FUNDAMENTA
MATHEMATICAE
187 (2005)

On the classification of inverse limits of tent maps

by

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Abstract. Let f_s and f_t be tent maps on the unit interval. In this paper we give a new proof of the fact that if the critical points of f_s and f_t are periodic and the inverse limit spaces (I, f_s) and (I, f_t) are homeomorphic, then $s = t$. This theorem was first proved by Kailhofer. The new proof in this paper simplifies the proof of Kailhofer. Using the techniques of the paper we are also able to identify certain isotopies between homeomorphisms on the inverse limit space.

1. Introduction. Given a continuous map f of a one-dimensional space to itself, one may form an inverse limit space by using f repeatedly as the bonding map. Spaces formed in this way commonly appear as attractors in dynamical systems [1, 2, 4, 8, 12, 21]. This motivates the study of such inverse systems. It is natural to try to determine when two such inverse limits are homeomorphic. In the case of solenoids, there is a well known characterization [1, 15]. Consider the inverse limit space for the inverse system where the inverse system spaces are each the interval and the bonding maps are each some tent map

$$f_s(x) = \min\{sx, s(1-x)\}$$

for $x \in [0,1]$ and $s \in [1,2]$. This inverse limit space has also been studied extensively. Any unimodal map without wandering intervals, restrictive intervals, or periodic attractors is conjugate to a tent map (see e.g. [16]). As conjugate maps have homeomorphic inverse limit spaces, the family of tent maps is more inclusive than it seems at first glance. Given parameters $s \neq t$ it is unknown whether the corresponding inverse limit spaces (I, f_s) and (I, f_t) could be homeomorphic where $I = [0,1]$. However, partial results exist [3, 6, 9, 11, 17, 20].

2000 *Mathematics Subject Classification*: Primary 54F15; Secondary 37E05, 37B45.
Key words and phrases: continuum, composant, inverse limit, tent family.

In this paper we work with tent maps for which $s \in [\sqrt{2}, 2]$ and the turning point is periodic, i.e. letting c denote the turning point, there is some positive integer n such that $f_s^n(c) = c$. In [13] and [14] Kailhofer proved the following result.

Theorem (Kailhofer). *Suppose that $s, t \in [\sqrt{2}, 2]$. Assume that the turning point is periodic for both f_s and f_t. Then X_s is homeomorphic to X_t if and only if $s = t$.*

In this theorem, X_s and X_t are the cores of (I, f_s) and (I, f_t), respectively. These will be defined in the next section. The theorem implies that if (I, f_s) and (I, f_t) are homeomorphic, then $s = t$ under the given assumptions. Related results appear in [3], [9], and [19].

One can extend the same result to the whole interval $s \in (1, 2]$ in the following way. For $s \in (1, \sqrt{2}]$, there are two intervals J_1 and J_2 in the core I_s of f_s with pairwise disjoint interiors such that $f_s^2|_{J_1}$ and $f_s^2|_{J_2}$ are topologically conjugate to $f_{s^2}|_{I_{s^2}}$. It follows that for $s \in (1, \sqrt{2}]$, (I_s, f_s) is determined by (I_{s^2}, f_{s^2}). Therefore, it is enough to consider tent maps with slopes in $(\sqrt{2}, 2]$.

In the present paper we give a simplified proof of Kailhofer's theorem. The proof in this paper uses some of the results in [13] together with some new results. One of the results proved in this paper is of particular interest in itself.

Isotopy Theorem. *Let $s \in (\sqrt{2}, 2)$. Let $I_s = [f_s^2(c), f_s(c)]$ be the core of f_s. Let $X_s = (I_s, f_s)$ be the inverse limit of the core. Let h be any homeomorphism of X_s. Then there is a positive integer n and an integer k such that h^n is isotopic to σ^k where σ is the shift map on X_s.*

A weakened version of this theorem will be proved in the early part of the paper. In the simplified proof of Kailhofer's theorem, we only need that a certain homeomorphism h permutes the composants of X_s in the same way as σ^k for some integer k. If h and σ^k were isotopic, then it would easily follow that the composants of X_s are permuted by them in the same way. It is only at the end of the paper that we actually show that h and σ^k are in fact isotopic.

2. Preliminaries. In this section we will recall some general definitions and known background results needed to state and prove the main results of this paper.

Let X be a topological space. X is an *arc* if there exists a homeomorphism from X onto $[0, 1]$. The *components* of X are the maximal connected subspaces of X. We define $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}_+ = \{1, 2, \ldots\}$, $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = [0, \infty)$.

A *continuum* is a compact connected metric space. Let X be a continuum. The *composant* of $x \in X$ is the union of all proper subcontinua of X that contain x. An *end continuum* in X is a subcontinuum T of X such that whenever $T \subset H$, $T \subset J$ for continua $H, J \subset X$, then either $H \subset J$ or $J \subset H$. A point $x \in X$ is an *endpoint* of X if $\{x\}$ is an end continuum in X. Note that endpoints are topological invariants.

Let $\{X_i, d_i\}_{i=0}^{\infty}$ be a collection of compact metric spaces with d_i bounded by 1, and such that for each i, $f_i : X_{i+1} \to X_i$ is a continuous map. The *inverse limit space* is

$$\{X_i, f_i\} = \Big\{\overline{x} = (x_0, x_1, \dots) \,\Big|\, \overline{x} \in \prod_{i=0}^{\infty} X_i,\ f_i(x_{i+1}) = x_i,\ i \in \mathbb{N}\Big\},$$

and has metric d given by

$$d(\overline{x}, \overline{y}) = \sum_{i=0}^{\infty} \frac{d_i(x_i, y_i)}{2^i}.$$

For each i, π_i denotes the projection map from $\prod_{i=0}^{\infty} X_i$ into X_i. An inverse limit space $\{X_i, f_i\}_{i=0}^{\infty}$ is a continuum if X_i is a continuum for every i [18, Theorem 2.4]. If $X_i = X$ and $f_i = f$ for all i, the inverse limit space is denoted (X, f), and the map $\sigma : (X, f) \to (X, f)$ defined by $\sigma(x_0, x_1, \dots) = (f(x_0), x_0, x_1, \dots)$ is known as the *shift homeomorphism* or as the *induced homeomorphism*.

A continuous map $f : [a, b] \to [a, b]$ is called *unimodal* if there exists a unique *turning* or *critical point*, c, such that $f|_{[a,c]}$ is increasing and $f|_{[c,b]}$ is decreasing. For each $x \in [a, b]$, the *forward itinerary* of x is $I(x) = b_0 b_1 b_2 \cdots$ where $b_i = R$ if $f^i(x) > c$, $b_i = L$ if $f^i(x) < c$, and $b_i = C$ if $f^i(x) = c$, with the convention that the itinerary stops after the first C. The itinerary of $f(c)$ is known as the *kneading sequence* of f and it is denoted $K(f)$.

The set of itineraries is given the *parity-lexicographical ordering* in the following way. Set $L < C < R$. Let $W = w_0 w_1 \cdots$ and $V = v_0 v_1 \cdots$ be two distinct itineraries and let k be the first index where the itineraries differ. If $k = 0$, then $W < V$ if and only if $w_0 < v_0$. If $k \geq 1$, and $w_0 w_1 \cdots w_{k-1} = v_0 v_1 \cdots v_{k-1}$ has an even number of R's, that is, has *even parity*, then $W < V$ if and only if $w_k < v_k$; if $w_0 w_1 \cdots w_{k-1} = v_0 v_1 \cdots v_{k-1}$ has an odd number of R's, that is, has *odd parity*, then $W < V$ if and only if $v_k < w_k$. It is known that the map $x \mapsto I(x)$ is monotone, that is, if $x < y$, then $I(x) < I(y)$ (cf. [10]).

The *modified forward itinerary* of $f(c)$, denoted $I'(f(c))$, is defined as follows. If $K(f) = a_0 a_1 \cdots$ is infinite, let $I'(f(c)) = K(f)$. If $K(f) = a_0 a_1 \cdots a_{n_0 - 2} C$, then $I'(f(c)) = (a_0 a_1 \cdots a'_{n_0-1})^{\infty}$, where a'_{n_0-1} replaces the terminal C in the kneading sequence and $a_0 a_1 \cdots a'_{n_0-1} < K(f)$ in the parity-lexicographical ordering.

DEFINITION 2.1. Let $f : I \to I$ be a unimodal map. Let $\overline{x} = (x_0, x_1, \cdots)$ be a point in the inverse limit space (I, f) of f. The *backward itinerary* of $\overline{x}$, denoted $B(\overline{x}) = b_0 b_1 \cdots$, is a sequence of R's and L's such that

(1) $b_i = R$ if $x_i \geq c$ and $b_i = L$ if $x_i \leq c$,
(2) if $x_i = c$ for some $i > 0$, then $b_0 b_1 \cdots b_{i-1} = a_{i-1} a_{i-2} \cdots a_1 a_0$, where $I'(f(c)) = a_0 a_1 \cdots$.

Define $B_f = \{B(\overline{x}) \mid \overline{x} \in (I, f)\}$.

REMARK 2.2. Suppose that c is periodic with period n_0. Let $\overline{x} \in (I, f)$. If $x_i \neq c$ for all $i \in \mathbb{N}$, or if $x_i = c$ for infinitely many $i \in \mathbb{N}$, then $\overline{x}$ has exactly one backward itinerary. If $x_i = c$ for finitely many $i \in \mathbb{N}$, then $\overline{x}$ has two backward itineraries that differ at only one coordinate, $\max\{i \in \mathbb{N} \mid x_i = c\}$.

Consider the one-parameter family of tent maps $f_s : I \to I$, $f_s(x) = \min\{sx, s(1-x)\}$, $x \in I$ and $s \in [\sqrt{2}, 2]$. The tent map f_s is unimodal for all $s \in [\sqrt{2}, 2]$. From now on, unless otherwise specified, consider $s \in (\sqrt{2}, 2]$ fixed such that the critical point c of f_s has period n_0. Write

$$c_i = f_s^i(c) \quad \text{and} \quad \overline{c}_i = (c_i, c_{i-1}, \ldots, c_1, c, c_{n_0-1}, \ldots, c_{i+1})^\infty$$

for $i = 0, 1, \ldots, n_0 - 1$. Set $I_L = [c_2, c], I_R = [c, c_1]$ and $I_s = [c_2, c_1]$. Then I_s is invariant under f_s and f_s is *locally eventually onto* on I_s, that is, for every nondegenerate interval $J \subset I_s$ there exists an $n > 0$ such that $f_s^n(J) = I_s$. The interval I_s is known as the *core* of f_s. The inverse limit space of (I, f_s) is equal to the union of $X_s = (I_s, f_s)$ with an open ray having X_s as its limit set. To denote the nth coordinate in the inverse system we use I_n instead of $(I_s)_n$. We know that X_s is indecomposable. Under the assumption that c is periodic every proper subcontinuum of X_s is an arc. Thus, every composant is a union of arcs.

The *composant* $C_{\overline{x}}$ of $\overline{x} \in X_s$ is the set of all points in X_s with backward itineraries eventually identical to $B(\overline{x})$, that is, $\overline{y} \in C_{\overline{x}}$ if and only if the backward itineraries of $\overline{x}$ and $\overline{y}$ differ in at most finitely many coordinates (Lemma 4.1).

3. Definitions and results from Kailhofer's paper. In this section we give several definitions introduced by Kailhofer and some of the results from her paper [13].

DEFINITION 3.1. Let $w = w_0 w_1 \cdots \in B_{f_s}$. Define

$$\mathcal{A}_w = \{\overline{x} \in X_s \mid \pi_i(\overline{x}) \in I_{w_i} \text{ for all } i \in \mathbb{N}\},$$

the set of points in X_s with backward itinerary w. Define

$$\mathcal{A}_w^n = \sigma^n(\mathcal{A}_w).$$

Note that $\mathcal{A}_w = \{\overline{x} \in X_s \mid B(\overline{x}) = w\}$.

REMARK 3.2. Each $\mathcal{A}_w$ is a nondegenerate arc contained in a single composant.

LEMMA 3.3 ([13, Lemma 4]). *Let $w \in B_{f_s}$. There exist $0 \leq i \neq j < n_0$ such that $\pi_0|_{\mathcal{A}_w}$ is a homeomorphism onto $[c_i, c_j]$.*

The proof of this lemma shows that for an interval $J \subset [c_2, c_1]$, $f_s(g_s(J)) \neq J$ if and only if c_3 is in the interior of J, where g_s is the inverse of the map f_s restricted on $[c_2, c]$, but for $n \geq n_0$, $f_s(g_s^n(J)) = g_s^{n-1}(J)$.

REMARK 3.4. Note that if $\overline{x} \in X_s$ is an endpoint of $\mathcal{A}_w$ and $\overline{x} \neq \overline{c}_i$ for any $i = 0, 1, \ldots, n_0 - 1$, then $\overline{x}$ has two backward itineraries.

The following lemma is well known and appears in several publications. Barge and Martin in [5] describe the basic construction of endpoints in (X, f).

LEMMA 3.5 ([13, Lemma 8]). *The endpoints of X_s are $\overline{c}, \overline{c}_1, \ldots, \overline{c}_{n_0-1}$.*

REMARK 3.6. Each composant C in X_s has the property that there is a continuous bijection either from $\mathbb{R}_+$ to C or from $\mathbb{R}$ to C.

DEFINITION 3.7. Let $p \in \mathbb{N}$ and $0 \leq j < n_0$. Define

$$\Phi_{p,j} = \{\overline{x} \in C_{\overline{c}} \mid \pi_{pn_0}(\overline{x}) = c_j\}, \qquad \Phi_p = \bigcup_{j=0}^{n_0-1} \Phi_{p,j}.$$

The elements of Φ_p are called *p-special points.*

DEFINITION 3.8. Let $n, m \in \mathbb{N}$. Define $E_n = \pi_n(\Phi_0)$ and

$$P_{n,m} = \left\{ z \in I_n \;\middle|\; \exists x, y \in E_n, \exists k \in \{0, 1, \ldots, 2^m\} \text{ such that } (x, y) \cap E_n = \emptyset \text{ and } z = \frac{kx + (2^m - k)y}{2^m} \right\}.$$

We see that E_n partitions I_n into finitely many intervals and $P_{n,m}$ refines that partition by dividing each interval into 2^m subintervals.

DEFINITION 3.9. Let $n, m \in \mathbb{N}$ and let $x \in P_{n,m}$. If $x \neq c_2$, set $y = \max\{w \in P_{n,m} \mid x > w\}$. If $x \neq c_1$, set $z = \min\{w \in P_{n,m} \mid x < w\}$. Define

$$\mathrm{l}_{n,m}^x = \begin{cases} (y, z) & \text{if } x \in (c_2, c_1), \\ [x, z) & \text{if } x = c_2, \\ (y, z] & \text{if } x = c_1. \end{cases}$$

Let

$$L_{n,m} = \{\mathrm{l}_{n,m}^x \mid x \in P_{n,m}\}, \qquad \mathcal{L}_{n,m} = \{l_{n,m}^x \mid l_{n,m}^x = \pi_n^{-1}(\mathrm{l}_{n,m}^x),\, x \in P_{n,m}\}.$$

Let $U = \{U_i\}_{i=1}^n$ be an open cover of a topological space X. Recall that the set U is a *chaining* of the space X if $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$.

Let $U = \{U_i\}_{i=1}^n$ and let $V = \{V_j\}_{j=1}^m$ be chainings of a topological space X. We say that the chaining U *refines* the chaining V, in symbols $U \prec V$, if for every $1 \le i \le n$, there is $1 \le j \le m$ such that $U_i \subset V_j$.

LEMMA 3.10 ([13, Lemma 16]). *Fix $n, m, i, j \in \mathbb{N}$. Then*

(1) *$L_{n,m}$ is a chaining of I_n.*
(2) *$\mathcal{L}_{n,m}$ is a chaining of X_s.*
(3) *$\mathcal{L}_{n,m} \prec \mathcal{L}_{i,j}$ if $n \ge i, m \ge j$.*
(4) *If $\overline{x} \in \Phi_0$, then there is a unique $l \in \mathcal{L}_{n,m}$ such that $\overline{x} \in l$.*
(5) *$\mathrm{mesh}(\mathcal{L}_{n,m}) \to 0$ as $n \to \infty$ and $m \to \infty$.*

DEFINITION 3.11. For each $p \in \mathbb{N}$, define

$$W_p = \{\overline{x} \in C_{\overline{c}} \mid \exists \overline{x} \in \mathcal{A}_v^{pn_0} \cap \mathcal{A}_w^{pn_0},\ v \ne w \in B_f\} \cup \{\overline{c}\}.$$

If $\overline{x} \in W_p$, then $\overline{x}$ is called a *p-wrapping point*. There is a natural order on the set of all p-wrapping points with $\overline{x} < \overline{y}$ if $h^{-1}(\overline{x}) < h^{-1}(\overline{y})$ for any continuous bijection $h : \mathbb{R}_+ \to C_{\overline{c}}$.

LEMMA 3.12. *Fix $p \in \mathbb{N}$. Then*

(1) *$W_p = \{\overline{x} \in C_{\overline{c}} \mid \exists n \ge pn_0$ such that $\pi_n(\overline{x}) = c\}$.*
(2) *$W_{p+1} \subset \varPhi_{p+1} \subset W_p$.*
(3) *$\sigma^{n_0}(W_p) = W_{p+1}$.*

EXAMPLE 3.1. Let T be the tent map with kneading sequence $RLRRC$. Figure 1 shows the p-wrapping points and the $(p+1)$-wrapping points of $C_{\overline{c}}$.

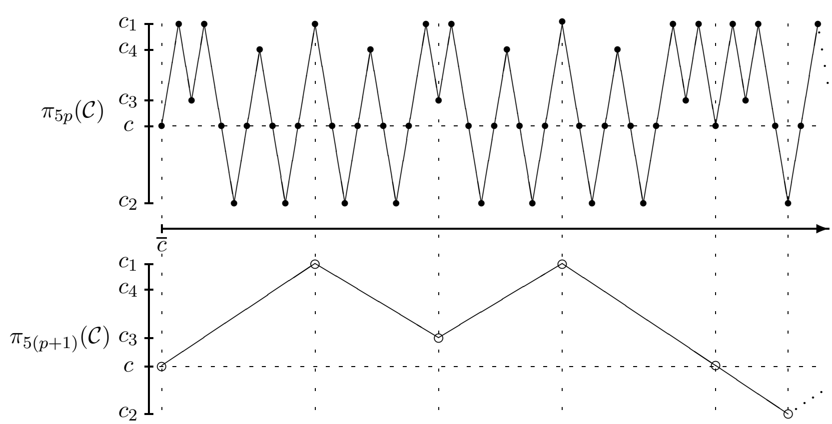

Fig. 1. The projections of the p-wrapping points $\bullet$ and the $(p+1)$-wrapping points $\circ$

PROPOSITION 3.13 ([13, Proposition 25]). *Fix $p, m, k \in \mathbb{N}$, $0 \le k < n_0$. If D is a component of $C_{\overline{c}} \cap l_{n,m}^{c_k}$, then the closure of D is an arc and D contains exactly one element of $\varPhi_{p,k}$.*

Definition 3.14. Let $\overline{x} \in W_p \backslash \{\overline{c}\}$. Let $k \in \mathbb{N}$ be such that $pn_0 + k = \max\{n \mid \pi_n(\overline{x}) = c\}$. Define the *p-level* of $\overline{x}$ by $L_p(\overline{x}) = k$. Set $L_p(\overline{c}) = \infty$.

The set $\{L_p(\overline{x}) \mid \overline{x} \in W_p \backslash \{\overline{c}\}\}$ is unbounded. Note that $\overline{x} \in W_{p+1}$ if and only if $L_p(\overline{x}) \geq n_0$.

Example 3.2. Let T be the tent map with kneading sequence $RLRRC$. Figure 2 shows the $5p$-projections of the p-wrapping points of $C_{\overline{c}}$, marked by $\bullet$, and the p-levels of the corresponding p-wrapping points of $C_{\overline{c}}$.

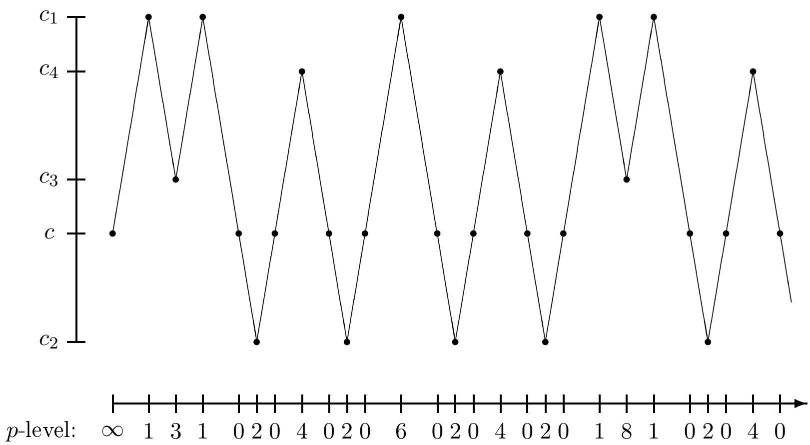

Fig. 2. The p-levels of the p-wrapping points of the composant of $\overline{c}$

Proposition 3.15 ([13, Proposition 29]). *Let $p \in \mathbb{N}$ and $\overline{w} < \overline{v}$ in $C_{\overline{c}}$ be such that $\pi_{pn_0}(\overline{w}) = \pi_{pn_0}(\overline{v})$. There exists a p-wrapping point $\overline{z}$ such that $\overline{w} < \overline{z} < \overline{v}$. Additionally, if both $\overline{w}$ and $\overline{v}$ are p-wrapping points, then there exists a p-wrapping point $\overline{z}$ such that $\overline{w} < \overline{z} < \overline{v}$ and $L_p(\overline{z}) > \min\{L_p(\overline{w}), L_p(\overline{v})\}$.*

Definition 3.16. Fix $p \in \mathbb{N}$. Let H be an arc in the composant of $\overline{c}$ with $\operatorname{Int}(H) \cap W_p = \{\overline{h}_1, \ldots, \overline{h}_{n-1}\}$ and $\partial H = \{\overline{h}_0, \overline{h}_n\}$.

(1) The arc H is *p-symmetric* if $\pi_{pn_0}(\overline{h}_0) = \pi_{pn_0}(\overline{h}_n)$ and $L_p(\overline{h}_i) = L_p(\overline{h}_{n-i})$ for all $0 < i < n$.
(2) The arc H is *p-pseudosymmetric* if $\pi_{pn_0}(\overline{h}_i) = \pi_{pn_0}(\overline{h}_{n-i})$ for all $0 \leq i \leq n$.

If H is p-pseudosymmetric or p-symmetric, then n is even and the *center* of H, denoted $\overline{\kappa}_H$, is the point $\overline{h}_{n/2}$.

Remark 3.17. Fix $p \in \mathbb{N}$ and let $H \subset C_{\overline{c}}$ be an arc. If H is p-pseudosymmetric, then H is q-pseudosymmetric for all $q < p$. If H is p-symmetric, then H is q-symmetric for all $q \in \mathbb{N}$ such that $qn_0 < pn_0 + L_p(\overline{\kappa}_H)$.

Proposition 3.18 ([13, Proposition 34]). *Let $p \in \mathbb{N}$ and $\overline{w} \in W_p \backslash \{\overline{c}\}$ such that $L_p(\overline{w}) \neq 0$. Let H be the union of all p-symmetric arcs with center $\overline{w}$. There exists a p-wrapping point $\overline{v} \in H$ such that $L_p(\overline{v}) > L_p(\overline{w})$. Furthermore, $\overline{v}$ is an endpoint of H.*

Remark 3.19. Let H be a p-symmetric arc in $C_{\overline{c}}$ and let $L = L_p(\overline{\kappa}_H)$. Proposition 3.18 implies that all the interior points in H have p-levels smaller than L, hence $\pi_{pn_0+L}|_H$ is a homeomorphism.

Definition 3.20. The set $\Phi_{p,0}$ partitions the composant of $\overline{c}$ into countably many arcs called *p-gaps*.

For any p-gap H, $c \notin \pi_{pn_0}(\mathrm{Int}(H))$ and $\pi_{pn_0}(\partial H) = \{c\}$. The intersection of any two p-gaps is at most one point.

Lemma 3.21. *For any $p \in \mathbb{N}$, a p-gap is p-symmetric.*

Proof. Fix $p \in \mathbb{N}$. Let H be a p-gap and $\partial H = \{\overline{y}, \overline{z}\}$. Let $\overline{x} \in \mathrm{Int}(H)$ be a p-wrapping point with largest p-level, say L. Suppose H is not p-symmetric. Then $f_s(\pi_{pn_0+L}(\overline{y})) \neq f_s(\pi_{pn_0+L}(\overline{z}))$, hence there is a p-wrapping point $\overline{w} \in \mathrm{Int}(H)$ such that $f_s(\pi_{pn_0+L}(\overline{w}))$ is equal to either $f_s(\pi_{pn_0+L}(\overline{y}))$ or $f_s(\pi_{pn_0+L}(\overline{z}))$. This implies that $\pi_{pn_0}(\overline{w}) = c$, which contradicts H being a p-gap. ∎

The proof of the previous lemma is longer than the one given by Kailhofer, but it is self-contained.

Definition 3.22. Fix $p, q \in \mathbb{N}$. Let G be a p-gap with $G \cap W_p = \{\overline{g}_0, \overline{g}_1, \ldots, \overline{g}_n\}$ and H be a q-gap with $H \cap W_q = \{\overline{h}_0, \overline{h}_1, \ldots, \overline{h}_m\}$. The gaps G and H are of the *same type* if $n = m$ and $\pi_{pn_0}(\overline{g}_i) = \pi_{qn_0}(\overline{h}_i)$ for all $0 \leq i \leq n$.

Proposition 3.23 ([13, Proposition 41]). *Fix $p, q \in \mathbb{N}$. Let G be a p-gap and H a q-gap. If $L_p(\overline{\kappa}_H) = L_q(\overline{\kappa}_G)$, then G and H are of the same type.*

Definition 3.24. Fix $p \in \mathbb{N}$ and let G be a p-gap. The arcs between two consecutive p-wrapping points in G are called *legs* of G.

The first p-gap in the composant of $\overline{c}$ is denoted F_p.

Lemma 3.25. *Fix $p \in \mathbb{N}$ and let G be a p-gap. Then*

(1) *The first leg of G contains a $(p-1)$-gap* [13, Lemma 46].
(2) *The first $(p-1)$-gap in G is of the same type as F_p* [13, Prop. 47].

Definition 3.26. Fix $p \in \mathbb{N}$. Define $\varphi = L_p(\overline{\kappa}_{F_p})$.

Remark 3.27. Since the type of F_p does not depend on p, φ does not depend on p. Since F_p is contained in the first leg of F_{p+1}, the center of F_p is not a $(p+1)$-wrapping point, hence $\varphi = L_p(\overline{\kappa}_{F_p}) < n_0$. Note also that $\pi_{pn_0}(\overline{\kappa}_{F_p}) = c_\varphi$.

Now, consider a homeomorphism $h : (I, f) \to (I, f)$ with $h(\overline{c}) = \overline{c}$. (If $h(\overline{c}) = \overline{c}_i$, where $0 < i < n_0$, consider the map $\overline{h} = \sigma^{-i} \circ h$.)

Fix $m, n, p, q \in \mathbb{N}$ such that $h(\mathcal{L}_{qn_0,n}) \prec \mathcal{L}_{pn_0,m}$. If $h(\overline{c}_j) = \overline{c}_i$ for $0 \le i, j < n_0$, then $h(l^{c_j}_{qn_0,n}) \subset l^{c_i}_{pn_0,m}$. This implies that $h(\Phi_{q,j}) \subset l^{c_i}_{pn_0,m}$. By Proposition 3.13, every component of $l^{c_j}_{qn_0,n}$ contains exactly one element of $\Phi_{q,j}$. Since two consecutive points of Φ_q lie in two different links, each component of $l^{c_i}_{pn_0,m}$ contains at most one element of $h(\Phi_{q,j})$. Thus, h induces a one-to-one map $h_{q,p} : \Phi_q \to \Phi_p$, defined as follows.

Definition 3.28. Fix $m, n, p, q \in \mathbb{N}$ such that $h(\mathcal{L}_{qn_0,n}) \prec \mathcal{L}_{pn_0,m}$. If $\overline{w} \in \Phi_{q,j}$ and $h(\overline{c}_j) = \overline{c}_i$ for $0 \le i, j < n_0$, then $h_{q,p}(\overline{w})$ is defined as the element of $\Phi_{p,i}$ that lies in the same component of $l^{c_i}_{pn_0,m}$ as $h(\overline{w})$.

If G is an arc in the composant of $\overline{c}$ with $\partial G = \{\overline{x}, \overline{y}\} \subset \Phi_q$, let $\widetilde{h}_{q,p}(G)$ be the arc between $h_{q,p}(\overline{x})$ and $h_{q,p}(\overline{y})$.

Theorem 3.29 ([13, Corollary 67]). *Fix positive integers m, n, p, q such that $h(\mathcal{L}_{qn_0,n}) \prec \mathcal{L}_{pn_0,m}$. If H is a q-pseudosymmetric arc in the composant of $\overline{c}$ with $\partial H \subset \Phi_q$, then $\widetilde{h}_{q,p}(H)$ is p-pseudosymmetric.*

Lemma 3.30 ([13, Lemma 68]). *Let $p \in \mathbb{N}$. Let G and H be distinct p-pseudosymmetric arcs in the composant of $\overline{c}$ such that $\overline{c} \in G$ and $\overline{c} \in H$. Then $G \subset H$ if and only if $L_p(\overline{\kappa}_G) < L_p(\overline{\kappa}_H)$.*

Theorem 3.31 ([13, Corollary 71]). *Fix $m, n, p, q, u, v \in \mathbb{N}$ such that $h(\mathcal{L}_{qn_0,n}) \prec \mathcal{L}_{pn_0,m} \prec h(\mathcal{L}_{un_0,v})$. If $\widetilde{h}_{q,p}(F_q) = F_t$ for some $t \in \mathbb{N}$, then $h_{q,p}(\Phi_{q+k,0}) = \Phi_{t+k,0}$ for all $k \in \mathbb{N}_+$.*

4. Main result. The following lemma is a well known result (see Brucks and Diamond [8] and Brucks and Bruin [7]).

Lemma 4.1. *Suppose that A is a proper subcontinuum of X_s. There is a nonnegative integer k such that $\pi_k|_A$ is a homeomorphism. In particular, A is an arc. Moreover, for any two points $\overline{x}, \overline{y} \in A$, $B(\overline{x})$ and $B(\overline{y})$ agree after the first k entries.*

Proof. If there is a nonnegative integer m such that for each $j > m$, $c \notin \pi_j(A)$, then $f_s|_{\pi_j(A)}$ is a homeomorphism for each $j > m$, and the conclusion follows easily. So, we may assume that $c \in \pi_j(A)$ for arbitrarily large integer j. Since c is periodic under f_s, it follows that for each nonnegative integer j, $\pi_j(A)$ contains at least one of the points $c_0, c_1, \ldots, c_{n_0-1}$. Since f_s is locally eventually onto, there is a nonnegative integer k such that for each integer $j > k$, $\pi_j(A)$ contains exactly one of the points $c_0, c_1, \ldots, c_{n_0-1}$. We complete the proof by showing that for each $j > k$ the element of $\{c_0, c_1, \ldots, c_{n_0-1}\}$ which is in $\pi_j(A)$ is an endpoint of $\pi_j(A)$. In particular, for each $j > k$, c is not in the interior of $\pi_j(A)$, so $f_s|_{\pi_j(A)}$ is a homeomorphism.

Suppose $j > k$. Since c is periodic, there is an integer $m > j$ such that $c_1 \in \pi_m(A)$. Since c_1 is an endpoint of I_s, it follows that c_1 is an endpoint of $\pi_m(A)$. Since c is not in the interior of $\pi_m(A)$, $f_s|_{\pi_m(A)}$ is a homeomorphism. Thus c_2 is an endpoint of $\pi_{m-1}(A)$. If $m-1 > j$, we may repeat this argument and conclude that c_3 is an endpoint of $\pi_{m-2}(A)$. By repeating the argument inductively, it follows that the element of $\{c_0, c_1, \ldots, c_{n_0-1}\}$ which is in $\pi_j(A)$ is an endpoint of $\pi_j(A)$. ∎

Let $\overline{x} \in X_s$. By Remark 3.6, there is a natural order on the elements of the composant of $\overline{x}$. With the order topology, $C_{\overline{x}}$ will be called the *unravelled* composant of $\overline{x}$.

REMARK 4.2. Note that the map $h_{q,p}$ is order-preserving.

We will put a specific metric on the unravelled composant which is derived from the inverse limit system. Let $\overline{x}, \overline{y}$ be in the same composant $C \subset X_s$. Then there is an arc $A \subset C$ with endpoints $\overline{x}$ and $\overline{y}$. By Lemma 4.1, there is a nonnegative integer k such that $\pi_k|_A$ is a homeomorphism. Define

$$\overline{d}(\overline{x}, \overline{y}) = s^k |\pi_k(\overline{x}) - \pi_k(\overline{y})|.$$

Note that if $m \geq k$, then $\overline{d}(\overline{x}, \overline{y}) = s^m |\pi_m(\overline{x}) - \pi_m(\overline{y})|$. Thus, $\overline{d}$ is well defined for every pair of points in the same composant C. We may consider $(C, \overline{d})$ either as $\mathbb{R}_+$ or $\mathbb{R}$ depending on whether C has an endpoint or not.

THEOREM 4.3. *Let $h_1, h_2 : X_s \to X_s$ be homeomorphisms which map $C_{\overline{c}}$ to itself. Suppose that there is $M \in \mathbb{N}_+$ such that $\overline{d}(h_1(\overline{z}), h_2(\overline{z})) \leq M$ for all $\overline{z} \in C_{\overline{c}}$. Then $h_1(C_{\overline{x}}) = h_2(C_{\overline{x}})$ for all $\overline{x} \in X_s$. Furthermore, for every $\overline{x} \in X_s$, $\overline{d}(h_1(\overline{x}), h_2(\overline{x})) \leq M$.*

Proof. Let $\overline{x} \in X_s$. If $\overline{x} \in C_{\overline{c}}$, then $h_1(C_{\overline{c}}) = h_2(C_{\overline{c}})$ by assumption.

Suppose that $\overline{x} \notin C_{\overline{c}}$. Since $C_{\overline{c}}$ is dense in X_s, there is a sequence $\{\overline{x}_n\}_{n=1}^{\infty}$ in $C_{\overline{c}}$ which converges to $\overline{x}$. Then $h_i(\overline{x}_n)$ converges to $h_i(\overline{x})$ for $i = 1, 2$. Consider the unique arcs $A_n \subset C_{\overline{c}}$ with endpoints $h_1(\overline{x}_n)$ and $h_2(\overline{x}_n)$. By assumption the length of A_n in $C_{\overline{c}}$ is less than or equal to M. Let $k > 0$ be an integer such that $M \leq s^k(c_1 - c_2)$. Then $\pi_k(A_n)$ is a proper subset of $[c_2, c_1]$ since the length of $\pi_k(A_n)$ is less than $M/s^k < c_1 - c_2$.

Let $\mathcal{C}(X_s)$ denote the space of nonempty subcontinua of X_s with the Hausdorff metric. Then $\pi_k : X_s \to [c_2, c_1]$ induces a continuous map $\pi_k : \mathcal{C}(X_s) \to \mathcal{C}([c_2, c_1])$. Since $\mathcal{C}(X_s)$ is a compact metric space, the sequence $\{A_n\}$ has a subsequence $\{A_{n_j}\}$ converging to some $A \in \mathcal{C}(X_s)$. Note that $h_i(\overline{x}_{n_j})$ converges to $h_i(\overline{x})$ for $i = 1, 2$. So, $h_i(\overline{x}) \in A$ for $i = 1, 2$. Since $\pi_k : \mathcal{C}(X_s) \to \mathcal{C}([c_2, c_1])$ is continuous, $\pi_k(A)$ has length at most $M/s^k < c_1 - c_2$. Thus, A must be a proper subcontinuum of X_s. Thus, $h_1(\overline{x})$ and $h_2(\overline{x})$ are in the same composant of X_s. This implies that $h_1(C_{\overline{x}}) = h_2(C_{\overline{x}})$.

Since $\overline{x}$ is arbitrary, the above also proves the last statement of the theorem. ■

Recall that the shift homeomorphism, $\sigma : X_s \to X_s$, is defined by

$$\sigma((x_0, x_1, \dots)) = (f_s(x_0), x_0, x_1, \dots).$$

LEMMA 4.4. *There is a positive integer B such that for any $p \in \mathbb{N}$ the number of legs in a p-gap is at most B.*

Proof. Since f_s is locally eventually onto, there is $K \in \mathbb{N}$ such that if J is an interval which contains two points in the orbit of c, then $f_s^K(J) = I_s$.

Fix $p \in \mathbb{N}$. Let H be a p-gap and $L = L_p(\overline{\kappa}_H)$. Let H_R be the arc connecting the center of H and the right endpoint of H. Then $\pi_{pn_0+L}(H_R)$ is an interval with one endpoint c and $\pi_{pn_0+L}|_{H_R}$ is a homeomorphism. Note also that

$$f_s^L(\pi_{pn_0+L}(H_R)) = \pi_{pn_0}(H_R) = \pi_{pn_0}(H)$$

is an interval with one endpoint c.

If $f_s^L|_{\pi_{pn_0+L}(H_R)}$ is linear, then there are at most two legs in H. Suppose $f_s^L|_{\pi_{pn_0+L}(H_R)}$ is not linear. There is a least $n \in \mathbb{N}$ such that $f_s^n(\pi_{pn_0+L}(H_R))$ contains two points in the orbit of c. This implies that H has at most 2^{L-n} legs. Since $\pi_{pn_0}(H_R)$ is a proper subset of $[c_2, c_1]$, we have $L - n < K$, hence the number of legs in H is at most 2^K. ■

REMARK 4.5. One might be led to conjecture that the number of distinct types of p-gaps in $C_{\overline{c}}$ is $n_0 - 1$ for any $p \in \mathbb{N}$. However, for s such that the kneading sequence is $RLLRRRLC$, there are at least eight p-gaps.

THEOREM 4.6. *Let $h : X_s \to X_s$ be a homeomorphism which maps each endpoint, $\overline{c}_i$ for $0 \le i \le n_0 - 1$, to itself. Then there exists an integer N and a positive number M such that $\overline{d}(h(\overline{x}), \sigma^N(\overline{x})) < M$ for all $\overline{x} \in C_{\overline{c}}$.*

Proof. For convenience of referral, two points of any subset of $C_{\overline{c}}$ are said to be *adjacent* in that set if the arc connecting those two points contains no other points of that set. Note that if $\overline{x}$ and $\overline{y}$ are adjacent in Φ_p, then $\overline{d}(\overline{x}, \overline{y}) < s^{pn_0}$.

By Lemma 3.10(5), given $u, v \in \mathbb{N}$, there are $p, m \in \mathbb{N}$ such that $\mathcal{L}_{pn_0,m} \prec h(\mathcal{L}_{un_0,v})$, and there are $q, r \in \mathbb{N}$ such that $h(\mathcal{L}_{qn_0,r}) \prec \mathcal{L}_{pn_0,m}$. Fix p, m, q, r, $u, v \in \mathbb{N}$ such that $h(\mathcal{L}_{qn_0,r}) \prec \mathcal{L}_{pn_0,m} \prec h(\mathcal{L}_{un_0,v})$.

Since F_q is q-symmetric, by Theorem 3.29, $\widetilde{h}_{q,p}(F_q)$ is p-pseudosymmetric and $h_{q,p}(\overline{\kappa}_{F_q}) = \overline{\kappa}_{\widetilde{h}_{q,p}(F_q)}$. Let $L = L_p(\overline{\kappa}_{\widetilde{h}_{q,p}(F_q)})$ and t be the largest positive integer with the property $tn_0 < pn_0 + L$. Obviously $t \ge p$. Since $h(\overline{c}_i) = \overline{c}_i$ for all $0 \le i < n_0$, we see that $\pi_{pn_0}(\overline{\kappa}_{\widetilde{h}_{q,p}(F_q)}) = \pi_{qn_0}(\overline{\kappa}_{F_q})$, which, by Remark 3.27, is equal to c_φ. From Definition 3.26, we have $L_t(\overline{\kappa}_{F_t}) = \varphi$. Thus

$$L_p(\overline{\kappa}_{F_t}) = L_t(\overline{\kappa}_{F_t}) + (t-p)n_0 = \varphi + (t-p)n_0 = L = L_p(\overline{\kappa}_{\widetilde{h}_{q,p}(F_q)}).$$

Hence, by Lemma 3.30,

$$F_t = \widetilde{h}_{q,p}(F_q).$$

By Theorem 3.31 it follows that for every $k \in \mathbb{N}_+$,

$$h_{q,p}(\Phi_{q+k,0}) = \Phi_{t+k,0}.$$

By Remark 4.2, $h_{q,p}$ is order-preserving on the set $\Phi_{t+k,0}$ for any $k \in \mathbb{N}_+$. From the definition of σ and since $\sigma^{(t-q)n_0}$ is order-preserving as well, it is easy to see that $\sigma^{(t-q)n_0}(\Phi_{q+k,0}) = \Phi_{t+k,0}$ for any $k \in \mathbb{N}_+$. Therefore, for every $\overline{x} \in \Phi_{q+1,0}$, we have $h_{q,p}(\overline{x}) = \sigma^{(t-q)n_0}(\overline{x})$. By Definition 3.28, for any $\overline{x} \in \Phi_{q+1,0}$, $h(\overline{x})$ lies between two adjacent p-special points, one of which is $h_{q,p}(\overline{x})$. Since the distance between two special points is less than s^{pn_0}, we have

$$\overline{d}(h(\overline{x}), \sigma^{(t-q)n_0}(\overline{x})) = \overline{d}(h(\overline{x}), h_{q,p}(\overline{x})) < s^{pn_0}$$

for any $\overline{x} \in \Phi_{q+1,0}$.

The length of any leg of a $(t+1)$-gap is bounded by $s^{(t+1)n_0}$ as $\pi_{(t+1)n_0}$ restricted to the leg is a homeomorphism. Since the number of legs in a $(t+1)$-gap is bounded by B by Lemma 4.4, it follows that the length of a $(t+1)$-gap is bounded. Namely, if $\overline{x}$ and $\overline{y}$ are the endpoints of a $(t+1)$-gap, then $\overline{d}(\overline{x}, \overline{y}) < l$, where

$$l = Bs^{(t+1)n_0}.$$

Let $N = (t-q)n_0$ and $M = s^{pn_0} + l$. Let $\overline{x} \in C_{\overline{c}}$. If $\overline{x} \in \Phi_{q+1,0}$, then

$$\overline{d}(h(\overline{x}), \sigma^N(\overline{x})) < s^{pn_0} < M.$$

If $\overline{x} \notin \Phi_{q+1,0}$, then there exist $\overline{y}$ and $\overline{z}$ adjacent in $\Phi_{q+1,0}$ such that the $(q+1)$-gap whose endpoints are $\overline{y}$ and $\overline{z}$, contains $\overline{x}$. As $\overline{y}, \overline{z} \in \Phi_{q+1,0}$, we have $h_{q,p}(\overline{y}) = \sigma^N(\overline{y})$ and $h_{q,p}(\overline{z}) = \sigma^N(\overline{z})$. Since h is a homeomorphism, the arc connecting $h(\overline{y})$ and $h(\overline{z})$ contains $h(\overline{x})$. Similarly, the arc connecting $\sigma^N(\overline{y})$ and $\sigma^N(\overline{z})$ contains $\sigma^N(\overline{x})$. Thus,

$$\overline{d}(h(\overline{x}), \sigma^N(\overline{x})) < \max\{\overline{d}(\sigma^N(\overline{x}), h(\overline{y})), \overline{d}(\sigma^N(\overline{x}), h(\overline{z}))\}.$$

Since σ^N sends a $(q+1)$-gap to a $(t+1)$-gap,

$$\overline{d}(\sigma^N(\overline{z}), \sigma^N(\overline{y})) < l.$$

Since $h(\overline{y})$ lies between two adjacent p-special points, one of which is $\sigma^N(\overline{y})$,

$$\overline{d}(h(\overline{y}), \sigma^N(\overline{y})) < s^{pn_0}.$$

Therefore

$$\begin{aligned}\overline{d}(\sigma^N(\overline{x}), h(\overline{y})) &\leq \overline{d}(\sigma^N(\overline{x}), \sigma^N(\overline{y})) + \overline{d}(\sigma^N(\overline{y}), h(\overline{y})) \\ &< \overline{d}(\sigma^N(\overline{z}), \sigma^N(\overline{y})) + \overline{d}(\sigma^N(\overline{y}), h(\overline{y})) < l + s^{pn_0} = M.\end{aligned}$$

Similarly, $\overline{d}(\sigma^N(\overline{x}), h(\overline{z})) < M$. Thus,

$$\overline{d}(\sigma^N(\overline{x}), h(\overline{x})) < M. \blacksquare$$

COROLLARY 4.7. *Let $h : X_s \to X_s$ be a homeomorphism which maps each endpoint, $\overline{c}_i$ for $0 \le i \le n_0 - 1$, to itself. Then there is an integer N such that $h(C_{\overline{x}}) = \sigma^N(C_{\overline{x}})$ for all $\overline{x} \in X_s$.*

We adopt the following notation. If k is a positive integer, we let $F(f_s^k)$ denote the number of fixed points of f_s^k in I_s.

LEMMA 4.8. *Suppose $\sqrt{2} < s < t < 2$ and for each of the tent maps f_s and f_t, the critical point is periodic with period n_0. Then*

$$F(f_s^{n_0}) < F(f_t^{n_0}).$$

Proof. Since each point in the orbit of the critical point is a fixed point of $f_t^{n_0}$ and the same holds for $f_s^{n_0}$, we need only consider fixed points of $f_t^{n_0}$ and $f_s^{n_0}$ which are not in the orbit of the critical point. Suppose y is such a fixed point of $f_s^{n_0}$. Then the forward itinerary $I(y)$ equals S^∞ for some sequence S of length n_0 of L's and R's. By [10, Theorem II.3.8] there is a fixed point z of $f_t^{n_0}$ with $I(z) = S^\infty$.

We complete the proof by showing that there is a sequence T of length n_0 of L's and R's such that there is a fixed point of $f_t^{n_0}$ with itinerary T^∞ but no fixed point of $f_s^{n_0}$ has this itinerary. The itinerary of $f_s(c)$ is of the form DC where D is a sequence of length $n_0 - 1$ of L's and R's. We can modify f_s to construct a unimodal map g with the same kneading sequence as f_s such that on the orbit of c, $g = f_s$, but for a small nondegenerate interval J with right endpoint $g(c)$ each point of J is periodic under g with period n_0. The itinerary of a point in J other than $g(c)$ is of the form T^∞, where T is a sequence of L's and R's of length n_0. Moreover, T is shift maximal and T is either DR or DL. It follows that no fixed point of $f_s^{n_0}$ has itinerary T^∞, but by [10, Theorem II.3.8] there is a fixed point of $f_t^{n_0}$ with itinerary T^∞. ■

LEMMA 4.9. *Let $s \in (\sqrt{2}, 2)$. For any integer m, the number of composants mapped to themselves by σ_s^m is $F(f_s^m)$.*

Proof. Without loss of generality we may assume that $m > 0$.

Our first claim is that there is at most one periodic point in each composant of X_s.

Suppose not. There is a composant C in X_s with at least two distinct periodic points of σ_s, say $\overline{x} = (x_0, x_1, \ldots)$ and $\overline{y} = (y_0, y_1, \ldots)$. Then there is a positive integer k such that $\sigma_s^k(\overline{x}) = \overline{x}$ and $\sigma_s^k(\overline{y}) = \overline{y}$. In particular, $f_s^k(x_{k-1}) = x_{k-1}$ and $f_s^k(y_{k-1}) = y_{k-1}$. Note that $x_{k-1} \ne y_{k-1}$. Since $\overline{x}$ and $\overline{y}$ are in the same composant, they have eventually the same backward itinerary. Thus, there is some positive integer N such that for all $n \ge N$, x_n and y_n are on the same side of c. (By this we mean either both $x_n \ge c$ and $y_n \ge c$, or both $x_n \le c$ and $y_n \le c$.) Since $\overline{x}$ and $\overline{y}$ are periodic, it follows that for each integer $j = 0, 1, \ldots, k-1$, x_j and y_j are on the same side of c.

Hence for each integer $j \geq 0$, $f_s^j(x_{k-1})$ and $f_s^j(y_{k-1})$ are on the same side of c. This is impossible since $x_{k-1} \neq y_{k-1}$ and f_s is a tent map with a slope $s > 1$. This proves the first claim.

Our next claim is that each composant mapped to itself by σ_s^m contains a fixed point of σ_s^m.

Suppose C is a composant in X_s with $\sigma_s^m(C) = C$. If $\overline{x}, \overline{y} \in C$, then

$$\overline{d}(\sigma_s^m(\overline{x}), \sigma_s^m(\overline{y})) = s^m \overline{d}(\overline{x}, \overline{y}).$$

Hence σ_s^{-m} is a contraction and has a fixed point. This proves our second claim.

It follows from the claims that the number of composants mapped to themselves by σ_s^m is equal to the number of points fixed by σ_s^m. By definition of σ_s, this number is equal to the number of fixed points of f_s^m. By definition, this number is $F(f_s^m)$. ∎

THEOREM 4.10. *Let $s, t \in (\sqrt{2}, 2)$ be such that f_s and f_t have periodic critical points. Then X_s and X_t are homeomorphic if and only if $s = t$.*

Proof. It is well known that if X_s and X_t are homeomorphic and the critical point of f_s is periodic, then the critical point of f_t is also periodic with the same period. Thus, there is no loss in generality in assuming that the period of the critical points for f_s and f_t are both periodic of period n_0. Suppose $s < t$. Assume there is a homeomorphism $g : X_s \to X_t$.

Consider the map $h : X_t \to X_t$, $h = g \circ \sigma_s^{n_0} \circ g^{-1}$. Then h is a homeomorphism, and it maps each composant with an endpoint to itself. By Corollary 4.7, there is an integer N such that $h(C_{\overline{x}}) = \sigma_t^N(C_{\overline{x}})$ for all $\overline{x} \in X_t$. Since $\sigma_s^{n_0}$ maps each composant with an endpoint to itself, the same is true for h. Thus σ_t^N also maps each endpoint of X_t to itself. By Lemma 4.9, the total number of composants mapped to themselves by $\sigma_s^{n_0}$ and hence by h is $F(f_s^{n_0})$. Thus, the same is true for σ_t^N. It follows that $|N| \geq n_0$ and n_0 divides $|N|$. But the number of composants mapped to themselves by σ_t^N is $F(f_t^N)$. Thus

$$F(f_s^{n_0}) = F(f_t^N).$$

On the other hand, since $s < t$, by Lemma 4.8, $F(f_s^{n_0}) < F(f_t^{n_0})$. Hence

$$F(f_t^N) < F(f_t^{n_0}),$$

which is a contradiction. ∎

5. Proof of the Isotopy Theorem. In this section we prove the Isotopy Theorem stated in the introduction. We have already shown that for any homeomorphism $g : X_s \to X_s$ such that g leaves all the endpoints $\{\overline{c}_i\}_{i=0}^{n_0-1}$ fixed, there is a k such that g and σ^k permute the composants of X_s in precisely the same way. It is clear that for any homeomorphism $h : X_s \to X_s$, there is an $n > 0$ such that h^n leaves $\overline{c}_i$ fixed for $i = 0, 1, \dots, n_0 - 1$. Let k be the integer such that h^n and σ^k permute the composants of X_s the same way. We now show that h^n and σ^k are actually isotopic.

The following lemma is a well known fact for the experts in this field.

LEMMA 5.1. *Suppose A is an arc in X_s not containing any endpoint of X_s. Then there is a neighborhood V of A homeomorphic to $C \times I$, where C is a Cantor set. The boundary of V will correspond to $C \times \{0,1\}$. Moreover, there is a positive integer m such that π_m maps each component of V homeomorphically onto its image in I_m.*

Proof. Let A be an arc in X_s not containing any endpoint of X_s. By the proof of Lemma 4.1, there is a positive integer m such that for each $k \geq m$, none of the points c_i are in $\pi_k(A)$. In particular, $\pi_k|_A$ is a homeomorphism. Let $\overline{z} \in A$, $\overline{z} = (z_0, z_1, \dots, z_m, \dots)$. Let $C = \{\overline{y} \in X_s \mid y_0 = z_0,\, y_1 = z_1, \dots, y_m = z_m\}$. Then C is compact, totally disconnected, and every point is a limit point. Therefore C is a Cantor set.

Let $J_m = \pi_m(A)$. Fix $\overline{y} \in C$. Since $J_m \cap \{c_0, c_1, \dots, c_{n_0-1}\} = \emptyset$, for this $\overline{y} \in C$, there is a sequence $\{J_i\}_{i=m}^{\infty}$ of intervals such that $y_i \in J_i$ for each $i \geq m$ and $f_s(J_{i+1}) = J_i$ for each $i \geq m$.

We can extend the sequence $\{J_i\}_{i=m}^{\infty}$ to $\{J_i\}_{i=0}^{\infty}$ by $J_0 = f_s^m(J_m), J_1 = f_s^{m-1}(J_m), \dots, J_{m-1} = f_s(J_m)$. Then for all $i = 0, 1, \dots,$ $f_s(J_{i+1}) = J_i$ and $y_i \in J_i$.

Now J_m is homeomorphic to

$$J(\overline{y}) = \varprojlim\{J_i, f_s\} \subset X_s$$

by the projection $\pi_m : X_s \to I_m$. Let $g_{\overline{y}} : J_m \to J(\overline{y})$ be the inverse of this homeomorphism.

Finally, let $\xi : C \times J_m \to X_s$ be defined by $\xi(\overline{y}, t) = g_{\overline{y}}(t)$. Then $V = \xi(C \times J_m)$ is the required neighborhood. ∎

REMARK 5.2. In the above proof let x be in the Cantor set C. Note that the points $\overline{z}_0$ and $\overline{z}_1$ corresponding to $(x, 0)$ and $(x, 1)$, respectively, are in the same composant. Moreover, $\overline{d}(\overline{z}_0, \overline{z}_1)$ does not depend on x. That is, the lengths of the components of V are all the same in the $\overline{d}$ metric.

DEFINITION 5.3. Suppose $\{D_i\}_{i=1}^{\infty}$ is a sequence of nonempty compact subsets of a metric space Y. Then $\limsup\{D_i\} = \{y \in Y \mid$ for some subsequence $\{D_{i_j}\}$ and $y_{i_j} \in D_{i_j}$, $\lim_{j\to\infty} y_{i_j} = y\}$.

We let $\overline{\ell}$ denote the length of an arc under the metric $\overline{d}$.

LEMMA 5.4. *Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of arcs in X_s. Suppose $A_i \to B$ in the Hausdorff metric. Suppose also that there is an $M > 0$ such that $\overline{\ell}(A_i) \leq M$ for all i. Then B is an arc and $\overline{\ell}(B) \leq M$.*

Proof. Let N be such that $Ms^{-N} \leq \ell(I_N)/2 = \ell(I_s)/2 = (f(c) - f^2(c))/2$. Then for every k, $\pi_N(A_k)$ has length at most $\ell(I_N)/2$. Since $A_k \to B$, $\pi_N(A_k) \to \pi_N(B)$. In particular, $\pi_N(B)$ is a proper subset of I_N. It follows that B is a proper subcontinuum of X_s. By Lemma 4.1, B is an arc.

Finally, choose j large enough so that $\pi_j|_B$ is a homeomorphism. Then for each k, $s^j\ell(\pi_j(A_k)) \leq M$, and hence $\overline{\ell}(B) = s^j\ell(\pi_j(B)) \leq M$. ∎

LEMMA 5.5. *Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of arcs in X_s with endpoints $\overline{a}_i$ and $\overline{b}_i$, respectively. Suppose that there is a positive number M such that $\overline{d}(\overline{a}_i, \overline{b}_i) \leq M$ for each i. Suppose also that the sequence $\{\overline{a}_i\}_{i=1}^{\infty}$ converges to some $\overline{a} \in X_s$. Then $B = \limsup\{A_i\}$ is an arc in X_s and $\overline{\ell}(B) \leq 2M$.*

Proof. Let $\overline{x} \in B = \limsup\{A_i\}$. Then there is a subsequence $\{A_{i_j}\}_{j=1}^{\infty}$ such that $A_{i_j} \to D \subset B$ in the Hausdorff metric with $\overline{x} \in D$. By Lemma 5.4, $\overline{\ell}(D) \leq M$. So, $\overline{d}(\overline{a}, \overline{x}) \leq M$. From this it follows that B must be a proper subcontinuum and thus an arc with the $\overline{\ell}$-length of B at most $2M$. ∎

LEMMA 5.6. *Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of arcs in X_s with endpoints $\overline{a}_i$ and $\overline{b}_i$, respectively. Suppose that $\overline{a}_i \to \overline{a}$ and $\overline{b}_i \to \overline{b}$. Suppose also that there is an $M > 0$ such that $\overline{d}(\overline{a}_i, \overline{b}_i) \leq M$ for all i. Then $\overline{a}$ and $\overline{b}$ are in the same composant of X_s. Let A denote the unique arc with endpoints $\overline{a}$ and $\overline{b}$. Suppose that $\limsup A_i$ does not contain an endpoint of X_s. Then $A_i \to A$ in the Hausdorff metric.*

Proof. By the proof of Theorem 4.3, $\overline{a}$ and $\overline{b}$ are in the same composant of X_s and $\overline{d}(\overline{a}, \overline{b}) \leq M$. Let A be the unique arc with endpoints $\overline{a}$ and $\overline{b}$. Let $B = \limsup\{A_i\}$. By Lemma 5.5, B is an arc with $\overline{\ell}(B) \leq 2M$. By assumption B does not contain an endpoint of X_s. So, let V be the neighborhood of B given by Lemma 5.1. Then there is an N such that for all $n \geq N$, $A_n \subset V$ since B is the $\limsup\{A_i\}$. Therefore for each $i \geq N$, $A_i \subset \{\overline{y}_i\} \times I$ for some $\overline{y}_i \in C$. Furthermore, A_i is the subinterval of $\{\overline{y}_i\} \times I$ joining the endpoints. Let $\overline{a}, \overline{b} \in \{\overline{y}\} \times I$. Then

$$\lim_{i\to\infty} A_i = A = B \subset \{\overline{y}\} \times I. \ ∎$$

DEFINITION 5.7. Consider $J \times C \subset \mathbb{R}^2$ where C is the standard middle-third Cantor set and $J = [-1, 1]$. Define an equivalence relation $\sim$ on $J \times C$ by $(t, 1) \sim (-t, 1)$ for all $t \in J$. Let $Q = J \times C/\sim$. We will think of Q as the union of two sets E and F defined in the following way. Let $E = (C \cup (-C)) \times [1, 2] \subset \mathbb{R}^2$. Let F be a Cantor set of semicircles with centers at $(0, 1)$ joining each point of $C \times \{1\}$ with the corresponding point of $-C \times \{1\}$. See Figure 3. Now in the Cantor set C, let C_0 be the set of points in C in the interval between 0 and $1/3$, inclusive. Let C_1 be the set of points in C between $2/3$ and $7/9$, inclusive. For higher k, let C_k be the subset of C containing the points between $(3^k - 1)/3^k$ and $(3^{k+1} - 2)/3^{k+1}$, inclusive.

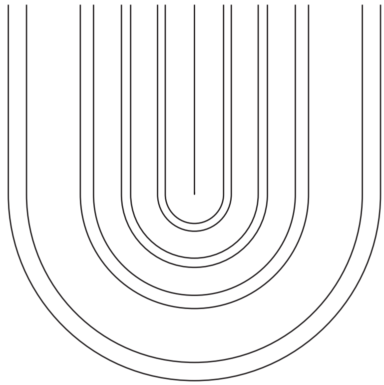

Fig. 3. Neighborhood of $\overline{c}$

Then $\{C_k\}_{i=0}^{\infty}$ is a disjoint collection of Cantor sets with $C_k \to 1$ in C and $C = \bigcup_{i=0}^{\infty} C_i \cup \{1\}$.

LEMMA 5.8. *Suppose that A is an arc in X_s which contains an endpoint of X_s. Then there is a neighborhood V of A homeomorphic to $Q = C \times J/\sim$.*

Proof. There is no loss of generality in assuming that $\overline{c}$ is the endpoint of A. Let A_w be the set of points in $C_{\overline{c}}$ with the same backward itinerary, w, and such that $\overline{c} \in A_w$. We know that A_w is a nondegenerate arc with $\overline{c}$ as one endpoint and some $\overline{z}$ as the other endpoint, and that $\pi_0|_{A_w}$ is a homeomorphism onto $[c, c_i]$ or $[c_i, c]$ for some $1 \leq i \leq n_0$.

Define

$$D_0 = \{\overline{x} \in X_s \mid \pi_{n_0}(\overline{x}) = c \text{ and } \pi_i(\overline{x}) \neq c \text{ for all } i > n_0\}.$$

The set D_0 is compact, totally disconnected and every point is a limit point, so D_0 is a Cantor set.

Let $\overline{x} \in D_0$. Then $\pi_{n_0}(\overline{x}) = c$ and $\pi_i(\overline{x}) \neq c$ for all $i > n_0$. There are two arcs $A_{\overline{x}}$ and $B_{\overline{x}}$ in X_s containing $\overline{x}$ as an endpoint, such that $\pi_{n_0}(A_{\overline{x}}) = \pi_{n_0}(A_w)$, and such that $\pi_{n_0}(A_{\overline{x}})$ and $\pi_{n_0}(B_{\overline{x}})$ are symmetric about c.

Similarly, for any $k \in \mathbb{N} \cup \{0\}$, define

$$D_k = \{\overline{x} \in X_s \mid \pi_{(k+1)n_0}(\overline{x}) = c \text{ and } \pi_i(\overline{x}) \neq c \text{ for all } i > (k+1)n_0\}.$$

Then, for any $k \in \mathbb{N} \cup \{0\}$, the set D_k is a Cantor set. For any $\overline{x} \in D_k$, there are two arcs $A_{\overline{x}}$ and $B_{\overline{x}}$ in X_s containing $\overline{x}$ as an endpoint, such that $\pi_{(k+1)n_0}(A_{\overline{x}}) = \pi_{(k+1)n_0}(A_w)$, and such that $\pi_{(k+1)n_0}(A_{\overline{x}})$ and $\pi_{(k+1)n_0}(B_{\overline{x}})$ are symmetric about c.

Let

$$V = \bigcup \left\{ A_{\overline{x}} \cup B_{\overline{x}} \;\middle|\; \overline{x} \in \bigcup_{k=0}^{\infty} D_k \right\} \cup A_w.$$

Let (a, b) be the open interval containing c such that $f^{n_0}((a, b))$ is $[c, c_i)$ or $(c_i, c]$, where $\pi_0(A_w)$ is $[c, c_i]$ or $[c_i, c]$. Then each point of $\pi_{n_0}^{-1}((a, b))$ is in V. Hence every point of A_w except $\overline{z}$ is an interior point of V.

Observe that $D = (\bigcup_{k=0}^{\infty} D_k) \cup \{\overline{c}\}$.

Define a map $h : V \to Q$ in the following way. For every $k \in \mathbb{N} \cup \{0\}$, h sends D_k homeomorphically onto C_k. For every $\overline{x} \in D_k$, $A_{\overline{x}}$ is mapped linearly onto $[-1, 0] \times \{h(\overline{x})\}$, and $B_{\overline{x}}$ is mapped linearly onto $[0, 1] \times \{h(\overline{x})\}$, and A_w is mapped linearly to $[-1, 0] \times \{1\}$. Then h is 1-1, continuous and onto, hence it is a homeomorphism.

Now the neighborhood V that we just created may not contain the given arc A. However, for $k > 1$, applying the shift map k times, $\sigma^{kn_0}(V)$, will create a longer and thinner neighborhood of the same form with $\bigcup_{k=0}^{\infty} \sigma^{kn_0}(V)$ dense in X_s. Thus, there will be some k for which $\sigma^{kn_0}(V)$ will contain A. ∎

REMARK 5.9. In the above proof $\overline{\ell}(A_w) = \overline{\ell}(A_{\overline{x}}) = \overline{\ell}(B_{\overline{x}})$ for every $\overline{x} \in D_k$ and every $k \in \mathbb{N} \cup \{0\}$. Furthermore, there are arbitrarily small neighborhoods of $\overline{c}$ homeomorphic to Q for which this is true.

THEOREM 5.10. *Suppose that h_1 and h_2 are homeomorphisms of X_s such that $h_1(\overline{c}) = h_2(\overline{c}) = \overline{c}$. Suppose also that there is an $M > 0$ such that $\overline{d}(h_1(\overline{y}), h_2(\overline{y})) \le M$ for each $\overline{y} \in C_{\bar{c}}$. Suppose that $\overline{x}_i \to \overline{x}$ in X_s. Let A_i be the unique arc joining $h_1(\overline{x}_i)$ and $h_2(\overline{x}_i)$. Let A be the unique arc joining $h_1(\overline{x})$ and $h_2(\overline{x})$. Then $A_i \to A$ in the Hausdorff metric.*

Proof. We assume the hypotheses and notation of the theorem.

CASE 1: *The composant containing $\overline{x}$ does not contain an endpoint.* In this case Lemma 5.6 applies since $\limsup\{A_i\}_{i=1}^{\infty}$ must be in the composant of $\overline{x}$ which does not contain an endpoint of X_s. Thus, we have $A_i \to A$ in this case.

CASE 2: *$\overline{x} \in C_{\bar{c}_i}$ for some i with $\overline{x} \neq \overline{c}_i$.* By Theorem 4.3, $h_1(\overline{x})$ and $h_2(\overline{x})$ are in the same composant, and this composant must be $C_{\bar{c}_j}$ for some j. Let $J = [\overline{e}, \overline{c}_j]$ be an arc in $C_{\bar{c}_j}$ such that $h_1(\overline{x}) \in J$, $h_2(\overline{x}) \in J$, $\overline{d}(\overline{e}, h_1(\overline{x})) > M+1$, $\overline{d}(\overline{e}, h_2(\overline{x})) > M+1$. Let V be a neighborhood of J as in Lemma 5.8.

Consider the arc $h_1^{-1}(J) \cup h_2^{-1}(J)$ in $C_{\bar{c}_i}$. Let W be a neighborhood of this arc as in Lemma 5.8. By shrinking V in the "vertical" direction if necessary, we may assume that $h_1^{-1}(V) \cup h_2^{-1}(V) \subset W$. Let K be a component of V which does not contain $\overline{c}_j$. By the *central point* of K we mean the unique point of K which corresponds to a point of the form $(0, y)$ in $[-1, 1] \times C$ as in Definition 5.7.

We may assume that $\overline{x}_n \in h_1^{-1}(V) \cap h_2^{-1}(V)$ for each n. Since $h_1(\overline{x}_n) \to h_1(\overline{x})$ and $h_1(\overline{x}_n) \to h_2(\overline{x})$, it follows from Remark 5.9 that for n sufficiently large, the $\overline{d}$-distance from $h_1(\overline{x}_n)$ to either endpoint of the component of V containing $h_1(\overline{x}_n)$ is greater than $M+1$. Since $\overline{d}(h_1(\overline{x}_n), h_2(\overline{x}_n)) \leq M$, it follows that $h_1(\overline{x}_n)$ and $h_2(\overline{x}_n)$ lie in the same component of V for n sufficiently large. Without loss of generality, we assume that this holds for each n.

For each positive integer n, let K_n denote the component of V which contains $h_1(\overline{x}_n)$ and $h_2(\overline{x}_n)$, and let $\overline{w}_n$ denote the central point of K_n. Let $\overline{y}_n = h_1^{-1}(\overline{w}_n)$ and $\overline{z}_n = h_2^{-1}(\overline{w}_n)$. Then $\overline{y}_n$ and $\overline{z}_n$ lie in the same component of W as $\overline{x}_n$. Since $\overline{x}_n \to \overline{x}$, $\overline{y}_n \to \overline{c}_i$, and $\overline{z}_n \to \overline{c}_i$, it follows that for n sufficiently large, $\overline{x}_n$ does not lie between $\overline{y}_n$ and $\overline{z}_n$ in a component of W. Again, we may assume that this holds for each n.

For each positive integer n, let $K_n = [\overline{a}_n, \overline{b}_n]$. We may assume that

$$\overline{d}(\overline{a}_n, \overline{w}_n) > M+1 \quad \text{and} \quad \overline{d}(\overline{b}_n, \overline{w}_n) > M+1$$

for each n.

We claim that for each positive n, $h_1(\overline{x}_n)$ and $h_2(\overline{x}_n)$ lie on the same side of $\overline{w}_n$ in K_n. We prove this by contradiction. Suppose that $h_1(\overline{x}_n)$ and $h_2(\overline{x}_n)$ lie on opposite sides of $\overline{w}_n$ for some n. Recall that $K_n = [\overline{a}_n, \overline{b}_n]$ and suppose without loss of generality that $h_1(\overline{x}_n)$ lies on the same side of $\overline{w}_n$ as $\overline{a}_n$. There is a point $\overline{p}_n \in W$ with $h_1(\overline{p}_n) = \overline{a}_n$. Moreover, $\overline{p}_n$, $\overline{x}_n$, $\overline{y}_n$, and $\overline{z}_n$ lie in the same component of W, and in this component, $\overline{p}_n$ is on one side of $\overline{x}_n$, while $\overline{y}_n$ and $\overline{z}_n$ are on the other side.

Now, using the monotonicity of h_2 on a component, we see that the arc in X_s with endpoints $h_1(\overline{p}_n)$ and $h_2(\overline{p}_n)$ contains both $\overline{a}_n$ and $\overline{w}_n$. This implies that $\overline{d}(h_1(\overline{p}_n), h_2(\overline{p}_n)) > M+1$. This is a contradiction and the claim is established. Since $A_i \subset V$ for each i, it follows from the special form of V and the claim that $A_i \to A$.

CASE 3: $\overline{x} = \overline{c}_i$ *for some* i. In this case A is just the point $\{\overline{c}_j\}$. This case is routine using the structure of the neighborhood of $\overline{c}$ given in Lemma 5.8. We leave the proof to the reader.

One of Cases 1–3 must hold so together they prove Theorem 5.10. ∎

THEOREM 5.11. *Suppose that $h_1, h_2 : X_s \to X_s$ are homeomorphisms which leave the endpoints of X_s fixed. Suppose that there is an $M > 0$ such that $\overline{d}(h_1(\overline{x}), h_2(\overline{x})) \leq M$ for all $\overline{x} \in C_{\overline{c}}$. Then h_1 and h_2 are isotopic.*

Proof. Let $H : X_s \times I \to X_s$ be defined in the following way.

Let $\overline{x} \in X_s$ and $t \in I$. By Theorem 4.3, there is a unique arc A_x connecting $h_1(\overline{x})$ and $h_2(\overline{x})$. Let $m \in \mathbb{N}$ be such that $\pi_m|_{A_x}$ is a homeomorphism into I_m. Let $g_m : \pi_m(A_x) \to A_x$ be the inverse of this homeomorphism. Let

$$H(\overline{x}, t) = g_m((1-t)\pi_m(h_1(\overline{x})) + t\pi_m(h_2(\overline{x}))).$$

If $\pi_k|_{A_x}$ is a homeomorphism, then

$$g_k((1-t)\pi_k(h_1(\overline{x})) + t\pi_k(h_2(\overline{x}))) \\ = g_m((1-t)\pi_m(h_1(\overline{x})) + t\pi_m(h_2(\overline{x}))).$$

So, $H(\overline{x}, t)$ is well defined. We now show that H is continuous.

Suppose that $(\overline{x}_i, t_i) \to (\overline{x}, t)$. Let A_i be the unique arc with endpoints $h_1(\overline{x}_i), h_2(\overline{x}_i)$. Then $h_1(\overline{x}_i) \to h_1(\overline{x})$ and $h_2(\overline{x}_i) \to h_2(\overline{x})$. So, if A is the unique arc connecting $h_1(\overline{x})$ and $h_2(\overline{x})$, whose existence is given by Theorem 4.3, then, by Theorem 5.10, $A_i \to A$ in the Hausdorff metric.

Case 1: *$\overline{x}$ is not an endpoint.* In this case the arc A connecting $h_1(\overline{x})$ and $h_2(\overline{x})$ does not contain an endpoint. Let V be a neighborhood of A of the form $V \approx C \times I$ with C a Cantor set and I an interval as in Lemma 5.1. Then there is an N such that for $n \geq N$, the arc A_n is contained in V. Now by Lemma 5.1, there is an m such that π_m is a homeomorphism of each component of V onto its image in I_m. Therefore for this m and for all $n \geq N$,

$$H(\overline{x}_n, t_n) = g_m((1-t_n)\pi_m(h_1(\overline{x}_n)) + t_n\pi_m(h_2(\overline{x}_n)))$$

and

$$H(\overline{x}, t) = g_m((1-t)\pi_m(h_1(\overline{x})) + t\pi_m(h_2(\overline{x}))).$$

So, clearly, $H(\overline{x}_n, t_n) \to H(\overline{x}, t)$.

Case 2: *$\overline{x}$ is an endpoint.* In this case $h_1(\overline{x}) = h_2(\overline{x}) = \overline{x}$ since the endpoints are assumed to be fixed. Therefore $A = \{\overline{x}\}$ and thus $\mathcal{A}_n \to \{\overline{x}\}$. This implies that $H(\overline{x}_n, t_n) \to \{\overline{x}\} = H(\overline{x}, t)$.

So, $H(\overline{x}, t)$ is a homotopy. We now show that it is an isotopy by showing that for each t, $h_t(\overline{x}) = H(\overline{x}, t)$ is one-to-one and onto.

First we show that h_t is one-to-one. Note that h_t permutes the composants of X_s the same way that h_1 and h_2 do. So, to show that h_t is one-to-one it will suffice to show that h_t restricted to a composant $C_{\overline{x}}$ is one-to-one. Now $C_{\overline{x}}$ is the arc-component of $\overline{x}$. This arc-component with the $\overline{d}$-metric is homeomorphic to either $\mathbb{R}$ or $\mathbb{R}_+$. Fix orderings on $C_{\overline{x}}$ and $C_{h_1(\overline{x})}$. Now h_1 and h_2 are homeomorphisms from $C_{\overline{x}}$ to $C_{h_1(\overline{x})}$ either preserving or reversing the orders of $C_{\overline{x}}$ and $C_{h_1(\overline{x})}$. However, since the $\overline{d}$-distance between h_1 and h_2 on $C_{\overline{x}}$ is bounded, these either both preserve the orders or both reverse the orders on $C_{\overline{x}}$ and $C_{h_1(\overline{x})}$ in the same way. Thus, $h_t|_{C_{\overline{x}}}$ is one-to-one.

To show that h_t is onto is similar. ∎

We now give the proof of the Isotopy Theorem as outlined at the beginning of this section.

Proof of the Isotopy Theorem. Let $h : X_s \to X_s$ be a homeomorphism. Let n be such that h^n leaves the endpoints of X_s fixed. By Theorem 4.6, there is an $M > 0$ and there is a $k \in \mathbb{Z}$ such that $\overline{d}(h^n(\overline{x}), \sigma^k(\overline{x})) \leq M$ for all $\overline{x} \in C_{\overline{c}}$. By Theorem 5.11, h^n and σ^k are isotopic. ∎

References

[1] J. M. Aarts and R. J. Fokkink, *The classification of solenoids*, Proc. Amer. Math. Soc. 111 (1991), 1161–1163.

[2] M. Barge, *Horseshoe maps and inverse limits*, Pacific J. Math. 121 (1986), 29–39.

[3] M. Barge and B. Diamond, *Homeomorphisms of inverse limit spaces of one-dimensional maps*, Fund. Math. 146 (1995), 171–187.

[4] M. Barge and S. Holte, *Nearly one-dimensional Hénon attractors and inverse limits*, Nonlinearity 8 (1995), 29–42.

[5] M. Barge and J. Martin, *Endpoints of inverse limit spaces and dynamics*, in: Continua with the Houston Problem Book, Lecture Notes in Pure and Appl. Math. 170, Dekker, New York, 1995, 165–182.

[6] M. Brown, *Some applications of an approximation theorem for inverse limits*, Proc. Amer. Math. Soc. 11 (1960), 478–483.

[7] K. Brucks and H. Bruin, *Subcontinua of inverse limit spaces of unimodal maps*, Fund. Math. 160 (1999), 219–246.

[8] K. Brucks and B. Diamond, *A symbolic representation of inverse limit spaces for a class of unimodal maps*, in: Continua with the Houston Problem Book, Lecture Notes in Pure and Appl. Math. 170, Dekker, New York, 1995, 207–226.

[9] H. Bruin, *Inverse limit spaces of post-critically finite tent maps*, Fund. Math. 165 (2000), 125–138.

[10] P. Collet and J.-P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems*, Birkhäuser, Boston, 1980.

[11] W. Dębski, *On topological types of the simplest indecompasable continua*, Colloq. Math. 49 (1985), 203–211.

[12] S. Holte, *Generalized horseshoe maps and inverse limits*, Pacific J. Math. 156 (1992), 297–305.

[13] L. Kailhofer, *A partial classification of inverse limit spaces of tent maps with periodic critical points*, Topology Appl. 123 (2002), 235–265.

[14] —, *A classification of inverse limit spaces of tent maps with periodic critical points*, Fund. Math. 177 (2003), 95–120.

[15] J. Keesling, *The group of homeomorphisms of a solenoid*, Trans. Amer. Math. Soc. 172 (1972), 119–131.

[16] W. de Melo and S. van Strien, *One-Dimensional Dynamics*, Springer, New York, 1993.

[17] J. Mioduszewski, *Mappings of inverse limits*, Colloq. Math. 10 (1963), 39–44.

[18] S. Nadler, *Continuum Theory. An Introduction*, Dekker, New York, 1992.

[19] R. C. Swanson and H. W. Volkmer, *Invariants of weak equivalence in primitive matrices*, Ergodic Theory Dynam. Systems 20 (2000), 611–626.

[20] W. T. Watkins, *Homeomorphic classification of certain inverse limit spaces with open bonding maps*, Pacific J. Math. 103 (1982), 589–601.
[21] R. F. Williams, *One-dimensional nonwandering sets*, Topology 6 (1967), 473–487.

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Received 15 April 2005;
in revised form 30 August 2005