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Closed graph multi-selections

by

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Abstract. A classical Lefschetz result about point-finite open covers of normal spaces is generalised by showing that every lower semi-continuous mapping from a normal space into the nonempty compact subsets of a metrizable space admits a closed graph multi-selection. Several applications are given as well.

1. Introduction. For a space Y, we will use 2^Y to denote the *power set* of Y, i.e. the set of all subsets of Y. Also, let

$$\mathscr{F}(Y) = \{S \in 2^Y : S \neq \emptyset \text{ and } S \text{ is closed}\},$$
$$\mathscr{C}(Y) = \{S \in \mathscr{F}(Y) : S \text{ is compact}\}.$$

Any relation $R \subset X \times Y$ can be represented as a map $\Phi_R : X \to 2^Y$ by letting $\Phi_R(x) = \{y \in Y : \langle x, y\rangle \in R\}$, $x \in X$. This map is usually called a *set-valued* (or *multi-valued*) *mapping*, and sometimes a *multifunction*. The converse is also true. To any set-valued mapping $\Phi : X \to 2^Y$ one can associate the relation

$$\mathsf{Graph}(\Phi) = \{\langle x, y\rangle \in X \times Y : y \in \Phi(x)\},$$

which is called the *graph* of Φ. Thus, $R = \mathsf{Graph}(\Phi_R)$.

The *inverse* relation $R^{-1} \subset Y \times X$ of a relation $R \subset X \times Y$ is defined by $R^{-1} = \{\langle y, x\rangle \in Y \times X : \langle x, y\rangle \in R\}$. Clearly, $(R^{-1})^{-1} = R$. In particular, for any set-valued mapping $\Phi : X \to 2^Y$ the *inverse* $\Phi^{-1} : Y \to 2^X$ is defined by

$$\mathsf{Graph}(\Phi^{-1}) = (\mathsf{Graph}(\Phi))^{-1},$$

and we always have $(\Phi^{-1})^{-1} = \Phi$. There is a more practical way to describe the inverse of a mapping $\Phi : X \to 2^Y$. Namely, for a set $B \subset Y$, let

$$\Phi^{-1}[B] = \{x \in X : \Phi(x) \cap B \neq \emptyset\}.$$

Then $\Phi^{-1}(y) = \Phi^{-1}[\{y\}]$ for every $y \in Y$. Finally, let us explicitly mention

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that we will make no difference between single-valued and singleton-valued mappings; such mappings represent the same relation.

A map $f : X \to Y$ is a *selection* (or a *single-valued selection*) for $\Phi : X \to 2^Y$ if $f(x) \in \Phi(x)$ for every $x \in X$. A set-valued mapping $\psi : X \to 2^Y$ is a *multi-selection* (or a *set-valued selection*) for $\Phi : X \to 2^Y$ if $\psi(x) \subset \Phi(x)$ for every $x \in X$. A mapping $\Phi : X \to 2^Y$ is *lower semi-continuous*, or l.s.c., if the set $\Phi^{-1}[U]$ is open in X for every open $U \subset Y$. A mapping $\Psi : X \to 2^Y$ is *upper semi-continuous*, or u.s.c., if the set

$$\Psi^{\#}[U] = X \setminus \Psi^{-1}[Y \setminus U] = \{x \in X : \Psi(x) \subset U\}$$

is open in X for every open $U \subset Y$. For convenience, we say that $\Psi : X \to 2^Y$ is *usco* if it is u.s.c. and nonempty-compact-valued.

This paper extends the ideas of [6, 7], and its main goal is to demonstrate that combinatorial properties of covers of topological spaces can be transformed with ease into multi-selections of set-valued mappings with point-images in completely metrizable spaces. A starting point is the following theorem of Michael's.

Theorem 1.1 ([10]). *If X is a paracompact space, Y is completely metrizable, and $\Phi : X \to \mathscr{F}(Y)$ is l.s.c., then there is a pair of mappings $\langle \varphi, \psi \rangle : X \to \mathscr{C}(Y)$ such that φ is l.s.c., ψ is u.s.c. and $\varphi(x) \subset \psi(x) \subset \Phi(x)$ for all $x \in X$.*

In this theorem, if Y is assumed to be a discrete space, then the stated selection property means that for every open cover $\mathscr{U}$ of X there is an open locally-finite cover $\{V_U : U \in \mathscr{U}\}$ of X such that $\overline{V_U} \subset U$ for all $U \in \mathscr{U}$ (see Proposition 3.1). That is, any Hausdorff space X (actually, a T_1-space X) which has this property with respect to discrete spaces will be paracompact, so Theorem 1.1 yields a characterisation of paracompactness. In the present paper, we prove the following theorem which transforms another well-known cover property (see Lemma 3.3) into the existence of multi-selections.

Theorem 1.2. *For a T_1-space X, the following are equivalent:*

(a) *X is normal.*
(b) *If Y is a metrizable space and $\Phi : X \to \mathscr{C}(Y)$ is l.s.c., then there exists a pair of mappings $\langle \varphi, \psi \rangle : X \to \mathscr{C}(Y)$ such that φ is l.s.c., ψ has a closed graph and $\varphi(x) \subset \psi(x) \subset \Phi(x)$ for all $x \in X$.*
(c) *If Y is a metrizable space, then every l.s.c. $\Phi : X \to \mathscr{C}(Y)$ admits a closed graph multi-selection.*

It is well-known that every paracompact space is normal, but the converse is not true. In the same way, every u.s.c. mapping into the closed subsets of a regular space has a closed graph, but not every mapping which has a closed graph is u.s.c.

The paper is organised as follows. In the next section are summarised some basic properties of closed graph mappings. Section 3 provides relations between cover properties and set-valued mappings with point-images in discrete spaces, in particular it contains the proof of Theorem 1.2 for a discrete Y (see Corollary 3.5). Section 4 provides the main interface between set-valued mappings with a discrete range and those with a completely metrizable one. The proof of Theorem 1.2 is finally accomplished in Section 5. Some related results and applications are obtained in the same section (see Theorems 5.1 and 5.2). Section 6 deals with some characterisations of paracompactness-like properties, in Theorems 6.1–6.3.

2. Some properties of closed graph mappings. If $\psi : X \to 2^Y$ and $x \in X$, then $\{x\} \times \psi(x) = (\{x\} \times Y) \cap \mathsf{Graph}(\psi)$. Thus, if X is a T_1-space and $\psi : X \to 2^Y$ has a closed graph, then ψ must be closed-valued. However, $\psi^{-1} : Y \to 2^X$ also has a closed graph. Hence, if Y is a T_1-space, then $\psi^{-1}(y)$ is closed in X for every $y \in Y$. This property holds for arbitrary compact subsets of Y. Namely, according to Kuratowski's theorem (see [4, 3.1.16]), we have the following observation.

PROPOSITION 2.1. *If Y is a Hausdorff space and $\psi : X \to 2^Y$ has a closed graph, then $\psi^{-1}[K]$ is closed in X for every compact $K \subset Y$. In particular, if Y is a compact Hausdorff space and $\psi : X \to 2^Y$ has a closed graph, then ψ is u.s.c.*

Our next observation deals with products of such mappings; its proof is also easy and is left to the reader.

PROPOSITION 2.2. *If each mapping $\psi_\alpha : X \to \mathscr{F}(Y_\alpha)$, $\alpha \in \mathscr{A}$, has a closed graph, then the product mapping $\psi(x) = \prod\{\psi_\alpha(x) : \alpha \in \mathscr{A}\}$, $x \in X$, also has a closed graph.*

For a set-valued mapping $\eta : Y \to 2^Z$ and $B \subset Y$, let

$$\eta[B] = \bigcup\{\eta(y) : y \in B\} = (\eta^{-1})^{-1}[B]. \tag{2.1}$$

We conclude this section with another construction of closed graph mappings.

PROPOSITION 2.3. *Let $\theta : X \to 2^Y$ have a closed graph, and let $\eta : Y \to 2^Z$ be such that its inverse $\eta^{-1} : Z \to 2^Y$ is usco. Then the composite mapping $\psi(x) = \eta[\theta(x)]$, $x \in X$, has a closed graph.*

Proof. Take a point $\langle x, z\rangle \notin \mathsf{Graph}(\psi)$. Then

$$z \notin \psi(x) = \eta[\theta(x)] = \bigcup\{\eta(y) : y \in \theta(x)\}.$$

By hypothesis, η^{-1} is nonempty-compact-valued, so $K = \eta^{-1}(z)$ is a nonempty compact subset of Y such that $(\{x\} \times K) \cap \mathsf{Graph}(\theta) = \emptyset$. Since

θ has a closed graph, there are open sets $U \subset X$ and $V \subset Y$ such that $\{x\} \times K \subset U \times V$ and $(U \times V) \cap \mathsf{Graph}(\theta) = \emptyset$. Since η^{-1} is u.s.c. and $\eta^{-1}(z) = K \subset V$, the set $W = (\eta^{-1})^{\#}[V]$ is open in Z and contains z. Thus, $U \times W$ is an open subset of $X \times Z$ such that $\langle x, z\rangle \in U \times W$ and $(U \times W) \cap \mathsf{Graph}(\psi) = \emptyset$. ∎

3. Multi-selections and cover properties. Any family $\mathscr{U} \subset 2^X$ of subsets of a space X can be represented as a set-valued mapping $\mathscr{U} : \mathbb{D} \to 2^X$ by considering $\mathscr{U}$ as an *indexed* collection $\{\mathscr{U}(d) : d \in \mathbb{D}\}$ of subsets of X. In this section, and in what follows, the set $\mathbb{D}$ of indices will always be endowed with the discrete topology. This assumption cannot help much in deriving properties of $\mathscr{U}$ from properties of the set-valued mapping $\mathscr{U} : \mathbb{D} \to 2^X$ because these properties are actually provided by the space X. The benefit of this assumption comes to light when we look at properties of the inverse mapping $\mathscr{U}^{-1} : X \to 2^{\mathbb{D}}$. Different parts of the following proposition were used implicitly by several authors (see, e.g., [2, 7, 12]).

PROPOSITION 3.1. *For a space X and a family $\mathscr{U} : \mathbb{D} \to 2^X$, the following hold:*

(a) *$\mathscr{U}$ is a cover of X iff $\mathscr{U}^{-1}$ is nonempty-valued (i.e., $\mathscr{U}^{-1} : X \to \mathscr{F}(\mathbb{D})$).*
(b) *$\mathscr{U}$ is an open cover of X iff $\mathscr{U}^{-1} : X \to \mathscr{F}(\mathbb{D})$ has an open graph (in particular, is l.s.c.).*
(c) *$\mathscr{U}$ is a point-finite cover of X iff $\mathscr{U}^{-1} : X \to \mathscr{C}(\mathbb{D})$.*
(d) *$\mathscr{U}$ is a locally-finite closed cover of X iff $\mathscr{U}^{-1} : X \to \mathscr{C}(\mathbb{D})$ and is u.s.c. (i.e., $\mathscr{U}^{-1}$ is usco).*

A family $\mathscr{V} : \mathbb{D} \to 2^X$ is an *indexed refinement* of $\mathscr{U} : \mathbb{D} \to 2^X$ if $\mathscr{V}(d) \subset \mathscr{U}(d)$ for every $d \in \mathbb{D}$; in the Introduction we then called the set-valued mapping $\mathscr{V}$ a multi-selection for $\mathscr{U}$. Moreover, $\mathscr{V} : \mathbb{D} \to 2^X$ is a multi-selection for $\mathscr{U} : \mathbb{D} \to 2^X$ if and only if $\mathsf{Graph}(\mathscr{V}) \subset \mathsf{Graph}(\mathscr{U})$, but the inverse mappings preserve the graph-inclusion. Hence, we have the following immediate observation.

PROPOSITION 3.2. *Let X be a set, and let $\mathscr{U} : \mathbb{D} \to 2^X$ and $\mathscr{V} : \mathbb{D} \to 2^X$ be families of subsets of X. Then $\mathscr{V}$ is an indexed refinement of $\mathscr{U}$ if and only if the mapping $\mathscr{V}^{-1} : X \to 2^{\mathbb{D}}$ is a multi-selection for $\mathscr{U}^{-1} : X \to 2^{\mathbb{D}}$.*

In this section, we are interested in the following result of Lefschetz [9] (see also [4]), and its possible interpretation in terms of set-valued mappings.

LEMMA 3.3 ([9]). *Let X be a normal space, and let $\mathscr{U} : \mathbb{D} \to 2^X$ be an open point-finite cover of X. Then X has an open cover $\mathscr{V} : \mathbb{D} \to 2^X$ such that $\overline{\mathscr{V}(d)} \subset \mathscr{U}(d)$ for every $d \in \mathbb{D}$.*

In view of Lemma 3.3, to any set-valued mapping $\varphi : X \to 2^Y$ we associate another one $\overline{\varphi} : X \to 2^Y$ defined by $\overline{\varphi}(x) = \overline{\varphi(x)}$, $x \in X$, which will play the role of *pointwise closure* of φ. We also associate to φ the *uniform closure* $\overline{\varphi}^{\mathbf{u}} : X \to 2^Y$ defined by $\mathsf{Graph}(\overline{\varphi}^{\mathbf{u}}) = \overline{\mathsf{Graph}(\varphi)}$. In general, $\overline{\varphi}$ and $\overline{\varphi}^{\mathbf{u}}$ may not coincide, but in the case of a discrete domain they do.

Proposition 3.4. *If $\psi : \mathbb{D} \to 2^X$ is a closed-valued mapping, then it has a closed graph. In particular, $\overline{\psi}^{\mathbf{u}} = \overline{\psi}$ whenever $\psi : \mathbb{D} \to 2^X$.*

Proof. Take a point $\langle d, x\rangle \in \overline{\mathsf{Graph}(\psi)}$ and a neighbourhood U of x in X. Then $(\{d\} \times U) \cap \mathsf{Graph}(\psi) \neq \emptyset$, and therefore $U \cap \psi(d) \neq \emptyset$. By hypothesis, $\psi(d)$ is closed in X. Hence, $x \in \psi(d)$. ∎

Now, we turn to the following characterisation of normality which is actually Theorem 1.2 for the case of a discrete space Y.

Corollary 3.5. *For a T_1-space X and a discrete space $\mathbb{D}$ with at least two distinct points, the following are equivalent:*

(a) *X is normal.*
(b) *Every l.s.c. mapping $\Phi : X \to \mathscr{C}(\mathbb{D})$ admits a pair of mappings $\langle \varphi, \psi\rangle : X \to \mathscr{C}(\mathbb{D})$ such that φ is l.s.c., ψ has a closed graph, and $\varphi(x) \subset \psi(x) \subset \Phi(x)$ for all $x \in X$.*
(c) *Every l.s.c. $\Phi : X \to \mathscr{C}(\mathbb{D})$ admits a closed graph multi-selection.*

Proof. (a)⇒(b). Suppose that X is normal and $\Phi : X \to \mathscr{C}(\mathbb{D})$ is l.s.c. By Proposition 3.1, $\mathscr{U} = \Phi^{-1} : \mathbb{D} \to 2^X$ is an open point-finite cover of X. Hence, by Lemma 3.3, X has an open cover $\mathscr{V} : \mathbb{D} \to 2^X$ such that $\overline{\mathscr{V}(d)} \subset \mathscr{U}(d)$ for all $d \in \mathbb{D}$. By Propositions 3.1, 3.2 and 3.4, the pair $\langle \varphi, \psi\rangle : X \to \mathscr{C}(\mathbb{D})$ is as required, where $\varphi = \mathscr{V}^{-1}$ and $\psi = \overline{\varphi}^{\mathbf{u}}$. The implication (b)⇒(c) is obvious. To show finally that (c)⇒(a), take disjoint closed sets $F_0, F_1 \subset X$, and define an l.s.c. mapping $\Phi : X \to \mathscr{C}(\{0,1\})$ by $\Phi(x) = \{i\}$ if $x \in F_i$, $i = 0, 1$, and $\Phi(x) = \{0,1\}$ otherwise. By (c), Φ has a closed graph multi-selection $\psi : X \to \mathscr{C}(\{0,1\})$. Since the discrete two-point space $\{0,1\}$ is compact, by Proposition 2.1, ψ is u.s.c. Thus, $U_i = \psi^{\#}[\{i\}]$, $i = 0, 1$, are disjoint open subsets of X such that $F_i \subset U_i$, $i = 1, 2$. ∎

4. Trees—an interface to completely metrizable spaces. A partially ordered set $(T, \preceq)$ is a *tree* if $\{s \in T : s \prec t\}$ is well-ordered for every $t \in T$, where "$s \prec t$" means that $s \preceq t$ and $s \neq t$. For a tree $(T, \preceq)$, we use $T(0)$ to denote the set of minimal elements of T. Given an ordinal α, if $T(\beta)$ is defined for every $\beta < \alpha$, then $T(\alpha)$ denotes the minimal elements of $T \setminus (T\restriction\alpha)$ where $T\restriction\alpha = \bigcup\{T(\beta) : \beta < \alpha\}$. The set $T(\alpha)$ is called the αth *level* of T, while the *height* of T is the least ordinal α such that $T\restriction\alpha = T$. We say that $(T, \preceq)$ is an *α-tree* if its height is α. In this case, a natural set-valued mapping associates to every $\beta < \alpha$ the corresponding level $T(\beta)$.

We will denote this mapping again by T (i.e., $T : \alpha \to 2^T$), and refer to it as the *level mapping* of $(T, \preceq)$. Clearly, each level mapping is nonempty-valued. A maximal linearly ordered subset of a tree $(T, \preceq)$ is called a *branch*, and $\mathscr{B}(T)$ is used to denote the set of all branches of T. Finally, let us recall that a tree $(T, \preceq)$ is *pruned* if every element of T has a successor in T, i.e. for every $s \in T$ there exists $t \in T$ with $s \prec t$. In these terms, an ω-tree $(T, \preceq)$ is pruned if each branch $\beta \in \mathscr{B}(T)$ is infinite.

Given a set D and a nonempty-valued mapping $T : \omega \to 2^D$, let $\Pi(T)$ be the set of all single-valued selections for T, i.e. $f \in \Pi(T)$ if and only if $f : \omega \to D$ and $f(n) \in T(n)$ for every $n < \omega$. In fact, $\Pi(T) = \prod\{T(n) : n < \omega\}$ and we will consider it as a topological space endowed with the Tikhonov product topology generated by the discrete topology on D. The resulting space is always a completely metrizable non-Archimedean space, and it is the Baire space of weight τ provided $\tau = |T(n)| \geq \omega$ for every $n < \omega$ (see [4]). We will refer to $\Pi(T)$ as a *generalised Baire space*. In these terms, any generalised Baire space $\Pi(T)$ is a closed subset of the Baire space of weight $\tau = |D|$.

To any ω-tree $(T, \preceq)$ we may associate the corresponding generalised Baire space $\Pi(T)$ generated by the level mapping $T : \omega \to 2^T$ of $(T, \preceq)$. If, moreover, $(T, \preceq)$ is also pruned, then $\beta \cap T(n) \neq \emptyset$ for every $\beta \in \mathscr{B}(T)$ and $n < \omega$. Hence, every branch $\beta \in \mathscr{B}(T)$ can be identified with a selection $\beta : \omega \to T$ for $T : \omega \to 2^T$ for which $\beta(n) \in \beta \cap T(n)$ and $\beta(n) \prec \beta(n+1)$, $n < \omega$. Thus, we may consider $\mathscr{B}(T)$ as a topological space endowed with the relative topology as a subspace of the generalised Baire space $\Pi(T)$. We will refer to this topology on $\mathscr{B}(T)$ as the *branch topology*, and to the resulting topological space as the *branch space*.

PROPOSITION 4.1 ([6]). *If $(T, \preceq)$ is a pruned ω-tree, then the branch space $\mathscr{B}(T)$ is a closed subset of the generalised Baire space $\Pi(T)$. In particular, $\mathscr{B}(T)$ is a completely metrizable non-Archimedean space, and it is compact if and only if all levels of $(T, \preceq)$ are finite.*

There is another way to deal with the branch space of a pruned ω-tree $(T, \preceq)$. Following Nyikos [13], for every $t \in T$, we let

$$\mathscr{O}(t) = \{\beta \in \mathscr{B}(T) : t \in \beta\}. \tag{4.1}$$

PROPOSITION 4.2 ([6]). *If $(T, \preceq)$ is a pruned ω-tree, then the family $\{\mathscr{O}(t) : t \in T\}$ is a base for the branch topology on $\mathscr{B}(T)$.*

A *path* π in a tree $(T, \preceq)$ is a subset $\pi \subset T$ which is linearly ordered by $\preceq$ and $s \in \pi$ whenever $s \preceq t$ and $t \in \pi$. Clearly, every maximal path is a branch. For a nonmaximal path $\pi \subset T$, the *node* of π in T is the subset $\mathsf{node}(\pi) \subset T$ defined by $t \in \mathsf{node}(\pi)$ if and only if $\pi = \{s \in T : s \prec t\}$. Let $\mathscr{N}(T)$ be the set of all nodes of T. If $\nu \in \mathscr{N}(T)$ for some ω-tree $(T, \preceq)$, then

$\nu = \mathsf{node}(\pi)$ for some finite path $\pi \subset T$. In case $\pi = \emptyset$, we have

$$\mathsf{node}(\emptyset) = \{t \in T : t \in T(0)\} = T(0).$$

Otherwise, if $\pi \neq \emptyset$, then $\pi = \{s \in T : s \preceq t\}$ where $t = \max_{\preceq} \pi$. In this case, we will say that the node $\nu \in \mathscr{N}(T)$ is *generated* by t, and will simply write $\nu = \mathsf{node}(t)$.

DEFINITION 4.3. Given a set X and a pruned ω-tree $(T, \preceq)$, a set-valued mapping $\mathscr{S} : T \to 2^X$ is a *sieve* on X if

(i) $X = \mathscr{S}[\mathsf{node}(\emptyset)]$,
(ii) $\mathscr{S}(t) = \mathscr{S}[\mathsf{node}(t)]$ for every $t \in T$.

Sieves provide the main interface between systems of covers and metrizable spaces. Namely, for a tree $(T, \preceq)$ and a mapping $\mathscr{S} : T \to 2^X$, we define another mapping $\Omega_{\mathscr{S}} : \mathscr{B}(T) \to 2^X$, called the *polar mapping*, by letting

$$\Omega_{\mathscr{S}}(\beta) = \bigcap \{\mathscr{S}(t) : t \in \beta\}, \qquad \beta \in \mathscr{B}(T). \tag{4.2}$$

The value $\Omega_{\mathscr{S}}(\beta)$ for a branch $\beta \in \mathscr{B}(T)$ is called the *polar* of β by $\mathscr{S}$. The inverse mapping $\Omega_{\mathscr{S}}^{-1} : X \to 2^{\mathscr{B}(T)}$ will be denoted by $\mho_{\mathscr{S}}$. Thus, for every $x \in X$, we have

$$\mho_{\mathscr{S}}(x) = \big\{\beta \in \mathscr{B}(T) : x \in \Omega_{\mathscr{S}}(\beta)\big\}. \tag{4.3}$$

The mapping $\mho_{\mathscr{S}}$ corresponding to a sieve $\mathscr{S} : T \to 2^X$ was studied in [6]. In order to state some of the results obtained there, let $\mathscr{P}$ be a property of *indexed* covers of topological spaces, and let $(T, \preceq)$ be an ω-tree. We shall say that a mapping $\mathscr{S} : T \to 2^X$ has the property $\mathscr{P}$, or is a $\mathscr{P}$ mapping, if each family $\mathscr{S} \restriction T(n) : T(n) \to 2^X$, $n < \omega$, has the property $\mathscr{P}$. We will be mainly interested in $\mathscr{P}$ *sieves*, when $\mathscr{P}$ is the property "locally-finite", "point-finite", etc.

LEMMA 4.4 ([6]). *Let X be a space, and let $\mathscr{S} : T \to 2^X$ be a sieve on X. Then $\mho_{\mathscr{S}} : X \to 2^{\mathscr{B}(T)}$ is always nonempty-valued. Moreover,*

(a) *$\mho_{\mathscr{S}} : X \to 2^{\mathscr{B}(T)}$ is compact-valued provided $\mathscr{S}$ is point-finite.*
(b) *$\mho_{\mathscr{S}} : X \to 2^{\mathscr{B}(T)}$ is l.s.c. provided $\mathscr{S}$ is open-valued.*
(c) *$\mho_{\mathscr{S}} : X \to 2^{\mathscr{B}(T)}$ is usco provided $\mathscr{S}$ is locally-finite and closed-valued.*

Here, we refine some of the results of Lemma 4.4, and also deal with the case when $\mho_{\mathscr{S}}$ has a closed graph. To prepare for this, following Choban and Nedev [3] (see also [12]), a mapping $\mathscr{S} : T \to 2^X$ defined on a pruned ω-tree $(T, \preceq)$ will be called a *semi-sieve* on X if

(i) $X = \mathscr{S}[T(n)]$ for every $n < \omega$,
(ii) $\mathscr{S}[\mathsf{node}(t)] \subset \mathscr{S}(t)$ for every $t \in T$.

Also, let us recall that a subset $S \subset T$ of a tree $(T, \preceq)$ is a *subtree* if $(S, \preceq)$ is itself a tree.

LEMMA 4.5. *Let X be a space, $\mathscr{S} : T \to 2^X$ be a semi-sieve on X and, for every $n < \omega$, let $\mathscr{S}_n = \mathscr{S} \restriction T(n) : T(n) \to 2^X$. Endowing T with the discrete topology, let $\Pi_{\mathscr{S}}(x) = \prod\{\mathscr{S}_n^{-1}(x) : n < \omega\}$, $x \in X$, be the corresponding product mapping from X to the subsets of the generalised Baire space $\Pi(T)$. Then*

(a) *$\mho_{\mathscr{S}}(x) = \Pi_{\mathscr{S}}(x) \cap \mathscr{B}(T)$ for every $x \in X$, and, in particular, $\mho_{\mathscr{S}}$ is always closed-valued.*
(b) *$\mho_{\mathscr{S}}$ is nonempty-valued provided $\mathscr{S}$ is point-finite.*

Proof. To show (a), take a point $x \in X$. By (4.2) and (4.3), $\beta \in \mho_{\mathscr{S}}(x)$ if and only if $x \in \Omega_{\mathscr{S}}(\beta) = \bigcap\{\mathscr{S}(t) : t \in \beta\}$. Moreover, for a map $\beta : \omega \to T$, we have $\beta \in \Pi_{\mathscr{S}}(x)$ if and only if $\beta(n) \in T(n)$ and $x \in \mathscr{S}(\beta(n))$, for all $n < \omega$. Consequently, for a branch $\beta \in \mathscr{B}(T)$, we find that $\beta \in \mho_{\mathscr{S}}(x)$ if and only if $\beta \in \Pi_{\mathscr{S}}(x)$. The second part of (a) follows from Proposition 4.1 and the fact that $\Pi_{\mathscr{S}} : X \to 2^{\Pi(T)}$ is always closed-valued. The proof of (b) follows an idea in the proof of [7, Lemma 3.2]. Namely, for a point $x \in X$, set $T(x) = \{t \in T : x \in \mathscr{S}(t)\}$. According to the definition of a semi-sieve, $(T(x), \preceq)$ is a subtree of $(T, \preceq)$ because $\mathscr{S}$ is order-preserving with respect to reverse inclusion. Also, $T(x)$ is infinite because, for every $n < \omega$, $x \in \mathscr{S}(t)$ for some $t \in T(n)$. Since each level of $T(x)$ is finite, by König's lemma (see Lemma 5.7 in Chapter II of [8]), $T(x)$ contains an infinite branch $\beta(x)$. Since $\beta(x)$ is also a branch in T, from (4.3) we conclude that $\beta(x) \in \mho_{\mathscr{S}}(x)$. ∎

By Lemma 4.4, the inverse polar mapping corresponding to a sieve $\mathscr{S} : T \to 2^X$ is always nonempty-valued. Thus, according to Propositions 2.2 and 3.4, Lemma 4.5 implies the following further property of this mapping.

COROLLARY 4.6. *Let X be a space, and let $\mathscr{F} : T \to 2^X$ be a closed-valued sieve on X. Then $\mho_{\mathscr{F}} : X \to 2^{\mathscr{B}(T)}$ is nonempty-valued and has a closed graph.*

Recall that a map $f : Z \to Y$ is *perfect* if it is a continuous closed map such that each $f^{-1}(y)$, $y \in Y$, is compact. Our next result is a consequence of Lemma 4.4, and represents the well-known fact that every completely metrizable space is a perfect image of a closed subset of the Baire space. As emphasised in the Introduction, we will make no difference between singleton-valued mappings $g : Z \to 2^Y$ and single-valued ones $g : Z \to Y$.

COROLLARY 4.7. *Every completely metrizable space Y has an open-valued locally-finite sieve $\mathscr{R} : T \to 2^Y$ such that the polar mapping $\Omega_{\overline{\mathscr{R}}} : \mathscr{B}(T) \to 2^Y$ is singleton-valued and perfect.*

Proof. Let d be a complete metric on Y compatible with the topology of Y. Since Y is paracompact, it has an open-valued locally-finite sieve $\mathscr{R} : T \to 2^Y$ such that $\mathscr{R}(t) \neq \emptyset$ and $\mathsf{diam}_d(\mathscr{R}(t)) < 2^{-n}$ for every $t \in T(n)$ and $n < \omega$. Since (Y, d) is complete, by Cantor's Theorem (see [4]) each polar $\Omega_{\overline{\mathscr{R}}}(\beta)$, $\beta \in \mathscr{B}(T)$, is a singleton and if $\Omega_{\overline{\mathscr{R}}}(\beta) \subset V$ for some open $V \subset Y$ and $\beta \in \mathscr{B}(T)$, then $\overline{\mathscr{R}}(t) = \overline{\mathscr{R}(t)} \subset V$ for some $t \in \beta$. Hence, $\Omega_{\overline{\mathscr{R}}}$ is continuous (see (4.1) and Proposition 4.2). Since $\overline{\mathscr{R}} : T \to \mathscr{F}(Y)$ is a closed-valued locally-finite sieve on Y, by Lemma 4.4 the inverse polar mapping $\mho_{\overline{\mathscr{R}}} : Y \to 2^{\mathscr{B}(T)}$ is usco. Hence, $\Omega_{\overline{\mathscr{R}}}$ is also perfect. ∎

A sieve $\mathscr{S} : T \to 2^Y$ on space Y is *complete* (see [1, 11]) if for every branch $\beta \in \mathscr{B}(T)$ and every nonempty centred (i.e., with the finite intersection property) family $\mathscr{F}$ of sets which refines $\{\mathscr{S}(t) : t \in \beta\}$ we have $\bigcap\{\overline{F} : F \in \mathscr{F}\} \neq \emptyset$. In other words, a sieve $\mathscr{S} : T \to 2^Y$ is complete if each family $\{\mathscr{S}(t) : t \in \beta\}$, $\beta \in \mathscr{B}(T)$, is a *compact* filter base (i.e., each ultrafilter containing it is convergent) [14]. If $\mathscr{S} : T \to 2^Y$ is a nonempty-valued complete sieve on a space Y, then for every branch $\beta \in \mathscr{B}(T)$, the polar $\Omega_{\overline{\mathscr{S}}}(\beta)$ is a nonempty compact subset of Y, and every open $V \supset \Omega_{\overline{\mathscr{S}}}(\beta)$ contains some $\overline{\mathscr{S}}(t)$ for $t \in \beta$ (see, e.g., [1, Proposition 2.10]). In terms of properties of set-valued mappings, this means that in this case the polar mapping $\Omega_{\overline{\mathscr{S}}} : \mathscr{B}(T) \to 2^Y$ is usco.

In [6], a sieve $\mathscr{S} : T \to 2^Y$ on the space Y was called *complete* if for every branch $\beta \in \mathscr{B}(T)$, the polar $\Omega_{\mathscr{S}}(\beta)$ is a nonempty compact subset of Y, and every open $V \supset \Omega_{\mathscr{S}}(\beta)$ contains some $\mathscr{S}(t)$ for $t \in \beta$. This corresponds to what is usually called an *open-valued strong complete sieve* $\mathscr{S} : T \to 2^Y$. Finally, recall that a space Y is *sieve-complete* if it has an open-valued complete sieve. Every regular sieve-complete space has an open-valued strong complete sieve (see [1, Proposition 2.10] and [11, Lemma 2.5]). Let us also remark that every Čech-complete space is sieve-complete, and it was shown in [1] (see also [11]) that the two concepts are equivalent in the presence of paracompactness.

Now, we also have the following observation for the case of paracompact Čech complete spaces which is very similar to Corollary 4.7.

Corollary 4.8. *Every paracompact Čech complete space Y has an open-valued locally-finite sieve $\mathscr{R} : T \to 2^Y$ such that both the polar mapping $\Omega_{\overline{\mathscr{R}}} : \mathscr{B}(T) \to 2^Y$ and its inverse $\mho_{\overline{\mathscr{R}}} : Y \to 2^{\mathscr{B}(T)}$ are usco.*

Proof. As mentioned above, Y has an open-valued complete sieve $\mathscr{S} : T \to 2^Y$, being sieve-complete. Since Y is paracompact, by [11, Lemma 2.2] there exists an open-valued locally-finite sieve $\mathscr{R} : T \to 2^Y$ on Y such that $\overline{\mathscr{R}}$ is a multi-selection for $\mathscr{S}$, i.e. $\overline{\mathscr{R}(t)} \subset \mathscr{S}(t)$ for every $t \in T$. We may assume that $\mathscr{R}(t) \neq \emptyset$ for every $t \in T$. Since $\mathscr{R}$ is also a complete sieve on Y, being a multi-selection for $\mathscr{S}$, as mentioned above, $\Omega_{\overline{\mathscr{R}}} : \mathscr{B}(T) \to 2^Y$ is usco. Since $\overline{\mathscr{R}} : T \to 2^Y$ is a closed-valued locally-finite sieve on Y, by Lemma 4.4 the inverse polar mapping $\mho_{\overline{\mathscr{R}}} : Y \to 2^{\mathscr{B}(T)}$ is also usco. ∎

5. Normality and closed graph multi-selections. In this section, we first prove Theorem 1.2.

Proof of Theorem 1.2. According to Corollary 3.5, it only suffices to show that (a)⇒(b). So, let X be a normal space, Y be metrizable, and let $\Phi : X \to \mathscr{C}(Y)$ be an l.s.c. mapping. Observe that $\Phi : X \to \mathscr{C}(Y) \subset \mathscr{C}(Z)$ remains l.s.c. whenever $Y \subset Z$. Hence, we may assume that Y is itself a completely metrizable space. Then, by Corollary 4.7, Y has an open-valued locally-finite sieve $\mathscr{R} : T \to 2^Y$ whose polar mapping $\Omega_{\overline{\mathscr{R}}} : \mathscr{B}(T) \to 2^Y$ is singleton-valued and perfect. Since Φ is l.s.c., the composite mapping $\mathscr{L} = \Phi^{-1} \circ \mathscr{R} : T \to 2^X$ defines an open-valued sieve on X. Since Φ is compact-valued and $\mathscr{R}$ is locally-finite, it now follows that $\mathscr{L}$ is point-finite. Then there exists an open-valued sieve $\mathscr{S} : T \to 2^X$ on X such that $\overline{\mathscr{S}}$ is a multi-selection for $\mathscr{L}$. Indeed, $\{\mathscr{L}(t) : t \in T(0)\}$ is an open and point-finite cover of X. Hence, by Lemma 3.3, X has an open cover $\{\mathscr{S}(t) : t \in T(0)\}$ such that $\overline{\mathscr{S}(t)} \subset \mathscr{L}(t)$ for every $t \in T(0)$. Take an element $s \in T(0)$. Then $\{\mathscr{L}(t) : t \in \mathsf{node}(s)\}$ is an open and point-finite cover of $\overline{\mathscr{S}(s)}$. Since $\overline{\mathscr{S}(s)}$ is also normal, by Lemma 3.3 it has an open cover $\{\mathscr{G}(t) : t \in \mathsf{node}(s)\}$ such that $\overline{\mathscr{G}(t)} \subset \mathscr{L}(t)$, $t \in \mathsf{node}(s)$. Set $\mathscr{S}(t) = \mathscr{G}(t) \cap \mathscr{S}(s)$, $t \in \mathsf{node}(s)$, and extend these arguments by induction. This completes the construction of the sieve $\mathscr{S} : T \to 2^X$. Let us show that the mappings

$$\varphi = \Omega_{\overline{\mathscr{R}}} \circ \mho_{\mathscr{S}} : X \to 2^Y \quad \text{and} \quad \psi = \Omega_{\overline{\mathscr{R}}} \circ \mho_{\overline{\mathscr{S}}} : X \to 2^Y$$

are as required. Since $\Omega_{\overline{\mathscr{R}}}$ is continuous, by Lemma 4.4, $\varphi : X \to \mathscr{C}(Y)$ and is l.s.c. Since both $\Omega_{\overline{\mathscr{R}}}$ and $\Omega_{\overline{\mathscr{R}}}^{-1} = \mho_{\overline{\mathscr{R}}}$ are usco, by Corollary 4.6 and Proposition 2.3, $\psi : X \to \mathscr{C}(Y)$ and has a closed graph. Finally, according to [6, Lemma 7.1], ψ is a multi-selection for Φ because $\Omega_{\overline{\mathscr{R}}}$ is singleton-valued. ∎

The rest of this section deals with related results and applications for normal spaces. In our next result, and in what follows, a mapping $\psi : X \to 2^Y$ is a *section* for $\Phi : X \to 2^Y$ (see [6]) if $\psi(x) \cap \Phi(x) \neq \emptyset$ for every $x \in X$.

THEOREM 5.1. *Let X be a normal space, Y be a paracompact Čech complete space, and let $\Phi : X \to \mathscr{C}(Y)$ be an l.s.c. mapping. Then Φ has a closed graph section $\psi : X \to \mathscr{C}(Y)$.*

Proof. According to Corollary 4.8, the space Y has an open-valued locally-finite sieve $\mathscr{R} : T \to 2^Y$ such that both the polar mapping $\Omega_{\overline{\mathscr{R}}} : \mathscr{B}(T) \to 2^Y$ and its inverse $\mho_{\overline{\mathscr{R}}} : Y \to 2^{\mathscr{B}(T)}$ are usco. Since Φ is l.s.c. and compact-valued and $\mathscr{R}$ is open-valued and locally-finite, the composition $\mathscr{L} = \Phi^{-1} \circ \mathscr{R}$:

$T \to 2^X$ is an open-valued point-finite sieve on X. We may now accomplish the proof in the same way as that of Theorem 1.2. Namely, exactly as in that proof construct an open-valued sieve $\mathscr{S} : T \to 2^X$ on X such that $\overline{\mathscr{S}}$ is a multi-selection for $\mathscr{L}$. Then the mapping $\psi = \Omega_{\overline{\mathscr{R}}} \circ \mho_{\overline{\mathscr{S}}} : X \to 2^Y$ is as required. Indeed, as shown in the previous proof, $\psi : X \to \mathscr{C}(Y)$ and has a closed graph. Finally, by [6, Lemma 7.1], $\psi(x) \cap \Phi(x) \neq \emptyset$ for every $x \in X$. ■

Let us remark that Theorem 5.1 can be deduced from Theorem 1.2 by using Frolík's result [5] (see also [4, 5.5.9(a)]) that, in this case, there exists a perfect map $g : Y \to Z$ of Y onto a completely metrizable space Z (see the proof of [6, Corollary 1.3]). Concerning the section constructed in this theorem, let us explicitly mention that if $\Phi : X \to \mathscr{C}(Y)$, then $\psi = \overline{\Phi}^{\mathbf{u}}$ has a closed graph but is not necessarily compact-valued. For instance, define an l.s.c. mapping $\Phi : [0,1] \to \mathscr{C}(\mathbb{R})$ by $\Phi(0) = \{0\}$ and $\Phi(x) = [-1/x, 1/x]$, $x \neq 0$. Then $\overline{\Phi}^{\mathbf{u}}(0) = \mathbb{R}$.

In our last result of this section, $\dim(X)$ means the *covering dimension* of X.

THEOREM 5.2. *For a T_1-space X, the following are equivalent:*

(a) *X is a normal space with* $\dim(X) = 0$.
(b) *Whenever Y is a metrizable space, every l.s.c. $\Phi : X \to \mathscr{C}(Y)$ has an l.s.c. closed graph multi-selection.*

Proof. (a)⇒(b). Suppose that X is a normal space with $\dim(X) = 0$, and $\Phi : X \to \mathscr{C}(Y)$ is an l.s.c. mapping for some metrizable space Y. As in the proof of Theorem 1.2, we may assume that Y is completely metrizable. Following the same proof, there exists an open-valued locally-finite sieve $\mathscr{R} : T \to 2^Y$ on Y such that the polar mapping $\Omega_{\overline{\mathscr{R}}} : \mathscr{B}(T) \to 2^Y$ is singleton-valued and perfect. Since $\dim(X) = 0$, there now exists a clopen-valued sieve $\mathscr{S} : T \to 2^X$ such that $\mathscr{S}(t) \subset \Phi^{-1}[\mathscr{R}(t)]$, $t \in T$. Briefly, $\{\Phi^{-1}[\mathscr{R}(t)] : t \in T(0)\}$ is an open and point-finite cover of X because Φ is l.s.c. and compact-valued. Hence, by Lemma 3.3, X has a closed cover $\{F_t : t \in T(0)\}$ such that $F_t \subset \Phi^{-1}[\mathscr{R}(t)]$ for every $t \in T(0)$. Since $\dim(X) = 0$, for every $t \in T(0)$ there exists a clopen set $\mathscr{S}(t) \subset X$ such that $F_t \subset \mathscr{S}(t) \subset \Phi^{-1}[\mathscr{R}(t)]$. We may now proceed by induction. Namely, if $s \in T(0)$, then $\{\Phi^{-1}[\mathscr{R}(t)] : t \in \mathsf{node}(s)\}$ is an open and point-finite cover of $\mathscr{S}(s)$. Exactly in the same way as before, we get a clopen cover $\{\mathscr{S}(t) : t \in \mathsf{node}(s)\}$ of $\mathscr{S}(s)$ such that $\mathscr{S}(t) \subset \Phi^{-1}[\mathscr{R}(t)]$ for $t \in \mathsf{node}(s)$. Having already constructed the sieve $\mathscr{S} : T \to 2^X$, the proof can be accomplished in the same manner as that of Theorem 1.2. Namely, as in that theorem, $\varphi = \Omega_{\overline{\mathscr{R}}} \circ \mho_{\mathscr{S}} : X \to \mathscr{C}(Y)$ is a multi-selection for Φ which, by Lemma 4.4, Corollary 4.6 and Proposition 2.3, is l.s.c. and has a closed graph.

(b)⇒(a). By Theorem 1.2, X is a normal space. To show that $\dim(X)=0$, take a closed set $F \subset X$ and an open set $V \subset X$ with $F \subset V$. Next, define an l.s.c. mapping $\Phi : X \to \mathscr{C}(\{0,1\})$ by $\Phi(x) = \{0\}$ if $x \in F$; $\Phi(x) = \{1\}$ if $x \notin V$; and $\Phi(x) = \{0,1\}$ otherwise. By (b), Φ has an l.s.c. closed-graph multi-selection $\varphi : X \to \mathscr{C}(\{0,1\})$. Since the discrete two-point space is compact, by Proposition 2.1, φ is u.s.c. Then $U = \varphi^{\#}[\{0\}]$ is a clopen subset of X such that $F \subset U \subset V$. Consequently, $\dim(X) = 0$. ∎

6. Paracompactness-like properties and closed graph multi-selections. In this section, we first characterise countably paracompact normal spaces by means of special closed graph multi-selections.

THEOREM 6.1. *For a T_1-space X, the following are equivalent:*

(a) *X is a countably paracompact normal space.*
(b) *Whenever Y is a completely metrizable separable space, every l.s.c. mapping $\Phi : X \to \mathscr{F}(Y)$ admits a pair of mappings $\langle \varphi, \psi \rangle : X \to \mathscr{C}(Y)$ such that φ is l.s.c., ψ has a closed graph and $\varphi(x) \subset \psi(x) \subset \Phi(x)$ for all $x \in X$.*

Proof. The implication (a)⇒(b) follows from a more general result of Choban in [2]. Namely, if X is a countably paracompact normal space, Y is a completely metrizable separable space and $\Phi : X \to \mathscr{F}(Y)$ is l.s.c., then, by [2, Theorem 11.2], there is an l.s.c. mapping $\varphi : X \to \mathscr{C}(Y)$ and an usco mapping $\psi : X \to \mathscr{C}(Y)$ such that $\varphi(x) \subset \psi(x) \subset \Phi(x)$ for all $x \in X$.

To show that (b)⇒(a), observe that X must be normal by Theorem 1.2. Take an open cover $\mathscr{U} : \mathbb{D} \to 2^X$ of X for some countable discrete space $\mathbb{D}$. Then, by Proposition 3.1, $\Phi = \mathscr{U}^{-1} : X \to \mathscr{F}(\mathbb{D})$ is l.s.c. and, by (b), Φ has an l.s.c. multi-selection $\varphi : X \to \mathscr{C}(\mathbb{D})$. By Propositions 3.1 and 3.2, $\varphi^{-1} : \mathbb{D} \to 2^X$ is an open and point-finite cover of X which refines $\mathscr{U}$. This implies that X is countably paracompact (see, for instance, [4, Theorem 5.2.6]). ∎

Our next results refine Choban's [2, Theorem 6.1] stating that a regular space X is weakly paracompact if and only if for every completely metrizable Y, every l.s.c. $\Phi : X \to \mathscr{F}(Y)$ has an l.s.c. multi-selection $\varphi : X \to \mathscr{C}(Y)$. In this connection, let us explicitly mention that there are non-regular weakly paracompact spaces (see, e.g., [4, Example 5.3.4]). Concerning the role of regularity in this result, we first prove the following theorem.

THEOREM 6.2. *For a Hausdorff space X, the following are equivalent:*

(a) *X is weakly paracompact.*
(b) *Whenever Y is a completely metrizable space, every l.s.c. mapping $\Phi : X \to \mathscr{F}(Y)$ has an l.s.c. section $\varphi : X \to \mathscr{C}(Y)$.*

Proof. To show that (a)⇒(b), we follow to some extent the corresponding proof in Theorem 1.2. Namely, suppose that X is weakly paracompact, Y is completely metrizable, and $\Phi : X \to \mathscr{F}(Y)$ is l.s.c. By Corollary 4.7, Y has an open-valued (locally-finite) sieve $\mathscr{R} : T \to 2^Y$ such that the polar mapping $\Omega_{\overline{\mathscr{R}}} : \mathscr{B}(T) \to 2^Y$ is singleton-valued and continuous. Then the composition $\mathscr{L} = \Phi^{-1} \circ \mathscr{R} : T \to 2^X$ defines an open-valued sieve on X because Φ is l.s.c. Since X is weakly paracompact, this implies that X has an open-valued point-finite semi-sieve $\mathscr{S} : T \to 2^X$ which is a multi-selection for $\mathscr{L}$. Indeed, $\{\mathscr{L}(t) : t \in T(0)\}$ is an open cover of X, so there exists an open point-finite cover $\{\mathscr{S}(t) : t \in T(0)\}$ of X such that $\mathscr{S}(t) \subset \mathscr{L}(t)$, $t \in T(0)$ (see, for instance, [4, Lemma 5.3.5]). For the same reason, there exists an open point-finite cover $\{\mathscr{S}(s) : s \in T(1)\}$ of X such that $\mathscr{S}(s) \subset \mathscr{L}(s) \cap \mathscr{S}(t)$ for every $t \in T(0)$ and $s \in \mathsf{node}(t)$. The construction of $\mathscr{S}$ is completed by induction. Now, by Proposition 3.1 and Lemma 4.5, the product mapping $\Pi_{\mathscr{S}} : X \to 2^{\Pi(T)}$ corresponding to the semi-sieve $\mathscr{S}$ is nonempty-compact-valued and l.s.c. as a product of l.s.c. compact-valued mappings. By Lemma 4.5 once again, we also have

$$\mho_{\mathscr{S}}(x) = \Pi_{\mathscr{S}}(x) \cap \mathscr{B}(T) \neq \emptyset \quad \text{for every } x \in X.$$

Since $\Omega_{\overline{\mathscr{R}}}$ is continuous, by [6, Lemma 7.1], $\psi = \Omega_{\overline{\mathscr{R}}} \circ \mho_{\mathscr{S}} : X \to \mathscr{C}(Y)$ is a multi-selection for Φ. By Proposition 4.1, the branch space is a closed subset of the generalised Baire space $\Pi(T)$. Since $\Pi(T)$ is a completely metrizable non-Archimedean space, there exists a retraction $r : \Pi(T) \to \mathscr{B}(T)$. Finally, define $\varphi = \Omega_{\overline{\mathscr{R}}} \circ r \circ \Pi_{\mathscr{S}} : X \to \mathscr{C}(Y)$, which is l.s.c. because r and $\Omega_{\overline{\mathscr{R}}}$ are continuous, and $\Pi_{\mathscr{S}}$ is l.s.c. Since r is a retraction, $\emptyset \neq \psi(x) \subset \varphi(x) \cap \Phi(x)$ for every $x \in X$. That is, φ is as required.

To show that (b)⇒(a), suppose $\mathscr{U} : \mathbb{D} \to 2^X$ is an open cover of X, and consider $\mathbb{D}$ as a discrete space. By Proposition 3.1, $\Phi = \mathscr{U}^{-1} : X \to \mathscr{F}(\mathbb{D})$ is l.s.c. and, by (b), Φ has an l.s.c. section $\varphi : X \to \mathscr{C}(\mathbb{D})$. However, Φ has an open graph, which implies that $\psi(x) = \varphi(x) \cap \Phi(x)$, $x \in X$, defines an l.s.c. multi-selection $\psi : X \to \mathscr{C}(\mathbb{D})$ for Φ. Finally, by Propositions 3.1 and 3.2, $\mathscr{V} = \psi^{-1} : \mathbb{D} \to 2^X$ is an open point-finite cover of X which is an indexed refinement of $\mathscr{U}$. ■

In the case of regular weakly paracompact spaces, we have the following slight generalization of the above mentioned [2, Theorem 6.1].

THEOREM 6.3. *For a regular space X, the following are equivalent:*

(a) *X is weakly paracompact.*
(b) *For every completely metrizable space Y and l.s.c. $\Phi : X \to \mathscr{F}(Y)$, there exists a completely metrizable (non-Archimedean) space Z, a continuous $g : Z \to Y$, an l.s.c. mapping $\varphi : X \to \mathscr{C}(Z)$ and a closed*

graph mapping $\psi : X \to \mathscr{F}(Z)$ *such that* φ *is a multi-selection for* ψ *while* $g \circ \psi$ *is a multi-selection for* Φ.

(c) *Whenever* Y *is a completely metrizable space, every l.s.c. mapping* $\Phi : X \to \mathscr{F}(Y)$ *has an l.s.c. multi-selection* $\varphi : X \to \mathscr{C}(Y)$.

Proof. It only suffices to show that (a)⇒(b). So, suppose that X is weakly paracompact, Y is completely metrizable, and $\Phi : X \to \mathscr{F}(Y)$ is l.s.c. Just like before, Y has an open-valued (locally-finite) sieve $\mathscr{R} : T \to 2^Y$ such that the polar mapping $\Omega_{\overline{\mathscr{R}}} : \mathscr{B}(T) \to 2^Y$ is singleton-valued and continuous. Hence, the composition $\mathscr{L} = \Phi^{-1} \circ \mathscr{R} : T \to 2^X$ defines an open-valued sieve on X because Φ is l.s.c. Then, by [6, Lemma 6.3] (see also the proof of [2, Theorem 6.1]), there exists a pruned ω-tree $(D, \preceq)$, a map $h : D \to T$ and an open-valued point-finite sieve $\mathscr{S} : D \to 2^X$ of X such that $\overline{\mathscr{S}(d)} \subset \mathscr{L}(h(d))$ for every $d \in D$, and $h(s) \prec h(t)$ for every $s, t \in D$ with $s \prec t$. According to [6, Proposition 6.2], h generates a continuous map $h_{\mathscr{B}} : \mathscr{B}(D) \to \mathscr{B}(T)$ such that $h(\beta)$ is cofinal in $h_{\mathscr{B}}(\beta)$ for every $\beta \in \mathscr{B}(D)$. Then take $Z = \mathscr{B}(D)$ and $g = \Omega_{\overline{\mathscr{R}}} \circ h_{\mathscr{B}} : Z \to Y$. Finally, let

$$\varphi = \mho_{\mathscr{S}} : X \to 2^Z \quad \text{and} \quad \psi = \mho_{\overline{\mathscr{S}}} : X \to 2^Z.$$

Since $\mathscr{S}$ is open-valued and point-finite, by Lemma 4.4, $\varphi : X \to \mathscr{C}(Y)$ and is l.s.c. By Corollary 4.6, $\psi : X \to \mathscr{F}(Z)$ and has a closed graph. Since $\Omega_{\overline{\mathscr{R}}}$ is singleton-valued and $\overline{\mathscr{S}(d)} \subset \mathscr{L}(h(d))$ for every $d \in D$, by [6, Lemma 7.1], $g(\psi(x)) \subset \Phi(x)$ for all $x \in X$. ∎

The following consequence is a well-known fact (see [4]).

COROLLARY 6.4. *Every normal weakly paracompact space is countably paracompact.*

Proof. Let Y be a complete metrizable separable space, and $\Phi : X \to \mathscr{F}(Y)$ be l.s.c. By Theorem 6.3, Φ has an l.s.c. multi-selection $\Psi : X \to \mathscr{C}(Y)$ because X is weakly paracompact. Since X is also normal, by Theorem 1.2 there exists a pair of mappings $\langle \varphi, \psi \rangle : X \to \mathscr{C}(Y)$ such that φ is l.s.c., ψ has a closed graph and $\varphi(x) \subset \psi(x) \subset \Psi(x) \subset \Phi(x)$ for every $x \in X$. Finally, by Theorem 6.1, X is countably paracompact. ∎

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