap S = \emptyset$ for each $i \in \mathbb{N}_0$.

Let $K_i^1 = I_{p_i}$ for $i \in \mathbb{N}_0$. By Lemma 2.5, we can choose closed intervals $K_i^j = [a_i^j, b_i^j]$, $(i,j) \in \mathbb{N}_0 \times \mathbb{N}$, such that

(i) $K_i^j \subset K_i^{j-1}$,

(ii) $f(K_i^j) = K_{i+1}^{j-1}$ and $\operatorname{conv}\{f(a_i^j), f(b_i^j)\} = K_{i+1}^{j-1}$,

(iii) $\operatorname{sign}_{f_S}(I_i^j) = \operatorname{sign}_f(K_i^j)$

(iv) for each $j \in \mathbb{N}$, the orders of $\{K_i^j\}_{i\in\mathbb{N}_0}$ and $\{I_i^j\}_{i\in\mathbb{N}_0}$ are the same.

Put $\mathcal{K}_i = \bigcap_{j\in\mathbb{N}} K_i^j$. Clearly $\mathcal{K}_i$ is a point or a closed interval. Using (i)–(iv) we can show as for $\mathcal{I}_i$ the following properties:

(v) $\mathcal{K}_{i(1)} \cap \mathcal{K}_{i(2)} = \emptyset$ for $i(1) \neq i(2)$ and $\mathcal{K}_i \cap S = \emptyset$, $f^i(\mathcal{K}_0) \subset I_{p_i}$ and $f^i(\mathcal{K}_0) = \mathcal{K}_i$ for each $i \in \mathbb{N}_0$,

(vi) the order of $\operatorname{orb}(\mathcal{K}_0, f)$ is the same as the order of $\operatorname{orb}(\mathcal{I}_0, f_S)$, which is the same as the order of $\operatorname{orb}(t, f_S)$.

Thus the sequence $\{\mathcal{K}_i\}_{i\in\mathbb{N}_0}$ satisfies the assumptions of Lemma 2.2. Therefore, we can find $T^* \subset \operatorname{conv} S$ for which $(T^*, f) \in [(T, f_S)]$. This proves Lemma 2.6. ■

3. Main results. Our goal in this section is to use the lemmas developed in the previous section to prove the main results. We begin with a statement that extends [ALM, Th. 2.6.13].

Theorem 3.1. *Let A, B be minimal patterns. Then the following conditions are equivalent.*

(i) *A forces B.*

(ii) *For some $(T, g) \in A$, g_T exhibits the pattern B.*

Proof. The implication (i)$\Rightarrow$(ii) is clear. The case when both patterns A, B are periodic is known [ALM].

Let A, B be minimal patterns, and suppose (ii). Let $f \in C(I)$ be any map that exhibits the pattern A, i.e. there is $S \subset I$ such that $(S, f) \in A$. Consider two maps: $f_S \in C(S, f)$ and g_T ensured by (ii).

If A is a periodic pattern then by [BCv, Th. 2.6] the maps f_S and g_T are topologically conjugate, hence they exhibit the same patterns. By (ii), f_S exhibits the pattern B, i.e. there is $T \subset \operatorname{conv} S$ such that $(T, f_S) \in B$. Notice that all assumptions of Lemma 2.6 are satisfied. Hence there exists $T^* \subset \operatorname{conv} S$ such that $(T^*, f) \in [(T, f_S)] = B$, i.e. f exhibits the pattern B. So $A \hookrightarrow B$ in this case.

Suppose that A is infinite. By assumption, g_T exhibits B. In order to use Lemma 2.6 again, we need to show that f_S also exhibits B. By Lemma 2.3, there are maps $h_1 \in \mathcal{H}(\operatorname{conv} S)$, $\widetilde{f} \in C(h_1(S), \widetilde{f})$, $h_2 \in \mathcal{H}(\operatorname{conv} T)$ and $\widetilde{g} \in C(h_2(T), \widetilde{g})$ such that $f_S, \widetilde{f}$, resp. $g_T, \widetilde{g}$ exhibit the same patterns. Moreover, $A = [(S, f)] = [(h_1(S), \widetilde{f})] = [(h_2(T), \widetilde{g})] = [(T, g)]$ and the pairs $(h_1(S), \widetilde{f}), (h_2(T), \widetilde{g})$ are canonical. By Lemma 2.4, the maps $\widetilde{f} \in C(\operatorname{conv} S)$ and $\widetilde{g} \in C(\operatorname{conv} T)$ are topologically conjugate. This implies that all four maps f_S, $\widetilde{f}$, $\widetilde{g}$, g_T exhibit the same patterns. In particular so do f_S, g_T. Now as above, Lemma 2.6 ensures the existence of $T^* \subset S$ such that $(T^*, f) \in [(T, f_S)] = B$, i.e. f exhibits the pattern B. Hence also in this case $A \hookrightarrow B$.

This proves the theorem. ∎

In [B] it is shown that the forcing relation on periodic (oriented) patterns is a partial ordering (see also [ALM, Th. 2.5.1]). In the next theorem we show that this also holds for a larger set of minimal (finite or infinite) patterns.

Theorem 3.2. *The forcing relation on minimal patterns is a partial ordering.*

Proof. Clearly, if A is a pattern, then $A \hookrightarrow A$ (reflexivity); if A, B, C are patterns such that $A \hookrightarrow B$ and $B \hookrightarrow C$, then $A \hookrightarrow C$ (transitivity). Thus it remains to prove the weak antisymmetry of the forcing relation.

Suppose that for patterns A, B, $A \hookrightarrow B$ and $B \hookrightarrow A$. If both patterns are periodic, then $A = B$ by Theorem 1.1.

Thus, let A be infinite and $A \neq B$. Take $(S, f) \in A$. We know that (S, f) is piecewise monotone. If $\widetilde{S} \subset \operatorname{conv} S$ is such that $(\widetilde{S}, f_S) \in A$, then since the modalities of the pairs $(S,f),(\widetilde{S},f_S)$ equal we see that $(s_0=\min S)$

$$\min\widetilde{S}<c=\max\{x\in\operatorname{conv}S: f_S \text{ is monotone on } [s_0,c]\}. \tag{4}$$

Using Theorem 3.1 and Lemma 2.6 repeatedly we can consider closed sets S_j, $j\in\mathbb{N}_0$, such that $S_0=S$ and

$$\begin{aligned}&\operatorname{conv}S_j\supset\operatorname{conv}S_{j+1},\qquad S_{j(1)}\cap S_{j(2)}=\emptyset\quad\text{for } j(1)\neq j(2),\\&(S_j,f_S)\in A.\end{aligned}\tag{5}$$

In particular, all orbits $\operatorname{orb}(s_j=\min S_j,f_S)$ have the same order. Set

$$s_\infty=\sup\{\min\widetilde{S}:(\widetilde{S},f_S)\in A\}.$$

Since for c defined in (4) we have $c\in S$, $f_S^k(c)<s_1=\min S_1$ for some $k\in\mathbb{N}$; this implies $s_\infty<c$. We can consider the sets S_j satisfying (5) and such that for $s_j=\min S_j$ we have

$$s_0<s_1<\ldots<s_j<\ldots<s_\infty<c,\qquad\lim_j s_j=s_\infty.$$

In any case $\operatorname{orb}(s_\infty,f_S)\cap S_0=\emptyset$. Now we distinguish two cases.

Case 1. Let us show $\operatorname{orb}(s_\infty,f_S)$ cannot be infinite. If it is, then since $s_\infty=\lim_j s_j$ and $(S_j,f_S)\in A$, the continuity of f_S shows that $\operatorname{orb}(s_\infty,f_S)$ has the same order as $\operatorname{orb}(s_0,f_S)$. If we put $\mathcal{K}_i=f_S^i(s_\infty)$ in Lemma 2.2, all conditions of that lemma are satisfied. Hence there is a set $T^*\subset\operatorname{conv}S$ such that $\max\mathcal{K}_0=s_\infty\le t^*=\min T^*$ and $(T^*,f_S)\in A$. Using Theorem 3.1 and Lemma 2.6 again we see that there is a set $T\subset\operatorname{conv}T^*$ such that $\min T^*<\min T$ and $(T,f_S)\in A$—a contradiction with the choice of s_∞.

Case 2. Finally we show that $\operatorname{orb}(s_\infty,f_S)$ cannot be finite. Suppose to the contrary that $\#\operatorname{orb}(s_\infty,f_S)\in\mathbb{N}$. Then there are $k\in\mathbb{N}_0$ and $n\in\mathbb{N}$ such that $f_S^k(s_\infty)\in\operatorname{Per}(f_S)$ and $\operatorname{per}(f_S^k(s_\infty))=n$. Let k,n be the least with this property. We can write

$$\operatorname{orb}(s_\infty,f_S)=\{s_\infty=p_1<\ldots<p_{k+n}\}.$$

I. First we verify that $p_1\notin\operatorname{Per}(f_S)$. If p_1 were periodic its period would be n. Then $f_S^{2n}(p_1)=p_1$ and since $\operatorname{orb}(s_\infty,f_S)\cap S_0=\emptyset$, the map f_S^{2n} is affine with slope greater than 1 on some $U_\varepsilon=(p_1-\varepsilon,p_1)$. But then for sufficiently large j we have $s_j\in U_\varepsilon$ and also $f_S^{2n}(s_j)<s_j$, which contradicts $s_j=\min S_j$.

II. Let us show that $p_2\in\operatorname{Per}(f_S)$; indeed, all orbits $\operatorname{orb}(s_j,f_S)$ have the same order and if $p_2\notin\operatorname{Per}(f_S)$ then $p_2=f_S^l(s_\infty)$ for $0<l<k$ and since $\lim_j s_j=s_\infty$, for $i>l$ we have

$$f_S^i(s_0)>f_S^l(s_0),$$

which is impossible for $(S_0,f_S)\in\mathcal{M}_\infty$.

III. The last situation that we have to disprove is the following: For $0 \leq m < n$, $f_S^{k+m}(p_1) = p_2 \in \mathrm{Per}(f_S)$ and $\mathrm{per}(p_2) = n$. Define $M = \{k + m + 2in : i \in \mathbb{N}_0\}$. In this case as for I we can show that for each $i \in \mathbb{N}_0$, $l \in \mathbb{N} \setminus M$,

$$f_S^{k+m+2in}(s_0) < f_S^l(s_0),$$

and then by the minimality of (S_0, f_S) either

$$\lim_i f_S^{k+m+2in}(s_0) = s_0 \quad \text{or} \quad \lim_i f_S^{k+m+4in}(s_0) = s_0.$$

Both cases imply $s_0 \in \mathrm{Per}(f_S)$—a contradiction.

Thus, $A = B$ and the proof of Theorem 3.2 is finished. ∎

References

[ALM] L. Alsedà, J. Llibre and M. Misiurewicz, *Combinatorial Dynamics and Entropy in Dimension One*, Adv. Ser. Nonlinear Dynam. 5, World Sci., Singapore, 1993.

[B] S. Baldwin, *Generalizations of a theorem of Sharkovskii on orbits of continuous real-valued functions*, Discrete Math. 67 (1987), 111–127.

[Bl1] A. Blokh, *The "spectral" decomposition for one-dimensional maps*, in: Dynamics Reported 4, Springer, 1995, 1–59.

[Bl2] —, *Rotation number for unimodal maps*, preprint # 58-94, MSRI, 1994.

[BCp] L. S. Block and W. A. Coppel, *Dynamics in One Dimension*, Lecture Notes in Math. 1513, Springer, Berlin, 1992.

[BCv] L. S. Block and E. M. Coven, *Topological conjugacy and transitivity for a class of piecewise monotone maps of the interval*, Trans. Amer. Math. Soc. 300 (1987), 297–306.

[BK] J. Bobok and M. Kuchta, *Invariant measures for maps of the interval that do not have points of some period*, Ergodic Theory Dynam. Systems 14 (1994), 9–21.

[P] C. Preston, *Iterates of Piecewise Monotone Mappings on an Interval*, Lecture Notes in Math. 1347, Springer, 1988.

[Y] X. Ye, *D-function of a minimal set and an extension of Sharkovskii's theorem to minimal sets*, Ergodic Theory Dynam. Systems 12 (1992), 365–376.

KM FSv. ČVUT
Thákurova 7
166 29 Praha 6, Czech Republic
E-mail: erastus@mbox.cesnet.cz

Received 7 June 2000;
in revised form 1 March 2001