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On the existence of universal covering spaces for metric spaces and subsets of the Euclidean plane

by

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Abstract. We prove several results concerning the existence of universal covering spaces for separable metric spaces. To begin, we define several homotopy-theoretic conditions which we then prove are equivalent to the existence of a universal covering space. We use these equivalences to prove that every connected, locally path connected separable metric space whose fundamental group is a free group admits a universal covering space. As an application of these results, we prove the main result of this article, which states that a connected, locally path connected subset of the Euclidean plane, $\mathbb{E}^2$, admits a universal covering space if and only if its fundamental group is free, if and only if its fundamental group is countable.

1. Introduction. The Hawaiian earring, the one-point compactification of a countably infinite set of disjoint open intervals, is one of the standard examples in homotopy theory since it is a one-dimensional, planar set which does not admit a universal covering space and whose fundamental group is uncountable and not a free group. Guided by this example, it is natural to ask what relationships exist between the properties of freedom and countability of the fundamental group of a space and existence of universal covering spaces for path connected spaces. In this article we explore these relationships, and find that, for example (Theorem 2.6), a connected, locally path connected, separable metric space which has a free fundamental group admits a universal covering space and that furthermore (Theorem 3.1) a connected, locally path connected subset of the Euclidean plane admits a universal cover if and only if it has a free fundamental group.

In [CF], Curtis and Fort show that any compact connected, locally path connected one-dimensional metric space is either semilocally simply connected, in which case it has a fundamental group which is free of countable rank, or has an uncountable fundamental group. In [CC2] this is generalized

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in several ways. In particular, it is shown there that if X a connected, locally path connected, one-dimensional separable metric space then X has a free fundamental group if and only if X has a countable fundamental group, if and only if X is locally simply connected, if and only if X has a universal cover. In the current article we prove a similar theorem, replacing the hypotheses that the space be one-dimensional and separable by the hypothesis that it be a subset of $\mathbb{R}^2$.

1.1. We will now discuss the terminology of the current article. The common thread connecting the approach of [CC2] with that of the current article is that planar sets and one-dimensional spaces share the property of being *homotopically Hausdorff*. This is a strong condition, which does not hold for subsets of $\mathbb{R}^3$, as we shall see below.

DEFINITION 1.1. A space X is *homotopically Hausdorff at* $x_0 \in X$ if for any essential closed curve, c, based at x_0 there is an open neighborhood U of x_0 so that c is not homotopic (rel endpoints) to a curve lying entirely in U. Furthermore, X is said to be *homotopically Hausdorff* if it is homotopically Hausdorff at each of its points.

The property of being homotopically Hausdorff intuitively says that closed curves in the space can be separated from the trivial closed curve by an open set. This intuition can be made rigorous by noting that the space of homotopy classes of curves in X emanating from x_0, sometimes denoted $\Omega(X, x_0)$, is Hausdorff at x_0 if and only if X is homotopically Hausdorff at x_0.

EXAMPLE 1.2. Define the *Hawaiian earring* as the union $H = \bigcup_{i\in\mathbb{N}} c_i$ of planar circles, c_i, tangent to the x-axis at the origin and of radius $1/i$. It is the simplest example of a one-dimensional Peano continuum which does not have free fundamental group [CC1]. Since H is one-dimensional, it is homotopically Hausdorff [CC2]. Let X be the cone over H, that is, $H \times [0,1]/H \times \{1\}$. Let $x = ((0,0),0)$ denote the basepoint of X. Let X_i, $i \in \{1,2\}$, be two copies of X with basepoints x_i. Let $Y = X_1 \cup X_2/\{x_1 = x_2\}$ be their amalgamated union (see Figure 1). It is shown in [CC1] that Y is a compact, connected subset of $\mathbb{R}^3$ which is not homotopically Hausdorff, and whose fundamental group is uncountable. Another interesting feature of the space Y is that it is a union of two contractible spaces along one point, but it is not itself contractible.

DEFINITION 1.3. If $i : X \to Y$ is an embedding of one path connected space into another then we say that X is a π_1*-retract* of Y if there exists a homomorphism $r : \pi_1(Y) \to \pi_1(X)$ so that the composition $ri_* : \pi_1(X) \to \pi_1(X)$ is an isomorphism. We say that r is a π_1*-retraction for* X *in* Y. Note that $\pi_1(X)$ is indeed a group-theoretic retract of $\pi_1(Y)$. We define a

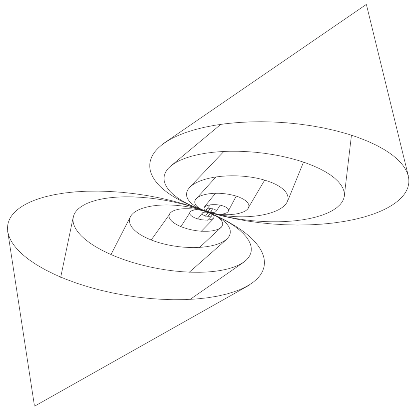

Fig. 1. The doubled cone over the Hawaiian earring

π_1-retract to be *tight* if it induces an isomorphism of fundamental groups. Similarly we say that X is a *neighborhood* π_1*-retract* of Y if X is a π_1-retract of one of its open neighborhoods in Y.

The above notion of π_1-retract is designed to be an analog of the notion of topological *retract*, where the property that a retract have a corresponding continuous retraction is replaced by the property that the fundamental group of a π_1-retract have an analogous group-theoretic retraction. To carry the analogy further, we now introduce a generalization of the notion of an *ANR* (an *absolute neighborhood retract*). Recall that a separable metric space is an ANR if it is a neighborhood retract of every separable metric space containing it as a closed subspace.

Definition 1.4. A separable metric space, X, is said to be an $AN\pi_1R$ (or an *absolute neighborhood* π_1*-retract*) if whenever X is a subspace of a separable metric space Y then X is a neighborhood π_1-retract in Y. Note that we do not require that X be a closed subset of Y.

We resist the temptation to define the analog of the notion of an AR (absolute retract) since this would merely correspond to the class of all simply connected, locally path connected, separable metric spaces.

1.2. The following is an outline of the results proven in this article. In Section 2 we prove:

Theorem 2.1. *If X is a connected, locally path connected, separable metric space then the following are equivalent:*

(1) *X admits a universal covering space.*
(2) *X is homotopically Hausdorff and $\pi_1(X)$ is countable.*
(3) *X is an $AN\pi_1R$.*
(4) *X is a tight π_1-retract of a Hilbert cube manifold.*

Using the tools of [CC2] along with the previous result and a number of nerve-theoretic lemmas from Appendix A we show:

THEOREM 2.6. *If X is a connected, locally path connected separable metric space with a fundamental group which is a free group then X admits a universal covering space.*

In Section 3, using standard techniques of planar topology, we prove:

THEOREM 3.4. *Every subset of $\mathbb{E}^2$ is homotopically Hausdorff.*

This combined with the previous results allows us to deduce the main result of this paper:

THEOREM 3.1. *If X is a connected, locally path connected subset of $\mathbb{E}^2$ then the following are equivalent:*

(1) *X admits a universal cover.*
(2) *X is locally simply connected.*
(3) *The fundamental group of X is countable.*
(4) *The fundamental group of X is a free group.*

The following result is needed in the proof of Theorem 2.1. Its proof constitutes Section 4.

THEOREM 4.1. *If X is a locally connected separable metric space and $\widetilde{X}$ is a covering space for X then $\widetilde{X}$ is metrizable. Furthermore, if $\widetilde{X}$ is connected, then it is separable.*

Finally, in Appendix A we prove a number of nerve-theoretic and planar topological lemmas needed in various parts of the paper.

1.3. We now mention recent work in the field which is related to the current article in order of decreasing generality.

In [S], Shelah proves that a connected, locally path connected, compact metric space has a fundamental group which is either finitely generated or uncountable. In [CC2], Cannon and Conner show that the fundamental group of a connected, locally path connected, separable one-dimensional metric space is a free group if and only if it is countable if and only if the space has a universal cover if and only if the space is locally simply connected. The main result, Theorem 3.1, in the current article is the analogous result for subsets of the plane. In [CCZ], we prove that any subset of $\mathbb{R}^2$ is aspherical (i.e., has trivial higher homotopy).

2. Fundamental groups of metric spaces

THEOREM 2.1. *If X is a connected, locally path connected, separable metric space then the following are equivalent:*

(1) *X admits a universal covering space.*
(2) *X is homotopically Hausdorff and $\pi_1(X)$ is countable.*
(3) *X is an $AN\pi_1R$.*
(4) *X is a tight π_1-retract of a Hilbert cube manifold.*

That (1) and (3) are equivalent is not surprising since a connected, locally path connected space has a universal cover if and only if it is $\mathbf{LC}^1$ (or, in another terminology, semilocally simply connected), and there is a well developed theory of ANR-like extensions of maps from spaces of prescribed dimension into spaces with $\mathbf{LC}^n$ properties (for instance, see van Mill [vM]). However, we are unaware of results in the literature which would imply the equivalence of (1) and (3).

Proof. We will first prove that (1) and (2) are equivalent and then show that (1) implies (3) implies (4) which, in turn, implies (1). We will state a number of the more interesting implications as lemmas.

The following result proves that (1) implies (2).

LEMMA 2.2. *The fundamental group of a path connected*, *locally connected*, *separable metric space which admits a universal cover is countable.*

Proof. Let X be a topological space satisfying the hypothesis, $\widetilde{X}$ be its universal cover and f be the corresponding covering map. Fix $x_0 \in X$. We assume by way of contradiction that the fundamental group of X is uncountable. Since the preimages under f of x_0 are in one-to-one correspondence with the elements of $\pi_1(X, x_0)$, $A = f^{-1}(x_0)$ is an uncountable subset of $\widetilde{X}$. Since $\widetilde{X}$ is a separable metric space by Theorem 4.1, there is an element of A which is a limit point of A. However, since f is a covering map there is an open set B in X containing x_0 whose inverse image under f is a disjoint collection of open sets each one mapped homeomorphically onto B by f. Thus A cannot contain a limit point of itself. ■

Recall that X will admit a universal cover if and only if it is *semilocally simply connected*. That (2) implies (1) follows immediately from the following result.

LEMMA 2.3. *If X is a locally path connected topological space which is first countable*, *homotopically Hausdorff but is not semilocally simply connected*, *then $\pi_1(X)$ is uncountable.*

Proof. Assume X is as in the hypothesis, fix $x_0 \in X$ such that X is not semilocally simply connected at x_0, and choose (B_i) to be a countable local basis for X at x_0.

Since X is not simply connected, we may choose $p_1 : I \to X$ to be an essential closed curve based at x_0. Let $S_0 = \{1\}$. Since X is homotopically

Hausdorff we may choose a neighborhood U_1 of x_0 contained in B_1 which does not contain a closed curve based at x_0 homotopic to p_1.

We will inductively define a sequence (U_i) of neighborhoods of x_0, and a sequence (p_i) of closed curves based at x_0 so that the image of each p_i is contained in U_i and such that each U_i is a subset of B_i. We set $\pi = \pi_1(X, x_0)$, and let π_i denote the image of $\pi_1(U_i, x_0)$ in π under the map induced by inclusion. If p_j is defined for $j < i$, we make the following definitions:

1. If α is a subset of $\{1, \ldots, i-1\}$, define w_α to be 1_π if $\alpha = \emptyset$ and otherwise to be $[p_{\alpha_1}] \circ \cdots \circ [p_{\alpha_k}]$ where $\alpha = \{\alpha_1, \ldots, \alpha_k\}$ and $\alpha_1 < \cdots < \alpha_k$. For example $w_{\{1\}} = [p_1]$.
2. Define S_{i-1} to be the set whose elements have the form $w_\alpha \circ w_\beta^{-1}$ or $w_\alpha^{-1} \circ w_\beta$ where α and β are two subsets of $\{1, \ldots, i-1\}$. For example, $S_1 = \{1_\pi, [p_1], [p_1]^{-1}\}$.

As part of the induction hypothesis we shall assume that we have chosen (p_j) and (U_j) so that $w_\alpha = w_\beta$ only if $\alpha = \beta$, and so that $S_{i-2} \cap \pi_{i-1} = \{1_\pi\}$.

Assume inductively that p_{i-1} and U_{i-1} have been defined. Since S_{i-1} is finite and X is homotopically Hausdorff, we may choose a neighborhood U_i about x_0 such that $U_i \subset U_{i-1} \cap B_i$ and $S_{i-1} \cap \pi_i = \{1_\pi\}$. Since X is not semilocally simply connected at x_0 we may choose $p_i : I \to X$ to be an essential closed curve based at x_0 such that $[p_i]$ lies in π_i.

Now suppose $\alpha, \beta \subset \{1, \ldots, i\}$ and $w_\alpha = w_\beta$. Then $p_{\alpha_1} \circ \cdots \circ p_{\alpha_k} \approx p_{\beta_1} \circ \cdots \circ p_{\beta_l}$. If $\alpha_k \neq i$ then $\beta_l \neq i$, since otherwise

$$p_i \approx p_{\beta_l} \approx (p_{\beta_1} \circ \cdots \circ p_{\beta_{l-1}})^{-1} \circ (p_{\alpha_1} \circ \cdots \circ p_{\alpha_k})$$

and so p_i would be homotopic to an element of S_{i-1}. Thus $w_\alpha, w_\beta \in S_{i-1}$ and by the inductive hypothesis $\alpha = \beta$. On the other hand, if $\alpha_k = \beta_l = i$ then $p_{\alpha_1} \circ \cdots \circ p_{\alpha_{k-1}} \circ p_i \approx p_{\beta_1} \circ \cdots \circ p_{\beta_{l-1}} \circ p_i$ implies $p_{\alpha_1} \circ \cdots \circ p_{\alpha_{k-1}} \approx p_{\beta_1} \circ \cdots \circ p_{\beta_{l-1}}$ and since each side is an element in S_{i-1} we have $\alpha - \{i\} = \beta - \{i\}$, thus $\alpha = \beta$. Hence the inductive step is complete.

Now for each subset, α, of the natural numbers, we define

$$w_\alpha = [p_{\alpha_1} \circ p_{\alpha_2} \circ \ldots] = \Big[\prod_{i=1}^{\infty} p_{\alpha_i}\Big].$$

Since $\bigcap_{i=1}^{\infty} U_i = \{x_0\}$, it follows that w_α is a well defined element of π. Suppose $w_\alpha = w_\beta$ for α and β two distinct subsets of the natural numbers. Let i be the least element of $(\alpha - \beta) \cup (\beta - \alpha)$ and, without loss of generality, assume $i \in \alpha$. Now, $\alpha_1 = \beta_1, \ldots, \alpha_k = \beta_k$, and $i = \alpha_{k+1} < \beta_{k+1}$. We let

$$Q = p_{\alpha_1} \circ \cdots \circ p_{\alpha_k} = p_{\beta_1} \circ \cdots \circ p_{\beta_k}$$

and note that $w_\alpha = [Q \circ p_i \circ Q_1]$ and $w_\beta = [Q \circ Q_2]$ where Q_1 and Q_2 can be chosen to be closed curves based at x_0 lying in U_{i+1}. Then $w_\alpha = w_\beta$

implies that $Q \circ p_i \circ Q_1 \approx Q \circ Q_2$ and thus $p_i \approx Q_2 \circ Q_1^{-1}$. However, p_i is not homotopic to a curve based at x_0 lying in U_{i+1}, yielding a contradiction.

Hence we have an injective map f from the set of subsets of the natural numbers, $\wp(\mathbb{N})$, to π given by $\alpha \mapsto w_\alpha$, whence π has cardinality no smaller than that of the continuum. ∎

We will now prove that (1) implies (3).

LEMMA 2.4. *If X is a connected, locally path connected, separable metric space which admits a universal covering space then X is an $AN\pi_1R$.*

Proof. Let X be embedded in the separable metric space Y and let $\widehat{U}$ be a neighborhood of X in Y. Since X is semilocally simply connected we may choose a cover C' of X by open sets whose images in the fundamental group of X are trivial and are each contained in $\widehat{U}$. Using Lemma A.3, we choose a locally finite open refinement C whose elements are path connected and such that $\pi_1(X)$ is isomorphic to $\pi_1(N(C))$. By Lemma A.1, we may choose a collection, $\widehat{C}$, of open sets in $\widehat{U} \subseteq Y$ covering X and which is compatible with C in the sense that the elements of c are in one-to-one correspondence with those of $\widehat{C}$ in such a way that $c = \widehat{c} \cap X$ for each c in C and so that any finite collection, $c_1, \dots, c_n$, of elements of C have a common point of intersection if and only if the corresponding elements, $\widehat{c}_1, \dots, \widehat{c}_n$, of $\widehat{C}$ have a common intersection. Clearly $N(\widehat{C})$ and $N(C)$, the nerves of $\widehat{C}$ and C respectively, are naturally isomorphic.

Let $U = \bigcup_{\widehat{c} \in C} \widehat{c}$ and $i : X \to U$ be the inclusion map. Now, we choose a partition of unity $\widehat{F} = \{f_{\widehat{c}}\}$ for the cover $\widehat{C}$. For each $c \in C$ let $f_c = f_{\widehat{c}}|c$. Clearly $F = \{f_c\}$ is a partition of unity corresponding to the cover C of X. Let $\widehat{p} : U \to N(\widehat{C})$ and $p : X \to N(C)$ be the maps (see Definition A.2) induced by $\widehat{F}$ and F respectively, and $q : N(\widehat{C}) \to N(C)$ be the obvious isomorphism. By construction, $p = q \circ \widehat{p} \circ i$, but, by Lemma A.3, the induced map $p_* : \pi_1(X) \to \pi_1(N(C))$ is an isomorphism. Thus $(p_*)^{-1} \circ (q \circ \widehat{p})_*$ is a π_1-retraction for X in U. ∎

To show that (3) implies (4), suppose X is an ANπ_1R. Since every separable metric space embeds in the Hilbert cube, we may chose an open set, U, in the Hilbert cube so that X is a π_1-retract of U. Fix $x_0 \in X$. Since U is an open set in the Hilbert cube, it is semilocally simply connected. Thus, by elementary covering space theory (see [M] for instance), there is a covering space $\widehat{U}$ and a covering map $c : (\widehat{U}, \widehat{x}_0) \to (U, x_0)$ so that $c_*(\pi_1(\widehat{U}, \widehat{x}_0)) = \pi_1(X, x_0)$. A standard result in covering space theory (again [M] is a good reference), states that i lifts to a map $\widehat{i} : (X, x_0) \to (\widehat{U}, \widehat{x}_0)$ so that $i = c \circ \widehat{i}$. Now, since i is an embedding, so is $\widehat{i}$. By the choice of $\widehat{U}$, it is evident that X is a tight π_1-retract of $\widehat{U}$. Finally, $\widehat{U}$ is a Hilbert cube manifold since it is a covering space of an open set in the Hilbert cube.

Finally, (4) implies (1) since any connected π_1-retract of a semilocally simply connected space (in this case a Hilbert cube manifold) is itself semilocally simply connected and thus admits a universal cover if it is connected and locally path connected. ∎

To prove the next result we will need to use one of the main tools of [CC2].

Theorem 2.5 ([CC2, Theorem 4.4]). *Let X be a topological space, let $f : \pi_1(X, x_0) \to L$ be a homomorphism to the group L, let $U_1 \supseteq U_2 \supseteq \cdots$ be a countable local basis for X at x_0, and let G_i be the image of the natural map of $\pi_1(U_i, x_0)$ into $\pi_1(X, x_0)$. Then*

(1) *If L is countable then the sequence $f(G_1) \supseteq f(G_2) \supseteq \cdots$ is eventually constant.*
(2) *If L is abelian with no infinitely divisible elements then $\bigcap_{i\in\mathbb{N}} f(G_i) = \{0_L\}$.*
(3) *If L is countable abelian with no infinitely divisible elements then $f(G_i) = \{0_L\}$ for some $i \in \mathbb{N}$.*

In [CC2] it is shown that if X is a second countable, locally path connected metric space with a free abelian fundamental group then X has a universal cover. We will use the previous theorem to prove a similar result for free groups:

Theorem 2.6. *If X is a connected, locally path connected separable metric space with a fundamental group which is a free group then X admits a universal covering space.*

Proof. First we apply [CC2, Theorem 5.1], which states that any free factor group of the fundamental group of a second countable, connected, locally path connected metric space has countable rank. Thus $\pi_1(X)$ is countable.

Let x_0 be a point in X. Let $U_1 \supseteq U_2 \supseteq \cdots$ be a countable local basis for X, let g_i be the natural map of $\pi_1(U_i, x_0)$ into $\pi_1(X, x_0)$, and let G_i be the image of g_i.

Let $C_0 = \pi_1(X, x_0)$, $C_1 = [C_0, C_0], \ldots, C_{i+1} = [C_i, C_i], \ldots$ be the standard commutator chain for $\pi_1(X, x_0)$. Since $\pi_1(X, x_0)$ is a free group, each group C_i/C_{i+1} is a free abelian group. Finally, let $f_n : \pi_1(X, x_0) \to \pi_1(X, x_0)/C_n$ be the natural homomorphism. Then by Theorem 2.5, the intersection of the images of the G_i's under f_1 is eventually trivial, $f_1(G_{i_1}) = \{0\}$ for some i_1. But $f_1(G_{i_1}) = G_{i_1}C_1/C_1$, whence $G_{i_1} \le C_1$. Then $f_2 g_{i_1} : \pi_1(U_{i_1}, x_0) \to C_1/C_2$. Applying Theorem 2.5 again, we see that there is a G_{i_2} which lies in C_2. By induction, for each j there is a G_{i_j} which lies in C_j. However, since $\pi_1(X, x_0)$ is a free group, $\bigcap_{j\in\mathbb{N}} C_j = \{1\}$ and thus $\bigcap_{j\in\mathbb{N}} G_j = \{1\}$. This implies that X is homotopically Hausdorff at x_0. Since x_0 was generic, X is homotopically Hausdorff.

Finally, we see that X is homotopically Hausdorff and has countable fundamental group, and so we may apply Theorem 2.1 to conclude that X admits a universal covering space. ■

3. Fundamental groups of planar sets. In this section we apply the results of the previous section to sets in the Euclidean plane to get the following result.

THEOREM 3.1. *If X is a connected locally path connected subset of $\mathbb{E}^2$ then the following are equivalent*:

(1) *X admits a universal cover.*
(2) *X is locally simply connected.*
(3) *The fundamental group of X is countable.*
(4) *The fundamental group of X is a free group.*

This theorem is related to Theorem 5.9 in [CC2] where it is shown that a second countable, connected, locally path connected, one-dimensional metric space has a universal cover if and only if it is locally simply connected if and only if it has a countable fundamental group if and only if it has a free fundamental group. Thus, we have replaced the hypothesis of being second countable and one-dimensional by the hypothesis of being a planar set and have obtained the same conclusion.

Proof. Theorem 2.6 shows that (4) implies (1). Thus to show that (1) and (4) are equivalent, we need only show that if X admits a universal cover then the fundamental group of X is a free group. By Theorem 2.1 we know that X is an ANπ_1R. Thus the fundamental group of X embeds in the fundamental group of an open set in the plane. Since such an open set is a noncompact 2-manifold, it has a fundamental group which is a free group. Since subgroups of free groups are free, the fundamental group of X is a free group.

We need the following technical lemma.

LEMMA 3.2. *Let X be a subset of $\mathbb{E}^2$ and N a closed disk in $\mathbb{E}^2$ whose boundary is not contained in X. If l_1, l_2 are closed curves in $X \cap \operatorname{int}(N)$ based at x_0 which are homotopic in X then there is a homotopy F between l_1 and l_2 whose image is contained in $X \cap N$.*

Proof. Let C denote the boundary of N, A denote the interior of N, let and let $p \in C \setminus X$. Let G be a homotopy (in X rel x_0) between l_1 and l_2. Let D be the component of $I^2 - G^{-1}(C)$ that contains $(0,0)$ in its boundary. Let B be the set of boundary components of D except for the boundary of the square (the component containing $(0,0)$). Since $l_1 \cup l_2 \subset A$ and I^2 is compact it follows that $\bigcup B$ is closed and $G(\bigcup B)$ is a closed subset of $C \cap X$. Given that $p \in C$ it follows that each component of $G(\bigcup B)$ is homeomorphic to a closed interval in the real numbers or is a point. If w is a component of $G(\bigcup B)$, let D_w be the component of $I^2 - (G^{-1}(w) \cap \bigcup B)$ that contains $(0,0)$. Then $G^{-1}(w) \cap \bigcup B$ is a closed subset of $I^2 - D_w$, hence by the Tietze extension theorem the map G, restricted to $G^{-1}(w) \cap \bigcup B$, can be extended to a continuous map G_w of $I^2 - D_w$ onto w. Note that if $G^{-1}(w)$ is not a separating set for I^2 then this is just the map G. Define F to be G on the closure of D and to be G_w on each $I^2 - D_w$ where w ranges over all components of $G(\bigcup B)$. That F is well defined follows from Lemma A.6. The only overlaps occur at points of $\bigcup B$ where $G = G_w$, and wherever F differs from G the image of F is an element of $G(\bigcup B)$, hence in $X \cap C$. Also note the image of F lies in $A \cup C$.

To show that F is continuous consider a point q in I^2. If $q \in I^2 - \bigcup B$, then continuity follows either from the continuity of G or of one particular G_w. If $q \in \bigcup B$, let w be the component of $G(\bigcup B)$ containing $F(q)$. If $F(q)$ is not an endpoint of w then a combination of the maps G and G_w is used to show F is continuous at q. If $F(q)$ is an endpoint of w then given an open set O containing $F(q)$ there exists an open subset $N \subset O$ containing $F(q)$ such that any component w' of $G(\bigcup B)$, $w' \neq w$, which intersects N is a subset of O. Continuity follows using N together with the continuity of G and G_w. Thus F is the desired homotopy. ∎

Since X is connected and locally path connected, (2) obviously implies (1). The next result proves that (1) implies (2).

LEMMA 3.3. *Let X be a subset of $\mathbb{E}^2$ which is locally path connected and semilocally simply connected. Then X is locally simply connected.*

Proof. Let x_0 be a point in X. If x_0 is in the interior of X, then clearly X is locally simply connected at x_0. Otherwise, choose a path connected neighborhood O of x_0 in $\mathbb{E}^2$ so that any closed curve in $O \cap X$ based at x_0 is nullhomotopic in X. We now choose a round closed Euclidean disk N contained in O about x_0 whose boundary is not contained in X (if this were impossible then X would contain a round disk about x_0 and thus x_0 would be interior to X). Since X is locally path connected, S, the path component of $\operatorname{int}(N) \cap X$ containing x_0, is open in X. We will show that S is simply connected.

Applying the previous result we find that any closed curve in S is nullhomotopic in $N \cap X$ by a nullhomotopy F. However we need to show that any such closed curve is actually nullhomotopic in S. Suppose l is a closed curve in S based at x_0. We have two dichotomous cases.

Case 1: There is a round closed subdisk N' of the interior of N which contains l so that the boundary of N' is not contained in X. Then we may apply the above argument to show that l is nullhomotopic in $N' \cap X$ and thus in S.

Case 2: There is a circle C in S which separates l from the boundary of S. In this case we project any images of F which are separated from x_0 by C radially onto C, obtaining a new nullhomotopy whose image lies entirely inside S. ∎

The next result and Theorem 2.1 together show that X admits a universal cover if and only if the fundamental group of X is countable, and thus (1) and (3) of Theorem 3.1 are equivalent.

Theorem 3.4. *Every subset of* $\mathbb{E}^2$ *is homotopically Hausdorff.*

Proof. Let $x_0 \in X \subset \mathbb{E}^2$. Let l_0 be a closed curve in X based at x_0 so that given any open set U containing x_0, l_0 is homotopic (in X rel x_0) to a closed curve lying entirely in U. If x_0 is interior to X then l_0 is homotopic to a closed curve whose image lies in an open set $U \subseteq X$ which is homeomorphic to a Euclidean disk and thus l_0 is nullhomotopic.

If x_0 is not interior to X then there is a sequence of points in $\mathbb{E}^2 - X$ which converges to x_0. If this is the case, let p_0 be a point in $\mathbb{E}^2 - X$ and for each natural number n pick a point p_n in $\mathbb{E}^2 - X$ so that the distance between p_n and x_0 is no more than the minimum of $1/n$ and one-half the distance between p_{n-1} and x_0, i.e.

$$p_n \in B_{x_0}\left(\min\left(\frac{1}{n}, \frac{1}{2}d(x_0, p_{n-1})\right)\right) \cap (\mathbb{E}^2 - X).$$

Let $\varepsilon_n = d(x_0, p_n)$ and choose a closed curve $l_n \subset B_{x_0}(\varepsilon_n)$ based at x_0 which is homotopic to l_0 (and hence to l_{n-1}). Note that l_{n-1} and l_n are both contained in the closed disk $B_{x_0}(\varepsilon_{n-1})$. Furthermore the boundary of $B_{x_0}(\varepsilon_{n-1})$ is not contained in X since, by the definition of ε_{n-1}, it contains the point p_{n-1} which was chosen to be an element of $\mathbb{E}^2 - X$. Applying Lemma 3.2, we may choose a homotopy F_n between l_n and l_{n-1} so that $F_n|_{I\times 1}$ is l_n, $F_n|_{I\times 0}$ is l_{n+1} and the image of F_n is contained in the closure of $B_{x_0}(\varepsilon_{n-1})$. We sequentially adjoin the homotopies F_i to form a homotopy F by defining $F(x,y) = F_n(x, 2^{n+1}y - 1)$ when $2^{-(n+1)} \le y \le 2^{-n}$, and $F(x,0) = x_0$. We claim that F is continuous.

Case 1: If $(x,y) \in I^2$ and $y > 0$ then continuity at (x,y) follows from the continuity of at most two of the functions F_{n-1} and F_n.

Case 2: If $(x,y) \in I^2$ and $y = 0$ then $F(x,y) = x_0$. Given any $\varepsilon > 0$ we may choose a k so that $\varepsilon_k < \varepsilon$. Now, for any $n > k$, the image of F_n is contained in $B_{x_0}(\varepsilon_n)$ and thus is a subset of $B_{x_0}(\varepsilon_k)$. It follows that any point in $B_{(x,y)}(2^{-(k+1)})$ would map to a point within ε_k and hence within ε of x_0.

Thus the closed curve l_0 is nullhomotopic and so the set X is homotopically Hausdorff. ∎

Thus we have completed the proof of Theorem 3.1. ∎

4. Metrization of covering spaces

THEOREM 4.1. *If X is a locally connected separable metric space and $\widetilde{X}$ is a covering space for X then $\widetilde{X}$ is metrizable. Furthermore, if $\widetilde{X}$ is connected, then it is separable.*

Proof. Assume that X and $\widetilde{X}$ are as above and f is a covering map. Because $\widetilde{X}$ is a covering space, for each point p of X we pick an open set B_p containing p such that $f^{-1}(B_p)$ could be thought of as a collection F_p of disjoint open sets each of which is homeomorphic to B_p using f restricted to that open set. For each p we choose such a collection F_p, and since X is regular, an open set C_p containing p whose closure is a subset of B_p. Since X is a locally connected separable metric space, X has a countable basis, D, such that each element is a connected open set. Also since if C is an open covering of X and D is a basis for X then $\{g \in D \mid \exists c \in C$ such that $g \subset c\}$ is a basis for X, we may choose a countable basis $G_1, G_2, \ldots$ for X such that each G_i is connected and a subset of C_p for some point p of X.

For each n pick a point p such that $G_n \subset C_p$, and let

$$L_n = \{y \cap f^{-1}(G_n) \mid y \in F_p\}.$$

Bing has shown ([B, Theorem 3]) that a regular topological space is metrizable if and only if it has a perfect screening, which we will now define.

DEFINITION 4.2. A *perfect screening* of a topological space X is a countable collection $\{L_1, L_2, \ldots\}$ of sets each of which is a discrete collection of open sets in X so that $\bigcup_{i\in\mathbb{N}} L_i$ is a basis for X. Here a *discrete* collection means one for which the following holds: every point in X is contained in an open neighborhood which intersects at most one of the elements of the collection.

Since regularity is a local property and X is regular, $\widetilde{X}$ is regular. Thus to finish the proof we need only check that $\{L_1, L_2, \ldots\}$ is a perfect screening (or in other terminology, a *σ-discrete basis*) for $\widetilde{X}$.

Given an n and a point $q \in \widetilde{X}$, if $f(q)$ is not an element of the closure of G_n pick an open set D in X containing q which does not intersect the closure of G_n; then $f^{-1}(D)$ is an open set which does not intersect any element of L_n and contains the point q. If $f(q)$ is in the closure of G_n, then

$f(q) \in C_p$ and q is an element of only one element y of F_p and y is an open set intersecting only one element of L_n. Thus L_n is a discrete collection of open sets.

Given any open set $D \subset \widetilde{X}$ and a point $q \in D$, let C be the unique element of $F_{f(q)}$ which contains q. Note that $f(D \cap C)$ is an open set containing $f(q)$, hence there exists an n such that $f(q) \in G_n \subset f(D \cap C)$. Since f restricted to C is a homeomorphism and G_n is a subset of $f(C)$, we see that $f^{-1}(G_n) \cap C = g$ and f is a homeomorphism between g and G_n. If p is the point associated with G_n, then F_p is a collection of disjoint open sets and since g is connected it intersects only one of the elements of F_p and hence is an element of L_n. Thus $\bigcup_{n=1}^{\infty} L_n$ is a basis and $\{L_1, L_2, \ldots\}$ is a perfect screening.

Now assume $\widetilde{X}$ is connected. The covering map is a local homeomorphism. Since X is separable it follows that $\widetilde{X}$ is locally separable. Since a locally separable connected metric space is separable, $\widetilde{X}$ is a separable metric space. ∎

Appendix A

LEMMA A.1. *If U is an open cover of the space X, and X is a subspace of the metric space Y, then there exists a collection, $U' = \{u' \mid u' \cap X = u,$ $u \in U\}$, of open sets in Y in one-to-one correspondence with the elements of U so that any finite collection $\{u_1, \ldots, u_n\}$ have a common point of intersection if and only if the corresponding elements of U', $\{u'_1, \ldots, u'_n\}$, have a common point. It follows that the nerves of U and U' are naturally isomorphic.*

Proof. For each $u \in U$ pick an open set O_u in Y such that $u = O_u \cap X$. For each $a \in u$ pick a real number δ_a such that the ball in Y centered at a of radius $2\delta_a$ is contained in O_u and let B_a be the ball in Y centered at a of radius δ_a. Let $u' = \bigcup_{a \in u} B_a$ and $U' = \{u' \mid u \in U\}$. We show that U' has the desired properties. If $s \in \bigcap_{i=1}^{k} u'_i$ then, by construction, for each u'_i we may choose $a_i \in u_i$ such that $s \in \bigcap_{i=1}^{k} B_{a_i}(\delta_{a_i})$. Choose j such that $\delta_{a_j} = \min(\delta_{a_1}, \ldots, \delta_{a_k})$. For each i,

$$d(a_j, a_i) \leq d(a_j, s) + d(s, a_i) < \delta_{a_j} + \delta_{a_i} \leq 2\delta_{a_i}.$$

Thus for all i, $a_j \in B_{a_i}(2\delta_{a_i}) \subseteq O_{u_i}$. However, by definition, $a_j \in X$ so that $a_j \in u_i$ for all i, which yields $a_j \in \bigcap_{i=1}^{k} u_i$. ∎

DEFINITION A.2. If X is a topological space and C is a locally finite open cover of X then a *partition of unity corresponding to C* is a collection $\{f_c\}_{c \in C}$ of nonnegative real-valued functions on X so that the support of f_c is contained in c for each $c \in C$ and $\sum_{c \in C} f(c)$ is the constant function with value 1. Given such a cover and a corresponding partition of unity we get an induced map from X to $N(C)$ (the nerve of the open cover C) in the following way: for each point in X, let

$$C_x = \{c \in C \mid x \in c\},$$

and map x into the simplex corresponding to C_x (such a simplex obviously exists since x is in the intersection of the C_x's) by using the values $\{f_c(x) \mid c \in C_x\}$ as barycentric coordinates.

LEMMA A.3. *If X is a connected, locally path connected separable metric space then X has a universal covering space if and only if whenever C' is an open cover of X, there is a locally finite refinement C of C' by path connected open sets so that given any corresponding partition of unity, the induced map from X to $N(C)$, the nerve of C, induces an isomorphism between fundamental groups; furthermore, if X is compact then C may be chosen to be finite.*

Proof. Clearly, if X has such a cover C then it follows that X is semilocally simply connected, since any closed curve contained in any element of the open cover must be mapped into the open star of a vertex and thus maps trivially into the fundamental group of $N(C)$ and so is trivial in $\pi_1(X)$. Conversely, assume that X has a universal cover. Let C' be a cover of X by elementary neighborhoods. Since X is a separable metric space we can construct a cover, C, of X by connected open sets which is a star-refinement of C' (i.e. given any element c of C there is an element of C' which contains every element of C which intersects c), and which is locally finite (and finite in the case that X is compact).

We remark that any closed curve which is contained in the union of two elements of C is nullhomotopic in X. In [Ca], Cannon calls such covers *two-set simple*. He shows that if X is a connected, locally path connected, separable metric space and C is a two-set simple cover by connected open sets then the fundamental group of X is isomorphic to the fundamental group of $N(C)$. However, our statement is somewhat more general than that of Cannon, and we refer the reader to [CC2] for the necessary generalization. ∎

LEMMA A.4. *Let D be a connected subset of a connected locally connected space S, and E be a component of $S - D$. Then D and E are not mutually separated and E does not separate S.*

Proof. Since S is connected, two cases are possible:

CASE 1: $S - E$ contains a limit point p of E. We claim that in this case $p \in D$. If not then $E \cup \{p\}$ is a connected subset of $S - D$, contradicting the assumption that E is a component of $S - D$.

CASE 2: E contains a limit point p of $S - E$. We claim that in this case p is a limit point of D. If p is not a limit point of D, there exists an open set O containing p but no points of D. Since S is locally connected there exists a connected open set N such that $p \in N \subset O$. Then $E \cup N$ is a connected subset of $S - D$, again contradicting the assumption that E is a component of $S - D$.

We do not use the fact that E does not separate S elsewhere in this article so we only mention that it is an easy consequence of the cases above. ∎

NOTATION. For the remainder of this article we shall denote the boundary of a set S by ∂S.

LEMMA A.5. *If D is as in the proof of Lemma* 3.2 *and E is any component of $I^2 - D$, then ∂E is connected.*

Proof. Assume that ∂E is not connected. Then it is the union of two mutually separated sets B_1 and B_2. Since B_1 and B_2 are compact there exists a positive distance δ between them. Cover B_1 with a finite number of neighborhoods of diameter $\delta/3$; then there exists a simple closed curve γ in the complement of $B_1 \cup B_2$ made up of segments of circles and segments of ∂I^2 bounding the neighborhoods which, by applying the Jordan Curve Theorem, must separate a component of B_1 from B_2 in $\mathbb{E}^2$. Since D and E are both connected and each meets both B_1 and B_2 they must each meet γ. Hence if $x \in E \cap \gamma$ and $y \in D \cap \gamma$ and xy is a subarc of γ from x to y then the supremum of the set $\{z \in xy \mid \text{the subarc } xz \text{ of } xy \text{ is a subset of } E\}$ is a boundary point of E. However, this is a contradiction since γ meets neither B_1 nor B_2 and thus cannot intersect ∂E. Note that we are making strong use of the Jordan Curve Theorem since this result is not true on a torus. ∎

LEMMA A.6. *If D is as in the proof of Lemma* 3.2 *and p is any point of $I^2 - D$, then exactly one of the boundary components of D has the property that it either separates p from* $(0,0)$ *in I^2 or contains p.*

Proof. Let E be the component of $I^2 - D$ containing p. Then either $p \in \partial E$ or ∂E separates p from $(0,0)$. We will now show that $\partial E \subseteq \partial D$. By Lemma A.5, ∂E is connected and so Lemma A.4 applies. Now by Lemma A.4, Case 2, we see that every point of ∂E is a limit point of D; however, such a point is also a limit point of $E \subseteq I^2 - D$ and thus is contained in ∂D. Consequently, ∂E is a subset of a unique boundary component K of D. Since $D \cup E$ is connected and ∂E is contained in K, no other boundary component of D can contain p or separate p from $(0,0)$. ∎

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