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Killing GCH everywhere by a cofinality-preserving forcing notion over a model of GCH

by

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Abstract. Starting from large cardinals we construct a pair $V_1 \subseteq V_2$ of models of ZFC with the same cardinals and cofinalities such that GCH holds in V_1 and fails everywhere in V_2.

1. Introduction. Easton's classical result showed that over any model of GCH, one can force any reasonable pattern of the power function $\lambda \mapsto 2^\lambda$ on the regular cardinals λ, preserving cardinals and cofinalities. Subsequently, much work has been done on the singular cardinal problem, whose aim is to characterize the patterns of the power function on all cardinals, including the singular ones. Typically in this work, large cardinals are used to obtain patterns of power function behaviour at singular cardinals after applying subtle forcings which change cofinalities or even collapse cardinals. This leads one to ask: Is it possible to obtain a failure of GCH everywhere by forcing over a model of GCH without changing cofinalities? If so, can one have a fixed finite gap in the resulting model, meaning that $2^\lambda = \lambda^{+n}$ for some finite $n > 1$ for all λ?

In this paper we prove the following theorem.

THEOREM 1.1. *Assume GCH + there exists a $(\kappa+4)$-strong cardinal κ. Then there is a pair $V_1 \subseteq V_2$ of models of ZFC such that:*

(a) *V_1 and V_2 have the same cardinals and cofinalities,*
(b) *GCH holds in V_1,*
(c) *$V_2 \models$ "$\forall\lambda$, $2^\lambda = \lambda^{+3}$".*

REMARK 1.2. In fact it suffices to have a Mitchell increasing sequence of extenders of length κ^+, each of them $(\kappa+3)$-strong. Thus the exact strength that we need for a fixed gap of 3 is a cardinal κ with $o(\kappa) = \kappa^{+3} + \kappa^+$. It is

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also easy to extend our result to an arbitrary finite gap n instead of 3. Then what we need is a cardinal κ with $o(\kappa) = \kappa^{+n} + \kappa^{+}$. We focus on the case $n = 3$ as it is typical of all cases $n \geq 3$ (the case $n = 2$ is easier).

The rest of this paper is devoted to the proof of this theorem. The proof is based on the extender-based Radin forcing developed by C. Merimovich in [M1], [M2]. We try to make the proof self-contained, thus we start with some preliminaries and facts from those papers, suitably modified for our purposes.

We now summarise the modifications of [M2] which are necessary to achieve our result. In [M2], one begins with a model V^* with a $(\kappa+4)$-strong cardinal κ and performs a (cofinality-preserving) reverse Easton preparation, which forces $2^\alpha = \alpha^{+3}$ for the first three successors of each inaccessible $\leq \kappa$. In the resulting model $V = V^*[G]$ one can construct suitable "guiding generics" for later use, which are in fact generics over a suitable inner model M of V which blow up the power sets of the first three successors of κ^{+3} and which collapse the image $i(\kappa)$ of κ to κ^{+6}, where $i : V \to M$ is a suitable elementary embedding. After this preparation, one performs an extender-based Radin forcing with interleaved collapses, using the guiding generics obtained through preparation. The result is a model with gap 3 everywhere below κ (i.e., $2^\alpha = \alpha^{+3}$ for all $\alpha < \kappa$). By truncating the universe at κ, one obtains gap 3 everywhere.

We would like to use a similar method, but we need to perform a preparation which preserves GCH below κ. Thus our first step is to obtain a model $V = V^*[G]$ which only forces $2^\alpha = \alpha^{+3}$ at the first three successors of κ and adds no new subsets of κ. Extra work is now required to show that in this model suitable guiding generics can be found to carry out the second step of Merimovich's construction. The result is again a model $V_2 = V[G][H]$ with gap 3 everywhere below κ (keeping κ inaccessible). We now form a model V_1 intermediate between $V[G]$ and V_2, essentially obtained by using the ordinary Radin forcing with interleaved collapses (using the collapsing part of the guiding generics). The model V_1 satisfies GCH below κ but has the same cofinalities below κ as the model V_2. This is verified using a suitable projection from Merimovich's extender-based Radin forcing with collapses into the ordinary Radin forcing with collapses.

We should mention that obtaining models $V_1 \subseteq V_2$ with the same *cardinals* (not the same cofinalities) and with GCH holding in V_1 but failing everywhere in V_2 is an easier result, as then we only need guiding generics for Cohen forcings, not for Lévy colllapses, and the second step of the forcing can consist of a cardinal-preserving (but of course not cofinality-preserving) Radin forcing. But to preserve cofinalities or to obtain the gap 3 behaviour of the power function it appears that the methods of this paper are needed to handle the necessary collapses.

2. Extender sequences. Suppose $j : V^* \to M^* \supseteq V^*_\lambda$ with $\operatorname{crit} j = \kappa$. Define an extender (with projections)

$$E(0) = \langle \langle E_\alpha(0) : \alpha \in A \rangle, \langle \pi_{\beta,\alpha} : \beta, \alpha \in A,\ \beta \geq_j \alpha \rangle \rangle$$

on κ by:

- $A = [\kappa, \lambda)$,
- $\forall \alpha \in A$, $E_\alpha(0)$ is the κ-complete ultrafilter on κ defined by

$$X \in E_\alpha(0) \Leftrightarrow \alpha \in j(X).$$

  We write $E_\alpha(0)$ as U_α.
- $\forall \alpha, \beta \in A$,

$$\beta \geq_j \alpha \Leftrightarrow \beta \geq \alpha \text{ and for some } f \in {}^\kappa\kappa,\ j(f)(\beta) = \alpha.$$

- $\beta \geq_j \alpha \Rightarrow \pi_{\beta,\alpha} : \kappa \to \kappa$ is such that $j(\pi_{\beta,\alpha})(\beta) = \alpha$.

Let us recall the main properties of $E(0)$ (see [G]):

(1) $\langle A, \leq_j \rangle$ is a κ^+-directed partial order,
(2) $\forall \alpha,\ \kappa \leq_j \alpha$,
(3) U_κ is a normal measure on κ,
(4) $\forall \alpha$, U_α is a P-point ultrafilter over κ, i.e. for any $f : \kappa \to \kappa$ there is $X \in U_\alpha$ such that $\forall \nu < \kappa$, $|X \cap f^{-1''}(\nu)| < \kappa$,
(5) $\pi^{-1''}_{\beta,\alpha}(X) \in U_\beta \Leftrightarrow X \in U_\alpha$,
(6) $\forall \alpha$, $\pi_{\alpha,\alpha} = \mathrm{id}$,
(7) $\forall \gamma \geq_j \beta \geq_j \alpha$ there is $X \in U_\gamma$ such that $\forall \nu \in X$, $\pi_{\gamma,\alpha}(\nu) = \pi_{\beta,\alpha}(\pi_{\gamma,\beta}(\nu))$,
(8) $\forall \gamma \geq_j \alpha, \beta$ where $\alpha \neq \beta$ there is $X \in U_\gamma$ such that $\forall \nu \in X$, $\pi_{\gamma,\alpha}(\nu) \neq \pi_{\gamma,\beta}(\nu)$,

Moreover the $\pi_{\alpha,\kappa}$'s can be chosen so that:

(9) $\forall \beta \geq_j \alpha$, $\forall \nu < \kappa$, $\pi_{\beta,\kappa}(\nu) = \pi_{\alpha,\kappa}(\pi_{\beta,\alpha}(\nu))$,
(10) $\forall \alpha, \beta$, $\forall \nu < \kappa$, $\pi_{\alpha,\kappa}(\nu) = \pi_{\beta,\kappa}(\nu)$; we denote the latter by ν^0.

Now suppose that we have defined the sequence $\langle E(\tau') : \tau' < \tau \rangle$. If $\langle E(\tau') : \tau' < \tau \rangle \notin M^*$ we stop the construction and set

$$\forall \alpha \in A, \quad \bar{E}_\alpha = \langle \alpha, E(0), \ldots, E(\tau'), \ldots : \tau' < \tau \rangle$$

and call $\bar{E}_\alpha$ an *extender sequence of length* τ ($\mathrm{l}(\bar{E}_\alpha) = \tau$).

If $\langle E(\tau') : \tau' < \tau \rangle \in M^*$ then we define an extender (with projections) $E(\tau) = \langle \langle E_{\langle \alpha, E(\tau') : \tau' < \tau \rangle}(\tau) : \alpha \in A \rangle$, $\langle \pi_{\langle \beta, E(\tau') : \tau' < \tau \rangle, \langle \alpha, E(\tau') : \tau' < \tau \rangle} : \beta, \alpha \in A$, $\beta \geq_j \alpha \rangle \rangle$ on V_κ by:

- $X \in E_{\langle \alpha, E(\tau'):\tau'<\tau\rangle}(\tau) \Leftrightarrow \langle \alpha, E(\tau') : \tau' < \tau\rangle \in j(X)$,
- for $\beta \geq_j \alpha$ in A, $\pi_{\langle \beta, E(\tau'):\tau'<\tau\rangle, \langle \alpha, E(\tau'):\tau'<\tau\rangle}(\langle \nu, d\rangle) = \langle \pi_{\beta,\alpha}(\nu), d\rangle$.

Note that $E_{\langle \alpha, E(\tau'):\tau'<\tau\rangle}(\tau)$ concentrates on pairs of the form $\langle \nu, d\rangle$ where $\nu < \kappa$ and d is an extender sequence. This makes the above definition correct.

We let the construction run until it stops due to the extender sequence not being in M^*.

Definition 2.1.

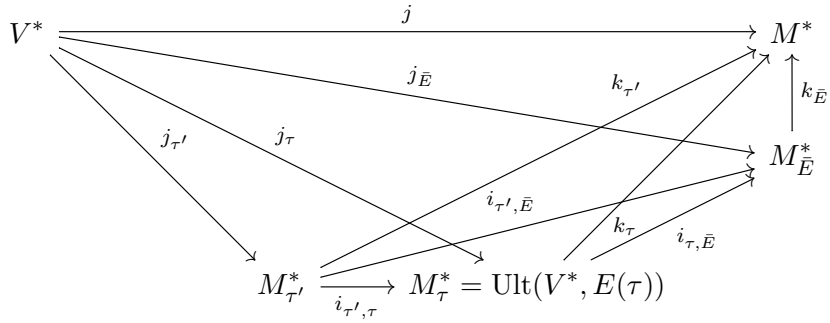
(1) $\bar{\mu}$ is an extender sequence if there are $j : V^* \to M^*$ and $\bar{\nu}$ such that $\bar{\nu}$ is an extender sequence derived from j as above (i.e. $\bar{\nu} = \bar{E}_\alpha$ for some α) and $\bar{\mu} = \bar{\nu}\restriction\tau$ for some $\tau \leq \mathrm{l}(\bar{\nu})$.
(2) $\kappa(\bar{\mu})$ is the ordinal of the beginning of the sequence (i.e. $\kappa(\bar{E}_\alpha) = \alpha$).
(3) $\kappa^0(\bar{\mu}) = (\kappa(\bar{\mu}))^0$ (i.e. $\kappa^0(\bar{E}_\alpha) = \kappa$)).
(4) The sequence $\langle \bar{\mu}_1, \dots, \bar{\mu}_n\rangle$ of extender sequences is ^{0}-*increasing* if $\kappa^0(\bar{\mu}_1) < \cdots < \kappa^0(\bar{\mu}_n)$.
(5) The extender sequence $\bar{\mu}$ is *permitted* to a ^{0}-increasing sequence $\langle \bar{\mu}_1, \dots, \bar{\mu}_n\rangle$ of extender sequences if $\kappa^0(\bar{\mu}_n) < \kappa^0(\bar{\mu})$.
(6) Notation: We write $X \in \bar{E}_\alpha$ iff $\forall \xi < \mathrm{l}(\bar{E}_\alpha), X \in E_\alpha(\xi)$.
(7) $\bar{E} = \langle \bar{E}_\alpha : \alpha \in A\rangle$ is an *extender sequence system* if there is $j : V^* \to M^*$ such that each $\bar{E}_\alpha$ is derived from j as above and $\forall \alpha, \beta \in A$, $\mathrm{l}(\bar{E}_\alpha) = \mathrm{l}(\bar{E}_\beta)$. Call this common length the *length* of $\bar{E}$, $\mathrm{l}(\bar{E})$.
(8) For an extender sequence $\bar{\mu}$, we use $\bar{E}(\bar{\mu})$ for the extender sequence system containing $\bar{\mu}$ (i.e. $\bar{E}(\bar{E}_\alpha) = \bar{E}$).
(9) $\mathrm{dom}(\bar{E}) = A$.
(10) $\bar{E}_\beta \geq_{\bar{E}} \bar{E}_\alpha \Leftrightarrow \beta \geq_j \alpha$.

3. Finding generic filters. Using GCH in V^* we construct an extender sequence system $\bar{E} = \langle \bar{E}_\alpha : \alpha \in \mathrm{dom}\,\bar{E}\rangle$ with $\mathrm{dom}\,\bar{E} = [\kappa, \kappa^{+3})$ and $\mathrm{l}(\bar{E}) = \kappa^+$ such that the ultrapower $j_{\bar{E}} : V^* \to M^*_{\bar{E}}$ (defined below) contains $V^*_{\kappa+3}$. Suppose that $\bar{E}$ is derived from an elementary embedding $j : V^* \to M^*$. Consider the following elementary embeddings $\forall \tau' < \tau < \mathrm{l}(\bar{E})$:

$$
\begin{aligned}
&j_\tau : V^* \to M^*_\tau \simeq \mathrm{Ult}(V^*, E(\tau)) = \{j_\tau(f)(\bar{E}_\alpha\restriction\tau) : f \in V^*\},\\
&k_\tau(j_\tau(f)(\bar{E}_\alpha\restriction\tau)) = j(f)(\bar{E}_\alpha\restriction\tau),\\
&i_{\tau',\tau}(j_{\tau'}(f)(\bar{E}_\alpha\restriction\tau')) = j_\tau(f)(\bar{E}_\alpha\restriction\tau'),\\
&\langle M^*_{\bar{E}}, i_{\tau,\bar{E}}\rangle = \lim \mathrm{dir}\langle\langle M^*_\tau : \tau < \mathrm{l}(\bar{E})\rangle, \langle i_{\tau',\tau} : \tau' \leq \tau < \mathrm{l}(\bar{E})\rangle\rangle.
\end{aligned}
$$

We demand that $\bar{E}\restriction\tau \in M^*_\tau$ for all $\tau < \mathrm{l}(\bar{E})$.

Thus we get the following commutative diagram:



Note that

- the critical point of those elementary embeddings originating in V^* is κ,
- the critical point of those elementary embeddings originating in other models is κ^{+4} as computed in that model.

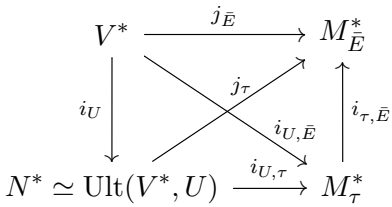
Thus we get

$$\begin{aligned}\operatorname{crit} i_{\tau',\tau} &= \operatorname{crit} k_{\tau'} = \operatorname{crit} i_{\tau',\bar{E}} = (\kappa^{+4})_{M^*_{\tau'}},\\ \operatorname{crit} k_\tau &= \operatorname{crit} i_{\tau,\bar{E}} = (\kappa^{+4})_{M^*_\tau},\\ \operatorname{crit} k_{\bar{E}} &= (\kappa^{+4})_{M^*_{\bar{E}}}.\end{aligned}$$

Each of these models catches $V^{M^*}_{\kappa^{+3}} = V^*_{\kappa^{+3}}$, hence computes κ^{+3} to be the same ordinal in all models. The larger τ is the more resemblance there is between M^*_τ and M^*. This can be verified by noting that

$$\kappa^{+4}_{M^*_{\tau'}} < j_{\tau'}(\kappa) < \kappa^{+4}_{M^*_\tau} < j_\tau(\kappa) < \kappa^{+4}_{M_{\bar{E}^*}} \le \kappa^{+4}_{M^*} \le \kappa^{+4}.$$

We also factor through the normal ultrafilter to get the commutative diagram



$$\begin{aligned}&U = E_\kappa(0),\\ &i_U : V^* \to N^* \simeq \mathrm{Ult}(V^*, U),\\ &i_{U,\tau}(i_U(f)(\kappa)) = j_\tau(f)(\kappa),\\ &i_{U,\bar{E}}(i_U(f)(\kappa)) = j_{\bar{E}}(f)(\kappa).\end{aligned}$$

N^* catches V^* only up to $V^*_{\kappa+1}$ and we have

$$\kappa^+ < \operatorname{crit} i_{U,\tau} = \operatorname{crit} i_{U,\bar{E}} = \kappa^{++}_{N^*} < i_U(\kappa) < \kappa^{++}.$$

We now define the forcings for which we will need "guiding generics".

Definition 3.1. Let

(1) $\mathbb{R}^{\mathrm{Col}}_U = \mathrm{Col}(\kappa^{+6}, i_U(\kappa))_{N^*}$,
(2) $\mathbb{R}^{\mathrm{Add},1}_U = \mathrm{Add}(\kappa^+, \kappa^{+4})_{N^*}$,

(3) $\mathbb{R}_U^{\mathrm{Add},2} = \mathrm{Add}(\kappa^{++}, \kappa^{+5})_{N^*}$,
(4) $\mathbb{R}_U^{\mathrm{Add},3} = \mathrm{Add}(\kappa^{+3}, \kappa^{+6})_{N^*}$,
(5) $\mathbb{R}_U^{\mathrm{Add},4} = (\mathrm{Add}(\kappa^{+4}, i_U(\kappa)^+) \times \mathrm{Add}(\kappa^{+5}, (i_U(\kappa)^{++})_{N^{*2}})$ $\times \mathrm{Add}(\kappa^{+6}, (i_U(\kappa)^{+3})_{N^{*2}}))_{N^*}$, where N^{*2} is the second iterate of V^* by U,
(6) $\mathbb{R}_U^{\mathrm{Add}} = \mathbb{R}_U^{\mathrm{Add},1} \times \mathbb{R}_U^{\mathrm{Add},2} \times \mathbb{R}_U^{\mathrm{Add},3} \times \mathbb{R}_U^{\mathrm{Add},4}$,
(7) $\mathbb{R}_U = \mathbb{R}_U^{\mathrm{Add}} \times \mathbb{R}_U^{\mathrm{Col}}$.

Remark 3.2. We have $(j_\tau(\kappa)^{++})_{M_\tau^*} = (j_\tau(\kappa)^{++})_{M_\tau^{*2}}$ and $(j_\tau(\kappa)^{+3})_{M_\tau^*} = (j_\tau(\kappa)^{+3})_{M_\tau^{*2}}$, where M_τ^{*2} is the second iterate of V^* by $E(\tau)$. Similarly $(j_{\bar E}(\kappa)^{++})_{M_{\bar E}^*} = (j_{\bar E}(\kappa)^{++})_{M_{\bar E}^{*2}}$ and $(j_{\bar E}(\kappa)^{+3})_{M_{\bar E}^*} = (j_{\bar E}(\kappa)^{+3})_{M_{\bar E}^{*2}}$, where $M_{\bar E}^{*2}$ is the second iterate of V^* by $\bar E$.

Definition 3.3. Let

(1) $\mathbb{R}_\tau^{\mathrm{Col}} = \mathrm{Col}(\kappa^{+6}, j_\tau(\kappa))_{M_\tau^*}$,
(2) $\mathbb{R}_\tau^{\mathrm{Add},1} = \mathrm{Add}(\kappa^{+}, \kappa^{+4})_{M_\tau^*}$,
(3) $\mathbb{R}_\tau^{\mathrm{Add},2} = \mathrm{Add}(\kappa^{++}, \kappa^{+5})_{M_\tau^*}$,
(4) $\mathbb{R}_\tau^{\mathrm{Add},3} = \mathrm{Add}(\kappa^{+3}, \kappa^{+6})_{M_\tau^*}$,
(5) $\mathbb{R}_\tau^{\mathrm{Add},4} = (\mathrm{Add}(\kappa^{+4}, j_\tau(\kappa)^+) \times \mathrm{Add}(\kappa^{+5}, j_\tau(\kappa)^{++})$ $\times \mathrm{Add}(\kappa^{+6}, j_\tau(\kappa)^{+3}))_{M_\tau^*}$,
(6) $\mathbb{R}_\tau^{\mathrm{Add}} = \mathbb{R}_\tau^{\mathrm{Add},1} \times \mathbb{R}_\tau^{\mathrm{Add},2} \times \mathbb{R}_\tau^{\mathrm{Add},3} \times \mathbb{R}_\tau^{\mathrm{Add},4}$,
(7) $\mathbb{R}_\tau = \mathbb{R}_\tau^{\mathrm{Add}} \times \mathbb{R}_\tau^{\mathrm{Col}}$.

Definition 3.4. Let

(1) $\mathbb{R}_{\bar E}^{\mathrm{Col}} = \mathrm{Col}(\kappa^{+6}, j_{\bar E}(\kappa))_{M_{\bar E}^*}$,
(2) $\mathbb{R}_{\bar E}^{\mathrm{Add},1} = \mathrm{Add}(\kappa^{+}, \kappa^{+4})_{M_{\bar E}^*}$,
(3) $\mathbb{R}_{\bar E}^{\mathrm{Add},2} = \mathrm{Add}(\kappa^{++}, \kappa^{+5})_{M_{\bar E}^*}$,
(4) $\mathbb{R}_{\bar E}^{\mathrm{Add},3} = \mathrm{Add}(\kappa^{+3}, \kappa^{+6})_{M_{\bar E}^*}$,
(5) $\mathbb{R}_{\bar E}^{\mathrm{Add},4} = (\mathrm{Add}(\kappa^{+4}, j_{\bar E}(\kappa)^+) \times \mathrm{Add}(\kappa^{+5}, j_{\bar E}(\kappa)^{++})$ $\times \mathrm{Add}(\kappa^{+6}, j_{\bar E}(\kappa)^{+3}))_{M_{\bar E}^*}$,
(6) $\mathbb{R}_{\bar E}^{\mathrm{Add}} = \mathbb{R}_{\bar E}^{\mathrm{Add},1} \times \mathbb{R}_{\bar E}^{\mathrm{Add},2} \times \mathbb{R}_{\bar E}^{\mathrm{Add},3} \times \mathbb{R}_{\bar E}^{\mathrm{Add},4}$,
(7) $\mathbb{R}_{\bar E} = \mathbb{R}_{\bar E}^{\mathrm{Add}} \times \mathbb{R}_{\bar E}^{\mathrm{Col}}$.

Also define the forcing notion $\mathbb{P}$ as follows:

$$\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3$$
$$= \mathrm{Add}(\kappa^+, (\kappa^{+4})_{M_{\bar E}^*}) \times \mathrm{Add}(\kappa^{++}, (\kappa^{+5})_{M_{\bar E}^*}) \times \mathrm{Add}(\kappa^{+3}, (\kappa^{+6})_{M_{\bar E}^*}) \ (^1)$$

and let $G = G_1 \times G_2 \times G_3$ be $\mathbb{P}$-generic over V^*. It is clear that $V^*[G]$ is a cofinality-preserving generic extension of V^* and that GCH holds in $V^*[G]$

(^{1}) Hence $\mathbb{P}$ is forcing isomorphic to $\mathrm{Add}(\kappa^+, \kappa^{+3}) \times \mathrm{Add}(\kappa^{++}, \kappa^{+3}) \times \mathrm{Add}(\kappa^{+3}, \kappa^{+3})$.

below and at κ. The forcing $\mathbb{P}$ is our (weak) "preparation forcing" (which preserves the GCH below κ). We set $V = V^*[G]$.

REMARK 3.5. (a) It is also possible to work with $\mathbb{P} = \mathrm{Add}(\kappa^+, \kappa^{+4}) \times \mathrm{Add}(\kappa^{++}, \kappa^{+5}) \times \mathrm{Add}(\kappa^{+3}, \kappa^{+6})$.

(b) We also require that $G_2 \times G_3$ contains some special element, that we will specify later (see the notes after Claim 3.13). The element will play the role of a master condition, and it will be used in the proof of Lemma 3.8(d).

LEMMA 3.6.

(a) $G_U = \langle i''_U G_1 \rangle \times \langle i''_U G_2 \rangle \times \langle i''_U G_3 \rangle$ *is* $\mathbb{P}_U = i_U(\mathbb{P})$*-generic over* N^*,
(b) $G_\tau = \langle j''_\tau G_1 \rangle \times \langle j''_\tau G_2 \rangle \times \langle j''_\tau G_3 \rangle$ *is* $\mathbb{P}_\tau = j_\tau(\mathbb{P})$*-generic over* M^*_τ,
(c) $G_{\bar E} = \langle \bigcup_{\tau < \mathrm{l}(\bar E)} i''_{\tau,\bar E} G_\tau \rangle$ *is* $\mathbb{P}_{\bar E} = j_{\bar E}(\mathbb{P})$*-generic over* $M^*_{\bar E}$.

Proof. (a) Suppose $D \in N^*$ is dense open in $\mathbb{P}_U$. Let $D = i_U(f)(\kappa)$ for some function $f \in V$ on κ. Then

$$D^* = \{\alpha < \kappa : f(\alpha) \text{ is dense open in } \mathbb{P}\} \in U.$$

Since $\mathbb{P}$ is κ^+-closed, $\bigcap_{\alpha \in D^*} f(\alpha)$ is dense open in $\mathbb{P}$. Let $p \in G \cap \bigcap_{\alpha \in D^*} f(\alpha)$. Then $i_U(p) \in G_U \cap D$.

(b) Suppose $D \in M^*_\tau$ is dense open in $\mathbb{P}_\tau$. Let $D = j_\tau(f)(\bar E_\alpha {\restriction} \tau)$ for some function $f \in V$ on V_κ. Then

$$D^* = \{\bar\nu \in V_\kappa : f(\bar\nu) \text{ is dense open in } \mathbb{P}\} \in E_\alpha(\tau).$$

Since $\mathbb{P}$ is κ^+-closed, $\bigcap_{\bar\nu \in D^*} f(\bar\nu)$ is dense open in $\mathbb{P}$. Let $p \in G \cap \bigcap_{\bar\nu \in D^*} f(\bar\nu)$. Then $j_\tau(p) \in G_\tau \cap D$.

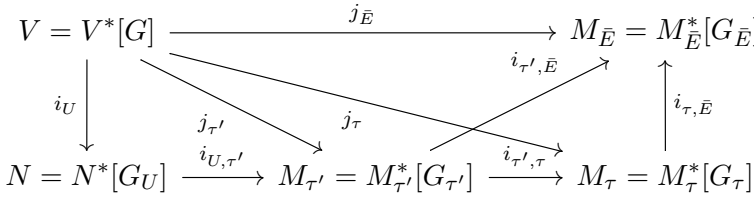
(c) Suppose $D \in M^*_{\bar E}$ is dense open in $\mathbb{P}_{\bar E}$. Let $\tau < \mathrm{l}(\bar E)$ and $D_\tau \in M^*_\tau$ be such that $D = i_{\tau,\bar E}(D_\tau)$. By elementarity, D_τ is dense open in $\mathbb{P}_\tau$. Let $p \in G_\tau \cap D_\tau$. Then $i_{\tau,\bar E}(p) \in G_{\bar E} \cap D$. ∎

The following lemma is now trivial.

LEMMA 3.7. *The generic filters above are such that*

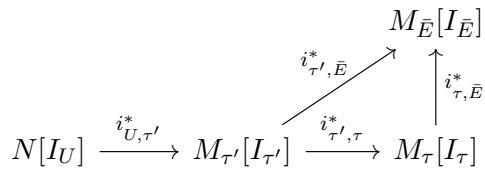
(a) $i''_U[G] \subseteq G_U$,
(b) $j''_\tau[G] \subseteq G_\tau$,
(c) $j''_{\bar E}[G] \subseteq G_{\bar E}$,
(d) $i''_{U,\tau'}[G_U] \subseteq G_\tau$,
(e) $i''_{\tau',\tau}[G_{\tau'}] \subseteq G_\tau$,
(f) $i''_{\tau,\bar E}[G_\tau] \subseteq G_{\bar E}$.

It then follows that we have the following lifting diagram.



Lemma 3.8. *In $V^*[G]$ there are I_U, I_τ and $I_{\bar{E}}$ such that*

(a) *I_U is R_U-generic over $N^*[G_U]$,*
(b) *I_τ is R_τ-generic over $M^*_\tau[G_\tau]$,*
(c) *$I_{\bar{E}}$ is $R_{\bar{E}}$-generic over $M^*_{\bar{E}}[G_{\bar{E}}]$,*
(d) *the generics are such that we have the lifting diagram*



Proof. We will prove the lemma in a sequence of claims.

Claim 3.9. *$I^{\mathrm{Add},1}_{\bar{E}} = G_1 \cap \mathbb{R}^{\mathrm{Add},1}_{\bar{E}}$ is $\mathbb{R}^{\mathrm{Add},1}_{\bar{E}}$-generic over $M^*_{\bar{E}}$.*

Proof. Suppose A is a maximal antichain of $\mathbb{R}^{\mathrm{Add},1}_{\bar{E}}$ in $M^*_{\bar{E}}$. Let $X = \bigcup\{\mathrm{dom}(p) : p \in A\}$. As $|A| \leq \kappa^+$, we have $|X| \leq \kappa^+$, and of course A is a maximal antichain of $\mathrm{Add}(\kappa^+, X)_{M^*_{\bar{E}}}$. For simplicity let us assume $|X| = \kappa^+$. Now we have $\mathrm{Add}(\kappa^+, X)_{M^*_{\bar{E}}} = \mathrm{Add}(\kappa^+, X)$ and hence A is a maximal antichain of $\mathrm{Add}(\kappa^+, X)$. It then follows that A is a maximal antichain of $\mathrm{Add}(\kappa^+, \kappa^{+4})$. Let $p \in G_1 \cap A$. Then $p \in I^{\mathrm{Add},1}_{\bar{E}} \cap A$. ∎

Claim 3.10. *$I^{\mathrm{Add},2}_{\bar{E}} = G_2 \cap \mathbb{R}^{\mathrm{Add},2}_{\bar{E}}$ is $\mathbb{R}^{\mathrm{Add},2}_{\bar{E}}$-generic over $M^*_{\bar{E}}$.*

Proof. Suppose A is a maximal antichain of $\mathbb{R}^{\mathrm{Add},2}_{\bar{E}}$ in $M^*_{\bar{E}}$. Let $X = \bigcup\{\mathrm{dom}(p) : p \in A\}$. As $|A| \leq \kappa^{++}$, we have $|X| \leq \kappa^{++}$, and of course A is a maximal antichain of $\mathrm{Add}(\kappa^{++}, X)_{M^*_{\bar{E}}}$. For simplicity let us assume $|X| = \kappa^{++}$. Now we have $\mathrm{Add}(\kappa^{++}, X)_{M^*_{\bar{E}}} = \mathrm{Add}(\kappa^{++}, X)$ and hence A is a maximal antichain of $\mathrm{Add}(\kappa^{++}, X)$. It then follows that A is a maximal antichain of $\mathrm{Add}(\kappa^{++}, \kappa^{+5})$. Let $p \in G_2 \cap A$. Then $p \in I^{\mathrm{Add},2}_{\bar{E}} \cap A$. ∎

Claim 3.11. *$I^{\mathrm{Add},3}_{\bar{E}} = G_3 \cap \mathbb{R}^{\mathrm{Add},3}_{\bar{E}}$ is $\mathbb{R}^{\mathrm{Add},3}_{\bar{E}}$-generic over $M^*_{\bar{E}}$.*

Proof. Suppose A is a maximal antichain of $\mathbb{R}^{\mathrm{Add},3}_{\bar{E}}$ in $M^*_{\bar{E}}$. Let $X = \bigcup\{\mathrm{dom}(p) : p \in A\}$. As $|A| \leq \kappa^{+3}$, we have $|X| \leq \kappa^{+3}$, and of course A is a maximal antichain of $\mathrm{Add}(\kappa^{+3}, X)_{M^*_{\bar{E}}}$. For simplicity let us assume $|X| = \kappa^{+3}$. Now we have $\mathrm{Add}(\kappa^{+3}, X)_{M^*_{\bar{E}}} = \mathrm{Add}(\kappa^{+3}, X)$ and hence A is

a maximal antichain of $\mathrm{Add}(\kappa^{+3}, X)$. It then follows that A is a maximal antichain of $\mathrm{Add}(\kappa^{+3}, \kappa^{+6})$. Let $p \in G_3 \cap A$. Then $p \in I_{\bar{E}}^{\mathrm{Add},3} \cap A$. ∎

It follows from the above claims that we have

Claim 3.12. $I_{\bar{E}}^{\mathrm{Add},1} \times I_{\bar{E}}^{\mathrm{Add},2} \times I_{\bar{E}}^{\mathrm{Add},3}$ *is* $\mathbb{R}_{\bar{E}}^{\mathrm{Add},1} \times \mathbb{R}_{\bar{E}}^{\mathrm{Add},2} \times \mathbb{R}_{\bar{E}}^{\mathrm{Add},3}$*-generic over* $M_{\bar{E}}^*$.

Claim 3.13. $I_U^{\mathrm{Add},1} = \langle i_{U,\bar{E}}^{-1''}(I_{\bar{E}}^{\mathrm{Add},1})\rangle$ *is* $\mathbb{R}_U^{\mathrm{Add},1}$*-generic over* N^*.

Proof. Let A be a maximal antichain of $\mathbb{R}_U^{\mathrm{Add},1}$ in N^*. Then $i_{U,\bar{E}}(A)$ is a maximal antichain of $\mathbb{R}_{\bar{E}}^{\mathrm{Add},1}$ in $M_{\bar{E}}^*$. Since $|A| \le \kappa^+$ and $\mathrm{crit}\, i_{U,\bar{E}} = \kappa_{N^*}^{++} > \kappa^+$, we have $i_{U,\bar{E}}(A) = i''_{U,\bar{E}}(A)$. Then $I_{\bar{E}}^{\mathrm{Add},1} \cap i''_{U,\bar{E}}(A) \neq \emptyset$, which implies $I_U^{\mathrm{Add},1} \cap A \neq \emptyset$. ∎

Now consider the forcing notion $\mathbb{R}_U^{\mathrm{Add},2} \times \mathbb{R}_U^{\mathrm{Add},3} \times \mathbb{R}_U^{\mathrm{Add},4} \times \mathbb{R}_U^{\mathrm{Col}}$. Working in $M_{\bar{E}}^*$, this forcing notion is κ^+-closed and there are only κ^+-many maximal antichains of it which are in N^*. Thus we can define a descending sequence $\langle \langle p_{\langle\alpha,\mathrm{Add},2\rangle}, p_{\langle\alpha,\mathrm{Add},3\rangle}, p_{\langle\alpha,\mathrm{Add},4\rangle}, p_{\langle\alpha,\mathrm{Col}\rangle}\rangle : \alpha < \kappa^+\rangle$ of conditions such that $I_U^{\mathrm{Add},2} \times I_U^{\mathrm{Add},3} \times I_U^{\mathrm{Add},4} \times I_U^{\mathrm{Col}} = \{p \in \mathbb{R}_U^{\mathrm{Add},2} \times \mathbb{R}_U^{\mathrm{Add},3} \times \mathbb{R}_U^{\mathrm{Add},4} \times \mathbb{R}_U^{\mathrm{Col}} : \exists \alpha < \kappa^+, \langle p_{\langle\alpha,\mathrm{Add},2\rangle}, p_{\langle\alpha,\mathrm{Add},3\rangle}, p_{\langle\alpha,\mathrm{Add},4\rangle}, p_{\langle\alpha,\mathrm{Col}\rangle}\rangle \le p\}$ is $\mathbb{R}_U^{\mathrm{Add},2} \times \mathbb{R}_U^{\mathrm{Add},3} \times \mathbb{R}_U^{\mathrm{Add},4} \times \mathbb{R}_U^{\mathrm{Col}}$-generic over N^*. Also note that this generic filter is in $M_{\bar{E}}^*$.

We may also note that $\{i_{U,\bar{E}}(p_{\langle\alpha,\mathrm{Add},2\rangle}, p_{\langle\alpha,\mathrm{Add},3\rangle}) : \alpha < \kappa^+\} \subseteq \mathbb{R}_{\bar{E}}^{\mathrm{Add},2} \times \mathbb{R}_{\bar{E}}^{\mathrm{Add},3}$, and since this forcing is κ^{++}-closed, there is $\langle p_{\langle\mathrm{Add},2\rangle}, p_{\langle\mathrm{Add},3\rangle}\rangle \in \mathbb{R}_{\bar{E}}^{\mathrm{Add},2} \times \mathbb{R}_{\bar{E}}^{\mathrm{Add},3}$ such that $\langle p_{\langle\mathrm{Add},2\rangle}, p_{\langle\mathrm{Add},3\rangle}\rangle \le i_{U,\bar{E}}(p_{\langle\alpha,\mathrm{Add},2\rangle}, p_{\langle\alpha,\mathrm{Add},3\rangle})$ for all $\alpha < \kappa^+$. We may suppose that $\langle p_{\langle\mathrm{Add},2\rangle}, p_{\langle\mathrm{Add},3\rangle}\rangle \in G_2 \times G_3$ (see Remark 3.5(b)).

Let $I_U = I_U^{\mathrm{Add},1} \times I_U^{\mathrm{Add},2} \times I_U^{\mathrm{Add},3} \times I_U^{\mathrm{Add},4} \times I_U^{\mathrm{Col}}$. It follows from the above results that I_U is $\mathbb{R}_U$-generic over N^*.

Claim 3.14.

(a) $I_\tau^{\mathrm{Add},1} = \langle i_{\tau,\bar{E}}^{-1''}(I_{\bar{E}}^{\mathrm{Add},1})\rangle$ *is* $\mathbb{R}_\tau^{\mathrm{Add},1}$*-generic over* M_τ^*,

(b) $I_\tau^{\mathrm{Add},2} = \langle i_{\tau,\bar{E}}^{-1''}(I_{\bar{E}}^{\mathrm{Add},2})\rangle$ *is* $\mathbb{R}_\tau^{\mathrm{Add},2}$*-generic over* M_τ^*,

(c) $I_\tau^{\mathrm{Add},3} = \langle i_{\tau,\bar{E}}^{-1''}(I_{\bar{E}}^{\mathrm{Add},3})\rangle$ *is* $\mathbb{R}_\tau^{\mathrm{Add},3}$*-generic over* M_τ^*.

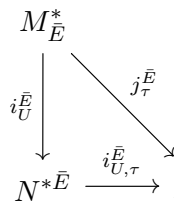
Proof. (a) Let A be a maximal antichain of $\mathbb{R}_\tau^{\mathrm{Add},1}$ in M_τ^*. Then $i_{\tau,\bar{E}}(A)$ is a maximal antichain of $\mathbb{R}_{\bar{E}}^{\mathrm{Add},1}$ in $M_{\bar{E}}^*$. Since $|A| \le \kappa^+$ and $\mathrm{crit}\, i_{\tau,\bar{E}} = \kappa_{M_\tau^*}^{+4} > \kappa^+$, we have $i_{\tau,\bar{E}}(A) = i''_{\tau,\bar{E}}(A)$. Then $I_{\bar{E}}^{\mathrm{Add},1} \cap i''_{\tau,\bar{E}}(A) \neq \emptyset$, which implies $I_\tau^{\mathrm{Add},1} \cap A \neq \emptyset$.

(b) Let A be a maximal antichain of $\mathbb{R}_\tau^{\mathrm{Add},2}$ in M_τ^*. Then $i_{\tau,\bar{E}}(A)$ is a maximal antichain of $\mathbb{R}_{\bar{E}}^{\mathrm{Add},2}$ in $M_{\bar{E}}^*$. Since $|A| \le \kappa^{++}$ and $\mathrm{crit}\, i_{\tau,\bar{E}} = \kappa_{M_\tau^*}^{+4} > \kappa^{++}$,

we have $i_{\tau,\bar{E}}(A)=i''_{\tau,\bar{E}}(A)$. Then $I^{\mathrm{Add},2}_{\bar{E}}\cap i''_{\tau,\bar{E}}(A)\neq\emptyset$, which implies $I^{\mathrm{Add},2}_{\tau}\cap A\neq\emptyset$.

(c) Let A be a maximal antichain of $\mathbb{R}^{\mathrm{Add},3}_{\tau}$ in M^*_τ. Then $i_{\tau,\bar{E}}(A)$ is a maximal antichain of $\mathbb{R}^{\mathrm{Add},3}_{\bar{E}}$ in $M^*_{\bar{E}}$. Since $|A|\le\kappa^{+3}$ and $\operatorname{crit} i_{\tau,\bar{E}}=\kappa^{+4}_{M^*_\tau}>\kappa^{+3}$, we have $i_{\tau,\bar{E}}(A)=i''_{\tau,\bar{E}}(A)$. Then $I^{\mathrm{Add},3}_{\bar{E}}\cap i''_{\tau,\bar{E}}(A)\neq\emptyset$, which implies $I^{\mathrm{Add},1}_{\tau}\cap A\neq\emptyset$. ∎

As $U\in M^*_{\bar{E}}$ and $E(\tau)\in M^*_{\bar{E}}$ for all $\tau<\mathrm{l}(\bar{E})$, we have the following diagram:



$$\begin{aligned}
&U=E_\kappa(0),\\
&i^{\bar{E}}_U: M^*_{\bar{E}}\to N^{*\bar{E}}\simeq \mathrm{Ult}(M^*_{\bar{E}},U),\\
&j^{\bar{E}}_\tau: M^*_{\bar{E}}\to M^{*\bar{E}}_\tau\simeq \mathrm{Ult}(M^*_{\bar{E}},E(\tau)),\\
&i^{\bar{E}}_{U,\tau}(i^{\bar{E}}_U(f)(\kappa))=j^{\bar{E}}_\tau(f)(\kappa).
\end{aligned}$$

Recall that $I^{\mathrm{Add},4}_U\times I^{\mathrm{Col}}_U\in M^*_{\bar{E}}$, which is $\mathbb{R}^{\mathrm{Add},4}_U\times\mathbb{R}^{\mathrm{Col}}_U$-generic over $N^{*\bar{E}}$.

Claim 3.15. *There is $I^{\mathrm{Add},4}_\tau\times I^{\mathrm{Col}}_\tau\in M^*_{\bar{E}}$ which is $\mathbb{R}^{\mathrm{Add},4}_\tau\times\mathbb{R}^{\mathrm{Col}}_\tau$-generic over M^*_τ.*

Proof. We follow the idea from [M2]. For this set

$$\begin{aligned}
\mathbb{R}^{\bar{E},\mathrm{Col}}_\tau&=\mathrm{Col}(\kappa^{+6},j^{\bar{E}}_\tau(\kappa))_{M^{*\bar{E}}_\tau},\\
\mathbb{R}^{\bar{E},\mathrm{Add},4}_\tau&=(\mathrm{Add}(\kappa^{+4},j^{\bar{E}}_\tau(\kappa)^+)\times\mathrm{Add}(\kappa^{+5},j^{\bar{E}}_\tau(\kappa)^{++})\\
&\quad\times\mathrm{Add}(\kappa^{+6},j^{\bar{E}}_\tau(\kappa)^{+3}))_{M^{*\bar{E}}_\tau},
\end{aligned}$$

$\mathbb{R}^{\bar{E},\mathrm{Add},4}_\tau\times\mathbb{R}^{\bar{E},\mathrm{Col}}_\tau$ and $\mathbb{R}^{\mathrm{Add},4}_\tau\times\mathbb{R}^{\mathrm{Col}}_\tau$ are coded in $V^{M^*_\tau}_{j_\tau(\kappa)+3}$, $V^{M^{*\bar{E}}_\tau}_{j^{\bar{E}}_\tau(\kappa)+3}$ respectively. $V^{M^*_\tau}_{j_\tau(\kappa)+3}$, $V^{M^{*\bar{E}}_\tau}_{j^{\bar{E}}_\tau(\kappa)+3}$ are determined by $V^{V^*}_{\kappa+3}$, $V^{M^*_{\bar{E}}}_{\kappa+3}$ (and $E(\tau)$, of course). As $E(\tau)\in M^*_{\bar{E}}$ and $V^{V^*}_{\kappa+3}=V^{M^*_{\bar{E}}}_{\kappa+3}$ we see that $\mathbb{R}^{\bar{E},\mathrm{Add},4}_\tau\times\mathbb{R}^{\bar{E},\mathrm{Col}}_\tau=\mathbb{R}^{\mathrm{Add},4}_\tau\times\mathbb{R}^{\mathrm{Col}}_\tau$.

By the same reasoning, each antichain of $\mathbb{R}^{\mathrm{Add},4}_\tau\times\mathbb{R}^{\mathrm{Col}}_\tau$ appearing in M^*_τ is also an anti-chain of $\mathbb{R}^{\bar{E},\mathrm{Add},4}_\tau\times\mathbb{R}^{\bar{E},\mathrm{Col}}_\tau$ appearing in $M^{*\bar{E}}_\tau$. Hence, if $I^{\mathrm{Add},4}_\tau\times I^{\mathrm{Col}}_\tau\in M^*_{\bar{E}}$ is an $\mathbb{R}^{\bar{E},\mathrm{Add},4}_\tau\times\mathbb{R}^{\bar{E},\mathrm{Col}}_\tau$-generic filter over $M^{*\bar{E}}_\tau$ then it is also $\mathbb{R}^{\mathrm{Add},4}_\tau\times\mathbb{R}^{\mathrm{Col}}_\tau$-generic over M^*_τ.

Let $I^{\mathrm{Add},4}_\tau\times I^{\mathrm{Col}}_\tau=\langle i''_{U,\tau}(I^{\mathrm{Add},4}_U\times I^{\mathrm{Col}}_U)\rangle$. We show that it is as required. So let $D\in M^{*\bar{E}}_\tau$ be dense open in $\mathbb{R}^{\bar{E},\mathrm{Add},4}_\tau\times\mathbb{R}^{\bar{E},\mathrm{Col}}_\tau$. Then $D=j^{\bar{E}}_\tau(f)(\bar{E}_\alpha{\restriction}\tau)$ for some function $f\in M^*_{\bar{E}}$ on V_κ. It then follows that in $M^*_{\bar{E}}$,

$$D^*=\{\bar\nu\in V_\kappa: f(\bar\nu)\text{ is dense open in }\mathbb{R}^{\mathrm{Add},4}_{\bar{E}}\times\mathbb{R}^{\mathrm{Col}}_{\bar{E}}\}\in E_\alpha(\tau).$$

It is easily seen that

$$B = \{\mu : |\{\bar{\nu} \in D^* : \kappa^0(\bar{\nu}) = \mu\}| \leq \mu^{+3}\} \in E_\kappa(0).$$

Thus for each $\mu \in B$ we can find $f^*(\mu)$ such that for all $\bar{\nu} \in D^*$ with $\kappa^0(\bar{\nu}) = \mu$ the set $f^*(\mu)$ is a dense open subset of $f(\bar{\nu})$ (in $M^*_{\bar{E}}$). Hence

$$N^{*\bar{E}} \models \text{“}i^{\bar{E}}_U(f^*)(\kappa) \text{ is dense open”},$$
$$M^{*\bar{E}}_\tau \models \text{“}j^{\bar{E}}_\tau(f^*)(\kappa) \subseteq j^{\bar{E}}_\tau(f)(\bar{E}_\alpha{\restriction}\tau)\text{”}.$$

So there is $g \in M^*_{\bar{E}}$ such that $i^{\bar{E}}_U(g)(\kappa) \in (I^{\mathrm{Add},4}_U \times I^{\mathrm{Col}}_U) \cap i^{\bar{E}}_U(f^*)(\kappa)$. It then follows that $j^{\bar{E}}_\tau(g)(\kappa) \in i^{\bar{E}''}_{U,\tau}(I^{\mathrm{Add},4}_U \times I^{\mathrm{Col}}_U) \cap j^{\bar{E}}_\tau(f^*)(\kappa)$. This means that $(I^{\mathrm{Add},4}_\tau \times I^{\mathrm{Col}}_\tau) \cap j^{\bar{E}}_\tau(f)(\bar{E}_\alpha{\restriction}\tau) \neq \emptyset$. The result follows. ∎

Now let $I_\tau = I^{\mathrm{Add},1}_\tau \times I^{\mathrm{Add},2}_\tau \times I^{\mathrm{Add},3}_\tau \times I^{\mathrm{Add},4}_\tau \times I^{\mathrm{Col}}_\tau$. It follows that I_τ is $\mathbb{R}_\tau$-generic over M^*_τ.

Claim 3.16. *$I^{\mathrm{Add},4}_{\bar{E}} \times I^{\mathrm{Col}}_{\bar{E}} = \langle \bigcup_{\tau<\mathrm{l}(\bar{E})} i''_{\tau,\bar{E}}(I^{\mathrm{Add},4}_\tau \times I^{\mathrm{Col}}_\tau)\rangle$ is $\mathbb{R}^{\mathrm{Add},4}_{\bar{E}} \times \mathbb{R}^{\mathrm{Col}}_{\bar{E}}$-generic over $M^*_{\bar{E}}$.*

Proof. Let D be a dense open subset of $\mathbb{R}^{\mathrm{Add},4}_{\bar{E}} \times \mathbb{R}^{\mathrm{Col}}_{\bar{E}}$ in $M^*_{\bar{E}}$. Let $\tau < \mathrm{l}(\bar{E})$ and $D_\tau \in M^*_\tau$ be such that $D = i_{\tau,\bar{E}}(D_\tau)$. By elementarity, D_τ is dense open in $\mathbb{R}^{\mathrm{Add},4}_\tau \times \mathbb{R}^{\mathrm{Col}}_\tau$. Let $\langle p, q\rangle \in (I^{\mathrm{Add},4}_\tau \times I^{\mathrm{Col}}_\tau) \cap D_\tau$. Then $\langle i_{\tau,\bar{E}}(p), i_{\tau,\bar{E}}(q)\rangle \in (I^{\mathrm{Add},4}_{\bar{E}} \times I^{\mathrm{Col}}_{\bar{E}}) \cap D$. ∎

Let $I_{\bar{E}} = I^{\mathrm{Add},1}_{\bar{E}} \times I^{\mathrm{Add},2}_{\bar{E}} \times I^{\mathrm{Add},3}_{\bar{E}} \times I^{\mathrm{Add},4}_{\bar{E}} \times I^{\mathrm{Col}}_{\bar{E}}$. It follows that $I_{\bar{E}}$ is $\mathbb{R}_{\bar{E}}$-generic over $M^*_{\bar{E}}$.

To summarize, so far we have shown the following:

- I_U is $\mathbb{R}_U$-generic over N^*,
- I_τ is $\mathbb{R}_\tau$-generic over M^*_τ,
- $I_{\bar{E}}$ is $\mathbb{R}_{\bar{E}}$-generic over $M^*_{\bar{E}}$.

Before continuing we recall Easton's lemma.

Lemma 3.17 (Easton's lemma). *Let λ be regular uncountable, and suppose that $\mathbb{P}$ satisfies the λ-c.c. and $\mathbb{Q}$ is λ-closed. Then*

(a) $\Vdash_{\mathbb{P}\times\mathbb{Q}}$ *“λ is a regular uncountable cardinal”,*
(b) $\Vdash_{\mathbb{Q}}$ *“$\mathbb{P}$ satisfies the λ-c.c.”,*
(c) $\Vdash_{\mathbb{P}}$ *“$\mathbb{Q}$ is λ-distributive”.*

Claim 3.18. *I_U is $\mathbb{R}_U$-generic over $N^*[G_U]$.*

Proof. First note that in N^* the forcing notions $\mathbb{R}_U$ and $\mathbb{P}_U$ are $i_U(\kappa)^+$-c.c. and $i_U(\kappa)^+$-closed respectively. Now let A be a maximal antichain of $\mathbb{R}_U$ in $N^*[G]$. By Easton's lemma $|A| \leq i_U(\kappa)$, hence again by Easton's lemma $A \in N^*$. It follows that $I_U \cap A \neq \emptyset$, as I_U is $\mathbb{R}_U$-generic over N^*. The result follows. ∎

By similar arguments we obtain

Claim 3.19. *I_τ is $\mathbb{R}_\tau$-generic over $M^*_\tau[G_\tau]$.*

Claim 3.20. *$I_{\bar{E}}$ is $\mathbb{R}_{\bar{E}}$-generic over $M^*_{\bar{E}}[G_{\bar{E}}]$.*

It remains to prove part (d) of Lemma 3.8. Before going into details let us recall a simple observation.

Claim 3.21.

(a) $V^{N^{*\bar{E}}}_{i^{\bar{E}}_U(\kappa)+3} = V^{N^*}_{i_U(\kappa)+3}$,

(b) $i^{\bar{E}}_{U,\tau} \restriction V^{N^{*\bar{E}}}_{i^{\bar{E}}_U(\kappa)+3} = i_{U,\tau} \restriction V^{N^*}_{i_U(\kappa)+3}$,

(c) $i^{\bar{E}}_{\tau',\tau} \restriction V^{N^{*\bar{E}}}_{i^{\bar{E}}_U(\kappa)+3} = i_{\tau',\tau} \restriction V^{N^*}_{i_U(\kappa)+3}$.

Claim 3.22. $i''_{U,\tau'}(I_U) \subseteq I_{\tau'}$.

Proof. (1) $i''_{U,\tau'}(I^{\mathrm{Add},1}_U) \subseteq I^{\mathrm{Add},1}_{\tau'}$: This follows from $i''_{U,\tau'}(I^{\mathrm{Add},1}_U) = i''_{U,\tau'}(\langle i^{-1''}_{U,\bar{E}}(I^{\mathrm{Add},1}_{\bar{E}})\rangle) \subseteq \langle i^{-1''}_{\tau',\bar{E}}(I^{\mathrm{Add},1}_{\bar{E}})\rangle = I^{\mathrm{Add},1}_{\tau'}$.

(2) $i''_{U,\tau'}(I^{\mathrm{Add},2}_U) \subseteq I^{\mathrm{Add},2}_{\tau'}$: It suffices to show $\forall \alpha < \kappa^+$, $i_{U,\tau'}(p_{\langle \alpha,\mathrm{Add},2\rangle}) \in I^{\mathrm{Add},2}_{\tau'}$. But we have $p_{\langle \mathrm{Add},2\rangle} \in I^{\mathrm{Add},2}_{\bar{E}}$ and $\forall \alpha < \kappa^+$, $p_{\langle \mathrm{Add},2\rangle} \leq i_{U,\bar{E}}(p_{\langle \alpha,\mathrm{Add},2\rangle})$. It then follows that $\forall \alpha < \kappa^+$, $i_{U,\tau'}(p_{\langle \alpha,\mathrm{Add},2\rangle}) = i^{-1}_{\tau',\bar{E}}(i_{U,\bar{E}}(p_{\langle \alpha,\mathrm{Add},2\rangle})) \geq i^{-1}_{\tau',\bar{E}}(p_{\langle \mathrm{Add},2\rangle})$. But now note that by our definition $i^{-1}_{\tau',\bar{E}}(p_{\langle \mathrm{Add},2\rangle}) \in I^{\mathrm{Add},2}_{\tau'}$. It then follows that $i_{U,\tau'}(p_{\langle \alpha,\mathrm{Add},2\rangle}) \in I^{\mathrm{Add},2}_{\tau'}$.

(3) $i''_{U,\tau'}(I^{\mathrm{Add},3}_U) \subseteq I^{\mathrm{Add},3}_{\tau'}$: By the same argument as in (2) using the fact that $p_{\langle \mathrm{Add},3\rangle} \in I^{\mathrm{Add},3}_{\bar{E}}$.

(4) $i''_{U,\tau'}(I^{\mathrm{Add},4}_U \times I^{\mathrm{Col}}_U) \subseteq I^{\mathrm{Add},4}_{\tau'} \times I^{\mathrm{Col}}_{\tau'}$: Trivial by the definition of $I^{\mathrm{Add},4}_{\tau'} \times I^{\mathrm{Col}}_{\tau'}$ and the previous claim.

The result follows. ∎

Claim 3.23. $i''_{\tau',\tau}(I_{\tau'}) \subseteq I_\tau$.

Proof. (1) $i''_{\tau',\tau}(I^{\mathrm{Add},1}_{\tau'}) \subseteq I^{\mathrm{Add},1}_\tau$: This follows from $i''_{\tau',\tau}(I^{\mathrm{Add},1}_{\tau'}) = i''_{\tau',\tau}(\langle i^{-1''}_{\tau',\bar{E}}(I^{\mathrm{Add},1}_{\bar{E}})\rangle) \subseteq \langle i^{-1''}_{\tau,\bar{E}}(I^{\mathrm{Add},1}_{\bar{E}})\rangle = I^{\mathrm{Add},1}_\tau$,

(2) $i''_{\tau',\tau}(I^{\mathrm{Add},2}_{\tau'}) \subseteq I^{\mathrm{Add},2}_\tau$: By the same argument as in (1),

(3) $i''_{\tau',\tau}(I^{\mathrm{Add},3}_{\tau'}) \subseteq I^{\mathrm{Add},3}_\tau$: By the same argument as in (1),

(4) $i''_{\tau',\tau}(I^{\mathrm{Add},4}_{\tau'} \times I^{\mathrm{Col}}_{\tau'}) \subseteq I^{\mathrm{Add},4}_\tau \times I^{\mathrm{Col}}_\tau$: Because $i''_{\tau',\tau}(I^{\mathrm{Add},4}_{\tau'} \times I^{\mathrm{Col}}_{\tau'}) = i''_{\tau',\tau}(\langle i^{\bar{E}''}_{U,\tau'}(I^{\mathrm{Add},4}_U \times I^{\mathrm{Col}}_U)\rangle) \subseteq \langle i^{\bar{E}''}_{U,\tau}(I^{\mathrm{Add},4}_U \times I^{\mathrm{Col}}_U)\rangle = I^{\mathrm{Add},4}_\tau \times I^{\mathrm{Col}}_\tau$.

The result follows. ∎

Claim 3.24. $i''_{\tau,\bar{E}}(I_\tau) \subseteq I_{\bar{E}}$.

Proof. (1) $i''_{\tau,\bar{E}}(I^{\mathrm{Add},1}_{\tau}) \subseteq I^{\mathrm{Add},1}_{\bar{E}}$: Trivial because we have $i''_{\tau,\bar{E}}(I^{\mathrm{Add},1}_{\tau}) = i''_{\tau,\bar{E}}(\langle i^{-1''}_{\tau,\bar{E}}(I^{\mathrm{Add},1}_{\tau})\rangle) \subseteq I^{\mathrm{Add},1}_{\bar{E}}$.

(2) $i''_{\tau,\bar{E}}(I^{\mathrm{Add},2}_{\tau}) \subseteq I^{\mathrm{Add},2}_{\bar{E}}$: As in (1).

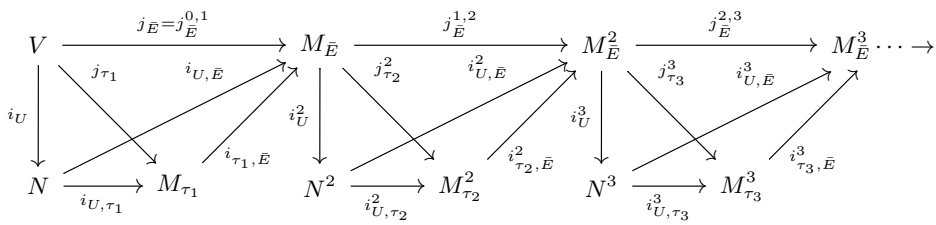
(3) $i''_{\tau,\bar{E}}(I^{\mathrm{Add},3}_{\tau}) \subseteq I^{\mathrm{Add},3}_{\bar{E}}$: As in (1).

(4) $i''_{\tau,\bar{E}}(I^{\mathrm{Add},4}_{\tau} \times I^{\mathrm{Col}}_{\tau}) \subseteq I^{\mathrm{Add},4}_{\bar{E}} \times I^{\mathrm{Col}}_{\bar{E}}$: Trivial by the definition of $I^{\mathrm{Add},4}_{\bar{E}} \times I^{\mathrm{Col}}_{\bar{E}}$.

The result follows. ∎

This completes the proof of Lemma 3.8. ∎

We iterate $j_{\bar{E}}$ ω-many times and consider the diagram



where

$$j^0_{\bar{E}} = \mathrm{id}, \quad j^n_{\bar{E}} = j^{0,n}_{\bar{E}}, \quad j^{m,n}_{\bar{E}} = j^{n-1,n}_{\bar{E}} \circ \cdots \circ j^{m+1,m+2}_{\bar{E}} \circ j^{m,m+1}_{\bar{E}}.$$

Let $R(-,-) = R^{\mathrm{Add}}(-,-) \times R^{\mathrm{Col}}(-,-)$ be a function such that

$$i^2_U(R^{\mathrm{Add}})(\kappa, i_U(\kappa)) = \mathbb{R}^{\mathrm{Add}}_U, \quad i^2_U(R^{\mathrm{Col}})(\kappa, i_U(\kappa)) = \mathbb{R}^{\mathrm{Col}}_U,$$

where i^2_U is the second iterate of i_U. Then we will have

$$i^2_U(R)(\kappa, i_U(\kappa)) = \mathbb{R}_U.$$

The following is trivial.

LEMMA 3.25.

(a) $j^2_{\bar{E}}(R^{\mathrm{Add}})(\kappa, j_{\bar{E}}(\kappa)) = \mathbb{R}^{\mathrm{Add}}_{\bar{E}}$,
(b) $j^2_{\bar{E}}(R^{\mathrm{Col}})(\kappa, j_{\bar{E}}(\kappa)) = \mathbb{R}^{\mathrm{Col}}_{\bar{E}}$,
(c) $j^2_{\bar{E}}(R)(\kappa, j_{\bar{E}}(\kappa)) = \mathbb{R}_{\bar{E}}$.

Cardinal structure and the power function in $N^*[I_U]$. The following lemma gives us everything that we need about the model $N^*[I_U]$.

LEMMA 3.26.

(a) *In $N^*[I_U]$ there are no cardinals in $[\kappa^{+7}, i_U(\kappa)]$ and all other N^*-cardinals are preserved.*
(b) *The power function differs from the power function of N^* at the following points:* $2^{\kappa^+} = \kappa^{+4}$, $2^{\kappa^{++}} = \kappa^{+5}$, $2^{\kappa^{+3}} = \kappa^{+6}$, $2^{\kappa^{+4}} = i_U(\kappa)^+$, $2^{\kappa^{+5}} = i_U(\kappa)^{++}$, $2^{\kappa^{+6}} = i_U(\kappa)^{+3}$.

Cardinal structure in $M^*_\tau[I_\tau]$ and $M^*_{\bar{E}}[I_{\bar{E}}]$. The following lifting says everything which we can possibly say:

$$\begin{array}{ccccc} & & & & M^*_{\bar{E}}[I_{\bar{E}}] \\ & & & \overset{i^*_{\tau',\bar{E}}}{\nearrow} & \uparrow{\scriptstyle i^*_{\tau,\bar{E}}} \\ N^*[I_U] & \xrightarrow{i^*_{U,\tau'}} & M^*_{\tau'}[I_{\tau'}] & \xrightarrow{i^*_{\tau',\tau}} & M^*_\tau[I_\tau] \end{array}$$

The forcing notion $\mathbb{P}_{\bar{E}}$, due to Merimovich, which we define later, adds a club to κ. For each pair ν_1, ν_2 of successive points in the club the cardinal structure and power function in the range $[\nu_1^+, \nu_2^{+3}]$ of the generic extension is analogous to the cardinal structure and power function in the range $[\kappa^+, j_{\bar{E}}(\kappa)^{+3}]$ of $M^*_{\bar{E}}[I_{\bar{E}}]$.

Cardinal structure in $N^*[I^{\mathrm{Col}}_U]$. The following lemma gives us everything that we need about the model $N^*[I^{\mathrm{Col}}_U]$.

Lemma 3.27.

(a) *In $N^*[I^{\mathrm{Col}}_U]$ there are no cardinals in $[\kappa^{+7}, i_U(\kappa)]$ and all other N^*-cardinals are preserved.*
(b) *GCH holds in $N^*[I^{\mathrm{Col}}_U]$.*

Cardinal structure in $M^*_\tau[I^{\mathrm{Col}}_\tau]$ and $M^*_{\bar{E}}[I^{\mathrm{Col}}_{\bar{E}}]$. The following lifting says everything which we can possibly say.

$$\begin{array}{ccccc} & & & & M^*_{\bar{E}}[I^{\mathrm{Col}}_{\bar{E}}] \\ & & & \overset{i^*_{\tau',\bar{E}}}{\nearrow} & \uparrow{\scriptstyle i^*_{\tau,\bar{E}}} \\ N^*[I^{\mathrm{Col}}_U] & \xrightarrow{i^*_{U,\tau'}} & M^*_{\tau'}[I^{\mathrm{Col}}_{\tau'}] & \xrightarrow{i^*_{\tau',\tau}} & M^*_\tau[I^{\mathrm{Col}}_\tau] \end{array}$$

The forcing notion $\mathbb{R}_{\bar{E}_\kappa}$, which we define later, adds a club to κ. For each pair ν_1, ν_2 of successive points in the club the cardinal structure and power function in the range $[\nu_1^+, \nu_2^{+3}]$ of the generic extension is the same as the cardinal structure and power function in the range $[\kappa^+, j_{\bar{E}}(\kappa)^{+3}]$ of $M^*_{\bar{E}}[I^{\mathrm{Col}}_{\bar{E}}]$.

4. Redefining extender sequences. As in [M2], in the prepared model $V = V^*[G]$ we define a new extender sequence system $\bar{F} = \langle \bar{F}_\alpha : \alpha \in \mathrm{dom}(\bar{F})\rangle$ by:

- $\mathrm{dom}(\bar{F}) = \mathrm{dom}(\bar{E})$,
- $\mathrm{l}(\bar{F}) = \mathrm{l}(\bar{E})$,
- $\leq_{\bar{F}} = \leq_{\bar{E}}$,
- $F(0) = E(0)$,

- $I(\tau) = I_\tau$,
- $\forall 0 < \tau < \mathrm{l}(\bar{F})$, $F(\tau) = \langle\langle F_\alpha(\tau) : \alpha \in \mathrm{dom}(\bar{F})\rangle, \langle \pi_{\beta,\alpha} : \beta, \alpha \in \mathrm{dom}(\bar{F}), \beta \geq_{\bar{F}} \alpha\rangle\rangle$ is such that

  $$X \in F_\alpha(\tau) \Leftrightarrow \langle \alpha, F(0), I(0), \ldots, F(\tau'), I(\tau'), \ldots : \tau' < \tau\rangle \in j_{\bar{E}}(X),$$

  and

  $$\pi_{\beta,\alpha}(\langle \xi, d\rangle) = \langle \pi_{\beta,\alpha}(\xi), d\rangle,$$

- $\forall \alpha \in \mathrm{dom}(\bar{F})$, $\bar{F}_\alpha = \langle \alpha, F(\tau), I(\tau) : \tau < \mathrm{l}(\bar{F})\rangle$.

Also let $I(\bar{F})$ be the filter generated by $\bigcup_{\tau<\mathrm{l}(\bar{F})} i''_{\tau,\bar{E}} I(\tau)$. Then $I(\bar{F})$ is $\mathbb{R}_{\bar{E}}$-generic over $M_{\bar{E}}$. Let us write $I(\bar{F}) = I^{\mathrm{Add}}(\bar{F}) \times I^{\mathrm{Col}}(\bar{F})$ corresponding to $\mathbb{R}_{\bar{E}} = \mathbb{R}^{\mathrm{Add}}_{\bar{E}} \times \mathbb{R}^{\mathrm{Col}}_{\bar{E}}$.

From now on we work with this new definition of extender sequence system and use $\bar{E}$ to denote it.

Definition 4.1.

(1) We write $T \in \bar{E}_\alpha$ iff $\forall \xi < \mathrm{l}(\bar{E}_\alpha)$, $T \in E_\alpha(\xi)$,
(2) $T \setminus \bar{\nu} = T \setminus V^*_{\kappa^0(\bar{\nu})}$,
(3) $T \restriction \bar{\nu} = T \cap V^*_{\kappa^0(\bar{\nu})}$.

We now define two forcing notions $\mathbb{P}_{\bar{E}}$ and $\mathbb{R}_{\bar{E}_\kappa}$.

5. Definition of the forcing notion $\mathbb{P}_{\bar{E}}$. This forcing notion, defined in the ground model $V = V^*[G]$, is essentially the forcing notion of [M2]. We give it in detail for completeness and later use. First we define a forcing notion $\mathbb{P}^*_{\bar{E}}$.

Definition 5.1. A condition p in $\mathbb{P}^*_{\bar{E}}$ is of the form

$$p = \{\langle \bar{\gamma}, p^{\bar{\gamma}}\rangle : \bar{\gamma} \in s\} \cup \{\langle \bar{E}_\alpha, T, f, F\rangle\}$$

where

(1) $s \in [\bar{E}]^{\leq \kappa}$, $\min \bar{E} = \bar{E}_\kappa \in s$,
(2) $p^{\bar{E}_\kappa} \in V^*_{\kappa^0(\bar{E})}$ is an extender sequence such that $\kappa(p^{\bar{E}_\kappa})$ is inaccessible (we allow $p^{\bar{E}_\kappa} = \emptyset$); write p^0 for $p^{\bar{E}_\kappa}$,
(3) $\forall \bar{\gamma} \in s \setminus \{\min(s)\}$, $p^{\bar{\gamma}} \in [V^*_{\kappa^0(\bar{E})}]^{<\omega}$ is a ^{0}-increasing sequence of extender sequences and $\max \kappa(p^{\bar{\gamma}})$ is inaccessible,
(4) $\forall \bar{\gamma} \in s$, $\kappa(p^0) \leq \max \kappa(p^{\bar{\gamma}})$,
(5) $\forall \bar{\gamma} \in s$, $\bar{E}_\alpha \geq \bar{\gamma}$,
(6) $T \in \bar{E}_\alpha$,
(7) $\forall \bar{\nu} \in T$, $|\{\bar{\gamma} \in s : \bar{\nu} \text{ is permitted to } p^{\bar{\gamma}}\}| \leq \kappa^0(\bar{\nu})$,
(8) $\forall \bar{\beta}, \bar{\gamma} \in s$, $\forall \bar{\nu} \in T$, if $\bar{\beta} \neq \bar{\gamma}$ and $\bar{\nu}$ is permitted to $p^{\bar{\beta}}$, $p^{\bar{\gamma}}$, then $\pi_{\bar{E}_\alpha,\bar{\beta}}(\bar{\nu}) \neq \pi_{\bar{E}_\alpha,\bar{\gamma}}(\bar{\nu})$,

(9) f is a function such that
  (9.1) $\mathrm{dom}(f) = \{\bar{\nu} \in T : \mathrm{l}(\bar{\nu}) = 0\}$,
  (9.2) $f(\nu_1) \in R(\kappa(p^0), \nu_1^0)$; if $p^0 = \emptyset$, then $f(\nu_1) = \emptyset$,

(10) F is a function such that
  (10.1) $\mathrm{dom}(F) = \{\langle \bar{\nu}_1, \bar{\nu}_2\rangle \in T^2 : \mathrm{l}(\bar{\nu}_1) = \mathrm{l}(\bar{\nu}_2) = \emptyset\}$,
  (10.2) $F(\nu_1, \nu_2) \in R(\nu_1^0, \nu_2^0)$,
  (10.3) $j^2_{\bar{E}}(F)(\alpha, j_{\bar{E}}(\alpha)) \in I(\bar{E})$.

We write $\mathrm{mc}(p)$, $\mathrm{supp}(p)$, T^p, f^p and F^p for $\bar{E}_\alpha$, s, T, f and F respectively.

Definition 5.2. For $p, q \in \mathbb{P}^*_{\bar{E}}$, we say p is a *Prikry extension* of q ($p \leq^* q$ or $p \leq^0 q$) iff

(1) $\mathrm{supp}(p) \supseteq \mathrm{supp}(q)$,
(2) $\forall \bar{\gamma} \in \mathrm{supp}(q)$, $p^{\bar{\gamma}} = q^{\bar{\gamma}}$,
(3) $\mathrm{mc}(p) \geq_{\bar{E}} \mathrm{mc}(q)$,
(4) $\mathrm{mc}(p) >_{\bar{E}} \mathrm{mc}(q) \Rightarrow \mathrm{mc}(q) \in \mathrm{supp}(p)$,
(5) $\forall \bar{\gamma} \in \mathrm{supp}(p) \setminus \mathrm{supp}(q)$, $\max \kappa^0(p^{\bar{\gamma}}) > \bigcup\bigcup j_{\bar{E}}(f^q)(\kappa(\mathrm{mc}(q)))$,
(6) $T^p \leq \pi^{-1''}_{\mathrm{mc}(p),\mathrm{mc}(q)} T^q$,
(7) $\forall \bar{\gamma} \in \mathrm{supp}(q)$, $\forall \bar{\nu} \in T^p$, if $\bar{\nu}$ is permitted to $p^{\bar{\gamma}}$, then
$$\pi_{\mathrm{mc}(p),\bar{\gamma}}(\bar{\nu}) = \pi_{\mathrm{mc}(q),\bar{\gamma}}(\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}(\bar{\nu})),$$
(8) $\forall \nu_1 \in \mathrm{dom}(f^p)$, $f^p(\nu_1) \leq f^q \circ \pi_{\mathrm{mc}(p),\mathrm{mc}(q)}(\nu_1)$,
(9) $\forall \langle \nu_1, \nu_2\rangle \in \mathrm{dom}(F^p)$, $F^p(\nu_1, \nu_2) \leq F^q \circ \pi_{\mathrm{mc}(p),\mathrm{mc}(q)}(\nu_1, \nu_2)$.

We are now ready to define the forcing notion $\mathbb{P}_{\bar{E}}$.

Definition 5.3. A condition p in $\mathbb{P}_{\bar{E}}$ is of the form
$$p = p_n^\frown \ldots^\frown p_0$$
where

- $p_0 \in \mathbb{P}^*_{\bar{E}}$, $\kappa^0(p_0^0) \geq \kappa^0(\bar{\mu}_1)$,
- $p_1 \in \mathbb{P}^*_{\bar{\mu}_1}$, $\kappa^0(p_1^0) \geq \kappa^0(\bar{\mu}_2)$,
- $\vdots$
- $p_n \in \mathbb{P}^*_{\bar{\mu}_n}$,

and $\langle \bar{\mu}_n, \ldots, \bar{\mu}_1, \bar{E}\rangle$ is a ^{0}-increasing sequence of extender sequence systems, that is, $\kappa^0(\bar{\mu}_n) < \cdots < \kappa^0(\bar{\mu}_1) < \kappa^0(\bar{E})$.

Definition 5.4. For $p, q \in \mathbb{P}_{\bar{E}}$, we say p is a *Prikry extension* of q ($p \leq^* q$ or $p \leq^0 q$) iff
$$p = p_n^\frown \ldots^\frown p_0, \quad q = q_n^\frown \ldots^\frown q_0$$
where

- $p_0, q_0 \in \mathbb{P}^*_{\bar{E}}$, $p_0 \leq^* q_0$,
- $p_1, q_1 \in \mathbb{P}^*_{\bar{\mu}_1}$, $p_1 \leq^* q_1$,

  $\vdots$
- $p_n, q_n \in \mathbb{P}^*_{\bar{\mu}_n}$, $p_n \leq^* q_n$.

Now let $p \in \mathbb{P}_{\bar{E}}$ and $\bar{\nu} \in T^p$. We define $p_{\langle\bar{\nu}\rangle}$, a one-element extension of p by $\bar{\nu}$.

DEFINITION 5.5. Let $p \in \mathbb{P}_{\bar{E}}$, $\bar{\nu} \in T^p$, $\kappa^0(\bar{\nu}) > \bigcup\bigcup j_{\bar{E}}(f^{p,\mathrm{Col}})(\kappa(\mathrm{mc}(p)))$, where $f^{p,\mathrm{Col}}$ is the collapsing part of f^p. Then $p_{\langle\bar{\nu}\rangle} = p_1^\frown p_0$ where

(1) $\mathrm{supp}(p_0) = \mathrm{supp}(p)$,
(2) $\forall \bar{\gamma} \in \mathrm{supp}(p_0)$,

$$p_0^{\bar{\gamma}} = \begin{cases} \pi_{\mathrm{mc}(p),\bar{\gamma}}(\bar{\nu}) & \text{if } \bar{\nu} \text{ is permitted to } p^{\bar{\gamma}} \text{ and } \mathrm{l}(\bar{\nu}) > 0, \\ \pi_{\mathrm{mc}(p),\bar{\gamma}}(\bar{\nu}) & \text{if } \bar{\nu} \text{ is permitted to } p^{\bar{\gamma}},\ \mathrm{l}(\bar{\nu}) = 0 \text{ and } \bar{\gamma} = \bar{E}_\kappa, \\ p^{\bar{\gamma}}{}^\frown\langle \pi_{\mathrm{mc}(p),\bar{\gamma}}(\bar{\nu})\rangle & \text{if } \bar{\nu} \text{ is permitted to } p^{\bar{\gamma}},\ \mathrm{l}(\bar{\nu}) = 0 \text{ and } \bar{\gamma} \neq \bar{E}_\kappa, \\ p^{\bar{\gamma}} & \text{otherwise,} \end{cases}$$

(3) $\mathrm{mc}(p_0) = \mathrm{mc}(p)$,
(4) $T^{p_0} = T^p \setminus \bar{\nu}$,
(5) $\forall \nu_1 \in T^{p_0}$, $f^{p_0}(\nu_1) = F^p(\kappa(\bar{\nu}), \nu_1)$,
(6) $F^{p_0} = F^p$,
(7) if $\mathrm{l}(\bar{\nu}) > 0$ then
  (7.1) $\mathrm{supp}(p_1) = \{\pi_{\mathrm{mc}(p),\bar{\gamma}}(\bar{\nu}) : \bar{\gamma} \in \mathrm{supp}(p) \text{ and } \bar{\nu} \text{ is permitted to } p^{\bar{\gamma}}\}$,
  (7.2) $p_1^{\pi_{\mathrm{mc}(p),\bar{\gamma}}(\bar{\nu})} = p^{\bar{\gamma}}$,
  (7.3) $\mathrm{mc}(p_1) = \bar{\nu}$,
  (7.4) $T^{p_1} = T^p \restriction \bar{\nu}$,
  (7.5) $f^{p_1} = f^p \restriction \bar{\nu}$,
  (7.6) $F^{p_1} = F^p \restriction \bar{\nu}$,
(8) if $\mathrm{l}(\bar{\nu}) = 0$ then
  (8.1) $\mathrm{supp}\, p_1 = \{\pi_{\mathrm{mc}(p),0}(\bar{\nu})\}$,
  (8.2) $p_1^{\pi_{\mathrm{mc}(p),0}(\bar{\nu})} = p^{\bar{E}_\kappa}$,
  (8.3) $\mathrm{mc}(p_1) = \bar{\nu}^0$,
  (8.4) $T^{p_1} = \emptyset$,
  (8.5) $f^{p_1} = f^p(\kappa(\bar{\nu}))$,
  (8.6) $F^{p_1} = \emptyset$.

We use $(p_{\langle\bar{\nu}\rangle})_0$ and $(p_{\langle\bar{\nu}\rangle})_1$ for p_0 and p_1 respectively. We also let $p_{\langle\bar{\nu}_1,\bar{\nu}_2\rangle} = (p_{\langle\bar{\nu}_1\rangle})_1^\frown (p_{\langle\bar{\nu}_1\rangle})_{0\langle\bar{\nu}_2\rangle}$ and so on.

The above definition is the key step in the definition of the forcing relation $\leq$.

DEFINITION 5.6. For $p, q \in \mathbb{P}_{\bar{E}}$, we say p is a 1-*point extension* of q ($p \leq^1 q$) iff

$$p = p_{n+1}^\frown \cdots^\frown p_0, \quad q = q_n^\frown \cdots^\frown q_0,$$

and there is $0 \leq k \leq n$ such that

- $\forall i < k$, $p_i, q_i \in \mathbb{P}^*_{\bar{\mu}_i}$, $p_i \leq^* q_i$,
- $\exists \bar{\nu} \in T^{q_k}$, $(p_{k+1})^\frown p_k \leq^* (q_k)_{\langle \bar{\nu} \rangle}$,
- $\forall i > k$, $p_{i+1}, q_i \in \mathbb{P}^*_{\bar{\mu}_i}$, $p_{i+1} \leq^* q_i$,

where $\bar{\mu}_0 = \bar{E}$.

DEFINITION 5.7. For $p, q \in \mathbb{P}_{\bar{E}}$, we say p is an *n-point extension* of q ($p \leq^n q$) iff there are $p^n, \ldots, p^0$ such that

$$p = p^n \leq^1 \cdots \leq^1 p^0 = q.$$

DEFINITION 5.8. For $p, q \in \mathbb{P}_{\bar{E}}$, we say p is an *extension* of q ($p \leq q$) iff there is some n such that $p \leq^n q$.

Suppose that H is $\mathbb{P}_{\bar{E}}$-generic over $V = V^*[G]$. For $\alpha \in \mathrm{dom}(\bar{E})$ set

$$C^\alpha_H = \{\max \kappa(p_0^{\bar{E}_\alpha}) : p \in H\}.$$

THEOREM 5.9.

(a) *$V[H]$ and V have the same cardinals $\geq \kappa$,*
(b) *κ remains strongly inaccessible in $V[H]$,*
(c) *C^α_H is unbounded in κ,*
(d) *C^κ_H is a club in κ,*
(e) *$\alpha \neq \beta \Rightarrow C^\alpha_H \neq C^\beta_H$,*
(f) *if $\lambda = \min(C^\kappa_H)$ and K is $\mathrm{Col}(\omega, \lambda^+)_{V[H]}$-generic over $V[H]$, then*

$$\mathrm{CARD}^{V[H][K]} \cap \kappa = (\lim(C^\kappa_H) \cup \{\mu^+, \ldots, \mu^{+6} : \mu \in C^\kappa_H\} \setminus \lambda^{++}) \cup \{\omega\},$$

(g) *$V[H][K] \models$ "$\forall \lambda \leq \kappa$, $2^\lambda = \lambda^{+3}$".*

Proof. Essentially the same as in [M2]. ∎

6. Definition of the forcing notion $\mathbb{R}_{\bar{E}_\kappa}$. We now define another forcing notion $\mathbb{R}_{\bar{E}_\kappa}$. It is essentially the Radin forcing corresponding to $\bar{E}_\kappa$ with interleaving collapses (see also [M1]).

DEFINITION 6.1. A condition in $\mathbb{R}_{\bar{E}_\kappa}$ is of the form

$$p = \langle \langle \bar{\gamma}_n, s^n, S^n, f^n, F^n \rangle, \ldots, \langle \bar{\gamma}_0, s^0, S^0, f^0, F^0 \rangle \rangle$$

where

(1) $\bar{\gamma}_n, \ldots, \bar{\gamma}_0$ are minimal extender sequences ([2]),
(2) $\bar{\gamma}_0 = \bar{E}_\kappa$,

([2]) An extender sequence $\bar{\gamma}$ is *minimal* if it has length 1 and $\kappa(\bar{\gamma}) = \kappa^0(\bar{\gamma})$.

(3) $\forall i \leq n-1$, $\kappa(\bar{\gamma}_{i+1}) < \kappa^0(\bar{\gamma}_i)$,
(4) $\forall i \leq n$, $S^i \in \bar{\gamma}_i$,
(5) $\forall i \leq n$, $s^i \in V_{\kappa^0(\bar{\gamma}_i)}$ is a minimal extender sequence such that $\kappa(s^i)$ is inaccessible,
(6) $\forall i \leq n$, f^i is a function such that
  (6.1) $\mathrm{dom}(f^i) = \{\bar{\nu} \in S^i : \mathrm{l}(\bar{\nu}) = 0\}$,
  (6.2) $f^i(\nu_1) \in R^{\mathrm{Col}}(\kappa(s^i), \nu_1^0)$,
(7) $\forall i \leq n, F^i$ is a function such that
  (7.1) $\mathrm{dom}(F^i) = \{\langle \bar{\nu}_1, \bar{\nu}_2 \rangle \in (S^i)^2 : \mathrm{l}(\bar{\nu}_1) = \mathrm{l}(\bar{\nu}_2) = 0\}$,
  (7.2) $F^i(\langle \nu_1, \nu_2 \rangle) \in R^{\mathrm{Col}}(\nu_1^0, \nu_2^0)$,
  (7.3) $j^2_{\bar{E}}(F^i)(\kappa(\bar{\gamma}_i), j_{\bar{E}}(\kappa(\bar{\gamma}_i)) \in I^{\mathrm{Col}}(\bar{E})$.

Definition 6.2. For $p, q \in \mathbb{R}_{\bar{E}_\kappa}$ we say p is a *Prikry extension* of q ($p \leq^* q$ or $p \leq^0 q$) iff p and q are of the form

$$p = \langle \langle \bar{\gamma}_n, s^n, S^n, f^n, F^n \rangle, \ldots, \langle \bar{\gamma}_0, s^0, S^0, f^0, F^0 \rangle \rangle,$$
$$q = \langle \langle \bar{\gamma}_n, t^n, T^n, g^n, G^n \rangle, \ldots, \langle \bar{\gamma}_0, t^0, T^0, g^0, G^0 \rangle \rangle,$$

where $\forall i \leq n$,

(1) $s^i = t^i$,
(2) $S^i \subseteq T^i$,
(3) $f^i \leq g^i$,
(4) $F^i \leq G^i$.

Definition 6.3. Let $p = \langle \langle \bar{\gamma}_n, s^n, S^n, f^n, F^n \rangle, \ldots, \langle \bar{\gamma}_0, s^0, S^0, f^0, F^0 \rangle \rangle \in \mathbb{R}_{\bar{E}_\kappa}$, and let $\langle \bar{\nu} \rangle \in S^i, \kappa^0(\bar{\nu}) > \bigcup\bigcup j_{\bar{E}}(f^i)(\kappa(\bar{\gamma}_i))$. We define $p_{\langle \bar{\nu} \rangle}$ as follows:

- if $\mathrm{l}(\bar{\nu}) > 0$, then
$$p_{\langle \bar{\nu} \rangle} = \langle \langle \bar{\gamma}_n, s^n, S^n, f^n, F^n \rangle, \ldots, \langle \bar{\gamma}_{i+1}, s^{i+1}, S^{i+1}, f^{i+1}, F^{i+1} \rangle, \langle \bar{\nu}, s^i, S^i \restriction \bar{\nu}, f^i \restriction \bar{\nu}, F^i \restriction \bar{\nu} \rangle, \langle \bar{\gamma}_i, \bar{\nu}, S^i \setminus \bar{\nu}, F^i(\kappa(\bar{\nu}, -)), F^i \rangle, \langle \bar{\gamma}_{i-1}, s^{i-1}, S^{i-1}, f^{i-1}, F^{i-1} \rangle, \ldots, \langle \bar{\gamma}_0, s^0, S^0, f^0, F^0 \rangle \rangle,$$
- if $\mathrm{l}(\bar{\nu}) = 0$, then
$$p_{\langle \bar{\nu} \rangle} = \langle \langle \bar{\gamma}_n, s^n, S^n, f^n, F^n \rangle, \ldots, \langle \bar{\gamma}_{i+1}, s^{i+1}, S^{i+1}, f^{i+1}, F^{i+1} \rangle, \langle \bar{\nu}^0, s^i, \emptyset, f^i(\kappa(\bar{\nu})), \emptyset \rangle, \langle \bar{\gamma}_i, \bar{\nu}, S^i \setminus \bar{\nu}, F^i(\kappa(\bar{\nu}, -)), F^i \rangle, \langle \bar{\gamma}_{i-1}, s^{i-1}, S^{i-1}, f^{i-1}, F^{i-1} \rangle, \ldots, \langle \bar{\gamma}_0, s^0, S^0, f^0, F^0 \rangle \rangle.$$

Definition 6.4. Let $p, q \in \mathbb{R}_{\bar{E}_\kappa}$, where

$$q = \langle \langle \bar{\gamma}_n, s^n, S^n, f^n, F^n \rangle, \ldots, \langle \bar{\gamma}_0, s^0, S^0, f^0, F^0 \rangle \rangle.$$

We say p is a 1-*point extension* of q ($p \leq^1 q$) iff there are i and $\langle \bar{\nu} \rangle \in S^i$ such that $p \leq^* q_{\langle \bar{\nu} \rangle}$.

DEFINITION 6.5. Let $p, q \in \mathbb{R}_{\bar{E}_\kappa}$. We say p is an *n-point extension* of q ($p \leq^n q$) iff there are $p^n, \ldots, p^0$ such that

$$p = p^n \leq^1 \cdots \leq^1 p^0 = q.$$

DEFINITION 6.6. Let $p, q \in \mathbb{R}_{\bar{E}_\kappa}$. We say p is an *extension* of q ($p \leq q$) iff there is n such that $p \leq^n q$.

Suppose G is $\mathbb{R}_{\bar{E}_\kappa}$-generic over V. Set

$$C = \{\kappa(s^0) : s^0 \text{ appears in some } p \in G\}.$$

THEOREM 6.7.

(a) *$V[G]$ and V have the same cardinals $\geq \kappa$,*
(b) *κ remains strongly inaccessible in $V[G]$,*
(c) *C is a club in κ,*
(d) *if $\lambda = \min(C)$ and K is $\mathrm{Col}(\omega, \lambda^+)_{V[G]}$-generic over $V[G]$, then*

$$\mathrm{CARD}^{V[G][K]} \cap \kappa = (\lim(C) \cup \{\gamma^+, \ldots, \gamma^{+6} : \gamma \in C\} \setminus \lambda^{++}) \cup \{\omega\},$$

(e) *$V[G][K] \models$ "GCH".*

Proof. Essentially the same as in [M1] and [M2]. ∎

7. Projection of $\mathbb{P}_{\bar{E}}$ into $\mathbb{R}_{\bar{E}_\kappa}$. We now define a projection

$$\pi : \mathbb{P}_{\bar{E}} \to \mathbb{R}_{\bar{E}_\kappa}$$

as follows. Suppose $p = p_n^\frown \ldots^\frown p_0$ where

- $p_0 \in \mathbb{P}^*_{\bar{E}}, \kappa^0(p_0^0) \geq \kappa^0(\bar{\mu}_1)$,
- $p_1 \in \mathbb{P}^*_{\bar{\mu}_1}, \kappa^0(p_1^0) \geq \kappa^0(\bar{\mu}_2)$,
- $\vdots$
- $p_n \in \mathbb{P}^*_{\bar{\mu}_n}$,

and $\langle \bar{\mu}_n, \ldots, \bar{\mu}_1, \bar{\mu}_0 \rangle$, where $\bar{\mu}_0 = \bar{E}$, is a ^{0}-increasing sequence of extender sequence systems. For each $i \leq n$ set $f^{p_i} = f^{p_i,\mathrm{Add}} \times f^{p_i,\mathrm{Col}}$ and $F^{p_i} = F^{p_i,\mathrm{Add}} \times F^{p_i,\mathrm{Col}}$, which correspond to $R = R^{\mathrm{Add}} \times R^{\mathrm{Col}}$. Given p as above, for each $i \leq n$, we have

(1) $j(f^{p_i,\mathrm{Col}})(\kappa(\mathrm{mc}(p_i))) \in V_{\kappa^0(\mathrm{mc}(p_i))}$; hence there is a function g^{p_i} such that

$$j(f^{p_i,\mathrm{Col}})(\kappa(\mathrm{mc}(p_i))) = j(g^{p_i})(\kappa^0(\mathrm{mc}(p_i))),$$

(2) $j_2(F^{p_i,\mathrm{Col}})(\kappa(\mathrm{mc}(p_i)), j(\kappa(\mathrm{mc}(p_i))))$ is in the generic filter constructed through the normal measure; hence there is a stronger condition

in the filter which is the image of a condition from the generic over the normal ultrapower, i.e. there is a function H^{p_i} such that

$$j_2(H^{p_i})(\kappa^0(\mathrm{mc}(p_i)), j(\kappa^0(\mathrm{mc}(p_i)))) \leq j_2(F^{p_i,\mathrm{Col}})(\kappa(\mathrm{mc}(p_i)), j(\kappa(\mathrm{mc}(p_i))))),$$

and

$$j_2(H^{p_i})(\kappa^0(\mathrm{mc}(p_i)), j(\kappa^0(\mathrm{mc}(p_i)))) \in I^{\mathrm{Col}}$$

and there is no weaker function H' satisfying this.

Let $(T^{p_i})^*$ be obtained from T^{p_i} by replacing extender sequences in T^{p_i} of length 0 with

$$\{\nu \in T^{p_i} : \mathrm{l}(\nu) = 0,\ f^{p_i,\mathrm{Col}}(\nu) = g^{p_i}(\pi_{\mathrm{mc}(p_i),0}(\nu))\}.$$

It follows from (1) that $(T^{p_i})^* \in \mathrm{mc}(p_i)$. Now let $(T^{p_i})^{**}$ be obtained from $(T^{p_i})^*$ by restricting extender sequences in $(T^{p_i})^*$ of length 0 to those $\nu_1 \in (T^{p_i})^*$ with $\mathrm{l}(\nu_1) = 0$ such that

$$\{\nu_2 \in (T^{p_i})^* : \mathrm{l}(\nu_2) = 0,\ H^{p_i}(\pi_{\mathrm{mc}(p_i),0}(\nu_1),\ \pi_{\mathrm{mc}(p_i),0}(\nu_2)) \leq F^{p_i,\mathrm{Col}}(\nu_1,\nu_2)\}$$

has measure one with respect to the normal measure determined by $\mathrm{mc}(p_i)$. Then by (2) we have $(T^{p_i})^{**} \in \mathrm{mc}(p_i)$. Let

$$\pi(p) = \langle\langle \min \bar{\mu}_n, p_n^0, A^{p_n}, g^{p_n}, H^{p_n}\rangle, \ldots, \langle \bar{E}_\kappa, p_0^0, A^{p_0}, g^{p_0}, H^{p_0}\rangle\rangle,$$

where $A^{p_i} = \{\pi_{\mathrm{mc}(p_i),0}(\bar{\nu}) : \bar{\nu} \in (T^{p_i})^{**}\}$. Note that $\pi(p) \in \mathbb{R}_{\bar{E}_\kappa}$ and π is well-defined.

LEMMA 7.1. *π is a projection, i.e.*

(a) $\pi(1_{\mathbb{P}_{\bar{E}}}) = 1_{\mathbb{R}_{\bar{E}_\kappa}}$,
(b) *π is order preserving,*
(c) *if $p \in \mathbb{P}_{\bar{E}}, q \in \mathbb{R}_{\bar{E}_\kappa}$ and $q \leq \pi(p)$ then there is $r \leq p$ in $\mathbb{P}_{\bar{E}}$ such that $\pi(r) \leq q$.*

Proof. Parts (a) and (b) are trivial; let us prove (c). Let $p \in \mathbb{P}_{\bar{E}}$, $q \in \mathbb{R}_{\bar{E}_\kappa}$, and suppose that $q \leq \pi(p)$. Let us suppose for simplicity that $p \in \mathbb{P}^*_{\bar{E}}$ ([3]). Let $\pi(p) = \langle\langle \bar{E}_\kappa, p^0, A^p, g^p, H^p\rangle\rangle$. Since $q \leq \pi(p)$, there is some k such that $q \leq^k \pi(p)$. We prove the lemma by induction on k.

First suppose that $k = 0$, so that q is a Prikry extension of $\pi(p)$. Let $q = \langle\langle \bar{E}_\kappa, t, T, g, G\rangle\rangle$. Then we have $t = p^0$, $T \subseteq A^p$, $g \leq g^p$ and $G \leq H^p$. Let $r \in \mathbb{P}^*_{\bar{E}}, r \leq^* p$ be obtained from p with the following changes:

- $T^r \subseteq \{\bar{\nu} \in (T^p)^{**} : \pi_{\mathrm{mc}(p),0}(\bar{\nu}) \in T\}$, $T^r \in \mathrm{mc}(p)$,
- $f^{r,\mathrm{Col}} \leq f^{p,\mathrm{Col}}$ is such that for all $\nu \in T^r$ of length 0,

$$g^r(\pi_{\mathrm{mc}(p),0}(\nu)) \leq g(\pi_{\mathrm{mc}(p),0}(\nu)),$$

([3]) In fact the general case follows from this special case using the factorization properties of $\mathbb{P}_{\bar{E}}$.

- $F^{r,\mathrm{Col}} \leq F^{p,\mathrm{Col}}$ is such that for all $\langle \nu_1, \nu_2 \rangle \in (T^r)^2$ with $\mathrm{l}(\nu_1) = \mathrm{l}(\nu_2) = 0$,
$$H^r(\pi_{\mathrm{mc}(p),0}(\nu_1), \pi_{\mathrm{mc}(p),0}(\nu_2)) \leq G(\pi_{\mathrm{mc}(p),0}(\nu_1), \pi_{\mathrm{mc}(p),0}(\nu_2)).$$

Then $\pi(r) = \langle\langle \bar{E}_\kappa, p^0, A^r, g^r, H^r \rangle\rangle$, where $A^r \subseteq T$ and for all ν, ν_1 and ν_2 such that their image under $\pi_{\mathrm{mc}(p),0}$ is in A^r, we have
$$g^r(\pi_{\mathrm{mc}(p),0}(\nu)) \leq g(\pi_{\mathrm{mc}(p),0}(\nu))$$
and
$$H^r(\pi_{\mathrm{mc}(p),0}(\nu_1), \pi_{\mathrm{mc}(p),0}(\nu_2)) \leq G(\pi_{\mathrm{mc}(p),0}(\nu_1), \pi_{\mathrm{mc}(p),0}(\nu_2)).$$
It follows that $\pi(r) \leq^* q$.

Now suppose that $k = 1$ (the general case $k \geq 1$ can be proved similarly). Let $\bar{\nu} \in A^p$ be such that $q \leq^* (\pi(p))_{\langle \bar{\nu} \rangle}$. Also let
$$q = \langle\langle \bar{\gamma}_1, t^1, T^1, g^1, G^1 \rangle, \langle \bar{\gamma}_0, t^0, T^0, g^0, G^0 \rangle\rangle.$$
Suppose e.g. that $\mathrm{l}(\bar{\nu}) > 0$ (the case $l(\bar{\nu}) = 0$ can be proved similarly). Then $(\pi(p))_{\langle \bar{\nu} \rangle} = \langle\langle \bar{\nu}, p^0, A^p \restriction \bar{\nu}, g^p \restriction \bar{\nu}, H^p \restriction \bar{\nu} \rangle, \langle \bar{E}_\kappa, \bar{\nu}, A^p \setminus \bar{\nu}, H^p(\kappa(\bar{\nu}, -)), H^p \rangle\rangle$. Let $\bar{\mu} \in T^p$ be an extender sequence system of size $(\kappa^0(\bar{\nu}))^{+3}$, obtained by the same elementary embedding generating $\bar{\nu}$, such that $\min \bar{\mu} = \bar{\nu}$ and let $(p)_{\langle \bar{\mu} \rangle} = p_1 \frown p_0$. Let $r = r_1 \frown r_0 \in \mathbb{P}_{\bar{E}}$ with $r \leq^* (p)_{\langle \bar{\mu} \rangle}$ be such that for $i \in \{0, 1\}$:

- $\mathrm{supp}(r_i) = \mathrm{supp}(p_i)$,
- $T^{r_i} \subseteq \{\bar{\nu} \in (T^{p_i})^{**} : \pi_{\mathrm{mc}(p_i),0}(\bar{\nu}) \in T^i\}$, $T^{r_i} \in \mathrm{mc}(p_i)$,
- $f^{r_i,\mathrm{Col}} \leq f^{p_i,\mathrm{Col}}$ is such that for all $\nu \in T^{r_i}$ with $\mathrm{l}(\nu) = 0$,
$$g^{r_i}(\pi_{\mathrm{mc}(p_i),0}(\nu)) \leq g^i(\pi_{\mathrm{mc}(p_i),0}(\nu)),$$
- $F^{r_i,\mathrm{Col}} \leq F^{p,\mathrm{Col}}$ is such that for all $\langle \nu_1, \nu_2 \rangle \in (T^{r_i})^2$ with $\mathrm{l}(\nu_1) = \mathrm{l}(\nu_2) = 0$,
$$H^{r_i}(\pi_{\mathrm{mc}(p_i),0}(\nu_1), \pi_{\mathrm{mc}(p_i),0}(\nu_2)) \leq G^i(\pi_{\mathrm{mc}(p_i),0}(\nu_1), \pi_{\mathrm{mc}(p_i),0}(\nu_2)).$$

Then $\pi(r) = \pi(r_1) \frown \pi(r_0)$ and as above $\pi(r_i) \leq^* \langle\langle \bar{\gamma}_i, t^i, T^i, g^i, G^i \rangle\rangle$, $i \in \{0, 1\}$. It follows that $\pi(r) \leq^* q$.

The lemma follows. ∎

8. Completing the proof. Finally in this section we complete the proof of Theorem 1.1. Let H be $\mathbb{P}_{\bar{E}}$-generic over V and let $H_0 = \langle \pi'' H \rangle$, the filter generated by $\pi'' H$. Then H_0 is $\mathbb{R}_{\bar{E}_\kappa}$-generic over V. Consider the clubs $C = \{\kappa(s^0) : s^0 \text{ appears in in some } p \in H_0\}$ and $C^\kappa_H = \{\kappa(p^0_0) : p \in H\}$. It is easily seen that $C = C^\kappa_H$. Let $\lambda = \min(C)$. Note that the forcing notions $\mathbb{P}_{\bar{E}}$ and $\mathbb{R}_{\bar{E}_\kappa}$ add no new bounded subsets to λ^+, hence $\mathrm{Col}(\omega, \lambda^+)_{V[H_0]} = \mathrm{Col}(\omega, \lambda^+)_{V[H]}$, and hence if K is $\mathrm{Col}(\omega, \lambda^+)_{V[H]}$-generic over $V[H]$ then K is $\mathrm{Col}(\omega, \lambda^+)_{V[H_0]}$-generic over $V[H_0]$. Let
$$V_1 = V_\kappa^{V[H_0][K]}, \quad V_2 = V_\kappa^{V[H][K]}.$$

It follows that V_1 and V_2 are models of ZFC. We show that the pair (V_1, V_2) satisfies the requirements of the theorem.

(a) V_1 and V_2 have the same cardinals: This is trivial, since

$$\begin{aligned}\mathrm{CARD}^{V_1} &= (\lim(C) \cup \{\mu^+, \dots, \mu^{+6} : \mu \in C\} \setminus \lambda^{++}) \cup \{\omega\} \\ &= (\lim(C^\kappa_H) \cup \{\mu^+, \dots, \mu^{+6} : \mu \in C^\kappa_H\} \setminus \lambda^{++}) \cup \{\omega\} \\ &= \mathrm{CARD}^{V_2}.\end{aligned}$$

(b) V_1 and V_2 have the same cofinalities: This is again trivial, since changing the cofinalities depends on the length of the extender sequence system used and not on its size.

(c) $V_1 \models$ "GCH": by Theorem 6.7(e).

(d) $V_2 \models$ "$\forall\lambda$, $2^\lambda = \lambda^{+3}$": by Theorem 5.9(g).

Theorem 1.1 follows.

Problem 8.1. Is it possible to kill GCH everywhere, preserving cofinalities, adding just a single real? (Allowing cofinalities, but not cardinalities, to change, this was accomplished in [FG].)

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