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Fundamental groups of one-dimensional spaces

by

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Abstract. Let X be a metrizable one-dimensional continuum. We describe the fundamental group of X as a subgroup of its Čech homotopy group. In particular, the elements of the Čech homotopy group are represented by sequences of words. Among these sequences the elements of the fundamental group are characterized by a simple stabilization condition. This description of the fundamental group is used to give a new algebro-combinatorial proof of a result due to Eda on continuity properties of homomorphisms from the fundamental group of the Hawaiian earring to that of X.

1. Introduction. In the 1950s Curtis and Fort [6, 7, 8] studied properties of fundamental groups of locally complicated spaces. Starting with the work of Cannon and Conner as well as Eda and Kawamura at the turn of the millennium (see e.g. [2, 11]) the investigation of fundamental groups of such spaces got a new impetus. Meanwhile, properties of fundamental groups of one-dimensional (cf. for instance [1, 3, 9, 10]) and planar (see [5, 12]) spaces were derived. Especially the description of such fundamental groups in terms of words turned out to be useful. Cannon and Conner gave such a description for the fundamental group of the Hawaiian earring (see Figure 1, left), and in Akiyama et al. [1] we gave a representation of the fundamental group $\pi(\triangle)$ of the Sierpiński gasket $\triangle$ (see Figure 1, right) in terms of words. Since $\triangle$ is a one-dimensional subset of $\mathbb{R}^2$, it is known from Eda and Kawamura [11] that $\pi(\triangle)$ can be embedded in the Čech homotopy group $\check{\pi}(\triangle)$, which is known to be a projective limit of free groups. In [1] we were able to endow the projective limit defining $\check{\pi}(\triangle)$ with a word structure. Moreover, we could characterize the elements of the subgroup $\pi(\triangle)$ by a simple stabilizing condition. Recently, Diestel and Sprüssel [9] provided descriptions of Freudenthal compactifications of locally finite connected graphs by similar means.

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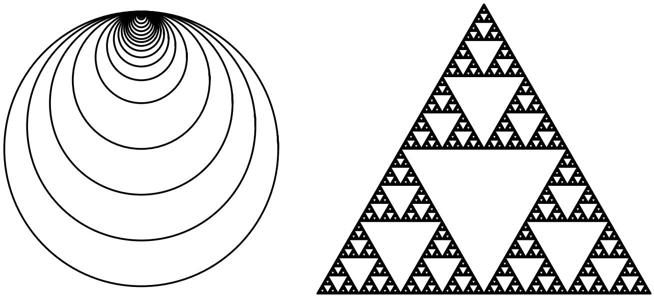

Fig. 1. The Hawaiian earring (left) and the Sierpiński gasket (right)—two well studied examples of locally complicated spaces.

The first aim of this paper is to extend this kind of description to a large class of spaces. Indeed, we are able to describe the fundamental group of any metrizable one-dimensional continuum X in terms of words. As an important technical tool we use a slight modification of a handle body construction employed by Cannon and Conner [3]. In particular, with the help of this construction we equip the space X with a structure that allows us to encode loops in X by words. While in the construction for the Sierpiński gasket $\triangle$ the letters correspond to (local) cut points of $\triangle$, in our setting letters represent (local) cut sets. This generalization turns out appropriate to extend the approach in [1] for the special case of the Sierpiński gasket to the class of all metrizable one-dimensional continua.

The difference of our treatment compared with other approaches to this topic is twofold: Firstly, we refrain from describing a loop by (an infinite sequence of) edges but instead we use a sequence (indexed by the approximation level) of finite words whose letters correspond to the (local) cut sets the loop crosses. Each word provides information which areas (separated by the cut set letters) the loop traverses. In combination with the handle body construction this finer and finer approximation to the loop as well as to the space X from outside turns out to do the right job. It avoids complications occurring when approximating the loop by edge sequences and the space from inside, where usually a topological closure operation is involved. The second new ingredient concerns the use of semigroups instead of groups. It is due to the fact that the word sequences describing loops carry a natural projective semigroup structure and homotopy of loops is reflected by appropriate cancelation rules applied to semigroup words. Altogether, the semigroup structure provides the crucial tool to identify those elements in $\check{\pi}(X)$ which correspond to homotopy classes of X.

In the second part of the paper our description of the fundamental group is applied in order to give a quite elementary algebro-combinatorial proof of a result due to Eda [10]. We show that each homomorphism from the fundamental group of the Hawaiian earring E to the fundamental group of a metrizable one-dimensional continuum X is induced by a continuous mapping $\psi : E \to X$ (Theorem 5.10). Furthermore, we obtain an "infinite homomorphism property" for such homomorphisms (Theorem 5.3).

The paper is organized as follows. In Section 2 we define the handle bodies and establish some preliminary results necessary for the proof of our first main result. As indicated above, some steps are similar to the case of the Sierpiński gasket, other parts need different ideas in order to capture the considerably more general situation. In Section 3 we state our description of the fundamental group (Theorem 3.2) and finish its proof. This result contains a simple criterion for an element of the Čech homotopy group to belong to the fundamental group of a given space. Moreover, it allows one to find a canonical "shortest" representative for each element of the fundamental group. At the end of that section we indicate how our handle body construction applies to the Sierpiński carpet (sometimes also called Menger curve) as an example. Section 4 contains cancelation rules for the words in the fundamental group. These rules are important in Section 5 where we prove Eda's result on homomorphisms mentioned above by means of our word description of the fundamental group. At the beginning of Section 5 for the convenience of the reader we provide guidelines to our proof of Eda's theorem which requires some technical effort.

2. Definition of the handles. Throughout this paper let X be a metrizable one-dimensional continuum([1]). Then (see Hurewicz and Wallman [15] or Cannon and Conner [3]) X can be embedded in the three-dimensional Euclidean space and represented as the intersection of handle bodies H_n, $n \in \mathbb{N}$, such that

$$H_0 \supset H_1 \supset H_2 \supset \cdots \supset \bigcap_{n\in\mathbb{N}} H_n = X.$$

Each handle body H_n consists of finitely many 0-handles joined by finitely many 1-handles. The 0-handles as well as the 1-handles are compact subsets of $\mathbb{R}^3$ homeomorphic to a closed ball. The diameter of each of these handles is bounded from above by $1/n$ in the maximum norm $\|\cdot\|_\infty$. Each 1-handle h is attached to two adjacent 0-handles by an attaching disk. These attaching disks are separated by an intermediate belt disk $B(h)$ contained in the 1-handle. This construction shows that H_n can be realized as a CW complex in $\mathbb{R}^3$. Without loss of generality we assume that each 1-handle in H_n has nonempty intersection with X and that each 0-handle is attached to at least

([1]) Note that in view of Urysohn's metrization theorem for compact spaces, metrizability is equivalent to second-countability.

one 1-handle (see Figure 2 for an example). Thus the connectedness of X implies that H_n is connected.

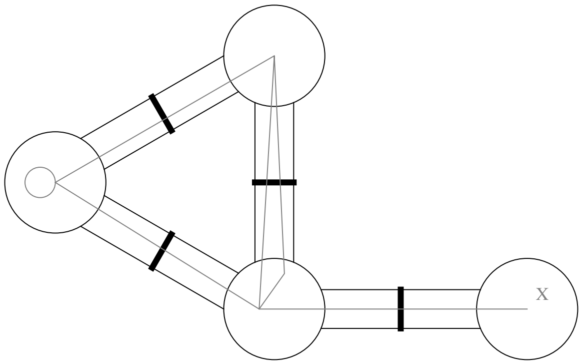

Fig. 2. An example of a handle body. The set X is indicated in gray. At this level n, the big triangle in X is "seen" to be a nontrivial loop in the handle body, while the small circle on the left as well as the thin triangle in the center are not captured by this handle body. To capture them, a finer handle body (i.e., a larger value of n) is needed.

Consider a fixed 0-handle h in H_n. Observe that the union U of all the belt disks of the 1-handles attached to h forms a separator of H_n. The *star of* h, $\mathrm{St}(h)$, is the component of $H_n \setminus U$ containing h. Note that each belt disk of H_n is contained in the boundary of exactly two stars.

With H_n we associate a graph $\langle V_n, E_n \rangle$ where the set V_n of vertices consists of the 0-handles of H_n, and two vertices are connected by an edge in E_n if and only if the associated 0-handles are connected by a 1-handle. Thus the edges are in a one-to-one correspondence to the 1-handles of H_n. Note that the graph $\langle V_n, E_n \rangle$ can be drawn in $\mathbb{R}^3$ as a deformation retract of H_n in the following way.

For every 1-handle h of H_n choose a simple arc in h joining the attaching disks. By the CW structure of the handle body there is a deformation retraction of the 1-handle onto the union of this simple arc and the two attaching disks. This can be done in such a way that $B(h)$ is retracted to a single point b_h, which we will call the *midpoint* of $B(h)$. By the Homotopy Extension Theorem for CW complexes this retraction can be performed for each 1-handle of H_n separately. Next, for every 0-handle h we choose an arbitrary point m_h (called the *midpoint* of h) in the interior of h and arcs connecting m_h with the end point of each arc contained in the attached retracted 1-handles. The 0-handle h can be deformation retracted onto these arcs. Again, by the Homotopy Extension Theorem for CW complexes this retraction can be performed for each 0-handle of H_n separately. The result of all these deformation retractions is the deformation retraction r_n which deformation retracts H_n onto the drawing of $\langle V_n, E_n\rangle$.

In the following we assume without loss of generality that H_n is defined in such a way that $\langle V_n, E_n\rangle$ does not contain cycles of length ≤ 2. Indeed, cycles of length ≤ 2 can easily be ruled out by splitting a 1-handle by an intermediate 0-handle at certain places.

Now we explicate how H_{n+1} is embedded in H_n. For each n the handle body H_{n+1} lies in the interior of H_n and if a handle h' of H_{n+1} intersects the belt disk $B(h)$ of a 1-handle h of H_n then we may assume that h' is a 1-handle of H_{n+1} and $h' \cap B(h) = B(h')$. In this case we call $B(h)$ a *predecessor* of $B(h')$.

Next we will describe loops with base point $x_0 \in X$. The base point x_0 is assumed to be contained in a belt disk of H_0 and, as $x_0 \in X$, also in a belt disk of H_n for each $n \geq 0$; indeed, without loss of generality we assume that x_0 is the midpoint of each of these belt disks.

For fixed n consider a loop f_n in the pointed space (H_n, x_0). The word $\sigma_n(f_n)$ representing f_n is defined over the alphabet

$$D_n := \{B(h) \mid B(h) \cap X \neq \emptyset,\ h \text{ a 1-handle in } H_n\}$$

in the following way. The pre-images $\{f_n^{-1}(B) \mid B \in D_n\}$ form a finite family of disjoint compact subsets of the interval $[0,1]$. Therefore this family is separated, i.e., there is $m \in \mathbb{N}$ such that for all $i \in \{1,\dots,m\}$ the set $f_n^{-1}(B) \cap [(i-1)/m, i/m]$ is nonempty for at most one $B = B_i$. We list these letters B_i as i increases and in the arising sequence we cancel out consecutive repetitions of letters. Thus we obtain a finite word $\sigma_n(f_n) := B_1 \dots B_k$ over D_n which is independent of the chosen m and contains all belts the loop f_n traverses in the right ordering.

Indeed, since $X \subseteq H_n$ for all $n \in \mathbb{N}$, for a loop $f \in (X, x_0)$ the word $\sigma_n(f)$ is defined for all $n \in \mathbb{N}$ and represents f at approximation level n.

We define the following relation $\sim_n$ on D_n. For $B_1, B_2 \in D_n$ we write $B_1 \sim_n B_2$ if and only if $B_1 \neq B_2$ and there is a 0-handle h in H_n such that $B_1, B_2 \subseteq \overline{\mathrm{St}(h)}$. We call a word $B_1 \dots B_k$ over D_n *admissible* if

(1) $B_1 = B_k$ and $x_0 \in B_1$,
(2) $B_i \sim_n B_{i+1}$ $(1 \leq i \leq k-1)$.

For each loop f based at x_0 the word $\sigma_n(f)$ is obviously admissible.

We now associate with each admissible word $\omega_n = B_1 \dots B_k$ over D_n a *canonical loop* $L(\omega_n)$ in (H_n, x_0). It is defined as follows. Since $B_i \sim B_{i+1}$ and $\langle V_n, E_n\rangle$ has no cycles of order 2 there is a unique 0-handle attached to the 1-handles corresponding to B_i and B_{i+1}. Let m_i be the midpoint of this 0-handle. Connect x_0 with m_0 and then m_i with m_{i+1} $(i \in \{0,\dots,k-1\})$ and finally m_{k-1} with x_0 by arcs contained in the graph $\langle V_n, E_n\rangle$. The parametrization of this loop $L(\omega_n)$ will mostly be irrelevant. In places where it becomes important (e.g. in the proof of Proposition 3.1) this will be made explicit. Obviously, $\sigma_n(L(\omega_n)) = \omega_n$.

If $\omega_n = B_1 \dots B_k$ satisfies only condition (2) a *canonical path* $L(\omega_n)$ is associated with ω_n in the same way. To keep the notation simple, the loop (or path, respectively) $L(\omega_n)$ will also be denoted by ω_n.

PROPOSITION 2.1. *Let $f : [0,1] \to H_n$ be a loop based at x_0. Then f and the canonical loop $\sigma_n(f)$ are homotopic in H_n.*

Proof. First note that f is homotopic to $r_n \circ f$, where r_n is the deformation retraction of H_n onto $\langle V_n, E_n \rangle$. Let $\sigma_n(f) = B_1 \dots B_k$. For every $i \in \{1, \dots, k\}$ there is a maximal interval $[s_i, t_i]$ such that $r_n \circ f(s_i) = r_n \circ f(t_i) = r_n(B_i)$, $r_n(f([s_i, t_i]) \cap \bigcup_{B \in D_n} B) = \{r_n(B_i)\}$ and $0 = s_1 \le t_1 < s_2 \le t_2 < \dots < s_k \le t_k = 1$. This means that the path $r_n \circ f([s_i, t_i])$ is contained in $\mathrm{St}(h_1) \cup B_i \cup \mathrm{St}(h_2)$ where h_1 and h_2 are the two 0-handles with $\overline{\mathrm{St}}(h_1) \cap \overline{\mathrm{St}}(h_2) = B_i$. By our assumptions on the graph $\langle V_n, E_n \rangle$ associated with H_n the set $\mathrm{St}(h_1) \cup B_i \cup \mathrm{St}(h_2)$ is simply connected, and hence the restriction $r_n \circ f \upharpoonright [s_i, t_i]$ is homotopic to the constant path in $r_n(B_i)$.

Moreover, the conditions on s_i and t_i imply that $r_n \circ f([t_i, s_{i+1}])$ is a subset of $r_n(\mathrm{St})$ where St is the star of H_n whose closure contains B_i and B_{i+1}, and hence $r_n \circ f \upharpoonright [t_i, s_{i+1}]$ is homotopic to the canonical path between $r_n(B_i)$ and $r_n(B_{i+1})$.

Putting the pieces together we obtain the assertion. ∎

The set of all admissible words over D_n is called S_n. If we endow S_n with the operation "$\cdot$" defined by concatenation of words where the first letter of the second word is omitted, we obtain a semigroup $(S_n, \cdot)$.

For each $n \ge 1$ define a mapping $\gamma_n : S_n \to S_{n-1}$ where for $\omega_n = B_1 \dots B_k \in S_n$ the image $\gamma_n(\omega_n)$ is defined as follows. Among the letters of ω_n we omit those which have no predecessor and replace each of the others by its predecessor. Finally, we cancel consecutive repetitions of letters. Obviously, the resulting word is admissible and therefore belongs to S_{n-1}. With these mappings γ_n ($n \ge 1$), which are easily seen to be compatible with concatenation, we get a projective limit of semigroups $\varprojlim S_n := \{(\omega_n)_{n \ge 0} \mid \gamma_k(\omega_k) = \omega_{k-1} \text{ for all } k \ge 1\}$. For $n > k$ the mapping $\gamma_{nk} : S_n \to S_k$ denotes the composition $\gamma_{k+1} \circ \dots \circ \gamma_n$.

Let $S(X, x_0)$ be the set of all loops in X based at x_0. The set $S(X, x_0)$ is a groupoid with respect to the concatenation of loops. Consider a loop $f \in S(X, x_0)$. Then, obviously, $\gamma_n(\sigma_n(f)) = \sigma_{n-1}(f)$. Thus each sequence $(\sigma_n(f))_{n \ge 0}$ is contained in the projective limit $\varprojlim S_n$ and we may define the map

$$\sigma : S(X, x_0) \to \varprojlim S_n, \qquad f \mapsto (\sigma_n(f))_{n \ge 0},$$

which is a groupoid homomorphism.

Our next aim is to describe how the homotopy of two loops f and g is reflected in their word representations $\sigma_n(f)$ and $\sigma_n(g)$. To this end we define the following equivalence relation $\equiv_n$ on S_n.

An elementary move on subwords of words in S_n consists of substitutions of the form

$$B_1B_2B_3 \leftrightarrow B_1B_3 \quad (\text{if } B_1 \neq B_3) \quad \text{or} \quad B_1B_2B_1 \leftrightarrow B_1$$

where B_1, B_2 and B_3 are all contained in the closure of a star $\mathrm{St}(h)$ for a 0-handle $h \in H_n$ (see Figure 3). We say that two words ω_n and ω_n' in S_n are equivalent, and write $\omega_n \equiv_n \omega_n'$, if ω_n' can be obtained from ω_n by finitely many elementary moves.

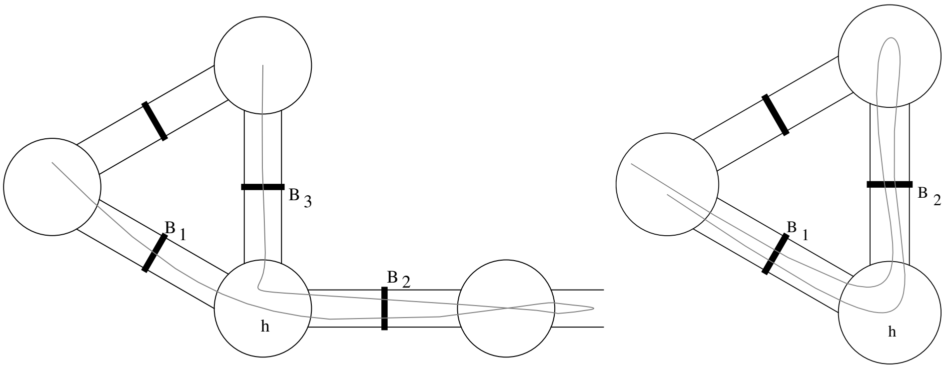

Fig. 3. The left path demonstrates the elementary move $B_1B_2B_3 \leftrightarrow B_1B_3$. The path on the right illustrates $B_1B_2B_1 \leftrightarrow B_1$.

We call a word *reduced* if it does not contain three consecutive letters of the form $B_1B_2B_3$ where B_1, B_2 and B_3 are all contained in the closure of a star $\mathrm{St}(h)$ for a 0-handle $h \in H_n$. Let G_n be the set of reduced words in S_n.

PROPOSITION 2.2.

(1) *Every $\equiv_n$ equivalence class of S_n contains a unique reduced word. Thus the mapping* $\mathrm{Red}_n : S_n \to G_n$ *which assigns to each ω_n the reduced word in its $\equiv_n$ class is well defined.*

(2) *The operation*

$$* : G_n \times G_n \to G_n, \qquad (\omega_n, \omega_n') \mapsto \mathrm{Red}_n(\omega_n \cdot \omega_n'),$$

is a group operation on G_n.

(3) *The group $(G_n, *)$ is isomorphic to the fundamental group $\pi(H_n, x_0)$ with the isomorphism $\varphi_n : [f]_n \mapsto \mathrm{Red}_n(\sigma_n(f))$ where $f : [0,1] \to H_n$ is a loop based at x_0 and $[f]_n$ is the homotopy class of f in H_n.*

(4) *The reduction map* $\mathrm{Red}_n : S_n \to G_n$ *is a semigroup epimorphism, i.e., $(G_n, *)$ is isomorphic to* $(S_n/\ker(\mathrm{Red}_n), \cdot)$.

Proof. Note that $\pi(H_n, x_0) \cong \pi(\langle V_n, E_n\rangle, x_0)$ since $\langle V_n, E_n\rangle$ is a deformation retract of H_n. Furthermore, $\pi(\langle V_n, E_n\rangle, x_0)$ is isomorphic to a free group F generated by the edges not contained in a fixed spanning tree of $\langle V_n, E_n\rangle$ (see [17, Corollary 7.35]). To each product $g_1 \dots g_k$ of generators of F we can associate a unique word by connecting the edges g_i by intermediate unique paths in the spanning tree. Obviously, the word obtained is reduced. On the other hand, reversing this process, two different reduced words give rise to two different products of generators of F and therefore correspond to two nonhomotopic paths. Thus we obtain a bijective correspondence between reduced words and homotopy classes of (H_n, x_0).

To prove (1) we start with an arbitrary word in S_n and apply elementary moves until we arrive at a reduced word. This shows that any $\equiv_n$ class contains at least one reduced word. However, if two different reduced words were $\equiv_n$ equivalent they could be transformed into each other by elementary moves. Thus the loops corresponding to the reduced words would be homotopic, contrary to the above mentioned bijection between reduced words and homotopy classes.

The above arguments imply that the operation $*$ is compatible with the group operation in $\pi(H_n, x_0)$, which proves (2) and (3). Note that φ_n is well defined due to Proposition 2.1. Assertion (4) follows immediately from the definition of $*$.

For a related proof see [1, Proposition 2.3]. ∎

Now we are going to define a projective limit on the groups $(G_n, *)$, $n \in \mathbb{N}$, and relate it to the semigroup limit $\varprojlim S_n$.

Proposition 2.3.

(1) *For $n \geq 1$ the map*

$$\delta_n : G_n \to G_{n-1}, \qquad \omega_n \mapsto \mathrm{Red}_{n-1}(\gamma_n(\omega_n)),$$

is a group homomorphism.

(2) *Set*

$$\varprojlim G_n := \{(\omega_n)_{n\geq 0} \mid \delta_k(\omega_k) = \omega_{k-1} \text{ for all } k \geq 1\}.$$

Then the mapping

$$\mathrm{Red} : \varprojlim S_n \to \varprojlim G_n, \qquad (\omega_n)_{n\geq 0} \mapsto (\mathrm{Red}_n(\omega_n))_{n\geq 0},$$

is a well defined semigroup homomorphism.

Proof. (1) Let $\omega_n, \omega_n' \in G_n$. A direct calculation yields

$$\delta_n(\omega_n * \omega_n') = \mathrm{Red}_{n-1}(\gamma_n(\mathrm{Red}_n(\omega_n \cdot \omega_n')))$$

and

$$\delta_n(\omega_n) * \delta_n(\omega_n') = \mathrm{Red}_{n-1}(\gamma_n(\omega_n \cdot \omega_n')).$$

Since for each $\alpha_n \in S_n$ we know that α_n and $\gamma_n(\alpha_n)$ are homotopic in H_{n-1}, and α_n and $\mathrm{Red}_n(\alpha_n)$ are homotopic in H_n, we deduce that $\gamma_n(\mathrm{Red}_n(\omega_n \cdot \omega_n'))$ and $\gamma_n(\omega_n \cdot \omega_n')$ are homotopic in H_{n-1}. Thus Proposition 2.2(1) implies that δ_n is a homomorphism.

(2) is an immediate consequence of the commutativity of the diagram

$$\begin{tikzcd} S_n \arrow[r, "\gamma_n"] \arrow[d, "\mathrm{Red}_n"'] & S_{n-1} \arrow[d, "\mathrm{Red}_{n-1}"] \\ G_n \arrow[r, "\delta_n"] & G_{n-1} \end{tikzcd} \tag{2.1}$$

which follows in a straightforward manner. ∎

REMARK 2.4. Note that in contrast to the setting of the Sierpiński gasket in [1], G_{n-1} can contain loops that are no longer present in G_n. Thus, in our general setting, the mappings δ_n need not be surjective.

We now consider the Čech homotopy group $\check{\pi}(X, x_0)$. For a definition we refer to Mardešić and Segal [16] ([2]).

PROPOSITION 2.5. *The Čech homotopy group $\check{\pi}(X, x_0)$ is isomorphic to* $\varprojlim G_n$.

Proof. A proof of this proposition is in essence already contained in [3]. For the sake of completeness we briefly repeat the key arguments.

For a subset A of a metric space let $(A)_\varepsilon$ denote the ε-neighborhood of A. Now we consider

$$U_n := \{(\mathrm{St}(h))_{\varepsilon_n} \mid h \text{ is a 0-handle of } H_n\}$$

where ε_n with $\lim_n \varepsilon_n = 0$ is chosen in such a way that

$$\overline{\mathrm{St}(h_1)} \cap \overline{\mathrm{St}(h_2)} \neq \emptyset \iff (\mathrm{St}(h_1))_{\varepsilon_n} \cap (\mathrm{St}(h_2))_{\varepsilon_n} \neq \emptyset$$

for all 0-handles h_1, h_2 of H_n. The family $(U_n)_{n\geq 0}$ is cofinal in the set of all finite open coverings of X since each 1-handle of H_n has nonempty intersection with X. From this construction we conclude that the nerve ([3]) of U_n is a deformation retract of H_n and thus by Proposition 2.2(3) the group G_n is the fundamental group of this nerve. This implies the result. ∎

REMARK 2.6. Note that the projective limit of fundamental groups of handle bodies occurring in the proof of [3, Theorem 5.11] is strongly related to our construction. Indeed, this projective limit contains the Čech homotopy group of (X, x_0) as a subgroup. The converse inclusion may fail in the

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([2]) Note that in [16] the Čech homotopy group is called the *shape group*.

([3]) For the definition of *nerve* see Hatcher [13, p. 257].

setting of [3] since there it is not assumed that each 1-handle of H_n has nonempty intersection with X. A special case of this construction is already contained in [8, Section 3].

From Proposition 2.5 we get the following result.

PROPOSITION 2.7. *The mapping*

$$\varphi : \pi(X, x_0) \to \varprojlim G_n, \qquad [f] \mapsto \mathrm{Red}(\sigma(f)),$$

is a group monomorphism.

Proof. This follows by combining Proposition 2.5 and the fact that the fundamental group of a one-dimensional continuum can be embedded in its Čech homotopy group in a canonical way (cf. [11, Theorem 1.1] and [3, Theorem 5.11]). ∎

Summing up we arrive at the following theorem.

THEOREM 2.8. *The fundamental group $\pi(X, x_0)$ of a metrizable one-dimensional continuum (X, x_0) is isomorphic to a subgroup of the Čech homotopy group $\check{\pi}(X, x_0) \cong \varprojlim G_n$. Moreover, the following diagram commutes:*

$$\begin{tikzcd} S(X, x_0) \arrow[r, "\sigma"] \arrow[d, "{[\cdot]}"'] & \varprojlim S_n \arrow[d, "\mathrm{Red}"] \\ \pi(X, x_0) \arrow[r, "\varphi_n"] & \varprojlim G_n \end{tikzcd}$$

Our aim is now to describe the range of φ, which provides a description of $\pi(X, x_0)$ as a subgroup of the projective limit $\varprojlim G_n$ of free groups.

3. Word description of the fundamental group. We associate with a fixed element $(\omega_n)_{n\ge 0} = (B_{n1} \dots B_{nk_n})_{n\ge 0}$ in $\varprojlim S_n$ a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ with vertex set $\mathcal{V}$ and the set $\mathcal{E}$ of directed edges. We think of the graph $\mathcal{G}$ as organized in rows of horizontally ordered vertices: in the nth row, $n \ge 0$, we have for every letter appearing in the word ω_n a corresponding vertex, i.e., $\mathcal{V} = \{(n, j) \mid n \ge 0,\ 1 \le j \le k_n\}$. Edges connect certain vertices in row n to vertices in row $n + 1$, namely, $((n, i), (n + 1, j)) \in \mathcal{E}$ if and only if B_{ni} is a predecessor of $B_{n+1,j}$ and in the course of γ_{n+1} that maps ω_{n+1} to ω_n the belt disk $B_{n+1,j}$ is mapped to B_{ni}. Consequently, any vertex (n, i) in row n has at least one successor in row $n + 1$, and the vertex (n, i) has a predecessor in row $n - 1$ if and only if the letter $B_{ni} \in D_n$ has a predecessor in D_{n-1}.

The graph $\mathcal{G}$ is used in the proof of the following proposition.

Proposition 3.1. *For every* $(\omega_n)_{n\ge0} \in \varprojlim S_n$ *there exists a loop* $f \in S(X,x_0)$ *such that* $\mathrm{Red}(\sigma(f)) = \mathrm{Red}((\omega_n)_{n\ge0})$, *i.e.*, $\mathrm{ran}(\mathrm{Red}\circ\sigma) = \mathrm{ran}(\mathrm{Red})$.

Proof. Let $(\omega_n)_{n\ge0} = (B_{n1}\dots B_{nk_n})_{n\ge0}$ be a fixed element of $\varprojlim S_n$. We will inductively define a sequence of functions $f_n : [0,1] \to H_n$, $n \ge 0$, such that f_n parametrizes the canonical loop associated with ω_n.

We start with $n = 0$, $\omega_0 = B_{01}\dots B_{0k_0}$, and divide $[0,1]$ into $2k_0 - 1$ subintervals of equal length by the points

$$0 = u_{01} < v_{01} < u_{02} < v_{02} < \dots < u_{0k_0} < v_{0k_0} = 1.$$

Define $f_0(t)$ to be constant and equal to the midpoint of the belt disk B_{0i} for $t \in [u_{0i}, v_{0i}]$, $1 \le i \le k_0$, and f_0 to parametrize the canonical path of the word $B_{0i}B_{0,i+1}$ for $t \in [v_{0i}, u_{0,i+1}]$, $1 \le i < k_0$. Obviously $\sigma_0(f_0) = \omega_0$.

Suppose f_n is already defined in such a way that $f_n(t)$ is equal to the midpoint of B_{ni} for $t \in [u_{ni}, v_{ni}]$, $1 \le i \le k_n$, f_n is the canonical path of the word $B_{ni}B_{n,i+1}$ for $t \in [v_{ni}, u_{n,i+1}]$, $1 \le i < k_n$, and thus $\sigma_k(f_n) = \gamma_{nk}(\sigma_n(f_n)) = \gamma_{nk}(\omega_n) = \omega_k$ for all $k \le n$. We now explain in detail how to define $f_{n+1}(t)$ for $t \in [u_{n1}, v_{n1}]$ and $t \in [v_{n1}, u_{n2}]$. In the equality $\gamma_{n+1}(\omega_{n+1}) = \omega_n$ we analyze the action of γ_{n+1} on the individual letters of ω_{n+1}: Figure 4 shows a part of the graph $\mathcal{G}$ we have

$$\begin{tikzcd}[column sep=small]
& B_{n1} \arrow[dl] \arrow[dr] & & & & & & B_{n2} \arrow[dl] \arrow[dr] & & \dots \\
B_{n+1,1} & \dots & B_{n+1,i_1} & B_{n+1,i_1+1} & \dots & B_{n+1,i_2} & B_{n+1,i_2+1} & \dots & B_{n+1,i_3} & \dots
\end{tikzcd}$$

Fig. 4

associated with $(\omega_n)_{n\ge0}$ at the beginning of this section and has the following interpretation: $B_{n+1,1}$ and B_{n+1,i_1} are the first and last letter in ω_{n+1} that are mapped to the first letter B_{n1} of ω_n by γ_{n+1}, respectively; B_{n+1,i_1+1} up to B_{n+1,i_2} have no predecessor in D_n and disappear by applying γ_{n+1}.

Now we define $f_{n+1}(t)$ for $t \in [u_{n1}, v_{n1}]$ analogously to f_0 in $[0,1]$: divide $[u_{n1}, v_{n1}]$ into $2i_1 - 1$ subintervals of equal length and define f_{n+1} in these subintervals alternately to be constant and equal to the midpoint of $B_{n+1,i}$ for $1 \le i \le i_1$, and to be the canonical path of the word $B_{n+1,i}B_{n+1,i+1}$ for $1 \le i \le i_1 - 1$.

Next, the interval $[v_{n1}, u_{n2}]$ is divided into $2(i_2 - i_1) + 1$ subintervals. Here f_{n+1} alternately is equal to the canonical path of the word $B_{n+1,i}B_{n+1,i+1}$ for $i_1 \le i \le i_2$, and is constant and equal to the midpoint of $B_{n+1,i}$ for $i_1 + 1 \le i \le i_2$.

In the same manner we proceed with the remaining intervals and obtain a loop f_{n+1} satisfying our requirements.

We compare f_n with f_{n+1}. For $1 \leq i \leq k_n$:

$$t \in [u_{ni}, v_{ni}] \,:\, \begin{cases} f_n(t) & \text{constant and equal to the midpoint of } B_{ni}, \\ f_{n+1}(t) & \text{stays in the union of } B_{ni} \text{ and the two stars} \\ & \text{of } H_n \text{ containing } B_{ni} \text{ in their closure,} \end{cases}$$

and for $1 \leq i \leq k_n - 1$:

$$t \in [v_{ni}, u_{n,i+1}] \,:\, \begin{cases} f_n(t) & \text{equal to the canonical path} \\ & \text{of the word } B_{ni}B_{n,i+1}, \\ f_{n+1}(t) & \text{stays in the star of } H_n \text{ containing} \\ & B_{ni} \text{ and } B_{n,i+1} \text{ in its closure.} \end{cases}$$

Summing up we obtain $\|f_n - f_{n+1}\|_\infty \leq 3/n$ where $\|\cdot\|_\infty$ denotes the maximum norm for $t \in [0,1]$. Consequently, f_n converges for $n \to \infty$ uniformly to a continuous $f : [0,1] \to X$.

By construction we have $f_m(u_{ni}) \in B_{ni}$, $1 \leq i \leq k_n$, for all $m \geq n$, and thus also $f(u_{ni}) \in B_{ni}$, $1 \leq i \leq k_n$. This means that $\sigma_n(f)$ contains at least all letters appearing in the word ω_n in proper order, but it may happen that $\sigma_n(f)$ contains further letters from D_n between the B_{ni}, and that some of the B_{ni} appear more than once. To illustrate this we consider the interval $[u_{ni}, u_{n,i+1}]$ (see also Figure 5): let St_1 and St_2 be the two

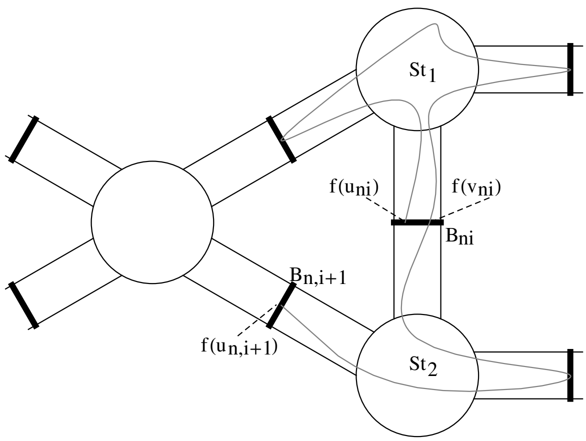


Fig. 5

stars containing B_{ni} in their closures. f_{n+1} and all f_m with $m \geq n+1$ stay for $t \in (u_{ni}, u_{n,i+1})$ in the interior of the (simply connected) union of the closures of the two stars $\mathrm{int}(\overline{\mathrm{St}_1} \cup \overline{\mathrm{St}_2})$ of H_n (interior as a subset of H_n). This implies that $f = \lim_{m\to\infty} f_m$ stays in the union of the closed stars $\overline{\mathrm{St}_1} \cup \overline{\mathrm{St}_2}$. Hence, $\sigma_n(f\restriction[u_{ni}, u_{n,i+1}]) = B_{ni}Q_{j_1}Q_{j_2}\ldots Q_{j_l}B_{n,i+1}$, $l \geq 0$, where the Q_{j_k} are contained in the set of belts $\{Q_1, \ldots, Q_L\}$ associated with the stars St_1 and St_2. However, since $f([u_{ni}, u_{n,i+1}]) \subseteq \overline{\mathrm{St}_1} \cup \overline{\mathrm{St}_2}$, all the possibly occurring letters $Q_{j_1} \ldots Q_{j_l}$ cancel out in the reduction process and we obtain $\mathrm{Red}_n(\sigma_n(f{\upharpoonright}[u_{ni}, u_{n,i+1}])) = B_{ni}B_{n,i+1}$, and hence altogether $\mathrm{Red}_n(\sigma_n(f)) = \mathrm{Red}_n(\omega_n)$. ∎

Theorem 2.8 implies that $\pi(X, x_0)$ can be considered as a subgroup of $\varprojlim G_n$. Now we characterize the elements of this subgroup and thus describe $\pi(X, x_0)$.

THEOREM 3.2. *An element* $(\omega_n)_{n\geq 0}$ *of* $\varprojlim G_n$ *is in* $\mathrm{ran}(\varphi) = \varphi(\pi(X, x_0))$ *and therefore represents an element of* $\pi(X, x_0)$ *if and only if for all* $k \geq 0$ *the sequence* $(\gamma_{nk}(\omega_n))_{n\geq k}$ *is eventually constant.*

In what follows, n_k is an index for which $\gamma_{nk}(\omega_n) = \gamma_{n_k k}(\omega_{n_k})$ for all $n \geq n_k$.

REMARK 3.3. Since the Freudenthal compactification of a locally finite connected graph is a metrizable one-dimensional continuum this result contains the main result of [9] (see [9, Theorem 15]) as a special case.

Recall that γ_{nk} is the composition $\gamma_{k+1} \circ \gamma_{k+2} \circ \cdots \circ \gamma_n : S_n \to S_k$. Analogously we define δ_{nk} to be the composition of the corresponding δ_i's.

The proof of Theorem 3.2 runs along the same lines as in the case of the Sierpiński gasket [1, Section 3.2]. However, in order to make the presentation self-contained we recall some of the details.

Let $P_1 \ldots P_m$, $Q_1 \ldots Q_k$ be two words over some alphabet. We write $P_1 \ldots P_m \preceq Q_1 \ldots Q_k$ if there exists $\alpha : \{1, \ldots, m\} \to \{1, \ldots, k\}$, α injective and order preserving, such that $P_i = Q_{\alpha(i)}$ for all $i \in \{1, \ldots, m\}$.

LEMMA 3.4. *Let* $\omega_n, \omega'_n \in S_n$. *Then*

(1) $\mathrm{Red}_n(\omega_n) \preceq \omega_n$,
(2) $\omega_n \preceq \omega'_n$ *implies* $\gamma_{nk}(\omega_n) \preceq \gamma_{nk}(\omega'_n)$ *for all* $k \leq n$,
(3) *if* $(\omega_k)_{k\geq 0} \in \varprojlim G_n$ *then* $\gamma_{nk}(\omega_n) \preceq \gamma_{n+1,k}(\omega_{n+1})$ *for all* $k \leq n$.

Proof. The assertions (1) and (2) follow from the definitions of Red_n and γ_{nk} by direct calculation; (3) is a consequence of (1) and (2). ∎

We want to point out that by means of Proposition 3.1 the remaining part of the proof of Theorem 3.2 can be performed purely in terms of words in G_n and S_n and does not have to deal with loops in (X, x_0). It consists merely in collecting the facts that we have proved up to now.

Proof of Theorem 3.2. We start by proving the sufficiency of the given condition. Put $\bar{\omega}_k = \gamma_{nk}(\omega_n)$, which is well defined for $n \geq n_k$, $k \geq 0$, where n_k is defined after the statement of Theorem 3.2. We show that

(i) $(\bar{\omega}_k)_{k\geq 0} \in \varprojlim S_n$, and
(ii) $\mathrm{Red}\,(\bar{\omega}_k)_{k\geq 0} = (\omega_n)_{n\geq 0}$.

For $k \geq 1$ and $n \geq \max\{n_k, n_{k-1}\}$ we obtain $\gamma_k(\bar{\omega}_k) = \gamma_k(\gamma_{nk}(\omega_n)) = \gamma_{n,k-1}(\omega_n) = \bar{\omega}_{k-1}$. This shows (i).

Next we prove for $\omega_n \in G_n$ that $\delta_{nk}(\omega_n) = \mathrm{Red}_k \circ \gamma_{nk}(\omega_n)$: by (2.1) we get $\delta_i \circ \mathrm{Red}_i = \mathrm{Red}_{i-1} \circ \gamma_i$ for all $i \geq 1$. Iterated application of this identity leads immediately to the claimed relation. Using this property, for $k \geq 0$ and $n \geq n_k$, we infer $\mathrm{Red}_k(\bar{\omega}_k) = \mathrm{Red}_k(\gamma_{nk}(\omega_n)) = \delta_{nk}(\omega_n) = \omega_k$, which proves (ii).

Due to Proposition 3.1 we can find $f \in S(X, x_0)$ such that $\mathrm{Red}(\sigma(f)) = \mathrm{Red}(\bar{\omega}_k)_{k\geq 0} = (\omega_n)_{n\geq 0}$ and thus, using Theorem 2.8, we get

$$(\omega_n)_{n\geq 0} = \mathrm{Red}(\sigma(f)) = \varphi([f]).$$

Now we prove the necessity of the condition. Suppose $(\omega_n)_{n\geq 0} \in \mathrm{ran}(\varphi)$. Since by Theorem 2.8, $\mathrm{ran}(\varphi) = \mathrm{ran}(\mathrm{Red} \circ \sigma)$, there exists $f \in S(X, x_0)$ with $\mathrm{Red}(\sigma(f)) = (\omega_n)_{n\geq 0}$. Then for all $k \geq 0$ and all $n \geq k$ we have

$$\sigma_k(f) = \gamma_{nk}(\sigma_n(f)) \succeq \gamma_{nk}(\mathrm{Red}_n(\sigma_n(f))) = \gamma_{nk}(\omega_n)$$

where we used (1) and (2) of Lemma 3.4. By (3) of that lemma we get

$$\gamma_{nk}(\omega_n) \preceq \gamma_{n+1,k}(\omega_{n+1}) \preceq \cdots \preceq \sigma_k(f),$$

hence $(\gamma_{nk}(\omega_n))_{n\geq k}$ is eventually constant.

This completes the proof of Theorem 3.2. ∎

For a word ω let $|\omega|$ denote the number of letters of ω; we call $|\omega|$ the *length* of ω.

We point out that by Theorem 3.2 we do not only represent an element $[f] \in \pi(X, x_0)$ by the sequence $\mathrm{Red}(\sigma(f))$ of group words. Indeed, this theorem also yields a unique representative of $[f]$ at the semigroup level which corresponds to a distinguished loop $f^* \in [f]$ that is minimal in the sense that

$$|\sigma_k(f^*)| = \min\{|\sigma_k(g)| : g \in [f]\}$$

for all $k \in \mathbb{N}$. Intuitively this means that f^* hits a belt disk of level k only if it is really necessary for a loop to belong to the homotopy class $[f]$. In the proof of Proposition 3.5 we will construct this loop f^*. Moreover, we will relate f^* explicitly to the stabilization condition in Theorem 3.2. For this purpose we set

$$\overline{\sigma}_k([f]) := \lim_{n\to\infty} \gamma_{nk}(\mathrm{Red}_n(\sigma_n(f))).$$

This is well defined as the limit exists due to Theorem 3.2 and since $\mathrm{Red}_n(\sigma_n(f))$ does not depend on the representative of the homotopy class $[f]$.

The sequence $\overline{(\omega_n)_{n\geq 0}} := (\gamma_{n_k k}(\omega_n))_{k\geq 0}$ with n_k as defined after the statement of Theorem 3.2 is called the *stabilized sequence* of $(\omega_n)_{n\geq 0} \in \varphi(\pi(X, x_0))$. Let $(\bar{\omega}_n)_{n\geq 0}$, $(\bar{\omega}'_n)_{n\geq 0}$ be two stabilized sequences. The *stabilized* *product* is defined by

$$(\bar{\omega}_n)_{n\geq 0} * (\bar{\omega}'_n)_{n\geq 0} := \overline{(\mathrm{Red}_n(\bar{\omega}_n \cdot \bar{\omega}'_n))}_{n\geq 0}.$$

Thus the product of two stabilized sequences is formed by concatenation and reduction at every level, followed by stabilization.

We collect some properties of f^* and $\overline{\sigma}_k$.

PROPOSITION 3.5. *For an arbitrary loop f in (X,x_0) we have:*

(1) $(\overline{\sigma}_n([f]))_{n\geq 0}$ *is an element of* $\varprojlim S_n$.
(2) *There exists $f^* \in [f]$ such that $|\sigma_k(f^*)| = \min\{|\sigma_k(g)| : g \in [f]\}$ for all $k \in \mathbb{N}$. Indeed, we even have $\overline{\sigma}_k([f]) \preceq \sigma_k(f^*) \preceq \sigma_k(g)$ for each $g \in [f]$.*
(3) *For any two loops $f, g \in S(X,x_0)$ we have*

$$(\overline{\sigma}_n([fg]))_{n\geq 0} = (\overline{\sigma}_n([f]))_{n\geq 0} * (\overline{\sigma}_n([g]))_{n\geq 0},$$

where the product on the right hand side is the stabilized product.

REMARK 3.6. (a) Note that the inequality $\overline{\sigma}_k([f]) \preceq \sigma_k(f^*)$ in Proposition 3.5(2) can be strict. This is due to the fact that $\overline{\sigma}_k([f])$ can be *incomplete* in a sense discussed after the proof of the proposition.

(b) By Proposition 3.5(3) the stabilized product can be interpreted as the group operation "$*$" on $\varphi(\pi(X,x_0))$ in terms of the stabilized sequences. This justifies the use of the same symbol "$*$" for this operation.

Proof of Proposition 3.5. (1) is property (i) in the proof of Theorem 3.2. Now we prove (2). To construct the loop f^* we proceed in the same way as in the proof of Proposition 3.1. Let f_k be the canonical loop corresponding to $\overline{\sigma}_k([f])$ with parametrization on the intervals $[u_{ki}, v_{ki}]$ as specified in that proof. Then f_k converges uniformly to a loop f^* in (X,x_0) and we obtain $\sigma_k(f_k) \preceq \sigma_k(f^*)$. By construction of f_k we have $\sigma_k(f_k) = \overline{\sigma}_k([f])$ and so $\overline{\sigma}_k([f]) \preceq \sigma_k(f^*)$. Finally, we have to prove that $\sigma_k(f^*) \preceq \sigma_k(g)$ for all $g \in [f]$. For $g \in [f]$, $i \geq 0$ and sufficiently large n we have

$$\sigma_{k+i}(g) = \gamma_{n,k+i}(\sigma_n(g)) \succeq \gamma_{n,k+i}(\mathrm{Red}_n(\sigma_n(g))) = \overline{\sigma}_{k+i}([f]) = \sigma_{k+i}(f_{k+i}).$$

Let $\overline{\sigma}_k([f]) =: P_1 \dots P_L$. For any additional letter Q that might occur between P_r and P_{r+1} in $\sigma_k(f^*)$ there exists a sequence of letters Q_i occurring in $\sigma_{k+i}(f_{k+i})$ between the level $k+i$ successors P_{ir} and $P_{i,r+1}$ of P_r and P_{r+1}, respectively, such that the distance between Q_i and Q tends to 0 as $i \to \infty$ (here we used again that $f_k \to f^*$ uniformly; recall that letters are belts). Since $\sigma_{k+i}(g) \succeq \sigma_{k+i}(f_{k+i})$ the letter Q_i also appears in $\sigma_{k+i}(g)$ between P_{ri} and $P_{r+1,i}$. Therefore, g traverses all the belts Q_i and thus also the belt Q after passing P_r and before passing P_{r+1} and we obtain $P_1 \dots P_r Q P_{r+1} \dots P_L \preceq \sigma_k(g)$. In this way we can argue inductively to prove that each letter occurring in $\sigma_k(f^*)$ also occurs in $\sigma_k(g)$ in the respective position. This yields $\sigma_k(f^*) \preceq \sigma_k(g)$.

(3) is a direct consequence of the definition of the stabilized product. One just has to use the fact that $\mathrm{Red}_n(\overline{\sigma}_n([f])) = \mathrm{Red}_n(\sigma_n(f))$, which is item (ii) in the proof of Theorem 3.2. ∎

An element $(w_n)_{n\geq 0} \in \varprojlim S_n$ is called *complete* if the corresponding graph $\mathcal{G}$ defined at the beginning of the present section has the property that any irrational cut in the horizontally ordered set of branches converges to a point that is not contained in a belt disk. As in [1, Section 3], one can prove that the complete elements in $\varprojlim S_n$ are exactly the elements in the range of σ, i.e., the complete elements can be represented in the form $(\sigma_k(g))_{k\geq 0}$ for some $g \in S(X, x_0)$.

Note that in general $(\overline{\sigma}_k(f))_{k\geq 0}$ is not complete and we only have $\overline{\sigma}_k([f]) \preceq \sigma_k(f^*)$. Indeed, $(\sigma_k(f^*))_{k\geq 0}$ is the *completion* of $(\overline{\sigma}_k([f]))_{k\geq 0}$ in the sense that it is the minimal (with respect to "$\preceq$") complete element of $\varprojlim S_n$ containing $(\overline{\sigma}_k([f]))_{k\geq 0}$.

In the following example we consider a loop f where $(\overline{\sigma}_k([f]))_{k\geq 0}$ is incomplete. This situation occurs if there is a sequence of "holes" in the space X that converges to a point of a given belt. This constellation cannot be avoided for certain X. In particular, each handle body construction for a *bad set* X (in the sense of [4]) gives rise to loops f with incomplete $(\overline{\sigma}_k([f]))_{k\geq 0}$.

Example 3.7. Let X be the one-dimensional space depicted in Figure 6 and let x_0 be the base point. Note that the "holes" in X accumulate at x_2. Moreover, we choose the handle body construction at each level k

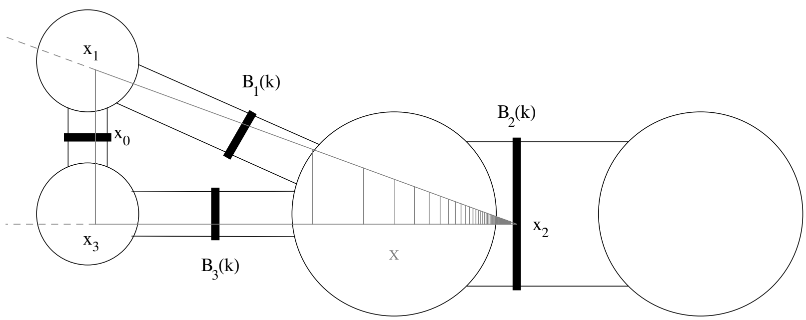
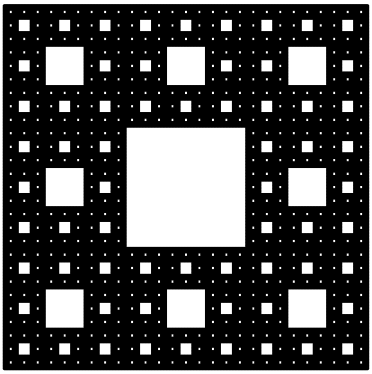


Fig. 6. An illustration of a situation that leads to an incomplete sequence $(\overline{\sigma}_k([f]))_{k\geq 0}$

in the way indicated in Figure 6. In particular, the belt $B_2(k)$ contains the point x_2. We choose f to be the loop that traverses the triangle $x_1x_2x_3x_1$ once. Now, as $B_1(k), B_2(k), B_3(k)$ lie in the same star, in the reduced description $\mathrm{Red}_k(\sigma_k(f))$ the letter $B_2(k)$ does not occur for any $k \in \mathbb{N}$. Thus $B_2(k)$ is not contained in $\overline{\sigma}_k([f])$ for all k. On the other hand, $B_2(k)$ is obviously contained in $\sigma_k(f)$. This shows that $\sigma_k(f) \neq \overline{\sigma}_k([f])$, and hence $(\overline{\sigma}_k([f]))_{k\geq 0}$ is not complete.

The next example is devoted to the Sierpiński carpet.

EXAMPLE 3.8. The well known Sierpiński carpet M is depicted on the left hand side of Figure 7. On the right hand side of this figure a handle body construction for this set is visualized. This construction can be performed in an analogous way at each approximation level and can be used to give a description of the fundamental group of M in terms of words (for the Sierpiński gasket such a description is detailed in [1]).



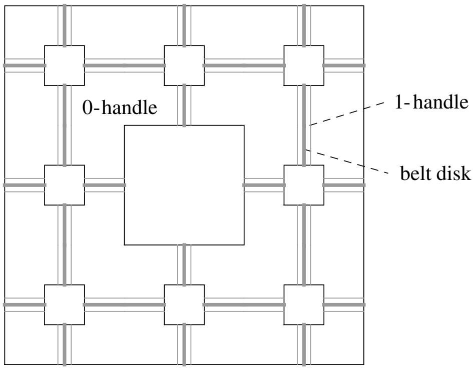


Fig. 7. The Sierpiński carpet (left) and its handle body approximation H_2 (right)

4. Cancelation. As before, let X be a metrizable one-dimensional continuum and $x_0 \in X$. In this section we collect some properties of the multiplication of elements of $\varprojlim G_n$ and their corresponding stabilized sequences. The results split into several lemmas, and later we will mainly use Lemma 4.3.

Let $P_L \dots P_1$ and $Q_1 \dots Q_M$ be two elements of G_m. We consider the possible reductions in the product $(P_L \dots P_1) * (Q_1 \dots Q_M)$. By definition $P_1 = Q_1$ is equal to the belt containing the base point x_0. Observe that $P_L \dots P_1$ as well as $Q_1 \dots Q_M$ are already reduced. Thus reduction is only possible at the point where the two words are concatenated. The group multiplication "$*$" on G_m can naturally be extended to subwords w, w' of group words provided that the last letter of w lies in the same star as the first letter of w'. We will make use of this extension throughout the remaining part of the paper. Also, in this setting the operation "$*$" means concatenation followed by reduction.

We start with the following *reduction algorithm* for the group operation "$*$". Note that with this extended notation for "$*$" we may write

$$(P_L \dots P_1) * (Q_1 \dots Q_M) = (P_L \dots P_1) * (Q_2 \dots Q_M).$$

Now we have to deal with the following cases:

(i) The word $P_L \dots P_1 Q_2 \dots Q_M$ is already reduced.

(ii) P_1, Q_2, Q_3 lie in the same star. This is impossible because, since $P_1 = Q_1$, it would imply that Q_1, Q_2, Q_3 are in the same star, which contradicts the fact that $Q_1 \dots Q_M$ is reduced.

(iii) P_2, P_1, Q_2 lie in the same star. Then

$$(P_L \dots P_1) * (Q_2 \dots Q_M) = (P_L \dots P_2) * (Q_2 \dots Q_M).$$

(a) If $P_2 \neq Q_2$ then P_3, P_2, Q_2 and P_2, Q_2, Q_3 are not in the same star, since otherwise P_1, P_2, P_3 or Q_1, Q_2, Q_3 would be in the same star, which is false. Hence, in this case

$$(P_L \dots P_1) * (Q_1 \dots Q_M) = P_L \dots P_2 Q_2 \dots Q_M.$$

(b) If $P_2 = Q_2$ then we have

$$(P_L \dots P_1) * (Q_1 \dots Q_M) = (P_L \dots P_2) * (Q_2 \dots Q_M)$$

and we may proceed iteratively in the same manner as before.

This algorithm shows that essential cancelation is only possible if a suffix of the first word is a mirror image of a prefix of the second word, i.e., if $Q_1 = P_1$, $Q_2 = P_2$, and so on.

We make this precise in the following lemma.

Lemma 4.1. *Let $P_L \dots P_1, Q_1 \dots Q_M \in G_m$. Then the operation "$*$" is given by the following procedure: Take ℓ maximal such that $P_1 \dots P_\ell = Q_1 \dots Q_\ell$. Then*

$$(P_L \dots P_1) * (Q_1 \dots Q_M) = \begin{cases} P_L \dots P_2 P_1 Q_2 \dots Q_M & \text{if } \ell = 1 \text{ and } P_2, Q_1, Q_2 \text{ do not lie in the same star,} \\ P_L \dots P_{\ell+1} Q_{\ell+1} \dots Q_M & \text{if } \ell = 1 \text{ and } P_2, Q_1, Q_2 \text{ lie in the same star, or } 2 \leq \ell < \min\{L, M\}, \\ P_L \dots P_\ell & \text{if } \ell = M, \\ Q_\ell \dots Q_M & \text{if } \ell = L. \end{cases}$$

This lemma follows immediately from the above considerations.

Now we want to use the formula in Lemma 4.1 as a definition of an operation which is also defined for semigroup words. Indeed, we define a new operation $\circledast : S_m \times S_m \to S_m$ as in Lemma 4.1 with one exception: if $2 \leq \ell < \min\{L, M\}$ it may happen for $P_L \dots P_1, Q_1 \dots Q_M \in S_m$ that $P_{\ell+1}$ is not a neighbor of $Q_{\ell+1}$, thus we define in this case

$$(P_L \dots P_1) \circledast (Q_1 \dots Q_M) = \begin{cases} P_L \dots P_{\ell+1} P_\ell Q_{\ell+1} \dots Q_M & \text{if } P_{\ell+1} \text{ is not a} \\ & \text{neighbor of } Q_{\ell+1}, \\ P_L \dots P_{\ell+1} Q_{\ell+1} \dots Q_M & \text{if } P_{\ell+1} \text{ is a} \\ & \text{neighbor of } Q_{\ell+1}. \end{cases}$$

Note that the operation "$\circledast$" corresponds to concatenation followed by reduction on the interface. Moreover, "$\circledast$" agrees with "$*$" on G_n.

We now relate this operation to the stabilized product.

LEMMA 4.2. *Let $(\bar{\omega}'_n)_{n\geq 0}, (\bar{\omega}''_n)_{n\geq 0}$ be two stabilized sequences and let*

$$(\bar{\omega}_n)_{n\geq 0} = (\bar{\omega}'_n)_{n\geq 0} * (\bar{\omega}''_n)_{n\geq 0}$$

be their stabilized product. Then on each level $k \in \mathbb{N}$ we have

$$\bar{\omega}_k \succeq \bar{\omega}'_k \circledast \bar{\omega}''_k. \tag{4.1}$$

In terms of the mapping $\overline{\sigma}_k$ and loops $f, g \in S(X, x_0)$ this reads

$$\overline{\sigma}_k([fg]) \succeq \overline{\sigma}_k([f]) \circledast \overline{\sigma}_k([g]).$$

Proof. Let $\bar{\omega}'_k = P_L \dots P_1$ and $\bar{\omega}''_k = Q_1 \dots Q_M$, let n be a "stabilizing index" satisfying $\gamma_{nk}(\mathrm{Red}_n(\bar{\omega}_n)) = \bar{\omega}_k$, $\gamma_{nk}(\mathrm{Red}_n(\bar{\omega}'_n)) = \bar{\omega}'_k$, $\gamma_{nk}(\mathrm{Red}_n(\bar{\omega}''_n)) = \bar{\omega}''_k$. Moreover, let $p = \mathrm{Red}_n(\bar{\omega}'_n)$, $q = \mathrm{Red}_n(\bar{\omega}''_n)$ be the reduced words of the sequences at level n.

Therefore, to show (4.1) we have to prove that $\gamma_{nk}(p * q) \succeq \gamma_{nk}(p) \circledast \gamma_{nk}(q)$. Let s be the maximal word with the property that $p = rs$ and $q = \widetilde{s}t$ (where $\widetilde{s}$ is the reversed word of s). In the following we work out the case $1 < |s| < \min\{|p|, |q|\}$; the remaining cases can be checked easily.

According to Lemma 4.1 we have

$$\gamma_{nk}(p * q) = \gamma_{nk}(rt) = \gamma_{nk}(r) \cdot \gamma_{nk}(t). \tag{4.2}$$

Moreover,

$$\gamma_{nk}(p) \circledast \gamma_{nk}(q) = \gamma_{nk}(r)\gamma_{nk}(s) \circledast \gamma_{nk}(\widetilde{s})\gamma_{nk}(t) = \gamma_{nk}(r) \circledast \gamma_{nk}(t).$$

For the last equality note that by (4.2), $\gamma_{nk}(r)\gamma_{nk}(t)$ is an admissible word, hence in $\gamma_{nk}(r)\gamma_{nk}(s) \circledast \gamma_{nk}(\widetilde{s})\gamma_{nk}(t)$ no letter from the part $\gamma_{nk}(s)$ or its reverse remains.

Summing up this means that our assertion is equivalent to $\gamma_{nk}(r) \cdot \gamma_{nk}(t) \succeq \gamma_{nk}(r) \circledast \gamma_{nk}(t)$, and the latter is obvious. ■

We use Lemma 4.2 to prove the following inequality for the lengths of stabilized products.

Lemma 4.3. *Let f, g be loops in (X, x_0). Then*

$$|\overline{\sigma}_k([fg])| \geq \big| |\overline{\sigma}_k([f])| - |\overline{\sigma}_k([g])| \big|. \tag{4.3}$$

Proof. Due to Lemma 4.2 and the definition of $\circledast$ we have

$$|\overline{\sigma}_k([fg])| \geq |\overline{\sigma}_k([f]) \circledast \overline{\sigma}_k([g])| \geq \big| |\overline{\sigma}_k([f])| - |\overline{\sigma}_k([g])| \big|. \ \blacksquare$$

5. Continuity of homomorphisms. As above, let X be a metrizable one-dimensional continuum and $x_0 \in X$. In this section we provide a new proof of a result of Eda [10, Theorem 1.1] which states that each homomorphism h from the fundamental group of the Hawaiian earring E to $\pi(X, x_0)$ is induced by a continuous map from E to X. The methods we have developed in the previous sections enable us to give an almost purely algebro-combinatorial proof of this result (though topological intuitions are helpful to understand the idea). Before we go into details we give an outline of our strategy.

We employ the following notation. Let $o \in E$ be the point contained in all loops of E and let C_n be the elements of $\pi(E, o)$ associated with the nth largest loop of E, $n \in \mathbb{N}$. First, one has to better understand the structure of a group homomorphism $h : \pi(E, o) \to F$ (in most cases $F = \pi(X, x_0)$ is the fundamental group of the space X) defined on the fundamental group $\pi(E, o)$ of E.

Many auxiliary results (from Lemma 5.1 to Proposition 5.4) are devoted to the observation that the (algebraic) property of h to be a homomorphism has remarkable consequences which can be interpreted as continuity properties of h. An important role is played by a theorem of Higman (Lemma 5.1) which states that $h : \pi(E, o) \to F$ does not depend on small circles if F is free, i.e., all C_n with n sufficiently large and, even more, all admissible infinite compositions of such C_n's have trivial image. As a consequence (due to Eda, cf. Lemma 5.2) each homomorphism $h : \pi(E, o) \to \pi(X, x_0)$ is uniquely determined by its values on the loops C_n. From this we derive as a byproduct Theorem 5.3, expressing that h is compatible with the inverse group limit involved. For the remaining parts Proposition 5.4 is crucial. It asserts that for elements $a \in \pi(E, o)$ which are small in the above sense the image $h(a)$ is also uniformly small in an appropriate sense, namely: there is a finite upper bound for the number of letters in $\overline{\sigma}_m(h(a))$ if a is restricted to the condition $\mathrm{Red}_{n_0}(\overline{\sigma}_{n_0}(a)) = e$ for sufficiently large $n_0 = n_0(m)$. A main tool in the proof of Proposition 5.4 is Lemma 4.3.

The continuity interpretation from the preceding paragraph suggests that, for $n \to \infty$, $h(C_n)$ tends to the homotopy class of the constant loop in a specific way. Loosely speaking, the imagined picture behind is that for large

n the minimal representative h_n of the homotopy class $h(C_n)$ can be decomposed into a path t from x_0 to some point x^*, followed by a small loop y_n based at x^* and then the converse path t^{-1} of t, that is, $h_n = ty_nt^{-1}$, where the path t does not depend on n. The technical effort to make this intuition rigorous is substantial and requires the considerations from Propositions 5.5 to 5.9. Proposition 5.5 essentially shows that, given any approximation level, for large enough n the digital representation of y_n at this level requires no more than one letter, so indeed y_n is small. Proposition 5.7 takes care of the fact that for increasing n the possible variation in the combinatorial fine structure is small and completely under control. Proposition 5.8 guarantees in a combinatorial way the existence of t and, as a consequence, of x^*. Proposition 5.9 shows that for $n \to \infty$ the loops y_n based at x^* tend to the constant loop.

With these auxiliary tools it is more or less straightforward to prove Theorem 5.10. Given any homomorphism $h : \pi(E, o) \to \pi(X, x_0)$ consider the point x^* and the loops y_n according to the above construction. Appropriate parametrizations of C_n and y_n produce a continuous mapping $\psi : E \to X$ which induces a homomorphism $\psi_* : \pi(E, o) \to \pi(X, x^*)$. With this homomorphism we finally obtain $h = \chi_t \circ \psi_*$ where $\chi_t : \pi(X, x^*) \to \pi(X, x_0)$, $[f] \mapsto [tft^{-1}]$.

Now we start to pursue the program outlined so far. Let W_n be the set of subwords of elements of S_n and define $\varprojlim W_n$ with bonding maps defined analogous to γ_{nk}. With no risk of confusion, these maps will again be called γ_{nk}. Recall that $|\omega|$ denotes the number of letters of the word ω, and $\widetilde{\omega}$ its reversed word; Λ is the empty word. Moreover, in each group we denote the neutral element by e.

In the following we will use a basic result of Higman [14, Theorem 1] (see also Eda [10, Lemma 3.1]).

Lemma 5.1. *Let F be an arbitrary free group and F_n be the* (*free*) *subgroup of $\pi(E, o)$ generated by the n largest loops $C_1, \dots, C_n$ of the Hawaiian earring. For each homomorphism $h : \pi(E, o) \to F$ there exist $k_0 \in \mathbb{N}$ and a homomorphism $\bar{h}$ from F_{k_0} to F such that $h = \bar{h} \circ q_{k_0}$ where q_{k_0} is the canonical epimorphism of $\pi(E, o)$ onto F_{k_0}.*

Next, we mention the following result of Eda [10, Lemma 3.15]. It is an immediate consequence of Lemma 5.1 and the fact that $\pi(X, x_0) \hookrightarrow \check{\pi}(X, x_0)$ (see [11]).

Lemma 5.2. *Let (X, x_0) be a metrizable one-dimensional continuum. If two homomorphisms h and h' from $\pi(E, o)$ to $\pi(X, x_0)$ coincide on all C_n then they are equal. Consequently,* $\mathrm{ran}(h)$ *is finitely generated if and only if the kernel of h contains almost all C_n, $n \in \mathbb{N}$.*

Recall that any element in $\pi(E,o)$ can be represented in the form $(C_{\alpha(i)})_{i\in I}$ where $(I,\leq)$ is a countable linearly ordered set and $\alpha : I \to \mathbb{N}$ is such that $\alpha^{-1}(n)$ is a finite subset of I for all $n \in \mathbb{N}$ (cf. [2]).

Before we state our next result, which can be interpreted as an "infinite homomorphism property", we have to define infinite products in $\varprojlim G_n$. Let $(I,\leq)$ be a countable linearly ordered set and $((\omega_{i\ell})_{\ell\geq 0})_{i\in I}$ be a family (indexed by I) of elements in $\varprojlim G_n$ with the property that for all $\ell \geq 0$ there exists a finite subset I_ℓ of I such that for all $i \in I \setminus I_\ell$ we have $\omega_{i\ell} = e$. In this case we define

$$\mathop{*}_{i\in I} (\omega_{i\ell})_{\ell\geq 0} = \Big(\mathop{*}_{i\in I_\ell} \omega_{i\ell} \Big)_{\ell\geq 0}.$$

Note that since $\omega_{i,\ell-1} \neq e$ implies $\omega_{i\ell} \neq e$ we have

$$\delta_\ell \Big(\mathop{*}_{i\in I_\ell} \omega_{i\ell} \Big) = \mathop{*}_{i\in I_\ell} \omega_{i,\ell-1} = \mathop{*}_{i\in I_{\ell-1}} \omega_{i,\ell-1},$$

hence the product is an element of $\varprojlim G_n$.

If $(\omega_{i\ell})_{\ell\geq 0}$ lies in $\varphi(\pi(X,x_0))$ for all $i \in I$ and moreover the product $*_{i\in I}(\omega_{i\ell})_{\ell\geq 0}$ is in $\varphi(\pi(X,x_0))$, we can extend this notion of an infinite product also to the corresponding elements in $\pi(X,x_0)$.

Theorem 5.3. *Let (X,x_0) be a metrizable one-dimensional continuum. Then for each homomorphism h from $\pi(E,o)$ to $\pi(X,x_0)$ and for each element $(C_{\alpha(i)})_{i\in I} \in \pi(E,o)$ the product $*_{i\in I}\, h(C_{\alpha(i)})$ is a well defined element in $\pi(X,x_0)$ and we have*

$$h((C_{\alpha(i)})_{i\in I}) = \mathop{*}_{i\in I} h(C_{\alpha(i)}).$$

Proof. We have to show that the product $(v_\ell)_{\ell\geq 0} := *_{i\in I}\, \varphi(h(C_{\alpha(i)}))$ is well defined in $\varprojlim G_n$. For this purpose we set $(\omega_{n\ell})_{\ell\geq 0} = \varphi(h(C_n))$ for each $n \in \mathbb{N}$. For $\ell \in \mathbb{N}$ let $p_\ell : \varprojlim G_n \to G_\ell$ denote the canonical projection in the projective limit and $h_\ell = p_\ell \circ \varphi \circ h : \pi(E,o) \to G_\ell$. Lemma 5.1 applied to h_ℓ implies that there exists k_ℓ with the following property: For any countable linearly ordered set $(J,\leq)$ and $\beta : J \to \mathbb{N}$ with $|\beta^{-1}(k)| < \infty$ we have

$$h_\ell((C_{\beta(j)})_{j\in J}) = h_\ell((C_{\beta(j)})_{j\in J_\ell}) = \mathop{*}_{j\in J_\ell} h_\ell(C_{\beta(j)})$$

where $J_\ell := \bigcup_{k<k_\ell} \beta^{-1}(k)$. In particular, we deduce for all $k \geq k_\ell$ that $\omega_{k\ell} = h_\ell(C_k) = h_\ell(e) = e$, and thus

$$h_\ell((C_{\alpha(i)})_{i\in I}) = h_\ell((C_{\alpha(i)})_{i\in I_\ell}) = \mathop{*}_{i\in I_\ell} h_\ell(C_{\alpha(i)}) = \mathop{*}_{i\in I_\ell} \omega_{\alpha(i)\ell}$$

with $I_\ell := \bigcup_{k<k_\ell} \alpha^{-1}(k)$. Now we obtain $v_\ell = p_\ell(*_{i\in I}(\omega_{\alpha(i)\ell'})_{\ell'\geq 0}) = *_{i\in I_\ell}\, \omega_{\alpha(i)\ell} = h_\ell((C_{\alpha(i)})_{i\in I})$, which shows that $(v_\ell)_{\ell\geq 0}$ as an (infinite) product

is well defined in $\varprojlim G_n$ and moreover

$$\mathop{*}_{i\in I}\varphi(h(C_{\alpha(i)})) = (v_\ell)_{\ell\geq 0} = (h_\ell((C_{\alpha(i)})_{i\in I}))_{\ell\geq 0} = \varphi(h(C_{\alpha(i)})_{i\in I}).$$

Transferring this equality back to $\pi(X,x_0)$ with φ^{-1} we are done. ∎

Let $m\in\mathbb{N}$ be fixed. The following proposition shows that the number of level m letters in words corresponding to $h(a)\in\pi(X,x_0)$ is uniformly bounded provided that $a\in\pi(E,o)$ contains only loops which are sufficiently small.

PROPOSITION 5.4 (cf. [10, Lemma 3.11]). *Let $h:\pi(E,o)\to\pi(X,x_0)$ be a homomorphism. Then for all $m\in\mathbb{N}$ there exists $n_0=n_0(m)$ such that*

$$\sup\{|\overline{\sigma}_m(h(a))| \mid a\in\pi(E,o) \textit{ with } \mathrm{Red}_{n_0}(\overline{\sigma}_{n_0}(a))=e\}<\infty.$$

Proof. The proof is done by contradiction. Suppose there exists $m\in\mathbb{N}$ such that for all $n\in\mathbb{N}$,

$$\sup\{|\overline{\sigma}_m(h(a))| \mid a\in\pi(E,o) \text{ with } \mathrm{Red}_n(\overline{\sigma}_n(a))=e\}=\infty.$$

Then we may choose $a_0,a_1,\ldots\in\pi(E,o)$ in such a way that for each $i\in\mathbb{N}$ we have

(i) $\mathrm{Red}_i(\overline{\sigma}_i(a_i))=e$,
(ii) $|\overline{\sigma}_m(h(a_i))|>|\overline{\sigma}_m(h(a_{i-1}))|$.

Note that because of (i) and Theorem 3.2 for an arbitrary sequence $0\leq j_0<j_1<j_2<\cdots$ the product $a_{j_0}a_{j_1}a_{j_2}\ldots$ is an element of $\pi(E,o)$.

Let $i_0=1$, $\ell_0=1$ and for $r\geq 0$ define i_{r+1} and ℓ_{r+1} inductively in the following way. Suppose $i_0,\ldots,i_r$ and $\ell_0,\ldots,\ell_r$ are already chosen; then there exists $i_{r+1}>i_r$ such that

(iii) $2|\overline{\sigma}_m(h(a_{i_1}\ldots a_{i_r}))|<|\overline{\sigma}_m(h(a_{i_{r+1}}))|$ (by (ii)),
(iv) $\mathrm{Red}_{\ell_r}(\overline{\sigma}_{\ell_r}(h(a_{i_{r+1}}a_{j_0}a_{j_1}\ldots)))=e$ for all sequences $(j_0,j_1,\ldots)$ with $i_{r+1}<j_0<j_1<\cdots$ (by Lemma 5.1).

Now choose $\ell_{r+1}>\ell_r$ such that

(v) $\overline{\sigma}_m(h(a_{i_0}\ldots a_{i_{r+1}}))=\gamma_{\ell_{r+1}m}(\mathrm{Red}_{\ell_{r+1}}(\overline{\sigma}_{\ell_{r+1}}(h(a_{i_0}\ldots a_{i_{r+1}}))))$.

Using (4.3) assertion (iii) implies that

(vi) $|\overline{\sigma}_m(h(a_{i_0}\ldots a_{i_r}))|<|\overline{\sigma}_m(h(a_{i_0}\ldots a_{i_{r+1}}))|$.

In the following we consider the element $a:=a_{i_0}a_{i_1}a_{i_2}\ldots\in\pi(E,o)$. Since $1=\ell_0<\ell_1<\cdots$ there exists $r\geq 1$ such that

$$\overline{\sigma}_m(h(a))=\gamma_{\ell_r m}(\mathrm{Red}_{\ell_r}(\overline{\sigma}_{\ell_r}(h(a))))=\gamma_{\ell_{r+1}m}(\mathrm{Red}_{\ell_{r+1}}(\overline{\sigma}_{\ell_{r+1}}(h(a)))).$$

With this choice of r we obtain

$$\begin{aligned}
|\overline{\sigma}_m(h(a))| &= |\gamma_{\ell_r m}(\mathrm{Red}_{\ell_r}(\overline{\sigma}_{\ell_r}(h(a))))| \\
&= |\gamma_{\ell_r m}(\mathrm{Red}_{\ell_r}(\overline{\sigma}_{\ell_r}(h(a_{i_0}\dots a_{i_r}))) \\
&\qquad * \mathrm{Red}_{\ell_r}(\overline{\sigma}_{\ell_r}(h(a_{i_{r+1}}a_{i_{r+2}}\dots))))| \\
&\overset{\text{(iv)}}{=} |\gamma_{\ell_r m}(\mathrm{Red}_{\ell_r}(\overline{\sigma}_{\ell_r}(h(a_{i_0}\dots a_{i_r}))))| \\
&\overset{\text{(v)}}{=} |\overline{\sigma}_m(h(a_{i_0}\dots a_{i_r}))| \\
&\overset{\text{(vi)}}{<} |\overline{\sigma}_m(h(a_{i_0}\dots a_{i_{r+1}}))| \\
&\overset{\text{(v)}}{=} |\gamma_{\ell_{r+1} m}(\mathrm{Red}_{\ell_{r+1}}(\overline{\sigma}_{\ell_{r+1}}(h(a_{i_0}\dots a_{i_{r+1}}))))| \\
&\overset{\text{(iv)}}{=} |\gamma_{\ell_{r+1} m}(\mathrm{Red}_{\ell_{r+1}}(\overline{\sigma}_{\ell_{r+1}}(h(a_{i_0}\dots a_{i_{r+1}}))) \\
&\qquad * \mathrm{Red}_{\ell_{r+1}}(\overline{\sigma}_{\ell_{r+1}}(h(a_{i_{r+2}}a_{i_{r+3}}\dots))))| \\
&= |\gamma_{\ell_{r+1} m}(\mathrm{Red}_{\ell_{r+1}}(\overline{\sigma}_{\ell_{r+1}}(h(a))))| = |\overline{\sigma}_m(h(a))|.
\end{aligned}$$

Since this is absurd we get the desired contradiction. ∎

In the next proposition we have to investigate the elements $h(C_n)$ in more detail.

Proposition 5.5. *Fix $m \in \mathbb{N}$, choose $n_0 = n_0(m)$ as in Proposition 5.4 and for $n \geq n_0$ write $\overline{\sigma}_m(h(C_n))$ in the form $\overline{\sigma}_m(h(C_n)) = p_n q_n \widetilde{p}_n$ with $p_n := P_{n1}\dots P_{nJ_n}$, $J_n \geq 0$, and $q_n := Q_{n0}Q_{n1}\dots Q_{nK_n}Q_{n0}$ such that $K_n \geq -1$ is as small as possible. Furthermore, let $\omega_n = (\omega_{n\ell})_{\ell\geq 0} = \varphi(h(C_n))$, and for all ℓ with $\gamma_{\ell m}(\omega_{n\ell}) = p_n q_n \widetilde{p}_n$ let $q_{n\ell}$ be the largest subword of $\omega_{n\ell}$ which is projected to (the central part) q_n by $\gamma_{\ell m}$, i.e., satisfies $\gamma_{\ell m}(q_{n\ell}) = q_n$.*

Then there exists $\ell_0 = \ell_0(n,m)$ such that for all $\ell \geq \ell_0$ the word $\omega_{n\ell}$ can be written as

$$\omega_{n\ell} = p_{n\ell} q_{n\ell} \widetilde{p}_{n\ell}. \tag{5.1}$$

Moreover, $q_n = Q_{n0}$, i.e., the canonical path associated with $q_{n\ell}$ is contained in the union of two stars of level m linked by Q_{n0}.

Remark 5.6. Concerning the notation in Proposition 5.5 note that

(1) the word p_n may be empty whereas q_n always contains at least one letter,

(2) $K_n = -1$ means that $q_n = Q_{n0}$, and, due to the definition of q_n, the cases $K_n = 0$ (q_n is not admissible) and $K_n = 1$ (the minimality condition on K_n is violated) cannot occur.

Proof of Proposition 5.5. The assertions are trivially true for $h(C_n) = e$. Thus we may assume that $h(C_n) \neq e$. Recall that n_0 is chosen as in Proposition 5.4 depending on the fixed level m and let ℓ_0 satisfy $\gamma_{\ell_0 m}(\omega_{n\ell_0}) =$

$\overline{\sigma}_m(h(C_n))$. By the definition of $q_{n\ell}$ the word $\omega_{n\ell}$ has a well defined representation of the form $\omega_{n\ell} = p_{n\ell}q_{n\ell}p'_{n\ell}$ such that $\gamma_{\ell m}(p_{n\ell}) = \gamma_{n\ell}(p'_{n\ell}) = P_{n1}\ldots P_{nJ_n}$. We prove the proposition by showing the following two assertions for all $\ell \geq \ell_0$:

(i) $p'_{n\ell} = \widetilde{p}_{n\ell}$,
(ii) $K_n = -1$.

For (i), assume $p'_{n\ell} \neq \widetilde{p}_{n\ell}$ for some $\ell \geq \ell_0$. (Note that this implies that at least one of the words $p_{n\ell}, p'_{n\ell}$ is nonempty and thus $J_n \geq 1$.) Then we have

$$\begin{aligned}\overline{\sigma}_m(h(C_n^2)) &= \gamma_{\ell m}(p_{n\ell}q_{n\ell}p'_{n\ell} * p_{n\ell}q_{n\ell}p'_{n\ell})\\ &\succeq P_{n1}\ldots(P_{nJ_n}Q_{n0}\ldots Q_{nK_n}Q_{n0})(P_{nJ_n}Q_{n0}\ldots Q_{nK_n}Q_{n0})P_{nJ_n}\ldots P_{n1},\end{aligned}$$

where the inequality is due to the assumption $p'_{n\ell} \neq \widetilde{p}_{n\ell}$, which implies that from the part $p'_{n\ell} * p_{n\ell}$ at least two successors of the letter P_{nJ_n} in level ℓ remain and possible further cancelations with $q_{n\ell}$ on the left or on the right (which can occur if $p'_{n\ell}$ is a suffix of $\widetilde{p}_{n\ell}$, or vice versa) stop as soon as successors of Q_{n0} in $q_{n\ell}$ appear.

Iterating this procedure we get

$$\begin{aligned}\overline{\sigma}_m(h(C_n^j)) &= \gamma_{\ell m}(\omega_{n\ell}^j)\\ &\succeq P_{n1}\ldots P_{n,J_{n-1}}(P_{nJ_n}Q_{n0}Q_{n1}\ldots Q_{nK_n}Q_{n0})^j P_{nJ_n}\ldots P_{n1}.\end{aligned}$$

Since the length of the right hand side is not bounded in j this contradicts Proposition 5.4. Thus $p'_{n\ell} = \widetilde{p}_{n\ell}$ and (i) is shown for $\ell \geq \ell_0$.

Now we prove (ii). By (i) and Lemma 4.2 we have

$$\begin{aligned}\overline{\sigma}_m(h(C_n^2)) &= \gamma_{\ell m}(p_{n\ell}q_{n\ell}\widetilde{p}_{n\ell} * p_{n\ell}q_{n\ell}\widetilde{p}_{n\ell})\\ &\succeq (P_{n1}\ldots P_{nJ_n}Q_{n0}\ldots Q_{nK_n}Q_{n0}P_{nJ_n}\ldots P_{n1})\\ &\quad \circledast (P_{n1}\ldots P_{nJ_n}Q_{n0}\ldots Q_{nK_n}Q_{n0}P_{nJ_n}\ldots P_{n1}).\end{aligned}$$

Suppose $K_n \geq 2$. Note that by the minimality of K_n we have $Q_{nK_n} \neq Q_{n1}$. There occur two (slightly) different cases: Q_{nK_n} can be a neighbor of Q_{n1} or not. We work out in detail the first case, the latter can be treated similarly (4). In any of the two cases we have $Q_{nK_n}Q_{n0} \neq \widetilde{Q_{n0}Q_{n1}}$. Therefore, if Q_{nK_n} is a neighbor of Q_{n1} we obtain

$$\overline{\sigma}_m(h(C_n^2)) \succeq P_{n1}\ldots P_{nJ_n}Q_{n0}\ldots Q_{nK_n}Q_{n1}\ldots Q_{nK_n}Q_{n0}P_{nJ_n}\ldots P_{n1}.$$

Iteration yields

$$\overline{\sigma}_m(h(C_n^j)) = \gamma_{\ell m}(\omega_{n\ell}^j) \succeq P_{n1}\ldots P_{nJ_n}Q_{n0}(Q_{n1}\ldots Q_{nK_n})^j Q_{n0}P_{nJ_n}\ldots P_{n1}.$$

This contradicts Proposition 5.4, and thus $K_n = -1$, which yields (ii). ∎

(4) The only difference is that in the latter case, between Q_{nK_n} and Q_{n1} the letter Q_{n0} has to be added.

In the following proposition we will compare the *tails* $p_{n\ell}$ of $\omega_{n\ell}$ when ℓ is fixed and n varies.

PROPOSITION 5.7. *With notation as in Proposition* 5.5, *write* $q_{n\ell}$ *in the form* $q_{n\ell} = r_{n\ell}s_{n\ell}\widetilde{r_{n\ell}}$ *with* $r_{n\ell}$ *maximal. Then for all* $n, n' \geq n_0 = n_0(m)$ *and for all* $\ell \geq \max\{\ell_0(n,m), \ell_0(n',m)\}$ *with* $\omega_{n\ell}, \omega_{n'\ell} \neq e$ *we have:*

(1) $p_{n'\ell}$ *is a prefix of* $p_{n\ell}$ *or vice versa, and moreover* $\big||\gamma_{\ell m}(p_{n\ell})| - |\gamma_{\ell m}(p_{n'\ell})|\big| \leq 1$.

(2) *If* $p_{n'\ell}$ *is a prefix of* $p_{n\ell}$ *and* $|\gamma_{\ell m}(p_{n\ell})| - |\gamma_{\ell m}(p_{n'\ell})| = 1$, *i.e.*, $\gamma_{\ell m}(p_{n\ell}) = P_{n1} \dots P_{nJ_n}$ *and* $\gamma_{\ell m}(p_{n'\ell}) = P_{n1} \dots P_{n,J_n-1}$, *then*

  (a) $\gamma_{\ell m}(\omega_{n'\ell}) = P_{n1} \dots P_{n,J_n-1}P_{nJ_n}P_{n,J_n-1} \dots P_{n1}$, *i.e.*, $Q_{n'0} = P_{nJ_n}$,

  (b) $p_{n\ell}$ *is a prefix of* $p_{n'\ell}r_{n'\ell}$ *and in* $\widetilde{p}_{n\ell} * (p_{n'\ell}r_{n'\ell}s_{n'\ell})$ *only the first letter is a successor of a letter from* D_m.

(3) *If* $p_{n'\ell}$ *is a prefix of* $p_{n\ell}$ *and* $|\gamma_{\ell m}(p_{n\ell})| = |\gamma_{\ell m}(p_{n'\ell})|$, *i.e.*, $\gamma_{\ell m}(p_{n\ell}) = \gamma_{\ell m}(p_{n'\ell}) = P_{n1} \dots P_{nJ_n}$, *then* $Q_{n0} = Q_{n'0}$ *and* $p_{n\ell} = p_{n'\ell}$.

Proof. (1) We first deal with the case that $p_{n'\ell}$ is the empty word Λ, i.e., $J_{n'} = 0$. Then we have $\overline{\sigma}_m(h(C_{n'})) = Q_{n'0}$ and $\omega_{n'\ell} = q_{n'\ell} = r_{n'\ell}s_{n'\ell}\widetilde{r}_{n'\ell}$. Since $\omega_{n'\ell} \neq e$ we know that $s_{n'\ell}$ contains at least three letters.

Now assume $J_n = |\gamma_{\ell m}(p_{n\ell})| \geq 2$ and consider the element

$$(\omega_{n\ell} * \omega_{n'\ell})^2 = (p_{n\ell}q_{n\ell}\widetilde{p}_{n\ell}) * (r_{n'\ell}s_{n'\ell}\widetilde{r}_{n'\ell}) * (p_{n\ell}q_{n\ell}\widetilde{p}_{n\ell}) * (r_{n'\ell}s_{n'\ell}\widetilde{r}_{n'\ell}).$$

In particular, we study cancelation in the part $\widetilde{p}_{n\ell} * (r_{n'\ell}s_{n'\ell}\widetilde{r}_{n'\ell}) * p_{n\ell}$: This amounts to a conjugation of the nontrivial loop $r_{n'\ell}s_{n'\ell}\widetilde{r}_{n'\ell}$, and due to the fact that $r_{n'\ell}s_{n'\ell}\widetilde{r}_{n'\ell}$ contains only successors of a single letter from D_m the reduction process stops—at the latest—at the last occurrence of a level ℓ successor of P_{n2} in $\widetilde{p}_{n\ell}$ and at the first occurrence of the same successor of P_{n2} in $p_{n\ell}$, respectively, and in between there remain at least three letters which all lie in the two m-stars attached to $Q_{n'0}$. So, when we apply $\gamma_{\ell m}$ we obtain

$$\gamma_{\ell m}((\underbrace{p_{n\ell}}_{\downarrow\, P_{nJ_n}}\,\underbrace{q_{n\ell}}_{\downarrow\, Q_{n0}}\,\underbrace{\widetilde{p}_{n\ell}) * (r_{n'\ell}s_{n'\ell}\widetilde{r}_{n'\ell}) * (p_{n\ell}}_{\downarrow\, P_{nJ_n}}\,\underbrace{q_{n\ell}}_{\downarrow\, Q_{n0}}\,\underbrace{\widetilde{p}_{n\ell}) * (r_{n'\ell}s_{n'\ell}\widetilde{r}_{n'\ell})}_{\downarrow\, P_{nJ_n}})$$
$$\succeq P_{nJ_n}(Q_{n0}P_{nJ_n})^2.$$

By iteration we get $|\overline{\sigma}_m(h((C_nC_{n'})^i))| \geq 2i + 1$, which contradicts Proposition 5.4, hence $J_n \leq 1$ and (1) is proved in the special case $p_{n'\ell} = \Lambda$.

Next we deal with the case $p_{n\ell}, p_{n'\ell} \neq \Lambda$, i.e., $J_n, J_{n'} \geq 1$, and we assume that neither $p_{n'\ell}$ is a prefix of $p_{n\ell}$ nor vice versa. We consider $\omega_{n\ell} * \omega_{n'\ell} = (p_{n\ell}r_{n\ell}s_{n\ell}\widetilde{r}_{n\ell}\widetilde{p}_{n\ell}) * (p_{n'\ell}r_{n'\ell}s_{n'\ell}\widetilde{r}_{n'\ell}\widetilde{p}_{n'\ell})$. Due to our assumption at the inner part $\widetilde{p}_{n\ell} * p_{n'\ell}$ we get $\widetilde{p}_{n\ell} * p_{n'\ell} = P^{(\ell)}_{nJ_n}sP^{(\ell)}_{n'J_{n'}}$ where $P^{(\ell)}_{nJ_n}$ and $P^{(\ell)}_{n'J_{n'}}$ are level ℓ successors of P_{nJ_n} and $P_{n'J_{n'}}$, respectively, and s is a word which can be empty if $P^{(\ell)}_{nJ_n} \neq P^{(\ell)}_{n'J_{n'}}$. Obeying Lemma 4.1 the cancelation stops here, and $\widetilde{r}_{n\ell}$ on the left and $r_{n\ell}$ on the right remain unchanged. Applying $\gamma_{\ell m}$ we obtain

$$\gamma_{\ell m}(\omega_{n\ell} * \omega_{n'\ell}) = \gamma_{\ell m}(\underbrace{p_{n\ell}}_{\substack{\downarrow\\ P_{nJ_n}}} \underbrace{r_{n\ell}s_{n\ell}\widetilde{r}_{n\ell}}_{\substack{\downarrow\\ Q_{n0}}} \underbrace{P^{(\ell)}_{nJ_n} s P^{(\ell)}_{n'J_{n'}}}_{\substack{\downarrow\\ P_{nJ_n}\ldots P_{n'J_{n'}}}} \underbrace{r_{n'\ell}s_{n'\ell}\widetilde{r}_{n'\ell}}_{\substack{\downarrow\\ Q_{n'0}}} \underbrace{p_{n'\ell}}_{\substack{\downarrow\\ P_{n'J_{n'}}}})$$

$$\succeq P_{nJ_n} Q_{n0} P_{nJ_n} \ldots P_{n'J_{n'}} Q_{n'0} P_{n'J_{n'}}.$$

Iterating this we end up with $|\overline{\sigma}_m(h((C_nC_{n'})^i))| \geq 4i$, contrary to Proposition 5.4.

So now we may suppose that $p_{n\ell}, p_{n'\ell} \neq \Lambda$ and without loss of generality $p_{n'\ell}$ is a prefix of $p_{n\ell}$. Assume $|\gamma_{\ell m}(p_{n\ell})| - |\gamma_{\ell m}(p_{n'\ell})| = j \geq 2$. Then $\widetilde{p}_{n\ell} * p_{n'\ell} = \widetilde{t}_{n\ell}$ where $t_{n\ell}$ is a suffix of $p_{n\ell}$ beginning with a level ℓ successor of P_{n,J_n-j}, and further containing successors of P_{n,J_n-k}, $0 \leq k \leq j-1$. Using this we get

$$\begin{aligned}(\omega_{n\ell} * \omega_{n'\ell})^2 &= ((p_{n\ell}q_{n\ell}\widetilde{p}_{n\ell}) * (p_{n'\ell}r_{n'\ell}s_{n'\ell}\widetilde{r}_{n'\ell}\widetilde{p}_{n'\ell}))^2\\ &= (p_{n\ell}q_{n\ell}\widetilde{t}_{n\ell}) * (r_{n'\ell}s_{n'\ell}\widetilde{r}_{n'\ell}) * (t_{n\ell}q_{n\ell}\widetilde{t}_{n\ell}) * (r_{n'\ell}s_{n'\ell}\widetilde{r}_{n'\ell}\widetilde{p}_{n'\ell}),\end{aligned}$$

and we can proceed in the same way as in the first part of this proof (case $p_{n'\ell} = \Lambda$) to show that $|\overline{\sigma}_m(h((C_nC_{n'})^i))|$ is not bounded for $i \to \infty$, a contradiction. Thus $|\gamma_{\ell m}(p_{n\ell})| - |\gamma_{\ell m}(p_{n'\ell})| \leq 1$ and (1) is proved.

(2)(a) Let as before $\widetilde{p}_{n\ell} * p_{n'\ell} = \widetilde{t}_{n\ell}$ and assume $Q_{n'0} \neq P_{nJ_n}$. Now we have

$$\begin{aligned}\omega_{n\ell} * \omega_{n'\ell} &= (p_{n\ell}q_{n\ell}\widetilde{p}_{n\ell}) * (p_{n'\ell}r_{n'\ell}s_{n'\ell}\widetilde{r}_{n'\ell}\widetilde{p}_{n'\ell})\\ &= (p_{n\ell}q_{n\ell}\widetilde{t}_{n\ell}) * (r_{n'\ell}s_{n'\ell}\widetilde{r}_{n'\ell}\widetilde{p}_{n'\ell}).\end{aligned}$$

Note that $\widetilde{t}_{n\ell}$ begins with a successor of P_{nJ_n} and due to our assumption this letter does not appear in $r_{n'\ell}$. On the other hand $r_{n'\ell}s_{n'\ell}$ contains a successor of $Q_{n'0}$ which does not appear in $\widetilde{t}_{n\ell}$. Since in the reduction process in the course of a group product only letters cancel out which appear in both factors (cf. Lemma 4.1) we get

$$\gamma_{\ell m}(\omega_{n\ell} * \omega_{n'\ell}) \succeq P_{n1} \ldots P_{nJ_n} Q_{n0} P_{nJ_n} Q_{n'0} P_{n,J_n-1} \ldots P_{n1},$$

and again we conclude that $|\overline{\sigma}_m(h((C_nC_{n'})^i))|$ is not bounded for $i \to \infty$, a contradiction. Thus we have proved $Q_{n'0} = P_{nJ_n}$.

(2)(b) With notation as before we have $\omega_{n'\ell} = p_{n'\ell}r_{n'\ell}s_{n'\ell}\widetilde{r}_{n'\ell}\widetilde{p}_{n'\ell}$ and $\omega_{n\ell} = p_{n'\ell} * (t_{n\ell}r_{n\ell}s_{n\ell}\widetilde{r}_{n\ell}\widetilde{p}_{n\ell})$. Now we consider

$$\omega^i_{n'\ell} * \omega_{n\ell} = (p_{n'\ell}r_{n'\ell}s^i_{n'\ell}\widetilde{r}_{n'\ell}) * (t_{n\ell}r_{n\ell}s_{n\ell}\widetilde{r}_{n\ell}\widetilde{p}_{n\ell}),$$

where the exponent $i \in \mathbb{N}$ will be specified later. Concerning the cancelations in the product we quote the following properties:

(I) The word $r_{n\ell}s_{n\ell}\widetilde{r}_{n\ell}$ contains a successor of Q_{n0} and the first occurrence of such a letter is either in $r_{n\ell}$ or $s_{n\ell}$. Such a letter does not occur in $r_{n'\ell}s^i_{n'\ell}\widetilde{r}_{n'\ell}$ since this word among successors of letters from D_m only contains successors of $Q_{n'0} = P_{nJ_n}$ and we have $P_{nJ_n} \neq Q_{n0}$.

(II) We choose $i = i_0$ so large that $|s^{i-1}_{n'\ell}| > |t_{n\ell}r_{n\ell}s_{n\ell}|$. This is possible since due to $\omega_{n'\ell} \neq e$ we have $|s_{n'\ell}| \geq 3$.

(III) By Lemma 4.1 we know that in a product $a * b$ of two reduced words a and b the number of letters canceling out is the same for a and b and that a letter P from a can cancel out only if P also appears in b at the corresponding position.

In view of (I)–(III) we obtain

$$\omega^i_{n'\ell} * \omega_{n\ell} = p_{n'\ell}r_{n'\ell}s_{n'\ell}\dots s^{(1)}_{n\ell}\widetilde{r}_{n\ell}\widetilde{p}_{n\ell}$$

where $s^{(1)}_{n\ell}$ is a suffix of $s_{n\ell}$ and

$$\begin{aligned}\gamma_{\ell m}(\omega^i_{n'\ell} * \omega_{n'\ell}) &= \gamma_{\ell m}(\underbrace{p_{n'\ell}}_{\substack{\downarrow\\ P_{n1}\dots P_{n,J_n-1}}} \quad \underbrace{r_{n'\ell}s_{n'\ell}}_{\substack{\downarrow\\ P_{nJ_n}}}\dots \underbrace{s^{(1)}_{n\ell}\widetilde{r}_{n\ell}}_{\substack{\downarrow\\ Q_{n0}}} \quad \underbrace{\widetilde{p}_{n\ell}}_{\substack{\downarrow\\ P_{nJ_n}\dots P_{n1}}})\\ &= P_{n1}\dots P_{nJ_n}Q_{n0}P_{nJ_n}\dots P_{n1}.\end{aligned}$$

In view of Proposition 5.5, $\omega^i_{n'\ell} * \omega_{n\ell}$ must have the form $\omega^i_{n'\ell} * \omega_{n\ell} = p^{(i)}_{n\ell}q^{(i)}_{n\ell}\widetilde{p}^{(i)}_{n\ell}$ with the corresponding properties for $p^{(i)}_{n\ell}$ and $q^{(i)}_{n\ell}$ for all $i \geq i_0$.

Next we show that $s_{n'\ell}$ does not contain a successor of P_{nJ_n}. Assume the contrary; then by increasing i the last occurrence of a successor of P_{nJ_n} before the first occurrence of a successor of Q_{n0} in the word $\omega^i_{n'\ell} * \omega_{n'\ell}$ (up to this letter all letters belong to $p^{(i)}_{n\ell}$) can be made at an arbitrary distance from the beginning. On the other hand, the occurrence of successors of P_{nJ_n} on the rear end of $\omega^i_{n'\ell} * \omega_{n\ell}$ is not influenced by the choice of i. Therefore a representation in the form $\omega^i_{n'\ell} * \omega_{n\ell} = p^{(i)}_{n\ell}q^{(i)}_{n\ell}\widetilde{p}^{(i)}_{n\ell}$ with $|\gamma_{\ell m}(q^{(i)}_{n\ell})| = 1$ is not possible. We conclude that $s_{n'\ell}$ cannot contain a successor of P_{nJ_n} and thus does not contain a successor of any letter from D_m at all.

The argument in the last part shows that $\widetilde{p}^{(i)}_{n\ell} = \widetilde{p}_{n\ell}$ for all $i \geq i_0$ and we obtain

$$p_{n'\ell}r_{n'\ell}s_{n'\ell}\dots s^{(1)}_{n\ell}\widetilde{r}_{n\ell}\widetilde{p}_{n\ell} = p_{n\ell}q^{(i)}_{n\ell}\widetilde{p}_{n\ell}.$$

Comparing the prefixes of the left and the right side in this equation and taking into account that $s_{n'\ell}$ does not contain successors of P_{nJ_n} we deduce that $p_{n\ell}$ is a prefix of $p_{n'\ell}r_{n'\ell}$ and also $\widetilde{p}_{n\ell} * (p_{n'\ell}r_{n'\ell}s_{n'\ell})$ does not (except for the first letter) contain a successor of a letter from D_m.

(3) Assume $Q_{n'0} \neq Q_{n0}$; then with the same notation and similar arguments we get

$$\gamma_{\ell m}(\omega_{n'\ell} * \omega_{n'\ell}) = \gamma_{\ell m}\big(\big(\underbrace{p_{n'\ell}}_{\substack{\downarrow\\ P_{n1}\ldots P_{nJ_n}}} \underbrace{r_{n'\ell}s_{n'\ell}\widetilde{r}_{n'\ell}}_{\substack{\downarrow\\ Q_{n'0}}}\big) * \big(\underbrace{t_{n\ell}}_{\substack{\downarrow\\ P_{nJ_n}}} \underbrace{r_{n\ell}s_{n\ell}\widetilde{r}_{n\ell}}_{\substack{\downarrow\\ Q_{n0}}} \underbrace{\widetilde{p}_{n\ell}}_{\substack{\downarrow\\ P_{nJ_n}\ldots P_{n1}}}\big)\big)$$
$$\succeq P_{n1}\ldots P_{nJ_n}Q_{n'0}Q_{n0}P_{nJ_n}\ldots P_{n1}.$$

Thus $|\overline{\sigma}_m(h((C_nC_{n'})^i))| \to \infty$ for $i \to \infty$ contrary to Proposition 5.4, hence $Q_{n0} = Q_{n'0}$.

In the case $p_{n\ell} \neq p_{n'\ell}$ we would get

$$\gamma_{\ell m}(\omega_{n'\ell} * \omega_{n'\ell}) = P_{n1}\ldots P_{nJ_n}Q_{n0}P_{nJ_n}Q_{n0}P_{nJ_n}\ldots P_{n1},$$

which, once more, leads to a contradiction to Proposition 5.4. ∎

Employing the same notation as before we can consider the following two sets:

$$N_{m1} := \{n \geq n_0(m) \mid \overline{\sigma}_m(h(C_n)) = P_{n1}\ldots P_{nJ_n}Q_{n0}P_{nJ_n}\ldots P_{n1}\},$$
$$N_{m2} := \{n \geq n_0(m) \mid \overline{\sigma}_m(h(C_n)) = P_{n1}\ldots P_{n,J_n-1}P_{nJ_n}P_{n,J_n-1}\ldots P_{n1}\}.$$

We may choose the letters $P_{n1},\ldots,P_{nJ_n},Q_{n0}$ in such a way that always $N_{m1} \neq \emptyset$ whereas N_{m2} may be empty. Moreover, if N_{m1} is finite, we enlarge $n_0(m)$ and readjust the letters so that N_{m1} is infinite and $n_0(m) \in N_{m1}$. Proceeding inductively on m we may assume that $n_0(m) \leq n_0(m+1)$. According to Proposition 5.7 we have $N_{m1} \cup N_{m2} = \{n \in \mathbb{N} \mid n \geq n_0(m)\}$.

Now the dependence on m of $p_{n\ell}$ occurring in the statement of Proposition 5.5 becomes important. Note that $n_0, \ell_0, J_n, p_{n\ell}, q_{n\ell}$ in Propositions 5.5 and 5.7 depend on m while $\omega_{n\ell}$ is independent of m. Below we will indicate this dependence on m by using a superscript $^{(m)}$, e.g., $\omega_{n\ell} = p_{n\ell}^{(m)}q_{n\ell}^{(m)}\widetilde{p}_{n\ell}^{(m)}$.

By Proposition 5.7 for all $n, n' \geq n_0(m)$ satisfying $\omega_{n\ell}, \omega_{n'\ell} \neq e$ we have $p_{n\ell}^{(m)} = p_{n'\ell}^{(m)}$ if $n, n' \in N_{m1}$, and $p_{n\ell}^{(m)}$ is a prefix of $p_{n'\ell}^{(m)}r_{n'\ell}^{(m)}$ if $n \in N_{m1}$ and $n' \in N_{m2}$. Note that $n = n_0(m)$ satisfies $\omega_{n\ell} \neq e$ if $\overline{\sigma}_m(h(C_k)) \neq e$ for at least one $k \geq n_0(m)$.

So, for $\ell \geq \ell_0(n_0(m), m)$ we define $t_\ell^{(m)} := p_{n_0(m)\ell}^{(m)}$. Then for all $n \geq n_0(m)$ and $\ell \geq \ell_0(n, m)$ satisfying $\omega_{n\ell} \neq e$ we obtain a representation of the form $\omega_{n\ell} = t_\ell^{(m)}y_{n\ell}^{(m)}\widetilde{t}_\ell^{(m)}$ with $|\gamma_{\ell m}(y_{n\ell}^{(m)})| \leq 1$, and for $n \in N_{m1}$ we have $p_{n\ell}^{(m)} = t_\ell^{(m)}$.

Proposition 5.8. *For all $m \geq 0$ and $\ell \geq \max\{\ell_0(n_0(m), m), \ell_0(n_0(m+1), m+1)\}$ we have:*

(1) *$t_\ell^{(m)}$ is a prefix of $t_\ell^{(m+1)}$.*

(2) *$\widetilde{t}_\ell^{(m)} * t_\ell^{(m+1)}$ contains only letters which (as belts) lie in the two closed m-stars attached to $Q_{n0}^{(m)}$.*

(3) *For all $\ell' > \ell \geq \ell_0(n_0(m), m)$ we have $\delta_{\ell'\ell}(t_{\ell'}^{(m)}) = t_\ell^{(m)}$.*

Proof. (1) Since $n_0(m+1) \geq n_0(m)$, we have representations of the form

$$\omega_{n_0(m+1)\ell} = p^{(m)}_{n_0(m+1)\ell}\, q^{(m)}_{n_0(m+1)\ell}\, \widetilde{p}^{(m)}_{n_0(m+1)\ell} = p^{(m+1)}_{n_0(m+1)\ell}\, q^{(m+1)}_{n_0(m+1)\ell}\, \widetilde{p}^{(m+1)}_{n_0(m+1)\ell}.$$

Assuming that $p^{(m)}_{n_0(m+1)\ell}$ is not a prefix of $p^{(m+1)}_{n_0(m+1)\ell}$ immediately leads to the conclusion that $q^{(m+1)}_{n_0(m+1)\ell}$ contains successors of more than one letter from D_m and therefore also successors of more than one letter from D_{m+1}, which contradicts Proposition 5.5. Therefore $p^{(m)}_{n_0(m+1)\ell}$ is always a prefix of $p^{(m+1)}_{n_0(m+1)\ell}$. By definition we have $t^{(m+1)}_\ell = p^{(m+1)}_{n_0(m+1)\ell}$. Now we show that $t^{(m)}_\ell = p^{(m)}_{n_0(m+1)\ell}$, which yields (1). By the choice of $n_0(m+1)$, $|\overline{\sigma}_{m+1}(h(C_{n_0(m+1)}))|$ is maximal among all $|\overline{\sigma}_{m+1}(h(C_n))|$ for $n \geq n_0(m+1)$. Therefore also $|\overline{\sigma}_m(h(C_{n_0(m+1)}))| = |\gamma_{m+1}(\overline{\sigma}_{m+1}(h(C_{n_0(m+1)})))|$ is maximal among all $|\overline{\sigma}_m(h(C_n))|$ for $n \geq n_0(m+1)$. Since $|N_{m1}| = \infty$ we know that this maximum equals $|\overline{\sigma}_m(h(C_{n_0(m)}))|$. So with Proposition 5.7 we obtain

$$\gamma_{\ell m}(\omega_{n_0(m+1)\ell}) = \overline{\sigma}_m(h(C_{n_0(m+1)})) = \overline{\sigma}_m(h(C_{n_0(m)})),$$

and this implies $t^{(m)}_\ell = p^{(m)}_{n_0(m+1)\ell}$.

(2) From the representation we got in the proof of (1),

$$\omega_{n_0(m+1)\ell} = t^{(m)}_\ell\, q^{(m)}_{n_0(m+1)\ell}\, \widetilde{t}^{(m)}_\ell = t^{(m+1)}_\ell\, q^{(m+1)}_{n_0(m+1)\ell}\, \widetilde{t}^{(m+1)}_\ell,$$

we find that $\widetilde{t}^{(m)}_\ell * t^{(m+1)}_\ell = \widetilde{p}^{(m)}_{n\ell} * p^{(m+1)}_{n\ell}$ is a word beginning with a level ℓ successor of $P^{(m)}_{nJ_n}$ followed by a prefix of $q^{(m)}_{n\ell}$, which yields the assertion.

(3) follows immediately from

$$\delta_{\ell'\ell}(p^{(m)}_{n\ell'}\, q^{(m)}_{n\ell'}\, \widetilde{p}^{(m)}_{n\ell'}) = \delta_{\ell'\ell}(\omega_{n\ell'}) = \omega_{n\ell} = p^{(m)}_{n\ell}\, q^{(m)}_{n\ell}\, \widetilde{p}^{(m)}_{n\ell}$$

and the properties of $p^{(m)}_{n\ell} = t^{(m)}_\ell$ for $n \in N_{m1}$ proved in Proposition 5.5. ∎

If we now define $t^{(m)}_\ell := \delta_{\ell_0(n_0(m),m)\ell}(t^{(m)}_{\ell_0(n_0(m),m)})$ for $0 \leq \ell < \ell_0(m)$, by Proposition 5.8(3) we arrive at a sequence $(t^{(m)}_\ell)_{\ell\geq 0}$ satisfying $\delta_{\ell'\ell}(t^{(m)}_{\ell'}) = t^{(m)}_\ell$ for all $\ell' > \ell \geq 0$. Thus this sequence $(t^{(m)}_\ell)_{\ell\geq 0}$ corresponds to a canonical path $t^{(m)}$ from the base point x_0 to some point x^*_m lying in the belt $P^{(m)}_{nJ_n}$.

Due to Proposition 5.8(1) the path $t^{(m)}$ is a prefix section of the path $t^{(m+1)}$, and Proposition 5.8(2) implies that $t^{(m)}$ converges for $m \to \infty$ to a path t from the base point x_0 to some point $x^* = \lim_{m\to\infty} x^*_m$ in X. Property (2) also implies that x^* lies in one of the two closed m-stars attached to $Q^{(m)}_{n0}$ for all $m \geq 0$. This path t has a word representation of the form $(t_\ell)_{\ell\geq 0}$ such that $t^{(m)}_\ell$ is a prefix of t_ℓ and $\widetilde{t}^{(m)}_\ell * t_\ell$ can contain successors of at most three different letters from D_m, which are $P^{(m)}_{nJ_n^{(m)}}$, $Q^{(m)}_{n0}$ and another neighbor $P^{(m)}$ of $Q^{(m)}_{n0}$ in D_m which contains x^* (cf. Proposition 5.8(2)).

Let h_n denote the minimal loop representing the homotopy class $h(C_n)$ considered in Proposition 3.5(2). In the next proposition we will show that the path t is such that the loop $t^{-1}h_n\,t$ in (X,x^*) is homotopic to a loop that stays arbitrarily near to x^* when n tends to infinity.

PROPOSITION 5.9. *For n tending to infinity the minimal representative of the homotopy class of the loop $t^{-1}h_nt$ in $\pi(X,x^*)$ tends to the constant loop x^*.*

Proof. We show that for all $m\ge 0$ and for all $n\ge n_0(m)$ the word $\overline{\sigma}_m(t^{-1}h_n\,t)$ contains only letters which (as belts) lie in the two m-stars attached to $Q^{(m)}_{n0}$. This proves the assertion.

The loop $t^{-1}h_n\,t$ corresponds to the sequence $(\widetilde{t}_\ell*\omega_{n\ell}*t_\ell)_{\ell\ge 0}:=(x_\ell)_{\ell\ge 0}$. For $\ell\ge\ell_0(n,m)$ we have $x_\ell=\widetilde{t}_\ell*(t^{(m)}_\ell y^{(m)}_{n\ell}\widetilde{t}^{(m)}_\ell)*t_\ell$. Employing the considerations before Proposition 5.9 we obtain

$$\overline{\sigma}_m(t^{-1}h_n\,t)=\gamma_{\ell m}(x_\ell)\preceq P^{(m)}Q^{(m)}_{n0}P^{(m)}_{nJ_n^{(m)}}Q^{(m)}_{n0}P^{(m)}_{nJ_n^{(m)}}Q^{(m)}_{n0}P^{(m)}$$

and we are done. ∎

In the following main result of this section we use the conjugacy map $\chi_z:\pi(X,x^*)\to\pi(X,x_0)$, $\chi_z([f])=[zfz^{-1}]$, where z is a path from x_0 to x^*.

THEOREM 5.10 (Eda [10, Theorem 1.1]). *Let (X,x_0) be a metrizable one-dimensional continuum. Then for each homomorphism h from $\pi(E,o)$ to $\pi(X,x_0)$ there exists a point $x^*\in X$, a path t from x_0 to x^* and a continuous map $\psi:E\to X$ such that $h=\chi_t\circ\psi_*$, i.e., h is conjugate to the homomorphism $\psi_*:\pi(E,o)\to\pi(X,x^*)$ induced by ψ.*

If the range of h is not finitely generated, then x^* is unique and t is unique up to homotopy relative to the end points.

Proof. Let t be the path corresponding to the sequence $(t_\ell)_{\ell\ge 0}$ defined before Proposition 5.9 and h_n be the minimal representative of the homotopy class $h(C_n)$. We fix parametrizations $h_n(x)$ and $C_n(x)$, $x\in[0,1]$, of $t^{-1}h_n\,t$ and C_n, respectively, where we assume that $C_n(x)$ is injective. This can be used to define the mapping $\psi:E\to X$ by $\psi(C_n(x))=h_n(x)$.

First we consider the case where $\mathrm{ran}(h)$ is finitely generated. By Lemma 5.2, $h(C_n)=e$ is the neutral element for all but finitely many $n\in\mathbb{N}$. Then obviously ψ is continuous and $h=\psi_*$. In this case the result follows by setting $x^*=x_0$ and t the constant path in x_0.

Now assume that $\mathrm{ran}(h)$ is not finitely generated and without loss of generality $h(C_n)\ne e$ for all $n\in\mathbb{N}$. Proposition 5.9 implies that the sequence of paths $(t^{-1}h_n\, t)_{n\in\mathbb{N}}$ converges to the constant path x^*. This implies that ψ is continuous also in this case. Observing that

$$h(C_n) = [t\, t^{-1}\, h_n\, t\, t^{-1}] = \chi_t([t^{-1}\, h_n\, t]) = \chi_t(\psi_*(C_n))$$

proves the existence part of the assertion.

The uniqueness of x^*, ψ_* and t is easily derived in the same way as in the proof of [10, Theorem 1.1]. ∎

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References

[1] S. Akiyama, G. Dorfer, J. M. Thuswaldner, and R. Winkler, *On the fundamental group of the Sierpiński-gasket*, Topology Appl. 156 (2009), 1655–1672.

[2] J. W. Cannon and G. R. Conner, *The combinatorial structure of the Hawaiian earring group*, Topology Appl. 106 (2000), 225–271.

[3] J. W. Cannon and G. R. Conner, *On the fundamental groups of one-dimensional spaces*, Topology Appl. 153 (2006), 2648–2672.

[4] G. R. Conner and K. Eda, *Fundamental groups having the whole information of spaces*, Topology Appl. 146–147 (2005), 317–328.

[5] G. R. Conner and J. W. Lamoreaux, *On the existence of universal covering spaces for metric spaces and subsets of the Euclidean plane*, Fund. Math. 187 (2005), 95–110.

[6] M. L. Curtis and M. K. Fort, *Homotopy groups of one-dimensional spaces*, Proc. Amer. Math. Soc. 8 (1957), 577–579.

[7] M. L. Curtis and M. K. Fort, *The fundamental group of one-dimensional spaces*, Proc. Amer. Math. Soc. 10 (1959), 140–148.

[8] M. L. Curtis and M. K. Fort, *Singular homology of one-dimensional spaces*, Ann. of Math. (2) 69 (1959), 309–313.

[9] R. Diestel and P. Sprüssel, *The fundamental group of a locally finite graph with ends*, Adv. Math. 226 (2011), 2643–2675.

[10] K. Eda, *The fundamental groups of one-dimensional spaces and spatial homomorphisms*, Topology Appl. 123 (2002), 479–505.

[11] K. Eda and K. Kawamura, *The fundamental groups of one-dimensional spaces*, Topology Appl. 87 (1998), 163–172.

[12] H. Fischer and A. Zastrow, *The fundamental groups of subsets of closed surfaces inject into their first shape groups*, Algebr. Geom. Topol. 5 (2005), 1655–1676.

[13] A. Hatcher, *Algebraic Topology*, Cambridge Univ. Press, Cambridge, 2002.

[14] G. Higman, *Unrestricted free products, and varieties of topological groups*, J. London Math. Soc. 27 (1952), 73–81.

[15] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton Math. Ser. 4, Princeton Univ. Press, Princeton, NJ, 1941.

[16] S. Mardešić and J. Segal, *Shape Theory. The Inverse System Approach*, North-Holland Math. Library 26, North-Holland, Amsterdam, 1982.

[17] J. J. Rotman, *An Introduction to Algebraic Topology*, Grad. Texts in Math. 119, Springer, New York, 1988.

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