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Discrete n-tuples in Hausdorff spaces

by

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Abstract. We investigate the following three questions: Let $n \in \mathbb{N}$. For which Hausdorff spaces X is it true that whenever Γ is an arbitrary (respectively finite-to-one, respectively injective) function from $\mathbb{N}^n$ to X, there must exist an infinite subset M of $\mathbb{N}$ such that $\Gamma[M^n]$ is discrete? Of course, if $n = 1$ the answer to all three questions is "all of them". For $n \geq 2$ the answers to the second and third questions are the same; in the case $n = 2$ that answer is "those for which there are only finitely many points which are the limit of injective sequences". The answers to the remaining instances involve the notion of *n-Ramsey limit*. We also show that the class of spaces satisfying these discreteness conclusions for all n includes the class of F-spaces. In particular, it includes the Stone–Čech compactification of any discrete space.

1. Introduction. The simplest nontrivial case of Ramsey's Theorem [10] says that whenever the two-element subsets $[Y]^2$ of an infinite set Y are divided into finitely many classes, there must be an infinite subset Z of Y such that $[Z]^2$ is contained in one of those classes. The version below which we will be using here is slightly more general; for a proof see [10], or [6, Theorem 1.2] or [8, Theorem 18.2].

1.1. Theorem (Ramsey). *If n is an integer and the set $[Y]^n$ of n-element subsets of an infinite set Y is divided into finitely many classes, then there must be an infinite subset Z of Y such that $[Z]^n$ is contained in one of those classes.*

Ramsey's Theorem inspired a branch of mathematics known as *Ramsey Theory*. (See [6].)

In a recent paper [2] we were deriving some Ramsey-theoretic consequences of some algebraic results about the Stone–Čech compactification βW of a discrete free semigroup W. (These algebraic results [1] extended

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the Graham–Rothschild Parameter Sets Theorem [5].) In the process we needed to know whether, given a doubly indexed sequence $\langle x_{i,j}\rangle_{(i,j)\in\mathbb{R}\times\mathbb{R}}$ in βW with the property that $x_{i,j}\neq x_{k,l}$ when $(i,j)\neq(k,l)$, there must exist an infinite $Y\subseteq\mathbb{N}=\{1,2,\ldots\}$ such that $\langle x_{i,j}\rangle_{(i,j)\in Y\times Y}$ is discrete. We determined that the answer is "yes", and characterized those Hausdorff spaces X for which the corresponding statement with βW replaced by X remained valid.

After consulting with several experts we were surprised to find out that this result appears to be new. Consequently, we were motivated to investigate the following questions.

1.2. QUESTION. *Let $n\in\mathbb{N}$. For which Hausdorff spaces X is it true that whenever Γ is an arbitrary* (*respectively finite-to-one*, *respectively injective*) *function from $\mathbb{N}^n$ to X, there must exist some $M\in[\mathbb{N}]^\omega$ such that $\Gamma[M^n]$ is discrete?*

Of course, everyone learns in infancy that if $n=1$, the answer is "all of them". To answer the three forms of Question 1.2 for $n>2$ we introduce the notion of an *n-Ramsey limit*.

Section 2 consists of an introduction to n-Ramsey limits and development of some of their properties. In Section 4 we establish our main results. And in Section 5 we investigate some spaces that satisfy the conclusion of the first form of Question 1.2 for all n.

Given a set Y and a cardinal number κ, we set $[Y]^\kappa=\{A\subseteq Y:|A|=\kappa\}$. We take $\mathbb{N}$ to be the set of positive integers and ω to be the set of nonnegative integers. Also, ω is the first infinite cardinal, so that, given a set Y, $[Y]^\omega$ is the set of countably infinite subsets of Y. Given any set Y, we write $\mathcal{P}_{\mathrm{f}}(Y)$ for the set of finite nonempty subsets of Y. Also, given $f:X\to Y$ and $A\subseteq X$ we write $f[A]=\{f(x):x\in A\}$.

Viewing $[Y]^\omega$ for Y infinite as being ordered by inclusion, $\mathcal{D}\subseteq[Y]^\omega$ is *downward cofinal* if for each $Z\in[Y]^\omega$ there is $W\in[Z]^\omega\cap\mathcal{D}$, and $\mathcal{D}$ is *downward closed* if for all $W,Z\in[Y]^\omega$, if $W\subseteq Z$ and $Z\in\mathcal{D}$ then $W\in\mathcal{D}$. We will often use the fact that the intersection of finitely many sets which are downward cofinal and downward closed is also downward cofinal and downward closed and, consequently, nonempty.

Late in our investigation we realized that we did not know the answer to the versions of Question 1.2 that replace $\mathbb{N}^n$ by $[\mathbb{N}]^n$. The answer to these versions (which is "all of them") is presented in Section 3.

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2. Ramsey filters and Ramsey limits. Our main tool is a certain class of filters which we introduce now.

2.1. Definition. Let M be an infinite set and let $n \in \mathbb{N}$. The *n-Ramsey filter on* $[M]^n$ is $\mathcal{R}_n(M) = \{A \subseteq [M]^n : \text{for all } B \in [M]^\omega,\ [B]^n \cap A \neq \emptyset\}$. We let $\mathcal{R}_n = \mathcal{R}_n(\mathbb{N})$.

Of course, since we have called $\mathcal{R}_n(M)$ a filter, we should verify that it is a filter indeed. This verification also shows the reason for "Ramsey" in the name.

2.2. Lemma. *Let $n \in \mathbb{N}$ and let M be an infinite set. Then $\mathcal{R}_n(M)$ is a filter on $[M]^n$.*

Proof. The only nontrivial part of this assertion is that $\mathcal{R}_n(M)$ is closed under finite intersections. So let $A, B \in \mathcal{R}_n(M)$. Let

$$D_1 = [M]^n \setminus (A \cup B), \quad D_2 = A \setminus B, \quad D_3 = B \setminus A, \quad D_4 = A \cap B.$$

Let $E \in [M]^\omega$ and pick by Ramsey's Theorem $i \in \{1,2,3,4\}$ and $C \in [E]^\omega$ such that $[C]^n \subseteq D_i$. Since $[C]^n \cap A \neq \emptyset$ and $[C]^n \cap B \neq \emptyset$, we must have $i = 4$. ∎

2.3. Lemma. *Let M be an infinite set and let $A \in \mathcal{R}_n(M)$. For every $L \in [M]^\omega$ there exists $B \in [L]^\omega$ such that $[B]^n \subseteq A$.*

Proof. Pick by Ramsey's Theorem some $B \in [L]^\omega$ such that either $[B]^n \subseteq A$ or $[B]^n \cap A = \emptyset$. The latter alternative cannot hold. ∎

We shall be concerned with limits in a Hausdorff space determined by the filters $\mathcal{R}_n(M)$.

2.4. Definition. Let X be a Hausdorff space, let $n \in \mathbb{N}$, let $y \in X$, let M be an infinite set and let $\varphi : [M]^n \to X$. Then $y = \mathcal{R}_n(M)\text{-}\lim \varphi$ if and only if for every neighborhood U of y, $\varphi^{-1}[U] \in \mathcal{R}_n(M)$. We say that y is a *nontrivial n-Ramsey limit in X* if and only if there exists some $\varphi : [\mathbb{N}]^n \to X$ such that $y = \mathcal{R}_n\text{-}\lim \varphi$ and $\varphi^{-1}[\{y\}] \notin \mathcal{R}_n$.

Notice that $\mathcal{R}_n$-limits are unique if they exist. Notice also that if $f : \mathbb{N} \to X$ and $\varphi(\{m\}) = f(m)$, then the statements $y = \lim_{m\to\infty} f(m)$ and $y = \mathcal{R}_1\text{-}\lim \varphi$ are equivalent as $\mathcal{R}_1 = \{A \subseteq [\mathbb{N}]^1 : \{x \in \mathbb{N} : \{x\} \notin A\} \text{ is finite}\}$.

We omit the routine proof of the following observation.

2.5. Lemma. *Let X be a Hausdorff space, let $n \in \mathbb{N}$, let L and M be infinite sets, let $y \in X$ and let $\varphi : [L \cup M]^n \to X$. If $y = \mathcal{R}_n(M)\text{-}\lim \varphi$ and $L \setminus M$ is finite, then $y = \mathcal{R}_n(L)\text{-}\lim \varphi$.*

2.6. Lemma. *Let X be a Hausdorff space, let $n \in \mathbb{N}$, and let $y \in X$. If there exists $\varphi : [\mathbb{N}]^n \xrightarrow{1\text{-}1} X$ such that $y = \mathcal{R}_n\text{-}\lim \varphi$, then there exists $\tau : [\mathbb{N}]^{n+1} \xrightarrow{1\text{-}1} X$ such that $y = \mathcal{R}_{n+1}\text{-}\lim \tau$.*

Proof. Let $\gamma : \mathbb{N}^2 \xrightarrow{1\text{-}1} \mathbb{N}$ and define $\psi : [\mathbb{N}]^{n+1} \xrightarrow{1\text{-}1} [\mathbb{N}]^n$ as follows. Let $A \in [\mathbb{N}]^{n+1}$ and write $A = \{a_0, a_1, \ldots, a_n\}$ with $a_0 = \min A$. Let

$$\psi(A) = \{\gamma(a_0, a_1), \gamma(a_0, a_2), \ldots, \gamma(a_0, a_n)\}.$$

Now let $\tau = \varphi \circ \psi$. We claim that $y = \mathcal{R}_{n+1}\text{-}\lim \tau$. So let U be a neighborhood of y and let $B \in [\mathbb{N}]^\omega$. We need to show that there is some $A \in [B]^{n+1}$ such that $\tau(A) \in U$. Let $a_0 = \min B$ and let $C = \{\gamma(a_0, b) : b \in B \text{ and } b > a_0\}$. Then $C \in [\mathbb{N}]^\omega$, so pick $D = \{c_1, \ldots, c_n\} \in [C]^n$ such that $\varphi(C) \in U$. For $i \in \{1, \ldots, n\}$ pick $a_i \in B$ such that $a_i > a_0$ and $\gamma(a_0, a_i) = c_i$. Then $A = \{a_0, a_1, \ldots, a_n\} \in [B]^{n+1}$ and $\tau(A) = \varphi(D) \in U$. ∎

We shall have need of the *Canonical Ramsey Theorem* (Theorem 2.8). To state it conveniently, we introduce some notation.

2.7. Definition. Let F be a finite subset of $\mathbb{N}$. If $F = \emptyset$, then for all $A \subseteq \mathbb{N}$, $\nu_F(A) = \emptyset$. Otherwise, let $n = |F|$ and let $k = \max F$. Then $\nu_F : \{A \subseteq \mathbb{N} : |A| \geq k\} \to [\mathbb{N}]^n$ is defined as follows. Given $A \subseteq \mathbb{N}$ such that $|A| \geq k$, let $a_1 < \cdots < a_k$ be the first k members of A. Then $\nu_F(A) = \{a_i : i \in F\}$.

2.8. Theorem (Erdős and Rado). *Let $n \in \mathbb{N}$, let $C \in [\mathbb{N}]^\omega$, and let ψ be any function with domain containing $[C]^n$. Then there exist $M \in [C]^\omega$ and $F \subseteq \{1, \ldots, n\}$ such that for all $A, B \in [M]^n$, $\psi(A) = \psi(B)$ if and only if $\nu_F(A) = \nu_F(B)$.*

Proof. [3], or see [6, Theorem 5.3]. ∎

2.9. Lemma. *Let X be a Hausdorff space, let $n \in \mathbb{N}$, and let $y \in X$. If y is a nontrivial n-Ramsey limit in X, then there exists $\varphi : [\mathbb{N}]^n \xrightarrow{1\text{-}1} X \setminus \{y\}$ such that $y = \mathcal{R}_n\text{-}\lim \varphi$. In particular, y is also a nontrivial $(n+1)$-Ramsey limit in X.*

Proof. Pick $\eta : [\mathbb{N}]^n \to X$ such that $y = \mathcal{R}_n\text{-}\lim \eta$ and $\eta^{-1}[\{y\}] \notin \mathcal{R}_n$. Pick $C \in [\mathbb{N}]^\omega$ such that $[C]^n \cap \eta^{-1}[\{y\}] = \emptyset$.

Pick by Theorem 2.8 some $M \in [C]^\omega$ and some $F \subseteq \{1, \ldots, n\}$ such that for all $A, B \in [M]^n$, $\eta(A) = \eta(B)$ if and only if $\nu_F(A) = \nu_F(B)$.

We first note that $F \neq \emptyset$. Indeed, suppose $F = \emptyset$ and let x be the constant value of η on $[M]^n$. Since $M \subseteq C$, we have $x \neq y$, so pick a neighborhood U of y such that $x \notin U$. Then $[M]^n \cap \eta^{-1}[U] \neq \emptyset$, a contradiction.

Let $m = |F|$. Pick $N \in [M]^\omega$ with the property that for all $B \in [N]^m$ there is some $A \in [M]^n$ such that $B = \nu_F(A)$. (For example, N could consist of every nth member of M.) Let $\psi : [N]^m \to [M]^n$ be a function such that for each $B \in [N]^m$, we see that $\nu_F(\psi(B)) = B$. Then given $A \in [N]^n$, $\nu_F(\psi(\nu_F(A))) = \nu_F(A)$ so $\eta(\psi(\nu_F(A))) = \eta(A)$.

Let $\mu : \mathbb{N} \xrightarrow{\text{1-1}} N$ and define $\tau : [\mathbb{N}]^m \xrightarrow{\text{1-1}} X$ by $\tau(B) = \eta(\psi(\mu[B]))$. (To see that τ is injective, let $B, D \in [\mathbb{N}]^m$ and assume that $\tau(B) = \tau(D)$. Since $\eta(\psi(\mu[B])) = \eta(\psi(\mu[D]))$, we have $\mu[B] = \nu_F(\psi(\mu[B])) = \nu_F(\psi(\mu[D])) = \mu[D]$, so $B = D$.)

We claim that $y = \mathcal{R}_m\text{-}\lim \tau$. To see this, let U be a neighborhood of y and let $E \in [\mathbb{N}]^\omega$. Then $\mu[E] \in [N]^\omega$ so pick $A \in [\mu[E]]^n$ such that $\eta(A) \in U$. Let $B = \mu^{-1}[\nu_F(A)]$. Then $B \in [E]^m$ and $\tau(B) = \eta(\psi(\mu[B])) = \eta(\psi(\nu_F(A))) = \eta(A) \in U$.

Since $m \le n$, by Lemma 2.6 there exists $\varphi : [\mathbb{N}]^n \xrightarrow{\text{1-1}} X$ such that $y = \mathcal{R}_n\text{-}\lim \varphi$.

For the "in particular" conclusion, apply Lemma 2.6. ∎

The following lemma, in conjunction with Lemma 2.9, shows that the property of being a nontrivial $(n+1)$-Ramsey limit is strictly stronger than that of being a nontrivial n-Ramsey limit.

2.10. Lemma. *Let $n \in \mathbb{N}$, let $X = [\mathbb{N}]^{n+1} \cup \{\infty\}$, and let X have the topology $\mathcal{T} = \mathcal{P}([\mathbb{N}]^{n+1}) \cup \{\{\infty\} \cup A : A \in \mathcal{R}_{n+1}\}$. Then ∞ is a nontrivial $(n+1)$-Ramsey limit in X, but no point of X is a nontrivial n-Ramsey limit in X.*

Proof. Let $\iota : [\mathbb{N}]^{n+1} \to [\mathbb{N}]^{n+1}$ be the identity function. Then $\infty = \mathcal{R}_{n+1}\text{-}\lim \iota$.

Now suppose that ∞ is a nontrivial n-Ramsey limit and, by Lemma 2.9, pick $\varphi : [\mathbb{N}]^n \xrightarrow{\text{1-1}} X \setminus \{\infty\}$ such that $\infty = \mathcal{R}_n\text{-}\lim \varphi$.

Notice that for any $L \in [\mathbb{N}]^\omega$ there is some $B \in [\mathbb{N}]^\omega$ such that $[B]^{n+1} \subseteq \varphi[[L]^n]$. (We have $[L]^n \cap \varphi^{-1}[X \setminus \varphi[[L]^n]] = \emptyset$, so $X \setminus \varphi[[L]^n]$ is not a neighborhood of ∞ and thus $[\mathbb{N}]^{n+1} \setminus \varphi[[L]^n] \notin \mathcal{R}_{n+1}$.)

For each $t \in \{1, \ldots, n+1\}$, define $f_t : [\mathbb{N}]^n \to [\mathbb{N}]^n$ as follows. Given $A \in [\mathbb{N}]^n$, if $\varphi(A) = \{b_1, \ldots, b_{n+1}\}$ and $b_1 < \cdots < b_{n+1}$, then $f_t(A) = \varphi(A) \setminus \{b_t\}$. We note that f_t is not injective on $[L]^n$ for any $L \in [\mathbb{N}]^\omega$. To see this pick some $B \in [\mathbb{N}]^\omega$ such that $[B]^{n+1} \subseteq \varphi[[L]^n]$. Pick $b_1 < \cdots < b_{n+2}$ in B and let $E = \{b_1, \ldots, b_{n+2}\}$. Pick $C, D \in [L]^n$ such that $\varphi(C) = E \setminus \{b_t\}$ and $\varphi(D) = E \setminus \{b_{t+1}\}$. Then $f_t(C) = f_t(D) = E \setminus \{b_t, b_{t+1}\}$, while $C \neq D$ because $\varphi(C) \neq \varphi(D)$.

Let $L_0 = \mathbb{N}$ and, by Theorem 2.8, for $t \in \{1, \ldots, n+1\}$ inductively pick $L_t \in [L_{t-1}]^\omega$ and $G_t \subseteq \{1, \ldots, n\}$ such that for all $C, D \in [L_t]^n$, $f_t(C) = f_t(D)$ if and only if $\nu_{G_t}(C) = \nu_{G_t}(D)$. Since f_t is not injective on $[L_t]^n$, we have $G_t \neq \{1, \ldots, n\}$, so pick $\sigma(t) \in \{1, \ldots, n\} \setminus G_t$.

Pick $t \neq s$ in $\{1, \ldots, n+1\}$ such that $\sigma(t) = \sigma(s)$ and pick $a_1 < \cdots < a_{n+1}$ in L_{n+1}, let $A = \{a_1, \ldots, a_{n+1}\}$, let $B = A \setminus \{a_{\sigma(t)}\}$ and let $C = A \setminus \{a_{\sigma(t)+1}\}$. If $B = \{b_1, \ldots, b_n\}$, $C = \{c_1, \ldots, c_n\}$, $b_1 < \cdots < b_n$ and

$c_1 < \cdots < c_n$, then

$$b_i = \begin{cases} a_i & \text{if } i < \sigma(t), \\ a_{i+1} & \text{if } i \geq \sigma(t), \end{cases} \qquad c_i = \begin{cases} a_i & \text{if } i \leq \sigma(t), \\ a_{i+1} & \text{if } i > \sigma(t). \end{cases}$$

Therefore $b_i = c_i$ for all $i \in G_t$ and $b_i = c_i$ for all $i \in G_s$, and thus $f_t(B) = f_t(C)$ and $f_s(B) = f_s(C)$. But then $\varphi(B) = \varphi(C)$, contradicting the fact that φ is injective. ∎

3. Discrete images of n-element sets. We show in this section that any Hausdorff space X has the property that for any $n \in \mathbb{N}$ and any function from $[\mathbb{N}]^n$ to X there is an infinite $B \subseteq \mathbb{N}$ such that the image of $[B]^n$ is discrete.

We show first that it suffices to consider injective functions.

3.1. Lemma. *Let X be a Hausdorff space and assume that whenever $n \in \mathbb{N}$ and $\psi : [\mathbb{N}]^n \xrightarrow{1\text{-}1} X$ there exists $B \in [\mathbb{N}]^\omega$ such that $\psi[[B]^n]$ is discrete. Then whenever $n \in \mathbb{N}$ and $\psi : [\mathbb{N}]^n \to X$ there exists $B \in [\mathbb{N}]^\omega$ such that $\psi[[B]^n]$ is discrete.*

Proof. Let $n \in \mathbb{N}$ and let $\psi : [\mathbb{N}]^n \to X$. Pick by Theorem 2.8 some $F \subseteq \{1, \ldots, n\}$ and some $M \in [\mathbb{N}]^\omega$ such that for all $A, B \in [M]^n$, $\psi(A) = \psi(B)$ if and only if $\nu_F(A) = \nu_F(B)$. If $F = \emptyset$, then ψ is constant on $[M]^n$ and so $\psi[[M]^n]$ is discrete. If $F = \{1, \ldots, n\}$, then ψ is injective on $[M]^n$, so our assumption applies. Hence we may assume that $0 < |F| < n$.

Let $k = |F|$ and pick $L \in [M]^\omega$ such that for every $C \in [L]^k$ there is some $A \in [M]^n$ such that $C = \nu_F(A)$. (One may have L consist of every nth member of M.) Pick $\tau : [L]^k \to [M]^n$ such that for all $C \in [L]^k$, $\nu_F(\tau(C)) = C$. Let $\mu = \psi \circ \tau$. We claim that $\mu : [L]^k \xrightarrow{1\text{-}1} X$. To see this, let $C, D \in [L]^k$ and assume that $\mu(C) = \mu(D)$. Then $\psi(\tau(C)) = \psi(\tau(D))$ so $C = \nu_F(\tau(C)) = \nu_F(\tau(D)) = D$.

By assumption we may pick $B \in [L]^\omega$ such that $\mu[[B]^k]$ is discrete. To complete the proof it suffices to show that $\psi[[B]^n] \subseteq \mu[[B]^k]$. So let $D \in [B]^n$. Then $\nu_F(D) \in [B]^k$ and $\nu_F \circ \tau$ is the identity on $[L]^k$, so $\nu_F(D) = \nu_F(\tau(\nu_F(D)))$ and thus $\psi(D) = \psi(\tau(\nu_F(D))) = \mu(\nu_F(D)) \in \mu[[B]^k]$. ∎

We now establish two technical lemmas. The first is a simple combinatorial consequence of Ramsey's Theorem and is in the folklore.

3.2. Lemma. *Suppose Y is an infinite set and $k \in \omega$. If $h : [Y]^k \to \mathcal{P}(Y)$ has the property that $h(A) \not\subseteq A$ for all $A \in [Y]^k$ then there is an infinite $Z \subseteq Y$ such that $h(A) \not\subseteq Z$ for all $A \in [Z]^k$.*

Proof. Without loss of generality, $Y = \mathbb{N}$. Choose $g : [\mathbb{N}]^k \to \mathbb{N}$ so that $g(A) \in h(A) \setminus A$ for all $A \in [\mathbb{N}]^k$.

Let $i \in \mathbb{N}$ with $1 \le i \le k+1$. For $A \in [\mathbb{N}]^{k+1}$, let A^- be $A \setminus \{\text{the } i\text{th element of } A\}$. Let $C_1 = \{A \in [\mathbb{N}]^{k+1} : g(A^-) \text{ is the } i\text{th element of } A\}$ and let $C_0 = [\mathbb{N}]^{k+1} \setminus C_0$. Notice that there can be no infinite $Z \subseteq \mathbb{N}$ such that $[Z]^{k+1}$ is a subset of C_1. (Consider two elements of $[Z]^{k+1}$ which differ only at the ith element.) By Ramsey's Theorem, the collection $\mathcal{D}_i = \{Z \in [\mathbb{N}]^\omega : [Z]^{k+1} \subseteq C_0\}$ is downward cofinal. Moreover, $\mathcal{D}_i$ is clearly downward closed.

Let Z be in each of the collections $\mathcal{D}_i$ for $1 \le i \le k+1$. Then Z has the desired properties. ∎

3.3. Lemma. *Let X be a Hausdorff space, let $n \in \mathbb{N}$, and let $\psi : [\mathbb{N}]^n \xrightarrow{1\text{-}1} X$. For each $k \in \{0, 1, \dots, n-1\}$ and each $B \in [\mathbb{N}]^k$ define $f_B : [\mathbb{N} \setminus B]^{n-k} \to X$ by $f_B(C) = \psi(B \cup C)$. There exists $M \in [\mathbb{N}]^\omega$ such that for all $A \in [M]^n$, all $k \in \{0, 1, \dots, n-1\}$, all $B \in [M]^k$, and all $L \in [M]^\omega$, $\psi(A)$ is not $\mathcal{R}_{n-k}(L \setminus B)$-$\lim f_B$.*

Proof. We first show that we can choose $K \in [\mathbb{N}]^\omega$ such that for each $k \in \{0, 1, \dots, n-1\}$ and each $B \in [\mathbb{N}]^k$, either $\mathcal{R}_{n-k}(K \setminus B)$-$\lim f_B$ exists or for all $L \in [K]^\omega$, $\mathcal{R}_{n-k}(L \setminus B)$-$\lim f_B$ does not exist. To see this, enumerate $\bigcup_{k=0}^{n-1} [\mathbb{N}]^k$ as $\langle B_t \rangle_{t=1}^\infty$. Let $K_0 = \mathbb{N}$. Let $t \in \mathbb{N}$ and assume that K_{t-1} has been chosen. Let $k = |B_t|$. If there is some $K_t \in [K_{t-1} \setminus B_t]^\omega$ such that $\mathcal{R}_{n-k}(K_t)$-$\lim f_{B_t}$ exists, choose such. Otherwise let $K_t = K_{t-1}$. The induction being complete, choose $x_t \in K_t \setminus \{x_1, \dots, x_{t-1}\}$ for each $t \in \mathbb{N}$ and let $K = \{x_t : t \in \mathbb{N}\}$. Using Lemma 2.5 one easily establishes that K is as required.

For each $k \in \{0, 1, \dots, n-1\}$ choose $h_k : [K]^k \to \mathcal{P}(\mathbb{N})$ such that $h_k(B) \not\subseteq B$ and $h_k(B) = A$ if $\mathcal{R}_{n-k}(K \setminus B)$-$\lim f_B = \psi(A)$ (notice that since ψ is injective, if such A exists then it is unique). By the previous lemma, there is an $M \in [K]^\omega$ such that $h_k(B) \not\subseteq M$ for all $k \in \{0, 1, \dots, n-1\}$ and $B \in [M]^k$.

To see that M has the desired properties, argue by contradiction and assume that $A \in [M]^n$, $k \in \{0, 1, \dots, n-1\}$, $B \in [M]^k$, $L \in [M]^\omega$ and $\psi(A) = \mathcal{R}_{n-k}(L \setminus B)$-$\lim f_B$. By Lemma 2.5 and the choice of K, $\mathcal{R}_{n-k}(K \setminus B)$-$\lim f_B = \psi(A)$. This implies that $A = h_k(B) \not\subseteq M$, a contradiction. ∎

3.4. Theorem. *Let X be a Hausdorff space, let $n \in \mathbb{N}$, and let $\psi : [\mathbb{N}]^n \to X$. There exists $L \in [\mathbb{N}]^\omega$ such that $\psi[[L]^n]$ is discrete.*

Proof. By Lemma 3.1 we may assume that $\psi : [\mathbb{N}]^n \xrightarrow{1\text{-}1} X$. For each $k \in \{0, 1, \dots, n-1\}$ and each $B \in [\mathbb{N}]^k$ define $f_B : [\mathbb{N} \setminus B]^{n-k} \to X$ by $f_B(C) = \psi(B \cup C)$.

Pick $M_0 \in [\mathbb{N}]^\omega$ as guaranteed by Lemma 3.3. Inductively, let $m \in \mathbb{N}$ and assume that we have chosen $l_1 < \dots < l_{m-1}$ and infinite $M_0 \supseteq M_1 \supseteq \dots \supseteq M_{m-1}$ such that for each $t \in \{1, \dots, m-1\}$, if $A \in [\{l_1, \dots, l_t\}]^n$ then there is a neighborhood of $\psi(A)$ which is disjoint from $f_B[[M_t]^{n-k}]$ whenever $k \in \{0, 1, \ldots, n-1\}$ and $B \in [\{l_1, \ldots, l_t\}]^k$. Notice that we may also assume that $\psi(A')$ is not in the given neighborhood for the finitely many $A' \neq A$ in $[\{l_1, \ldots, l_t\}]^n$.

Let l_m be the least element of M_{m-1}. For each $A \in [\{l_1, \ldots, l_m\}]^n$, $k \in \{0, 1, \ldots, n-1\}$ and $B \in [\{l_1, \ldots, l_m\}]^k$, let $\mathcal{D}_{A,B} = \{M \in [M_{m-1} \setminus \{l_m\}]^\omega :$ there is a neighborhood of $\psi(A)$ which is disjoint from $f_B[[M]^{n-k}]\}$. By the choice of M_0, $\mathcal{D}_{A,B}$ is a downward cofinal and downward closed subset of $[M_{m-1} \setminus \{l_m\}]^\omega$. Choose M_m in the intersection of the finitely many $\mathcal{D}_{A,B}$.

The main induction being complete, let $L = \{l_m : m \in \mathbb{N}\}$. To see that L is as required, let $A \in [L]^n$. Let $m = \max\{t : l_t \in A\}$ and let U be a neighborhood of $\psi(A)$ which is disjoint from $f_B[[M_m]^{n-k}]$ whenever $k \in \{0, 1, \ldots, n-1\}$ and $B \in [\{l_1, \ldots, l_m\}]^k$. As noted above, we may also assume that $\psi(A')$ is not in U whenever $A' \neq A$ is in $[\{l_1, \ldots, l_m\}]^n$. Since $L \subseteq \{l_0, l_1, \ldots, l_m\} \cup M_m$, we have $U \cap \psi[[L]^n] = \{\psi(A)\}$. ∎

4. Discrete n-tuples. In this section we answer Question 1.2.

4.1. Definition. Given $k \leq n$ in $\mathbb{N}$ and $f : \{1, \ldots, n\} \to \{1, \ldots, k\}$, define $\mu_f : [\mathbb{N}]^k \to \mathbb{N}^n$ by taking $\mu_f(A)_j$ to be the $f(j)$th member of A.

For example, if $n=4$, $k=3$, $f(1)=f(3)=2$, $f(2)=1$, and $f(4)=3$, then $\mu_f(\{7, 10, 30\}) = (10, 7, 10, 30)$. We omit the routine proof of the following lemma.

4.2. Lemma. *If $n \in \mathbb{N}$, $\vec{l} \in \mathbb{N}^n$, $k = |\{l_1, \ldots, l_n\}|$, and*

$$f : \{1, \ldots, n\} \xrightarrow{\text{onto}} \{1, \ldots, k\}$$

is defined by $f(j) = i$ if and only if l_j is the ith member of $\{l_1, \ldots, l_n\}$, then $\mu_f(\{l_1, \ldots, l_n\}) = \vec{l}$.

4.3. Lemma. *Let X be a Hausdorff space, let $n \in \mathbb{N}$, and let x be an element of X which is not a nontrivial n-Ramsey limit point. If $\psi : [\mathbb{N}]^n \to X$, then for all $M \in [\mathbb{N}]^\omega$ there exist $L \in [M]^\omega$ and a neighborhood U of x with $U \cap \psi[[L]^n] \subseteq \{x\}$.*

Proof. Choose $\tau : \mathbb{N} \xrightarrow{1\text{-}1} M$. Define $\gamma : [\mathbb{N}]^n \to X$ by $\gamma(A) = \psi(\tau[A])$.

Assume first that $\gamma^{-1}[\{x\}] \in \mathcal{R}_n$. Pick by Ramsey's Theorem some $B \in [\mathbb{N}]^\omega$ such that $[B]^n \subseteq \gamma^{-1}[\{x\}]$ or $[B]^n \cap \gamma^{-1}[\{x\}] = \emptyset$. Since $\gamma^{-1}[\{x\}] \in \mathcal{R}_n$, the latter is impossible so $[B]^n \subseteq \gamma^{-1}[\{x\}]$. Let $L = \tau[B]$ and let $U = X$. If $A \in [L]^n$, then $\tau^{-1}[A] \in [B]^n$ so $\psi(A) = \psi(\tau[\tau^{-1}[A]]) = \gamma(\tau^{-1}[A]) = x$.

Now assume that $\gamma^{-1}[\{x\}] \notin \mathcal{R}_n$ and consequently that $x \neq \mathcal{R}_n\text{-}\lim \gamma$. Pick a neighborhood U of x such that $\gamma^{-1}[U] \notin \mathcal{R}_n$ and pick $B \in [\mathbb{N}]^\omega$ such that $[B]^n \cap \gamma^{-1}[U] = \emptyset$. Let $L = \tau[B]$. If $A \in [L]^n$, then $\tau^{-1}[A] \in [B]^n$ so $\psi(A) = \gamma(\tau^{-1}[A]) \notin U$. ∎

4.4. Lemma. *Let X be a Hausdorff space, let $n \in \mathbb{N}$, and let x be an element of X which is not a nontrivial n-Ramsey limit. If $\Gamma : \mathbb{N}^n \to X$, then for all $M \in [\mathbb{N}]^\omega$ there exist $L \in [M]^\omega$ and a neighborhood U of x with $U \cap \Gamma[L^n] \subseteq \{x\}$.*

Proof. Let $F = \{f : (\exists k \in \{1,\dots,n\})(f : \{1,\dots,n\} \xrightarrow{\text{onto}} \{1,\dots,k\})\}$. For each $f \in F$, pick k such that $f : \{1,\dots,n\} \xrightarrow{\text{onto}} \{1,\dots,k\}$ and let $\mathcal{D}_f = \{L \in [\mathbb{N}]^\omega :$ there is a neighborhood U of x such that $U \cap (\Gamma \circ \mu_f[[L]^k]) \subseteq \{x\}\}$. Since x is not a nontrivial k-Ramsey limit point of X for any $k \le n$ by Lemma 2.9, Lemma 4.3 implies that $\mathcal{D}_f$ is downward cofinal for each $f \in F$. Since each $\mathcal{D}_f$ is clearly downward closed, we can choose $L \in [M]^\omega$ in the intersection of the finitely many sets $\mathcal{D}_f$.

For each $f \in F$, there is a neighborhood U_f of x such that $U \cap (\Gamma \circ \mu_f[[L]^k]) \subseteq \{x\}$. Let $U = \bigcap_{f\in F} U_f$. To see that $U \cap \Gamma[L^n] \subseteq \{x\}$, let $\vec{l} \in L^n$ and define f as in Lemma 4.2. Let $k = |\{l_1,\dots,l_n\}|$. Then $\{l_1,\dots,l_n\} \in [L]^k$ so $\Gamma(\vec{l}) = \Gamma(\mu_f(\{l_1,\dots,l_n\})) \notin U_f \setminus \{x\}$. ∎

4.5. Lemma. *Let X be a Hausdorff space, let $n \in \mathbb{N} \setminus \{1\}$, and let x be an element of X which is not a nontrivial $(n-1)$-Ramsey limit. If $\Gamma : \mathbb{N}^n \to X$ and $G \in \mathcal{P}_{\mathrm{f}}(\mathbb{N})$ then for all $M \in [\mathbb{N}]^\omega$ there exist $L \in [M]^\omega$ and a neighborhood U of x with $U \cap \Gamma[(G \cup L)^n \setminus L^n] \subseteq \{x\}$.*

Proof. Let $F = \{f : (\exists H)(\emptyset \ne H \subsetneq \{1,\dots,n\}$ and $f : H \to G)\}$. Suppose $f \in F$, let $H = \mathrm{domain}(f)$, and list $\{1,\dots,n\} \setminus H$ in order as $j_1,\dots,j_k$. Define $\eta_f : \mathbb{N}^k \to \mathbb{N}^n$ by

$$\eta_f(\vec{y})_i = \begin{cases} f(i) & \text{if } i \in H, \\ y_s & \text{if } i = j_s, \end{cases}$$

for $i \in \{1,\dots,n\}$ and $\vec{y} \in \mathbb{N}^k$. Then $\Gamma \circ \eta_f : \mathbb{N}^k \to X$ and x is not a nontrivial k-Ramsey limit point of X, so by Lemma 4.4 the set $\mathcal{D}_f$ is downward cofinal where $\mathcal{D}_f = \{L \in [\mathbb{N}]^\omega :$ there is a neighborhood U of x such that $U \cap (\Gamma \circ \eta_f[L^k]) \subseteq \{x\}\}$. Since each $\mathcal{D}_f$ is clearly downward closed, we can pick $L \in [M]^\omega$ which is in each $\mathcal{D}_f$. For each $f \in F$ let $k = n - |\mathrm{domain}(f)|$ and pick a neighborhood U_f of x such that $U_f \cap (\Gamma \circ \eta_f[L^k]) \subseteq \{x\}$. Let $U' = \bigcap_{f\in F} U_f$.

Let $D = \{\vec{y} \in G^n : \Gamma(\vec{y}) \ne x\}$ and pick a neighborhood V of x such that for all $\vec{y} \in D$, $\Gamma(\vec{y}) \notin V$. Let $U = V \cap U'$. Now let $\vec{y} \in (G \cup L)^n \setminus L^n$ and assume that $\Gamma(\vec{y}) \ne x$. If $\vec{y} \in G^n$, then $\Gamma(\vec{y}) \notin V$. So assume that $\vec{y} \notin G^n$. Let $H = \{i : y_i \in G\}$. Define $f : H \to G$ by $f(i) = y_i$. Enumerate $\{1,\dots,n\} \setminus H$ in order as $j_1,\dots,j_k$. For $i \in \{1,\dots,k\}$, let $z_i = y_{j_i}$. Then $\vec{z} \in L^k$ so either $\Gamma \circ \eta_f(\vec{z}) = x$ or $\Gamma \circ \eta_f(\vec{z}) \notin U'$. Since $\eta_f(\vec{z}) = \vec{y}$, we have $\Gamma(\vec{y}) \notin U'$. ∎

4.6. LEMMA. *Let X be a Hausdorff space, let $n \in \mathbb{N}$, and let x be an element of X which is not a nontrivial n-Ramsey limit. If $\Gamma : \mathbb{N}^n \to X$ and G is a finite subset of $\mathbb{N}$, then for all $M \in [\mathbb{N}]^\omega$ there exist $L \in [M]^\omega$ and a neighborhood U of x with $U \cap \Gamma[(G \cup L)^n] \subseteq \{x\}$.*

Proof. If $G = \emptyset$, this is Lemma 4.4, so assume that $G \neq \emptyset$. If $n = 1$, pick a neighborhood V of x and $L \in [M]^\omega$ as guaranteed by Lemma 4.4 and pick a neighborhood U of x with $U \subseteq V$ such that $\Gamma(y) \notin U \setminus \{x\}$ for each $y \in G$.

Now assume that $n > 1$. Pick a neighborhood V of x and $N \in [M]^\omega$ as guaranteed by Lemma 4.4. By Lemma 2.9, x is not a nontrivial $(n-1)$-Ramsey limit point in X, so pick a neighborhood W of x and $L \in [N]^\omega$ as guaranteed by Lemma 4.5. Let $U = V \cap W$. ∎

The following theorem answers Question 1.2 for the case of arbitrary functions. It is a curiosity that, while the characterization given by Theorem 4.7 and its proof are simpler than the characterization and proof for the other two cases given in Theorem 4.9, we discovered those latter characterizations first. This is at least partly due to the fact that we were initially concerned with injective functions for our Ramsey-theoretic applications.

Our referee noted that "Theorems 3.4 and Theorem 4.7 show that the behavior of functions $f : [\mathbb{N}]^n \to X$ and $g : \mathbb{N}^n \to X$ is very different". The referee then suggested that we add statements (c) and (d) to Theorem 4.7, noting that "concerning discrete images, the behavior of the functions $\Gamma : \mathbb{N}^n \to X$ is the same as the behavior of functions $\Gamma : [\mathbb{N}]^n \to [X]^k$".

4.7. THEOREM. *Let X be a Hausdorff space and let $n \in \mathbb{N} \setminus \{1\}$. The following statements are equivalent*:

(a) *Whenever $\mathcal{G}$ is a finite collection of functions from $\mathbb{N}^n$ to X there exists $B \in [\mathbb{N}]^\omega$ such that $\bigcup_{\Gamma \in \mathcal{G}} \Gamma[B^n]$ is discrete.*
(b) *Whenever $\Gamma : \mathbb{N}^n \to X$ there exists $B \in [\mathbb{N}]^\omega$ such that $\Gamma[B^n]$ is discrete.*
(c) *Whenever $\mathcal{G}$ is a finite collection of functions from $[\mathbb{N}]^n$ to X there exists $B \in [\mathbb{N}]^\omega$ such that $\bigcup_{\psi \in \mathcal{G}} \psi[[B]^n]$ is discrete.*
(d) *Whenever ψ_1 and ψ_2 are functions from $[\mathbb{N}]^n$ to X there exists $B \in [\mathbb{N}]^\omega$ such that $\psi_1[[B]^n] \cup \psi_2[[B]^n]$ is discrete.*
(e) *There are no nontrivial n-Ramsey limit points in X.*

Proof. Clearly, (a) implies both (b) and (c), and (c) implies (d). A moment's thought shows that (b) implies (d): given ψ_1 and ψ_2 apply (b) to $\Gamma : \mathbb{N}^n \to X$ where $\Gamma(x_1, \ldots, x_n)$ is defined to be $\psi_1(\{x_1, \ldots, x_n\})$ if $x_1 < \cdots < x_n$, $\psi_2(\{x_1, \ldots, x_n\})$ if $x_1, \ldots, x_n$ are distinct but not in increasing order, and arbitrary otherwise.

(d)$\Rightarrow$(e). Suppose x is a nontrivial n-Ramsey limit point in X. By Lemma 2.9 pick $\psi_1 : [\mathbb{N}]^n \to X \setminus \{x\}$ such that $x = \mathcal{R}_n\text{-}\lim \psi$. Let $\psi_2 : [\mathbb{N}]^n \to X$ be the constant function with value x. For any $B \in [\mathbb{N}]^\omega$, x is a limit point of $\psi_1[[B]^n]$ by the choice of ψ_1 and $x \in \psi_2[[B]^n]$ implying that $\psi_1[[B]^n] \cup \psi_2[[B]^n]$ is not discrete.

(e)$\Rightarrow$(a). Let $C_0 = \mathbb{N}$, let $m \in \mathbb{N}$, and assume that we have chosen infinite $C_0 \supseteq C_1 \supseteq \cdots \supseteq C_{m-1}$ and distinct $a_1, \ldots, a_{m-1}$ such that for $\Gamma \in \mathcal{G}$ and $\vec{x} \in \{a_1, \ldots, a_{m-1}\}^n$ (if any) we have chosen a neighborhood $U^\Gamma_{\vec{x}}$ of $\Gamma(\vec{x})$ so that if $k = \max\{t : \text{some } x_i = a_t\}$, then $U^\Gamma_{\vec{x}} \cap \bigcup_{\Delta \in \mathcal{G}} \Delta[(\{a_1, \ldots, a_k\} \cup C_k)^n] \subseteq \{\Gamma(\vec{x})\}$. Pick $a_m \in C_{m-1} \setminus \{a_1, \ldots, a_{m-1}\}$. Let $T = \{a_1, \ldots, a_m\}^n \setminus \{a_1, \ldots, a_{m-1}\}^n$. By Lemma 4.6, for each $\Gamma \in \mathcal{G}$ and $\vec{x} \in T$ the set $\mathcal{D}^\Gamma_{\vec{x}}$ is downward cofinal, where $\mathcal{D}^\Gamma_{\vec{x}} = \{L \in [\mathbb{N}]^\omega : \text{there is a neighborhood } U \text{ of } \Gamma(\vec{x}) \text{ such that } U \cap \bigcup_{\Delta \in \mathcal{G}} \Delta[(\{a_1, \ldots, a_m\} \cup L)^n] \subseteq \{\Gamma(\vec{x})\}\}$. Since each $\mathcal{D}^\Gamma_{\vec{x}}$ is clearly downward closed, there is $C_m \in [C_{m-1}]^\omega$ which is in each $\mathcal{D}^\Gamma_{\vec{x}}$. For each $\Gamma \in \mathcal{G}$ and $\vec{x} \in T$ pick a neighborhood $U^\Gamma_{\vec{x}}$ of $\Gamma(\vec{x})$ such that $U^\Gamma_{\vec{x}} \cap \bigcup_{\Delta \in \mathcal{G}} \Delta[(\{a_1, \ldots, a_m\} \cup C_m)^n] \subseteq \{\Gamma(\vec{x})\}$.

The induction being complete, let $B = \{a_m : m \in \mathbb{N}\}$. Given $\vec{x} \in B^n$, let $m = \max\{t : \text{some } x_i = a_t\}$. Then for any $\vec{y} \in B^n$, $\vec{y} \in (\{a_1, \ldots, a_m\} \cup C_m)^n$ so $\Gamma(\vec{y}) \notin U^\Gamma_{\vec{x}} \setminus \{\Gamma(\vec{x})\}$. ∎

Note that the characterization of Theorem 4.7 is not valid in the case $n = 1$ because statement (a) holds in any Hausdorff space. We need one more lemma before we can address the other two parts of Question 1.2.

4.8. Lemma. *Let X be a Hausdorff space, let $K \in [\mathbb{N}]^\omega$, let $n \in \mathbb{N}$, and let $\Gamma : K^n \to X$. There exist $N \in [K]^\omega$ and a finite subset X_0 of X such that for all $x \in X \setminus X_0$ and all $M \in [N]^\omega$ there exist $L \in [M]^\omega$ and a neighborhood U of x with $U \cap \Gamma[L^n] = \emptyset$.*

Proof. Let $F = \{f : (\exists k \in \{1, \ldots, n\})(f : \{1, \ldots, n\} \xrightarrow{\text{onto}} \{1, \ldots, k\})\}$. Given $f \in F$, pick $k \in \{1, \ldots, n\}$ such that $f : \{1, \ldots, n\} \xrightarrow{\text{onto}} \{1, \ldots, k\}$ and let $\mathcal{D}_f = \{N \in [\mathbb{N}]^\omega : \text{either } \mathcal{R}_k(N)\text{-}\lim(\Gamma \circ \mu_f) \text{ exists or } \mathcal{R}_k(N')\text{-}\lim(\Gamma \circ \mu_f) \text{ does not exist for any } N' \in [N]^\omega\}$. Then each $\mathcal{D}_f$ is downward cofinal and downward closed, so pick $N \in [K]^\omega \cap \bigcap_{f \in F} \mathcal{D}_f$. Let $X_0 = \{x \in X : \text{there exist } f \in F \text{ and } k \in \{1, \ldots, n\} \text{ such that } x = \mathcal{R}_k(N)\text{-}\lim(\Gamma \circ \mu_f)\}$.

Assume that $x \in X \setminus X_0$ and $M \in [N]^\omega$. For each $f \in F$, let $\mathcal{E}_f = \{L \in [N]^\omega : \text{there is a neighborhood } U \text{ of } x \text{ with } U \cap (\Gamma \circ \mu_f[[L]^k]) = \emptyset\}$, where $f : \{1, \ldots, n\} \xrightarrow{\text{onto}} \{1, \ldots, k\}$. By the choice of N and X_0, each $\mathcal{E}_f$ is downward cofinal, and is clearly downward closed. Pick $L \in \bigcap_{f \in F} \mathcal{E}_f$ and a neighborhood U of x as guaranteed by the fact that $L \in \mathcal{E}_f$ for each $f \in F$. ∎

4.9. Theorem. *Let $n \in \mathbb{N}$ and let X be a Hausdorff topological space. The following statements are equivalent*:

(a) *Whenever* $\mathcal{G}$ *is a finite collection of finite-to-one functions from* $\mathbb{N}^{n+1}$ *to* X *there exists* $B \in [\mathbb{N}]^\omega$ *such that* $\bigcup_{\Gamma \in \mathcal{G}} \Gamma[B^{n+1}]$ *is discrete.*

(b) *Whenever* Γ *is an injective function from* $\mathbb{N}^{n+1}$ *to* X *there exists* $B \in [\mathbb{N}]^\omega$ *such that* $\Gamma[B^{n+1}]$ *is discrete.*

(c) *The set of nontrivial* n*-Ramsey limit points in* X *is finite.*

Proof. That (a) implies (b) is trivial.

(b)⇒(c). Let $Y = \{y \in X : y \text{ is a nontrivial } n\text{-Ramsey limit in } X\}$ and suppose that Y is infinite. Pick an infinite discrete subset M of Y and let $\langle y_t \rangle_{t=1}^\infty$ be an injective sequence in M such that $M \setminus \{y_t : t \in \mathbb{N}\}$ is infinite.

For each $t \in \mathbb{N}$, by Lemma 2.9 pick $\psi_t : [\mathbb{N}]^n \xrightarrow{\text{1-1}} X \setminus \{y_t\}$ such that $y_t = \mathcal{R}_n\text{-}\lim \psi_t$ and pick a neighborhood U_t of y_t such that $U_t \cap M = \{y_t\}$. Pick by Lemma 2.3 some $B_t \in [\mathbb{N}]^\omega$ such that $\psi_t[[B_t]^n] \subseteq U_t \setminus \{y_t\}$ and $\min B_t > t$. We shall define for each $t \in \mathbb{N}$ an increasing function $f_t : \{s \in \mathbb{N} : s > t\} \to B_t$ such that whenever $t < l_1 < \cdots < l_n$, $s < m_1 < \cdots < m_n$, and $s \neq t$, one has $\psi_t(\{f_t(l_1), \ldots, f_t(l_n)\}) \neq \psi_s(\{f_s(m_1), \ldots, f_s(m_n)\})$.

We remark that if X is regular, one may assume that $U_t \cap U_s = \emptyset$ when $s \neq t$, so that f_t can simply be taken to be any increasing function from $\{s \in \mathbb{N} : s > t\}$ to B_t. The following construction need only be used if X is not regular.

Let $f_1(2) = \min B_1$. Inductively, let $k \in \mathbb{N} \setminus \{1\}$ and assume that we have defined $f_t(k)$ for all $t < k$ such that whenever $t < l_1 < \cdots < l_n \le k$, $s < m_1 < \cdots < m_n \le k$, and $s \neq t$, one has that $\psi_t(\{f_t(l_1), \ldots, f_t(l_n)\}) \neq \psi_s(\{f_s(m_1), \ldots, f_s(m_n)\})$. We shall define $f_t(k+1)$ for $t \le k$. If $t+n > k+1$, let $f_t(k+1)$ be any member of B_t such that $f_t(k+1) > f_t(k)$ if $k > t$. We define $f_t(k+1)$ for $t \le k+1-n$ (if any) inductively. So let $t \in \{1, \ldots, k+1-n\}$ and assume that $f_s(k+1)$ has been defined for $s \in \{1, \ldots, t-1\}$ (if any). Let

$$F_t = \{\psi_s(\{f_s(m_1), \ldots, f_s(m_n)\}) : t < s < m_1 < \cdots < m_n \le k, \\ \text{or } t > s \text{ and } s < m_1 < \cdots < m_n \le k+1\}.$$

Then F_t is finite, so $\{x \in B_t : \psi_t(\{f_t(l_1), \ldots, f_t(l_{n-1}), x\}) \in F_t\}$ is finite for each choice of $l_1, \ldots, l_{n-1}$ with $t < l_1 < \cdots < l_{n-1} \le k$, hence we may pick $f_t(k+1) \in B_t$ such that $f_t(k+1) > f_t(k)$ if $k > t$ and

$$\{\psi_t(\{f_t(l_1), \ldots, f_t(l_{n-1}), f_t(k+1)\}) : t < l_1 < \cdots < l_{n-1} < k+1\} \cap F_t = \emptyset.$$

Having defined f_t for each t we now define $\Gamma : \mathbb{N}^{n+1} \xrightarrow{\text{1-1}} X$. First, for each t, let $\Gamma(t, t, \ldots, t) = y_t$. Next, given $t < l_1 < \cdots < l_n$, $\Gamma(t, l_1, \ldots, l_n) = \psi_t(\{f_t(l_1), \ldots, f_t(l_n)\})$. Extend Γ to the rest of $\mathbb{N}^{n+1}$ so that

$$\mathbb{N}^{n+1} \setminus (\{(t, t, \ldots, t) : t \in \mathbb{N}\} \cup \{(t, l_1, \ldots, l_n) : t < l_1 < \cdots < l_n\})$$

is mapped injectively into $M \setminus \{y_t : t \in \mathbb{N}\}$. (Notice that all of the values previously defined lie in $\bigcup_{t=1}^\infty U_t$.)

Now pick $C \in [\mathbb{N}]^\omega$ such that $\Gamma[C^{n+1}]$ is discrete. Let $t = \min C$. Since $y_t \in \Gamma[C^{n+1}]$, pick a neighborhood V of y_t such that $V \cap \Gamma[C^{n+1}] = \{y_t\}$. Now $f_t[C] \in [\mathbb{N}]^\omega$, so pick by Lemma 2.3 some $D \in [f_t[C]]^\omega$ such that $\psi_t[[D]^n] \subseteq V$. Pick $l_1 < \cdots < l_n$ in ${f_t}^{-1}[D]$ with $l_1 > t$. Then we have $\{f_t(l_1), \ldots, f_t(l_n)\} \in [D]^n$ so $\Gamma(t, l_1, \ldots, l_n) = \psi_t(\{f_t(l_1), \ldots, f_t(l_n)\}) \in V \setminus \{y_t\}$ while $(t, l_1, \ldots, l_n) \in C^{n+1}$, a contradiction.

(c)$\Rightarrow$(a). We will choose distinct $l_1, l_2, \ldots$ in $\mathbb{N}$ and infinite subsets $L_0 \supseteq L_1 \supseteq \cdots$ of $\mathbb{N}$ inductively such that for each $m \in \mathbb{N}$, $l_m \in L_{m-1}$, $l_m < \min L_m$, and whenever $\Gamma \in \mathcal{G}$ and $\vec{x} \in \{l_0, l_1, \ldots, l_m\}^{n+1}$ there is a neighborhood $U^\Gamma_{\vec{x}}$ of $\Gamma(\vec{x})$ such that $U^\Gamma_{\vec{x}} \cap \bigcup_{\Delta \in \mathcal{G}} \Delta[(\{l_0, l_1, \ldots, l_m\} \cup L_m)^{n+1}] = \{\Gamma(\vec{x})\}$. Fix $N \in [\mathbb{N}]^\omega$ and a finite $X_0 \subseteq X$ such that whenever $x \in X \setminus X_0$ and $M \in [N]^\omega$ there exist $L \in [M]^\omega$ and a neighborhood U of x with $U \cap \bigcup_{\Delta \in \mathcal{G}} \Delta[L^{n+1}] = \emptyset$. We may do this by Lemma 4.8. Since Γ is finite-to-one, we may also assume that for all $\vec{x} \in N^{n+1}$, $\Gamma(\vec{x})$ is not a nontrivial n-Ramsey limit point in X and $\Gamma(x) \notin X_0$.

Let $L_0 = N$, let $m \in \mathbb{N}$, and assume that l_t and L_t have been chosen for $t \in \{0, 1, \ldots, m-1\}$. Let l_m be the least element of L_{m-1}.

For the moment, fix $\Gamma \in \mathcal{G}$ and $\vec{x} \in \{l_0, l_1, \ldots, l_m\}^{n+1}$ in which l_m appears. Let $\mathcal{D}^\Gamma_{\vec{x}}$ consist of all $L \in [\mathbb{N}]^\omega$ such that for some neighborhood U of $\Gamma(\vec{x})$,

$$U \cap \bigcup_{\Delta \in \mathcal{G}} \Delta[(\{l_0, l_1, \ldots, l_m\} \cup L)^{n+1} \setminus L^{n+1}] = \{\Gamma(\vec{x})\}.$$

By Lemma 4.5, $\mathcal{D}^\Gamma_{\vec{x}}$ is downward cofinal in $[\mathbb{N}]^\omega$. Clearly, $\mathcal{D}^\Gamma_{\vec{x}}$ is also downward closed. Let $\mathcal{E}^\Gamma_{\vec{x}} = \{L \in [N]^\omega :$ for some neighborhood U of $\Gamma(\vec{x})$, $U \cap \bigcup_{\Delta \in \mathcal{G}} \Delta[L^{n+1}] = \emptyset\}$. By the choice of N, $\mathcal{E}^\Gamma_{\vec{x}}$ is downward cofinal in $[N]^\omega$. Clearly, $\mathcal{E}^\Gamma_{\vec{x}}$ is also downward closed.

Choose $L_m \in [L_{m-1} \setminus \{l_m\}]^\omega$ in the intersection of all $\mathcal{D}^\Gamma_{\vec{x}}$ and $\mathcal{E}^\Gamma_{\vec{x}}$. By the choice of L_m, for each $\Gamma \in \mathcal{G}$ and $\vec{x} \in \{l_0, l_1, \ldots, l_m\}^{n+1}$ an appropriate $U^\Gamma_{\vec{x}}$ can be chosen.

The induction being complete, let $B = \{l_m : m \in \mathbb{N}\}$.

To see that $\bigcup_{\Gamma \in \mathcal{G}} \Gamma[B^{n+1}]$ is discrete, let $\Gamma \in \mathcal{G}$ and $\vec{x} \in B^{n+1}$ and let

$$m = \max\{j : \text{some } x_i = l_j\}.$$

Since $B \subseteq \{l_1, \ldots, l_m\} \cup L_m$, we have $U^\Gamma_{\vec{x}} \cap \bigcup_{\Delta \in \mathcal{G}} \Delta[B^{n+1}] = \{\Gamma(\vec{x})\}$. ■

Notice that a consequence of Theorems 4.7 and 4.9 is that for any $n \in \mathbb{N} \setminus \{1\}$, if X is a Hausdorff space satisfying Theorem 4.7(b), then it also satisfies Theorem 4.9(b). On the other hand, if X is the one-point compactification of $\mathbb{N}$, then for any $n \in \mathbb{N}$, there is exactly one nontrivial n-Ramsey limit point of X, so statements (a) and (b) of Theorem 4.9 hold while (if $n > 1$) statement (b) of Theorem 4.7 does not hold.

Notice also that using Lemma 2.10 one easily sees that the conditions of Theorems 4.7 and 4.9 are strictly increasing in strength as n increases.

5. Spaces good for all n. Recall that our original motivation for studying Question 1.2 was our desire to establish that, at least in case $n = 2$ and Γ is injective, the Stone–Čech compactification of a discrete space satisfies the conclusion. In this section, we investigate the following class $\mathbf{D}$, showing in particular that if D is a discrete space, then $\beta D \in \mathbf{D}$.

5.1. DEFINITION. $\mathbf{D}$ is the class of all Hausdorff spaces with the property that whenever $n \in \mathbb{N}$ and $\Gamma : \mathbb{N}^n \to X$, there is some $B \in [\mathbb{N}]^\omega$ such that $\Gamma[B^n]$ is discrete.

Equivalently, by Theorem 4.7, $\mathbf{D}$ is the class of all Hausdorff spaces that have no nontrivial n-Ramsey limit points for all $n \in \mathbb{N}$.

5.2. THEOREM. *The class* $\mathbf{D}$ *is hereditary and is finitely productive.*

Proof. Trivially, $\mathbf{D}$ is hereditary. To see that $\mathbf{D}$ is finitely productive, let X and Y be members of $\mathbf{D}$. Let $n \in \mathbb{N}$ and let $\Gamma : \mathbb{N}^n \to X \times Y$. We can choose $A \in [\mathbb{N}]^\omega$ such that $\pi_1 \circ \Gamma[A^n]$ is discrete and then choose $B \in [A]^\omega$ such that $\pi_2 \circ \Gamma[B^n]$ is discrete. Then $\Gamma[B^n]$ is discrete. ∎

Notice that $\mathbf{D}$ is not infinitely productive. Indeed, an infinite product of spaces, each of which has more than one point, can never be in $\mathbf{D}$ because every point in a product of this kind is the limit of an injective convergent sequence. We also observe that if we modify the definition of $\mathbf{D}$ to require that whenever $n \in \mathbb{N}$ and $\Gamma : \mathbb{N}^n \xrightarrow{\text{1-1}} X$, there is some $B \in [\mathbb{N}]^\omega$ such that $\Gamma[B^n]$ is discrete, then the resulting class is not finitely productive. Indeed, if X and Y are Hausdorff spaces, $n \in \mathbb{N}$, and x is a nontrivial n-Ramsey limit in X, then for each $y \in Y$, (x, y) is a nontrivial n-Ramsey limit in $X \times Y$.

The class of spaces in the following theorem includes the class of F-spaces. (An *F-space* is a completely regular Hausdorff space with the property that disjoint cozero sets are completely separated. See [4] for a wealth of information about such spaces.) This follows from the well known fact, which is an immediate consequence of [4, Exercise 14N.5], that, if A and B are countable subsets of an F-space for which $\overline{A} \cap B = A \cap \overline{B} = \emptyset$, then $\overline{A} \cap \overline{B} = \emptyset$. So this class includes the Stone–Čech compactification of any F-space. In particular, it includes the Stone–Čech compactification of any discrete space. It also includes all spaces of the form $\beta S \setminus S$, where S is locally compact and σ-compact. That this class properly includes the class of F-spaces is shown in Theorem 5.4 below.

5.3. THEOREM. *Let X be a Hausdorff space with the property that whenever C and D are disjoint subsets of some countable discrete subset of X, one has* $\mathrm{cl}(C) \cap \mathrm{cl}(D) = \emptyset$. *Then X is a member of the class* $\mathbf{D}$.

Proof. Let $n \in \mathbb{N}$ and suppose we have a nontrivial n-Ramsey limit point x in X. Pick by Lemma 2.9 some $\psi : [\mathbb{N}]^n \xrightarrow{\text{1-1}} X \setminus \{x\}$ such that $x = \mathcal{R}_n\text{-}\lim \psi$. Choose by Theorem 3.4 some $B \in [\mathbb{N}]^\omega$ such that $\psi[[B]^n]$ is discrete. Then whenever $C \in [B]^\omega$ and U is a neighborhood of x, we have $[C]^n \cap \psi^{-1}[U] \neq \emptyset$. Partition B into disjoint infinite sets C and D. Then $x \in \mathrm{cl}(\psi[[C]^n]) \cap \mathrm{cl}(\psi[[D]^n])$, a contradiction. ∎

We remark that the hypothesis of Theorem 5.3 is not necessary for X to be in $\mathbf{D}$. For example, $\beta\mathbb{N} \times \beta\mathbb{N}$ is in $\mathbf{D}$ by Theorem 5.2, but it does not satisfy this hypothesis. (Pick any $p \in \beta\mathbb{N} \setminus \mathbb{N}$, let $C = \{p\} \times \mathbb{N}$, and let $D = \mathbb{N} \times \{p\}$. Then $C \cup D$ is discrete and $(p, p) \in \mathrm{cl}(C) \cap \mathrm{cl}(D)$.)

We thank Alan Dow for bringing the argument in the proof of the following theorem to our attention. In its essential details it first appeared in the concluding remarks of [9], where it is attributed to C. F. Mills.

5.4. Theorem. *There is a countable space X which satisfies the hypothesis of Theorem* 5.3 *and is not an F-space.*

Proof. Pick disjoint infinite subsets A and B of ω. By [9, Theorem 0.1] there exist a countable subspace Y of $\beta A \setminus A$, a countable subspace Z of $\beta B \setminus B$, a point $y \in \beta A \setminus Y$, and a point $z \in \beta B \setminus Z$ such that $y \in \mathrm{cl}(Y)$ and $z \in \mathrm{cl}(Z)$, but y is not in the closure of any discrete countable subset of $(\beta\omega \setminus \omega) \setminus \{y\}$ and z is not in the closure of any discrete countable subset of $(\beta\omega \setminus \omega) \setminus \{z\}$. Let X denote the quotient space of $Y \cup Z \cup \{y, z\}$ obtained by identifying y and z. Then X satisfies the hypothesis of Theorem 5.3. However, since $Y \cap \mathrm{cl}_X(Y') = \mathrm{cl}_X(Y) \cap Y' = \emptyset$ but $\mathrm{cl}_X(Y) \cap \mathrm{cl}_X(Y') \neq \emptyset$, X is not an F-space. ∎

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