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A topological characterization of holomorphic parabolic germs in the plane

by

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Abstract. J.-M. Gambaudo and É. Pécou introduced the "linking property" in the study of the dynamics of germs of planar homeomorphisms in order to provide a new proof of Naishul's theorem. In this paper we prove that the negation of the Gambaudo–Pécou property characterizes the topological dynamics of holomorphic parabolic germs. As a consequence, a rotation set for germs of surface homeomorphisms around a fixed point can be defined, and it turns out to be non-trivial except for countably many conjugacy classes.

1. Introduction. Let $\mathcal{H}^+$ be the set of orientation preserving homeomorphisms of the plane that fix 0, and let $h \in \mathcal{H}^+$. We are interested in the dynamics of the germ of h at 0. Imagine one wants to evaluate the "amount of rotation" in a neighbourhood V of 0 by looking at the way the orbit of some point $x \in V$ rotates around 0. Then two kinds of difficulties can arise:

- if the orbit of x leaves V after a small number of iterations, then the behaviour of x is not significant with respect to the local dynamics;
- if the orbit of x tends to the fixed point 0, then the rotation of x around 0 is not significant either, because it is not invariant under a continuous change of coordinates.

These difficulties have led Gambaudo and Pécou to introduce the "linking property" (see [GP, Pé]) which demands that inside each neighbourhood of 0 there exist arbitrarily long segments of orbits starting and ending not too close to 0. In this paper we prove that the only germs that do not share the linking property are the contraction, dilatation and holomorphic parabolic germs. To be more precise, let us define the *short trip property*, which is the negation of the Gambaudo–Pécou property, as follows.

DEFINITION 1. Let $f \in \mathcal{H}^+$. We say that f has the *short trip property* if there exists a neighbourhood V of the fixed point 0 such that for every

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neighbourhood W of 0, there exists an integer $N_W > 0$ such that for every segment of orbit $(x, f(x), \dots, f^n(x))$ which is included in V, and whose endpoints $x, f^n(x)$ are outside W, the length n is less than N_W.

Two homeomorphisms $f_1, f_2 \in \mathcal{H}^+$ are said to be *locally topologically conjugate* if there exists a homeomorphism $\varphi \in \mathcal{H}^+$ such that $f_2 = \varphi f_1 \varphi^{-1}$ on some neighbourhood of 0. We are interested in the local dynamics near the fixed point 0, thus we consider maps up to local conjugacy. Note that any local homeomorphism locally coincides with a homeomorphism defined on the whole plane, so that working with globally defined homeomorphisms is just a matter of convenience and does not alter the results (see [Ham] or [LR1, chapitre 2]). As a consequence, to prove that two homeomorphisms are locally topologically conjugate it suffices to construct the conjugacy on a neighbourhood of 0.

DEFINITION 2. Let $f \in \mathcal{H}^+$, and identify the plane with the complex plane $\mathbb{C}$. We say that f is a *locally holomorphic parabolic homeomorphism* (or just *parabolic*) if f is holomorphic on some neighbourhood of 0, $f'(0)$ is a root of unity, and for every positive n the map f^n is not locally equal to the identity.

Note that the hypothesis on $f'(0)$ amounts to saying that the differential of f is a rational rotation, and then the last hypothesis is equivalent to saying that f is not locally topologically conjugate to its differential. According to Camacho's version of the Leau–Fatou theorem, if $f \in \mathcal{H}^+$ is parabolic, then f is locally topologically conjugate to some map

$$z \mapsto e^{2i\pi p/q} z(1 + z^{qr}) \quad \text{with } p/q \in \mathbb{Q},\ q, r \geq 1.$$

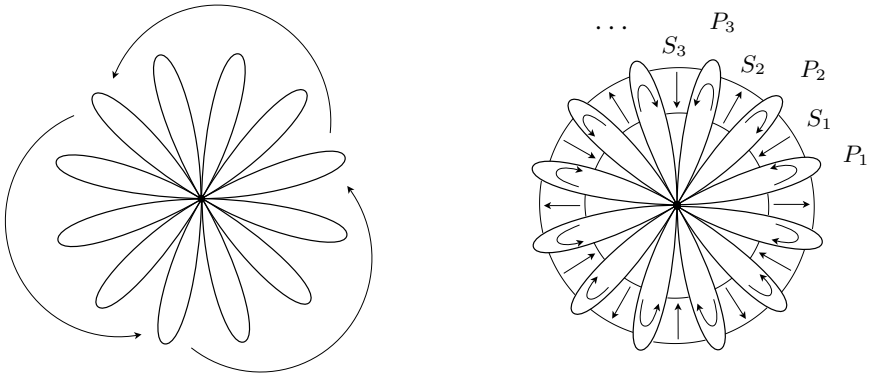
See [Cam], and Figure 1.

We can now state our theorem.

THEOREM 3. *Let f be an orientation preserving homeomorphism of the plane that fixes the point* 0. *Then f has the short trip property if and only if it is locally topologically conjugate to the contraction $z \mapsto \frac{1}{2}z$, to the dilatation $z \mapsto 2z$, or to a locally holomorphic parabolic homeomorphism.*

As a consequence there are only countably many conjugacy classes failing to have the Gambaudo–Pécou property.

In order to explain where Theorem 3 comes from, let us first discuss the Naishul theorem. In [GP] it was shown that the Gambaudo–Pécou property holds when f preserves area, and then this property is used to prove the Naishul theorem: *among area preserving homeomorphisms fixing* 0 *that are differentiable at* 0 *and whose differential is a rotation, the angle of the rotation is invariant under a local topological conjugacy.* Then the following nice generalization of the Naishul theorem is given by Gambaudo, Le Calvez and



qr periodic petals for f action of f^q

Fig. 1. Local topological dynamics of $f : z \mapsto e^{2i\pi p/q} z(1 + z^{qr})$; here $q = 3$, $p = 1$, $r = 2$, so that there are two orbits of attracting petals and two orbits of repulsive petals.

Pécou in [GLP]. As a generalization of differentiability at 0, they consider the homeomorphisms f for which the fixed point can be "blown up", i.e. replaced by an ideal circle in such a way that f can be extended to a circle homeomorphism (see the precise definition in [GLP]). They prove that for such homeomorphisms, *the Poincaré rotation number of the extended circle homeomorphism is invariant under a local topological conjugacy, unless f is a contraction or a dilatation.* The strategy of their proof is the following. If f has the Gambaudo–Pécou property, then one can use the arguments in [GP]. Now assume that f is *indifferent*, that is, f admits arbitrarily small non-trivial invariant compact connected sets K containing 0; then one can use Carathéodory's prime ends theory to associate a circle homeomorphism f_K to each such K, and use the rotation number of f_K to prove the topological invariance. Then one proves a last lemma asserting that *a germ which is not indifferent and does not have the Gambaudo–Pécou property must be a contraction or a dilatation.*

As a consequence of the Leau–Fatou theorem, parabolic maps are indifferent. Thus Theorem 3 can be seen as a generalization of this last lemma. Furthermore, it provides an alternative proof of the generalized Naishul theorem, avoiding the use of prime ends, as follows: we keep the arguments in [GP] to tackle homeomorphisms with the Gambaudo–Pécou property; then, in view of Theorem 3, it only remains to deal with parabolic homeomorphisms, for which the proof is easy because the local dynamics is fully understood.

More generally, in [LR2] we will define a local rotation set for any homeomorphism f in $\mathcal{H}^+$. This set is a subset of the extended line $\mathbb{R} \cup \{\pm\infty\}$, modulo integer translation, and it is a local topological conjugacy invariant.

Then Theorem 3 will entail that *the local rotation set is non-void as soon as f does not fall into the countably many conjugacy classes described by the theorem.*

One can also think of Theorem 3 as a local analogue of previous results showing that a simple dynamical property can imply a strong rigidity. The most striking result here is probably the Hiraide–Lewowicz theorem: an expansive homeomorphism on a compact surface is conjugate to a pseudo-Anosov homeomorphism (see [Hi, Le]). Closer to our setting, Kerékjártó has shown that an orientation preserving homeomorphism of a closed orientable surface whose singular set is totally disconnected is topologically conjugate to a conformal transformation (see [BK, Ke34a, Ke34b]). Thus, for instance, an orientation preserving homeomorphism f of the plane is conjugate to a translation if and only if it has no fixed point and the family $(f^n)_{n\geq 0}$ is equicontinuous at each point for the spherical metric.

In some sense, Theorem 3 highlights that it is easy to be locally conjugate to a locally parabolic homeomorphism: a homeomorphism that "looks like" a parabolic map will be conjugate to it. In contrast, the examples given in [BLR] reveal how difficult it is to be conjugate to the saddle homeomorphism $(2x, y/2)$, and in particular that it is not enough to preserve the hyperbolic foliation. A topological characterization can be given, but it must take into account the sophisticated *oscillating set* (see, in [BLR], the remark on Fig. 3 as well as part III).

2. Dynamics of parabolic germs. Propositions 12 and 13 below provide a first (classical) characterization of parabolic germs in terms of *attracting* and *repulsive sectors* and *invariant petals*.

2.1. *Contractions and attracting sectors.* We begin by characterizing the dynamics of contractions. Then we describe *attracting sectors*. Of course, similar results hold for dilatations and *repulsive sectors*, although we will not state them explicitly.

Let $f \in \mathcal{H}^+$. We will say that a sequence $(E_n)_{n\geq 0}$ of subsets of the plane *converges to* 0 if for every neighbourhood W of 0, all but finitely many terms of the sequence are included in W. The following result is very classical.

PROPOSITION 4. *Let $f \in \mathcal{H}^+$. Let D be a topological closed disc* ([1]) *which is a neighbourhood of* 0, *and suppose that the orbit $(f^n(D))_{n\geq 0}$ converges to* 0. *Then f is locally topologically conjugate to the contraction $z \mapsto \frac{1}{2}z$.*

Proof. By hypothesis there exists $n > 0$ such that $f^n(D) \subset \mathrm{Int}(D)$. Choose some decreasing finite sequence of topological closed discs D_i with $D_0 = D$, $\mathrm{Int}(D_i) \supset D_{i+1}$, and $\mathrm{Int}(D_{n-1}) \supset f^n(D_0)$. Consider the set

([1]) A *topological closed disc* is a set homeomorphic to the closed unit disc.

$$O = \mathrm{Int}(D_{n-1}) \cap \mathrm{Int}(f(D_{n-2})) \cap \cdots \cap \mathrm{Int}(f^{n-1}(D_0)).$$

Let U be the connected component of O containing the fixed point 0. The hypotheses on the D_i's entail that $\mathrm{Clos}(f(O)) \subset O$. Since $\mathrm{Clos}(f(U))$ is connected and contains 0, we deduce that $\mathrm{Clos}(f(U)) \subset U$. Furthermore, according to a theorem of Kerékjártó, the set $D' = \mathrm{Clos}(U)$ is a closed topological disc (see [Ke23, LCY]). This disc satisfies $f(D') \subset \mathrm{Int}(D')$.

Now the annulus $D' \setminus \mathrm{Int}(f(D'))$ is a "fundamental domain" for f, and can be used to construct a local topological conjugacy between f and the contraction. ∎

We will say that two sets S and S' *coincide in a neighbourhood of* 0, or *have the same germ at* 0, and we will write $S \overset{0}{=} S'$, if there exists a neighbourhood V of 0 such that $S \cap V = S' \cap V$.

Definition 5 (see Figure 2). An *attracting sector* is a topological closed disc S whose boundary contains 0, which coincides in a neighbourhood of 0 with its image $f(S)$, and whose orbit $(f^n(S))_{n\geq 0}$ converges to 0. The attracting sector is said to be *nice* if $f(S) \subset S$ and $S \setminus f(S)$ is connected. A (*nice*) *repulsive sector* is a (nice) attracting sector for f^{-1}.

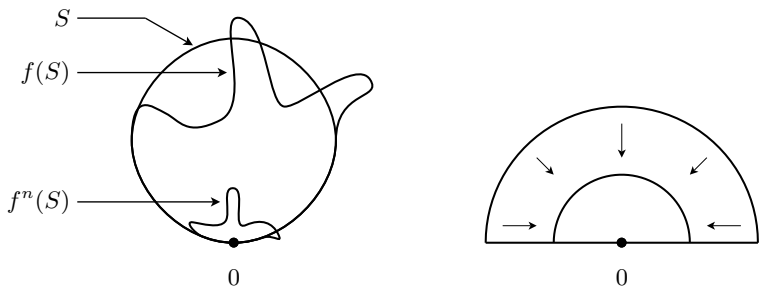


Fig. 2. Attracting and nice attracting sectors

Claim 6.

1. *If S is an attracting sector then there exists a nice attracting sector S', included in S, and having the same germ as S at* 0.
2. *If S' is a nice attracting sector for f, then there exists a homeomorphism Φ between S' and the half-disc $S_0 = \{|z| \leq 1, y \geq 0\}$ such that the conjugacy relation $\Phi f = \frac{1}{2}\Phi$ between f and the contraction $z \mapsto \frac{1}{2}z$ holds on S'.*

Remark 7. Here are some easy consequences of item 2 of the claim.

1. The sets $\Phi^{-1}([-1,0]), \Phi^{-1}([0,1])$ are called the *sides* of the nice attracting sector; they do not depend on the choice of Φ.

2. There exist arbitrarily small nice attracting sectors within S'; moreover, any pair of points x, y on both sides of S' are the endpoints of the sides of some nice attracting sector included in S'.
3. Any homeomorphism Φ between the union of the sides of S' and the segment $[-1,1]$, satisfying the conjugacy relation $\Phi f = \frac{1}{2}\Phi$, can be extended to a homeomorphism between S' and S_0 conjugating f and $z \mapsto \frac{1}{2}z$ as in item 2 of the claim.

Proof of Claim 6. Let S be an attracting sector. We first notice that *there exists an arc α included in the boundary of S, whose interior* ([2]) $\mathrm{Int}(\alpha)$ *contains the fixed point* 0, *and such that* $f(\alpha) \subset \mathrm{Int}(\alpha)$. Indeed, consider a one-to-one continuous map $\varphi : (-1,1) \to \partial S$ such that $\varphi(0) = 0$, and let $\hat{f} = \varphi^{-1} f \varphi$. Since $f(S) \overset{0}{=} S$, this map $\hat{f}$ is well defined on some neighbourhood V of 0 in $(-1,1)$, it is a homeomorphism from V to $\hat{f}(V) \subset (-1,1)$, and it fixes 0. Since f preserves the orientation, and $f(S) \overset{0}{=} S$, $\hat{f}$ also preserves the orientation. Furthermore, for every positive $x \in V$, we have $\hat{f}(x) < x$: otherwise the sequence $(\hat{f}^{-n}(x))_{n\geq 0}$ would be well defined and included in V, and then $\varphi(x)$ would be included in $f^n(\partial S)$ for every positive n, contradicting the hypothesis that $(f^n(S))_{n\geq 0}$ converges to 0. Similarly we get $\hat{f}(x) > x$ for every negative x, so $\hat{f}(\mathrm{Clos}(V)) \subset V$. Finally, the arc α is obtained as $\varphi(\mathrm{Clos}(V))$.

Consider the set

$$A := \bigcup_{n\geq 0} f^{-n}(\alpha).$$

This is clearly a continuous one-to-one image of the real line, and $f(A) = A$. By definition of an attracting sector, there exists an integer n_0 such that for every $n \geq n_0$, $f^n(S)$ is disjoint from the compact set $f^{-1}(\alpha) \setminus \mathrm{Int}(\alpha)$. Then $A \cap S = f^{-n_0}(\alpha) \cap S$. In particular, we can find a simple arc β such that $\alpha \cup \beta$ is a Jordan curve included in S, and whose intersection with A reduces to α.

Let D_0 be the topological closed disc bounded by $\alpha \cup \beta$. Then D_0 is included in S and coincides with S in a neighbourhood of 0, and $D_0 \cap A = \alpha$. Note that for every n, $f^n(D_0) \cap A = f^n(D_0 \cap A) = f^n(\alpha)$. The disc D_0 is clearly an attracting sector. Let n_0 be a positive integer such that for every $n \geq n_0$, $f^n(D_0)$ does not meet β. Thus $f^n(D_0) \subset \mathrm{Int}(D_0) \cup f^n(\alpha)$.

We can now find a closed topological disc $S' \subset D_0$, having the same germ at 0 as D_0, which is a nice attracting sector for f. For this we can make a construction similar to the proof of Proposition 4, with the following adaptations. Now we choose the topological discs D_i having the same germ at 0, containing α, and such that $\mathrm{Int}(D_i) \cup \alpha \supset D_{i+1}$ and $\mathrm{Int}(D_{n-1}) \cup f^n(\alpha)$

([2]) The interior of a curve is defined to be the curve minus its endpoints.

$\supset f^n(D_0)$. As before, we consider the set

$$O = \mathrm{Int}(D_{n-1}) \cap \mathrm{Int}(f(D_{n-2})) \cap \cdots \cap \mathrm{Int}(f^{n-1}(D_0)).$$

Now the open set U is defined to be the unique connected component of O that has the same germ at 0 as the D_i's, and S' is the closure of U. Then $f(S') \subset U \cup f^n(\alpha)$, and $f^n(S')$ contains $f^n(\alpha)$; and thus $S' \setminus f(S')$ is connected. The details are left to the reader.

The proof of item 2 is straightforward by using the fundamental domain $\mathrm{Clos}(S' \setminus f(S'))$. ∎

2.2. *Regular invariant petals.* We first recall a theorem of Kerékjártó ([Ke34b]).

THEOREM 8 (Kerékjártó). *Let f be a homeomorphism of the plane that preserves the orientation, and suppose that for any compact set K the orbit $(f^n(K))_{n\geq 0}$ converges to the point at infinity in the sphere $\mathbb{R}^2 \cup \{\infty\}$. Then f is topologically conjugate to the translation $z \mapsto z+1$.*

Sketch of proof. By considering the space of orbits, the problem can be brought into the realm of the classification of surfaces; then the conclusion follows from the fact that any (Hausdorff) surface without boundary, whose fundamental group is the group of integers, is homeomorphic to an infinite cylinder (see for example [AS]). ∎

DEFINITION 9. An *invariant petal* for f is a topological closed disc P whose boundary contains the fixed point 0, and such that $f(P) = P$. An invariant petal is called *regular* if for every compact set $K \subset P \setminus \{0\}$, the sequence $(f^n(K))_{n\geq 0}$ converges to 0.

REMARK 10. If P is a regular invariant petal then f has no fixed point on the topological line $\partial P \setminus \{0\}$. Thus we may endow this line with a *dynamical order* such that $f(x) > x$ for any $x \neq 0$ on ∂P. The petal will be called *direct* if this dynamical order is compatible with the topological (usual) orientation of ∂P as a Jordan curve of the oriented plane (for which the interior of P is "on the left" of ∂P); in the opposite case it will be called *indirect*.

An adaptation of the proof of Kerékjártó's theorem yields the following.

CLAIM 11. *Let P be a regular invariant petal for $f \in \mathcal{H}^+$. If P is direct then the restriction $f_{|P}$ is topologically conjugate, via an orientation preserving homeomorphism, to the restriction of the translation $z \mapsto z+1$ to the closed half-sphere $\{x+iy : y \geq 0\} \cup \{\infty\}$ of the Riemann sphere $\hat{\mathbb{C}}$. The same is true if P is indirect with $z+1$ replaced by $z-1$.*

2.3. *Characterization of parabolic homeomorphisms.* We can now characterize the local dynamics of parabolic homeomorphisms. For any set D the *maximal invariant subset* of D is the set $\bigcap_{n\in\mathbb{Z}} f^n(D)$ of points whose whole orbit is included in D.

PROPOSITION 12 (see Figure 1). *Let $f \in \mathcal{H}^+$. Fix some integer $l \geq 1$. Then f is locally topologically conjugate to $z \mapsto z(1+z^l)$ if and only if there exists a neighbourhood of 0 which is a topological closed disc D, called a* nice disc, *such that*

1. *the maximal invariant subset of D is the union of $2l$ regular invariant petals $P_1, \dots, P_{2l}$ whose pairwise intersections are $\{0\}$;*
2. *the sets $\partial D \cap P_i$ are connected, and the cyclic order of these sets along ∂D coincides with the order of the indices $i \in \mathbb{Z}/2l\mathbb{Z}$;*
3. *for every i, let S_i be the closure of the connected component of $D \setminus (P_1 \cup \dots \cup P_{2l})$ meeting both P_i and P_{i+1}; then S_i is a nice attracting sector for i odd and a nice repulsive sector for i even.*

The next statement takes into account a possible permutation of the petals.

PROPOSITION 13. *Let $f \in \mathcal{H}^+$ and suppose that for some positive integer n_0 the map f^{n_0} is locally topologically conjugate to a parabolic homeomorphism. Then so is f.*

The proofs are delayed until Section 4.

3. Proof of the theorem. From now on, f denotes an orientation preserving homeomorphism of the plane that fixes 0 and has the short trip property. For every set V we define the sets

$$W^{\mathrm{s}}(V) = \bigcap_{n \geq 0} f^{-n}(V) \quad \text{and} \quad W^{\mathrm{u}}(V) = \bigcap_{n \leq 0} f^{-n}(V).$$

We fix some open neighbourhood V of 0 as in the definition of the short trip property. Since our hypothesis is symmetric in time, both the above sets share the same properties, and we will usually restrict the study to $W^{\mathrm{s}}(V)$.

3.1. *Orbits.* The following lemma shows in particular that the orbits of points near 0 can only converge to 0 or escape from the neighbourhood V. Note that this lemma still holds in any dimension.

LEMMA 14.

1. *For every compact subset K of $W^{\mathrm{s}}(V) \setminus \{0\}$, the sequence $(f^n(K))_{n \geq 0}$ converges to 0.*
2. *The set $W^{\mathrm{s}}(V) \setminus \{0\}$ is open.*
3. *The set $W^{\mathrm{s}}(V) \cup W^{\mathrm{u}}(V)$ is a neighbourhood of 0.*
4. *If $W^{\mathrm{s}}(V)$ is a neighbourhood of 0 then f is locally topologically conjugate to $z \mapsto \frac{1}{2}z$; if $W^{\mathrm{u}}(V)$ is a neighbourhood of 0 then f is locally topologically conjugate to $z \mapsto 2z$.*

Proof. Let K be a compact subset of $W^{\mathrm{s}}(V) \setminus \{0\}$, and let W be some neighbourhood of 0 disjoint from K. Note that by definition of $W^{\mathrm{s}}(V)$, for

every positive n, $f^n(K) \subset V$. Now let N_W be given by the short trip property. Then the property forces $f^n(K) \subset W$ for every $n > N_W$. This proves item 1 of the lemma.

Let $x \neq 0$ be some point in $W^{\mathrm{s}}(V)$, and W a neighbourhood of 0 whose closure does not contain x, and such that $W \cup f(W) \subset V$. Let N_W be given by the short trip property. Let

$$O = \Big(\bigcap_{n=0}^{N_W} f^{-n}(V) \Big) \setminus \mathrm{Clos}(W).$$

This is an open set that contains x. We prove item 2 of the lemma by showing that $O \subset W^{\mathrm{s}}(V)$. To see this, let $y \in O$. By definition of N_W in the short trip property we have $f^{N_W}(y) \in W$. Then we claim that $f^n(y) \in W$ for every $n \geq N_W$, which will entail $y \in W^{\mathrm{s}}(V)$ as desired. Assume otherwise and let n_0 be the least integer after N_W such that $f^{n_0}(y) \notin W$. Since $f(W) \subset V$ we have $f^{n_0}(y) \in V$. The segment of orbit $y, \ldots, f^{n_0}(y)$ contradicts the definition of N_W. This completes the proof of item 2.

We consider again a neighbourhood W of 0 such that $W \cup f(W) \cup f^{-1}(W) \subset V$ and N_W given by the short trip property. We define the following neighbourhood of 0:

$$Z = \bigcap_{n=-N_W}^{N_W} f^{-n}(W).$$

We prove by contradiction that $Z \subset W^{\mathrm{s}}(V) \cup W^{\mathrm{u}}(V)$. Assume some $x \in Z$ belongs neither to $W^{\mathrm{s}}(V)$ nor to $W^{\mathrm{u}}(V)$. Since $W \subset V$, the orbit $(f^n(x))$ of x leaves W both in the past and in the future; but by definition of Z this cannot happen for n between $-N_W$ and N_W. Let r, s be the least positive integers such that $f^{-r}(x)$ and $f^s(x)$ do not belong to W; since $f(W) \cup f^{-1}(W) \subset V$, both points belong to $V \setminus W$ and again we have found a segment of orbit of length $r + s > 2N_W$ contradicting the definition of N_W.

Finally, we notice that item 4 is a consequence of item 1 and the topological characterization of contractions (Proposition 4 above). ∎

3.2. *Construction of the petals.* We still consider a homeomorphism $f \in \mathcal{H}^+$ with the short trip property, and from now on we assume that f is locally conjugate neither to the contraction $z \mapsto \frac{1}{2}z$ nor to the dilatation $z \mapsto 2z$. We aim to prove that f is locally conjugate to a parabolic homeomorphism by ultimately applying Propositions 12 and 13. The main task will be to construct the family of periodic petals. As a first approximation we will select a finite number of connected components of $W^{\mathrm{s}}(V) \cap W^{\mathrm{u}}(V) \setminus \{0\}$, hoping to find one petal inside each of these components.

We fix an open neighbourhood V of 0 as before, and we assume V is simply connected. According to item 3 of the previous lemma, we can

choose a topological closed disc D which is a neighbourhood of 0 included in $W^{\mathrm{s}}(V) \cup W^{\mathrm{u}}(V)$. According to item 4, since we excluded the cases of contractions and dilatations, D is included neither in $W^{\mathrm{s}}(V)$ nor in $W^{\mathrm{u}}(V)$. By compactness we can decompose ∂D into the concatenation of $2l \geq 2$ arcs $\alpha_1, \dots, \alpha_{2l}$ such that α_i is included in $W^{\mathrm{s}}(V)$ for i odd and in $W^{\mathrm{u}}(V)$ for i even. We make the following minimality hypothesis: *the number l is minimal among all such choices of topological closed discs D and decompositions of ∂D.*

For every i (integer modulo $2l$) the common endpoint x_i of α_{i-1} and α_i belongs to $W^{\mathrm{s}}(V) \cap W^{\mathrm{u}}(V)$. We denote by $\mathcal{C}_i$ the connected component of $W^{\mathrm{s}}(V) \cap W^{\mathrm{u}}(V) \setminus \{0\}$ that contains x_i. According to item 2 of Lemma 14, $\mathcal{C}_i$ is open. Let D' be a topological closed disc; since V is simply connected, if $\partial D' \subset V$ then $D' \subset V$. Applying this to the iterates of D', we see that if $\partial D' \subset W^{\mathrm{s}}(V) \cap W^{\mathrm{u}}(V)$, then $D' \subset W^{\mathrm{s}}(V) \cap W^{\mathrm{u}}(V)$. Since $W^{\mathrm{s}}(V)$ is not a neighbourhood of 0, we get the following consequence.

LEMMA 15. *Any connected component of $W^{\mathrm{s}}(V) \cap W^{\mathrm{u}}(V) \setminus \{0\}$ is open and simply connected. In particular, the sets $\mathcal{C}_i$ are homeomorphic to the plane. Furthermore, any topological closed disc D whose boundary is included in $\mathcal{C}_i \cup \{0\}$ is also included in $\mathcal{C}_i \cup \{0\}$.*

The next lemma is the fundamental step in the construction of the periodic petals. No dynamics is involved here; indeed, we will only need properties 2 and 3 from Lemma 14 on the topology of $W^{\mathrm{s}}(V)$ and $W^{\mathrm{u}}(V)$.

LEMMA 16. *For every i, the closure of $\mathcal{C}_i$ contains the fixed point* 0.

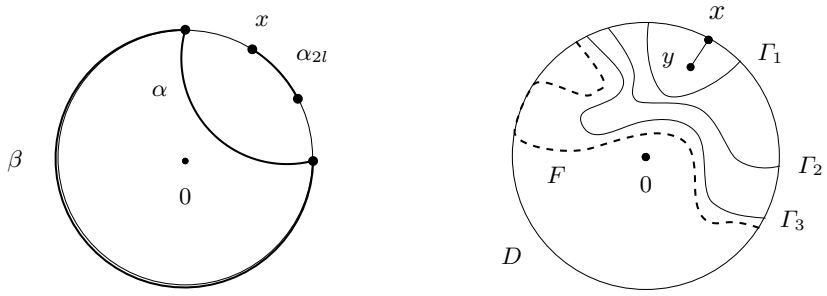
Proof. For notational simplicity we assume $i = 1$, and we write $\mathcal{C} = \mathcal{C}_1$ and $x = x_1 \in \alpha_{2l} \cap \alpha_1$. Using the Schoenflies theorem, up to a change of coordinates, we can assume that D is a euclidean closed disc.

We will argue by contradiction. Assuming that 0 does not belong to the closure of $\mathcal{C}$, we will construct a simple arc α with the following properties (we denote by $\partial\alpha$ the set of endpoints of α and set $\mathrm{Int}(\alpha) = \alpha \setminus \partial\alpha$):

1. $\mathrm{Int}(\alpha) \subset \mathrm{Int}(D)$, $\partial\alpha \subset \partial D$;
2. α separates ([3]) x from 0 in D;
3. either $\alpha \subset W^{\mathrm{s}}(V)$ and $\partial\alpha \cap W^{\mathrm{u}}(V) = \emptyset$, or $\alpha \subset W^{\mathrm{u}}(V)$ and $\partial\alpha \cap W^{\mathrm{s}}(V) = \emptyset$.

From this we will get a contradiction as follows (see Figure 3(a)). Assume for example that the first case of the last item holds. Let $1 \leq i_1 \leq i_2 \leq 2l$ be such that the endpoints of α are respectively in α_{i_1} and α_{i_2}. Since $\partial\alpha$ does not meet $W^{\mathrm{u}}(V)$, both i_1 and i_2 are odd, and in particular $1 \leq i_1 \leq i_2 < 2l$.

([3]) A set A *separates* two points in a set B if the two points belong to distinct connected components of $B \setminus A$.



(a) The arc α (b) The sequence Γ_k and the limit set F

Fig. 3. Proof of Lemma 16

Let $\beta \subset \partial D$ be the arc with the same endpoints as α and not containing x; then β is covered by $\alpha_{i_1} \cup \cdots \cup \alpha_{i_2}$, and from the second point we see that the Jordan curve $\alpha \cup \beta$ surrounds 0. Since $\alpha \cup \alpha_{i_1} \cup \alpha_{i_2} \subset W^{\mathrm{s}}(V)$, we can write $\alpha \cup \beta$ as the concatenation of $i_2 - i_1$ arcs, each included in $W^{\mathrm{s}}(V)$ or $W^{\mathrm{u}}(V)$. This contradicts the minimality hypothesis on l since $i_2 - i_1 < 2l$.

We now assume that $0 \notin \mathrm{Clos}(\mathcal{C})$ and turn to the construction of such an arc α. According to Lemma 15, there exists a homeomorphism $\Phi : \mathbb{R}^2 \to \mathcal{C}$. Let (D_k) be the sequence of images under Φ of the concentric discs with radius k and centre $\Phi^{-1}(x)$. Thus:

- $x \in D_1$,
- $D_k \subset \mathrm{Int}(D_{k+1})$,
- $\bigcup_{k\geq 0} D_k = \mathcal{C}$.

Let y be some point in $\mathrm{Int}(D)$ sufficiently near x so that the segment $[xy]$ is included in $D_1 \cap D$. For every k, the set $\partial D_k \cap \mathrm{Int}(D)$ is closed in $\mathrm{Int}(D)$ and separates y from 0 in $\mathrm{Int}(D)$ since $0 \notin D_k$. According to Theorem V.14.3 in [New], there exists a connected component of $\partial D_k \cap \mathrm{Int}(D)$ that also separates y from 0. Let Γ_k denotes the closure of this component; thus Γ_k is a subarc of the Jordan curve ∂D_k with $\mathrm{Int}(\Gamma_k) \subset \mathrm{Int}(D)$, $\partial\Gamma_k \subset \partial D$, and it separates x from 0 in D; in other words, it satisfies the first two properties above required for the arc α.

Remember that the space of compact connected subsets of D is compact under the Hausdorff metric. Thus, up to taking subsequences, we can assume that (Γ_k) converges to a compact connected set $F \subset D$ (see Figure 3(b)). Since $\Gamma_k \subset \partial D_k$, the set F is included in $\partial\mathcal{C}$. By assumption, $\partial\mathcal{C}$ does not contain 0, so neither does F. Then again F separates y from 0 in $\mathrm{Int}(D)$: if not, there would exist an arc in $\mathrm{Int}(D)$ from y to 0 missing F, but then this arc would also miss Γ_k for sufficiently large k, contrary to the property that Γ_k separates y from 0.

Since $\mathcal{C}$ is a connected component of the open set $W^{\mathrm{s}}(V) \cap W^{\mathrm{u}}(V) \setminus \{0\}$, its boundary $\partial\mathcal{C}$ is disjoint from this set. Thus F is disjoint from $W^{\mathrm{s}}(V) \cap W^{\mathrm{u}}(V)$; and since it is included in D it is covered by the two open sets $W^{\mathrm{s}}(V) \setminus \{0\}$ and $W^{\mathrm{u}}(V) \setminus \{0\}$. Since F is connected, it must be included in one of these sets, and disjoint from the other.

To fix ideas, suppose that $F \subset W^{\mathrm{s}}(V) \setminus W^{\mathrm{u}}(V)$. The arc Γ_k is included in $W^{\mathrm{s}}(V)$. Now this arc almost satisfies the three properties above required for the arc α: it only fails to have its endpoints outside $W^{\mathrm{u}}(V)$. To remedy this we notice that, up to taking sequences, the two sequences of endpoints $(\Gamma_k(0))$, $(\Gamma_k(1))$ converge to some $z_0, z_1 \in F \cap \partial D$. Since $F \subset W^{\mathrm{s}}(V) \setminus \{0\}$ we can choose $\varepsilon > 0$ so that the euclidean balls B_0, B_1 of radius ε and respective centres z_0, z_1 are included in $W^{\mathrm{s}}(V)$. For k large enough, Γ_k meets both balls, and then we construct the desired arc α by modifying Γ_k near its endpoints: we replace two small extreme subarcs of Γ_k, respectively included in B_0 and B_1, by two euclidean segments reaching z_0 and z_1. Note that since D is a euclidean disc, both segments, apart from their endpoints z_0, z_1, are included in $\mathrm{Int}(D)$. The endpoints z_0, z_1 of the resulting arc α are in F, thus outside $W^{\mathrm{u}}(V)$, and α has the third property while still satisfying the first two. As explained at the beginning of the proof, the existence of α contradicts the minimality of l. ∎

3.3. *Periodicity of the petals.* Unfortunately, we are not able to prove directly that the sets $\mathcal{C}_i$ of the previous section are periodic for f. To overcome this difficulty we will consider slightly larger sets $\mathcal{C}'_i$ which will turn out to be periodic. In the next section we will find a periodic petal inside each $\mathcal{C}'_i$.

We suppose that the closure of V is included in some neighbourhood V' of 0 which still has the short trip property. In other words, we apply the results of the previous sections with V small enough to meet this new assumption. We note that Lemmas 14 and 15 apply to V'. Obviously the inclusions $W^{\mathrm{s}}(V) \subset W^{\mathrm{s}}(V')$ and $W^{\mathrm{u}}(V) \subset W^{\mathrm{u}}(V')$ hold. Let the disc D and the sets $\mathcal{C}_i$ be defined from V as in the previous section. Each $\mathcal{C}_i$ is connected and included in $W^{\mathrm{s}}(V') \cap W^{\mathrm{u}}(V') \setminus \{0\}$, and thus included in a connected component of $W^{\mathrm{s}}(V') \cap W^{\mathrm{u}}(V') \setminus \{0\}$, which we denote by $\mathcal{C}'_i$.

LEMMA 17. *The sets $\mathcal{C}'_i$ are periodic*: *for every i there exists some positive integer q_i such that $f^{q_i}(\mathcal{C}'_i) = \mathcal{C}'_i$.*

Proof. We first note that the set $W^{\mathrm{s}}(V') \cap W^{\mathrm{u}}(V') \setminus \{0\}$ is invariant under f, and hence for every n, $f^n(\mathcal{C}'_i)$ is a connected component of that set.

We claim that for every i there exist infinitely many n such that $f^n(\mathcal{C}_i)$ meets the circle ∂D. Assuming the claim, we choose some point x, any neighbourhood of which contains points of $\partial D \cap f^{n_k}(\mathcal{C}_i)$ for infinitely many values

of n. Since x is a limit of points whose whole orbits are included in V, its orbit is included in $\mathrm{Clos}(V) \subset V'$; in other words, $x \in W^{\mathrm{s}}(V') \cap W^{\mathrm{u}}(V') \setminus \{0\}$. Let O be the connected component of this last set containing x. According to Lemma 15, O is open, and thus there exist infinitely many integers n such that $f^n(\mathcal{C}_i)$ meets O. For every such n, $f^n(\mathcal{C}'_i)$ is a connected component of $W^{\mathrm{s}}(V') \cap W^{\mathrm{u}}(V') \setminus \{0\}$ that meets O, thus it coincides with O. Hence we can find two integers $n_1 < n_2$ such that $f^{n_1}(\mathcal{C}'_i) = f^{n_2}(\mathcal{C}'_i)$, which proves that $\mathcal{C}'_i$ is periodic.

We prove the claim. By Lemma 16 the fixed point 0 belongs to the closure of $\mathcal{C}_i$. Since $\mathcal{C}_i \cup \{0\}$ is not a neighbourhood of 0, this point also belongs to the closure of $\partial\mathcal{C}_i$. Furthermore,

$$\begin{aligned}(\partial\mathcal{C}_i) \setminus \{0\} &\subset \partial(W^{\mathrm{s}}(V) \cap W^{\mathrm{u}}(V)) \setminus \{0\} \\ &\subset (W^{\mathrm{s}}(\mathrm{Clos}V) \cap W^{\mathrm{u}}(\mathrm{Clos}V)) \setminus (W^{\mathrm{s}}(V) \cap W^{\mathrm{u}}(V)).\end{aligned}$$

Consequently, for any $z \in \partial\mathcal{C}_i \setminus \{0\}$ there exists an integer n such that $f^n(z) \in \partial V$. Let (z_k) be a sequence in $\partial\mathcal{C}_i$ converging to 0. Then any sequence n_k such that $f^{n_k}(z_k) \in \partial V$ is unbounded, because the union of finitely many iterates of ∂V is a closed set which does not contain 0. For any k the set $f^{n_k}(\mathcal{C}_i)$ is connected, its closure contains 0 and meets ∂V, thus it also meets ∂D. This completes the proof of the claim. ∎

3.4. *Construction of the local conjugacy.* We finally define the petals. According to the previous lemma we can choose some $n_0 > 0$ such that $F = f^{n_0}$ leaves invariant every set $\mathcal{C}'_i$. In view of Proposition 13, Theorem 3 will follow from the fact that F is locally conjugate to a locally holomorphic parabolic homeomorphism. Let us prove this fact.

Recall that $\mathcal{C}'_i$ is homeomorphic to the plane (Lemma 15), and for any compact set $K \subset \mathcal{C}'_i$, the sequences $(f^n(K))_{n\geq 0}$ and $(f^n(K))_{n\leq 0}$ converge to 0 (Lemma 14). Consequently, Theorem 8 tells us that for every i the restriction of F to the invariant set $\mathcal{C}'_i$ is conjugate to the plane translation $\tau : z \mapsto z + 1$. Observe that any point is on a horizontal line

$$\Delta' = \bigcup_{n\in\mathbb{Z}} \tau^n(\delta')$$

with compact δ' (a horizontal segment). Bringing this line back under the conjugacy, we see that any point $x_i \in \mathcal{C}'_i$ is on a set $\Delta_i \subset \mathcal{C}'_i$ which is the union of the iterates under F of a compact set (an arc) included in $\mathcal{C}'_i$. By Lemma 14, $\Delta_i \cup \{0\}$ is a Jordan curve. Let P_i be the closed topological disc bounded by this curve (here we use the Schoenflies theorem). The last sentence of Lemma 15 entails that P_i is included in $\mathcal{C}'_i$, and then Lemma 14 shows that P_i is a regular invariant petal for F.

The curve α_i meets both petals P_i and P_{i+1}. Furthermore for i odd we have $\alpha_i \subset W^{\mathrm{s}}(V)$, so by Lemma 14, $(F^n(\alpha_i))_{n\geq 0}$ converges to 0, and

$(F^n(\alpha_i))_{n\leq 0}$ converges to 0 for i even. Thus the construction of a local conjugacy between F and $z \mapsto z(1+z^l)$ now boils down to the following lemma.

LEMMA 18 (see Figure 1). *Let $f \in \mathcal{H}^+$. Fix some integer $l \geq 1$. Assume the following hypotheses.*

1. *There exist $2l$ regular invariant petals $P_1, \dots, P_{2l}$ whose pairwise intersections are $\{0\}$.*
2. *There exists a topological closed disc D which is a neighbourhood of 0, and whose boundary is the concatenation of $2l$ arcs $\alpha_1, \dots, \alpha_{2l}$, each having one endpoint on P_i and the other on P_{i+1}.*
3. *For i odd the sequence $(f^n(\alpha_i))_{n\geq 0}$ converges to 0, and for i even the sequence $(f^n(\alpha_i))_{n\leq 0}$ converges to 0.*

Then f is locally conjugate to $z \mapsto z(1+z^l)$.

Note that we do not suppose that $\partial D \cap P_i$ is connected, nor that α_i does not meet some petal P_j with $j \neq i, i+1$, in contrast to Proposition 12. An important step of the proof will be to check that the petal indexation coincides with their cyclic order around 0.

Proof of Lemma 18. Consider a homeomorphism $f \in \mathcal{H}^+$ satisfying the hypotheses of the lemma. The arc α_i contains a minimal subarc α_i' connecting P_i to P_{i+1}; the endpoints of α_i' are respectively on P_i and P_{i+1}, and its interior $\mathrm{Int}(\alpha_i')$ is disjoint from P_i and P_{i+1}. Let i be odd, so that $(f^n(\alpha_i'))_{n\geq 0}$ converges to 0. Then we define an attracting sector S_i' as follows. We consider the curve obtained by concatenating the arc α_i', the subarc of ∂P_i from the endpoint of α to 0 following the dynamical orientation of ∂P_i, and the similar subarc on P_{i+1} (see Remark 10). This is a Jordan curve, it bounds a topological closed disc S_i'. Clearly $(f^n(\partial S_i'))_{n\geq 0}$ converges to 0, and thus so does $(f^n(S_i'))_{n\geq 0}$; in other words, S_i' is indeed an attracting sector. Note that as a consequence, it can contain neither P_i nor P_{i+1}, because an attracting sector contains no invariant set. We apply item 1 of Claim 6 to get a nice attracting sector $S_i \subset S_i'$ having the same germ as S_i' at 0. For i even we symmetrically define a repulsive sector S_i' and a nice repulsive sector S_i.

Since the petals are topological closed discs whose pairwise intersection is $\{0\}$, the Schoenflies theorem shows that the union of the petals is homeomorphic to the model pictured on the left of Figure 1; but we still have to prove that their cyclic order is as shown on the right of the figure (or the reverse one). For this we argue by contradiction. Suppose there exists some i such that the petals P_i and P_{i+1} are not adjacent: they are locally separated near 0 by the union of the other petals. Then there exists another petal P_j such that the sector S_i contains a neighbourhood of 0 in P_j; in other words, $\mathrm{Clos}(P_j \setminus S_i)$ is a compact subset of $P_j \setminus \{0\}$. Using Claim 11 that

describes the dynamics of f on P_j, we find a point $x \neq 0$ whose full orbit $\{f^n(x) : n \in \mathbb{Z}\}$ is included in $P_j \cap S_i$. But an attracting or repulsive sector contains no full orbit, which provides the desired contradiction.

Up to reversing the indexation, we may now assume that the petals are indexed in the positive cyclic order around 0 (so that the Schoenflies theorem provides an *orientation preserving* homeomorphism that sends each P_i onto the model of Figure 1). Since S_i contains neither P_i nor P_{i+1}, the dynamical order on the boundaries of the petals is as indicated in Figure 1: the petal P_i is direct for i odd and indirect for i even.

Up to replacing S_i with some smaller nice sector, we deduce that

1. for any i, j with $j \neq i, i+1$, we have $P_i \cap S_j = \{0\}$;
2. for any $i \neq j$, we have $S_i \cap S_j = \{0\} = f^{-1}(S_i) \cap S_j$.

Consider the set $D = P_1 \cup S_1 \cup \cdots \cup P_{2l} \cup S_{2l}$. Thanks to item 2 the maximal invariant subset of D is the union of the petals P_i. Thus D is a topological closed disc satisfying the hypotheses of Proposition 12. Now the lemma follows from the proposition. ∎

4. Proof of Propositions 12 and 13

Proof of Proposition 12. The fact that for the map $z \mapsto z(1 + z^l)$ there exists a topological closed disc D with properties 1–3 of the proposition is part of the proof of the Camacho–Leau–Fatou theorem (see [Cam, Mil]).

We turn to the proof of the reverse implication. We consider a homeomorphism $f \in \mathcal{H}^+$ and a disc D with properties 1–3 of the proposition. We have to prove that if $f' \in \mathcal{H}^+$ and a disc D' has the same properties (with the same l) then f and f' are locally topologically conjugate. Note that the union of all sectors S_i and petals P_i is equal to D.

Let i be an odd integer. Since S_i is an attracting sector between P_i and P_{i+1}, the petal P_i is direct, while P_{i+1} is indirect (see Remark 10). The same is true for f'. Thus according to Claim 11, the restrictions of f and f' to P_i and P'_i are conjugate. The conjugacies can be glued together to obtain an orientation preserving homeomorphism $\Phi : \bigcup P_i \to \bigcup P'_i$ which sends P_i onto P'_i and is a conjugacy between the restrictions of f and f'.

The image under Φ of $S_i \cap (P_i \cup P_{i+1})$ is not necessarily equal to $S'_i \cap (P'_i \cup P'_{i+1})$. But using item 2 of Claim 6 we can replace S_i and S'_i with smaller nice attracting sectors so that this equality becomes true (see item 2 of Remark 7). We can now use item 3 of Remark 7 to extend Φ to a homeomorphism between D and D', sending S_i onto S'_i and conjugating the restrictions of f and f'. We do this for every attracting or repulsive sector S_i. We further extend Φ to a homeomorphism of the plane. The conjugacy relation $f'\Phi = \Phi f$ is satisfied on $D \cap f^{-1}(D)$. This completes the proof of the proposition. ∎

To prove Proposition 13 we need a claim.

CLAIM 19. *Let Q_1, Q_2 be two invariant petals included in a regular invariant petal P for $F \in \mathcal{H}^+$. Then Q_1 meets Q_2, and there exists a unique connected component O of $\mathrm{Int}(Q_1) \cap \mathrm{Int}(Q_2)$ such that $F(O) = O$. Furthermore, the closure of O is a regular invariant petal for F.*

Proof. The first part is easily proved using the translation model given by Claim 11. The only difficulty in the second part is to check that the closure of O is indeed a topological closed disc. But this follows from the previously quoted result of Kerékjártó ([Ke23, LCY]). ∎

Also note that if $Q \subset P$ are two regular invariant petals and P is direct then Q is direct.

Proof of Proposition 13. Let $f^{n_0} = F$ be conjugate to a parabolic homeomorphism F_0. Up to increasing n_0, we can assume that $F_0'(0) = 1$, and thus F is locally conjugate to $z \mapsto z(1 + z^l)$ for some integer l. Let D be a nice disc for F, and let $\{P_1, \dots, P_{2l}\}$ be the family of petals associated with D, as given by Proposition 12.

For each i we choose a small invariant petal Q_i for F included in P_i. Since $f^{n_0} = F$ and Q_i is invariant for F, if Q_i is small enough then every iterate $f^n(Q_i)$ is included in D. Since $f^n(Q_i) \setminus \{0\}$ is connected and invariant for F, it is included in a connected component of the F-maximal invariant subset of $D \setminus \{0\}$, that is, $f^n(Q_i)$ is included in some petal P_j. Fix j and consider the finite family of all the petals $f^n(Q_i)$ for $n \in \mathbb{Z}$ and $i \in \mathbb{Z}/2l\mathbb{Z}$ which are included in P_j. We denote the intersection of their interiors by O_j. Applying Claim 19 inductively we see that the closure of O_j is a regular invariant petal for F; let us call it $\overline{P}_j$.

By construction the petals in the family $\{\overline{P}_j\}$ are permuted by f, their pairwise intersections are $\{0\}$, and their cyclic order around 0 is given by the cyclic order on the indices $j \in \mathbb{Z}/2l\mathbb{Z}$. Since f is an orientation preserving homeomorphism, there exists i_0 such that $f(\overline{P}_i) = \overline{P}_{i+i_0}$ for every i. Furthermore, since f respects the dynamical orders induced by F on the boundary of the petals, i_0 must be even. The order n_0' of the permutation $i \mapsto i + i_0$ is a divisor of n_0 (maybe proper). It is easy to see that there exists another nice disc $\overline{D}$ for F whose maximal invariant subset is the union of this family of petals. The nice attractive and repulsive sectors $\overline{S}_i$ for F, associated with $\overline{D}$, are clearly also attractive or repulsive sectors for $f^{n_0'}$, and according to Claim 6 we can find within each $\overline{S}_i$ a nice attracting or repulsive sector $\overline{\overline{S}}_i$ for $f^{n_0'}$ having the same germ at 0. Now the topological closed disc $\overline{\overline{D}}$ defined as the union of all petals $\overline{P}_i$ and sectors $\overline{\overline{S}}_i$ is a nice disc for $f^{n_0'}$, the hypotheses of Proposition 12 are satisfied, and $f^{n_0'}$ is conjugate to F.

Using these families of petals and sectors we are now in a position to construct a local conjugacy Φ between f and the model map $f_0 : z \mapsto e^{2i\pi i_0/2l} z(1+z^l)$. Note that $f_0^{n'_0}$ is conjugate to $z \mapsto z(1+z^l)$ and that f_0 permutes a family of regular invariant petals for $f_0^{n'_0}$. The construction of the conjugacy is similar to the one defined in the proof of Proposition 12. Here is the main difference: since the petals are permuted by f, we have first to define a conjugacy Φ between $f^{n'_0}$ and $f_0^{n'_0}$ on some petal $\overline{P}_i$, and then there is a unique way to extend it to the f-orbit of $\overline{P}_i$ to get a conjugacy between f and f_0. We do the same for every f-orbit of petals, and for every f-orbit of a sector. This completes the proof of Proposition 13. ∎

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