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On biorthogonal systems whose functionals are finitely supported

by

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Abstract. We show that for each natural number $n > 1$, it is consistent that there is a compact Hausdorff totally disconnected space K_{2n} such that $C(K_{2n})$ has no uncountable (semi)biorthogonal sequence $(f_\xi, \mu_\xi)_{\xi\in\omega_1}$ where μ_ξ's are atomic measures with supports consisting of at most $2n-1$ points of K_{2n}, but has biorthogonal systems $(f_\xi, \mu_\xi)_{\xi\in\omega_1}$ where μ_ξ's are atomic measures with supports consisting of $2n$ points. This complements a result of Todorcevic which implies that it is consistent that such spaces do not exist: he proves that its is consistent that for any nonmetrizable compact Hausdorff totally disconnected space K, the Banach space $C(K)$ has an uncountable biorthogonal system where the functionals are measures of the form $\delta_{x_\xi} - \delta_{y_\xi}$ for $\xi < \omega_1$ and $x_\xi, y_\xi \in K$. It also follows from our results that it is consistent that the irredundance of the Boolean algebra $\mathrm{Clop}(K)$ for a totally disconnected K or of the Banach algebra $C(K)$ can be strictly smaller than the sizes of biorthogonal systems in $C(K)$. The compact spaces exhibit an interesting behaviour with respect to known cardinal functions: the hereditary density of the powers K_{2n}^k is countable up to $k = n$ and it is uncountable (even the spread is uncountable) for $k > n$.

1. Introduction. If X is a Banach space and X^* is its dual, then $(x_i, x_i^*)_{i\in I} \subseteq X \times X^*$ is called a *biorthogonal system* if $x_i^*(x_i) = 1$ and $x_i^*(x_j) = 0$ if $i \neq j$ for each $i, j \in I$. If α is an ordinal, a transfinite sequence $(x_i, x_i^*)_{i<\alpha} \subseteq X \times X^*$ is called a *semibiorthogonal sequence* whenever $x_i^*(x_i) = 1$, $x_i^*(x_j) = 0$ for $j < i < \alpha$ and $x_i^*(x_j) \geq 0$ for $i < j < \alpha$.

Biorthogonal systems have always played an important role in the theory of Banach spaces ([9]) because all kinds of bases in Banach spaces are in particular the X-parts of biorthogonal systems ([20] and [21]). Semibiorthogonal sequences have been introduced quite recently ([2]) in connection with subsets of Banach spaces supported by all of their points ([17], [13], [8]).

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We will mainly deal with biorthogonal systems in real Banach spaces $C(K)$ of all real-valued continuous functions on a compact Hausdorff space K with the supremum norm. Its dual space is isometric to the Banach space $M(K)$ of all Radon measures on K with the variation norm, and so we will identify this dual with $M(K)$. If K is a compact Hausdorff space and $x \in K$, then δ_x denotes the functional on $C(K)$ defined by $\delta_x(f) = f(x)$ for all $f \in C(K)$.

This paper is motivated by the following question: *If there is an uncountable biorthogonal system $(f_\xi, \mu_\xi)_{\xi\in\omega_1}$ in $C(K)\times M(K)$, is there also one such that*

$$\mu_\xi = \delta_{x_\xi} - \delta_{y_\xi}$$

for some points $x_\xi, y_\xi \in K$? Following [4], we will call such biorthogonal systems *nice*.

The origin of this question is that in all concrete situations so far analyzed in the literature, the above question has a positive answer. Moreover, it happens for a good reason, namely, it follows from a recent result of Todorcevic that Martin's axiom together with the negation of the continuum hypothesis implies a positive answer for K totally disconnected. Indeed, analyzing the proof of Theorem 11 of [24], one gets two cases: the first case, when K is hereditarily separable, which is the main part of that proof and where the constructed biorthogonal system is nice; and the second case, when K is c.c.c. but contains a nonseparable subspace, in which case the proof of Theorem 10 of [24] provides the required nice system; if K is not c.c.c., one can easily obtain an uncountable nice biorthogonal system.

There is one more reason why nice biorthogonal systems appear frequently in the context of Banach spaces $C(K)$ and which makes them more meaningful. Namely, a family $(f_\alpha)_{\alpha\in\kappa}$ is the X-part of a nice biorthogonal system if and only if $(f_\alpha)_{\alpha\in\kappa}$ is irredundant in the Banach algebra $C(K)$, in the sense that no f_α belongs to the Banach subalgebra generated by the remaining elements. This is a consequence of the Stone–Weierstrass theorem. If K is totally disconnected and f_α's are the characteristic functions of clopen sets $A_\alpha \subseteq K$, we obtain the well-known notion of an *irredundant set* in a Boolean algebra, i.e., a set where no element belongs to the Boolean algebra generated by the remaining elements (see [14]). The *irredundance* of a Boolean algebra is the supremum of the cardinalities of irredundant sets.

To formulate our main results properly we need the following:

Definition 1.1. Let K be a compact Hausdorff space and $n \in \mathbb{N}$. We say that the functionals of a sequence $(f_\xi, \mu_\xi)_{\xi\in\omega_1} \subseteq C(K) \times M(K)$ are *n-supported* if each μ_ξ is an atomic measure whose support consists of no more than n points of K.

Theorem 1.2. *For each natural number $n > 1$, it is consistent that there is a compact Hausdorff totally disconnected space K_{2n} such that $C(K_{2n})$ has no uncountable semibiorthogonal sequence whose functionals are $2n-1$-supported, but has uncountable biorthogonal systems whose functionals are $2n$-supported.*

Moreover, K_{2n}^n is hereditarily separable but K_{2n}^{n+1} has an uncountable discrete subspace. Neither the Banach algebra $C(K_{2n})$ nor the Boolean algebra $\mathrm{Clop}(K_{2n})$ have an uncountable irredundant family. In particular, $C(K_4)$ has an uncountable biorthogonal system but it has no uncountable nice biorthogonal system.

This situation suggests many questions about the size of biorthogonal systems of various types in $C(K)$ spaces as well as in general Banach spaces. These more general discussions will appear elsewhere. In particular, we are unable to obtain K's such that $C(K)$ contains biorthogonal systems whose functionals are $2n+1$-supported but does not contain one whose functionals are $2n$-supported. The reason why some fundamental change in the approach would have to be taken to obtain such a space is shown in Lemma 3.9.

On the other hand, if $n = 1$ one has absolute results. If K is the split interval, then K is hereditarily separable, and so it cannot have an uncountable semibiorthogonal system whose functionals are 1-supported, but $C(K)$ has an uncountable nice biorthogonal system (see [7]).

It seems that our compact space is the first example showing that the hereditary density or spread of finite powers of a compact space may change its value from countable to uncountable arbitrarily high in $\mathbb{N}$. Such an example can be only consistent since, for example, under MA+$\neg$CH if K^3 is hereditarily separable for a compact K, then it is metrizable, and so all finite powers are hereditarily separable. This follows from the fact that under these assumptions there are no compact S-spaces ([22]), from the Katětov theorem ([10]) and from the fact that Lindelöf regular spaces are normal.

The paper is organized as follows: in the next section we discuss a general form of the compact spaces we construct and call them unordered N-split Cantor sets. They are versions of the split interval whose connection with biorthogonal systems in Banach spaces was already demonstrated in [7]. Section 3 is devoted to a generic construction of Boolean algebras whose Stone spaces are the K_{2n}'s. That is the only section that requires the knowledge of forcing. The partial order we use is a new modification of that of [11], which produced nonseparable $C(K)$'s with no uncountable semibiorthogonal sequences. Thus our spaces are quite controllable members of the class of compact spaces constructed in [1], [18], [19], [11]. In that section we also prove the existence of an uncountable discrete subspace of K_{2n}^{n+1} and an uncountable biorthogonal system in $C(K)$ whose functionals are $2n$-supported.

The section ends with Theorem 3.8, which expresses the random character of the compact space constructed. Later on we use this theorem to prove further properties of that space. Hence, a reader not familiar with forcing may use this theorem for other purposes and read only the following section. The last, fourth section is devoted to applications of Theorem 3.8 to prove that K_{2n}^n is hereditarily separable and that $C(K_{2n})$ has no uncountable semibiorthogonal sequences whose functionals are $2n-1$-supported.

We use standard notation: for a positive natural number n, we put $[n] = \{1,\dots,n\}$ and $n = \{0,\dots,n-1\}$. A^B denotes the set of all functions from B into A, and so if $2 = \{0,1\}$, then 2^ω denotes all infinite sequences with terms in $\{0,1\}$, while 2^n stands for functions from n into $\{0,1\}$; also, $2^{<\omega} = \bigcup\{2^n : n \in \mathbb{N}\}$ and $\langle s\rangle = \{x \in 2^\omega :\ s \subseteq x\}$ for $s \in 2^n$ and some $n \in \mathbb{N}$. If A, B are sets of ordinals, then $A < B$ means that $\alpha < \beta$ for any $\alpha \in A$ and any $\beta \in B$.

2. Unordered N-split Cantor sets. Fix a sequence $\mathcal{X} = \{x_\xi : \xi < \omega_1\} \subseteq 2^\omega$ of distinct elements and $N \in \mathbb{N}$. Let

$$K_N = (2^\omega \setminus \mathcal{X}) \cup (\mathcal{X} \times [N])$$

and define

$$V_s = (\langle s\rangle \cap (2^\omega \setminus \mathcal{X})) \cup ((\langle s\rangle \cap \mathcal{X}) \times [N]).$$

Definition 2.1. A family $(A_{\xi,i} : \xi < \omega_1,\ i \in [N])$ of subsets of K_N is called an *N-splitting family* if it satisfies the following conditions:

(1) $(x_\xi, i) \in A_{\xi,i} \subseteq K_N$ for each $\xi < \omega_1$ and $i \in [N]$;
(2) for each $\xi < \omega_1$ the sets $A_{\xi,i}$ are pairwise disjoint;
(3) for each $\xi < \omega_1$ we have $K_N = A_{\xi,1} \cup \dots \cup A_{\xi,N}$;
(4) if $\eta < \xi$, then there are $k \in \mathbb{N}$ and $j \in [N]$ such that $A_{\eta,i} \cap V_{x_\eta|k} \subseteq A_{\xi,j} \cap V_{x_\eta|k}$;
(5) if $\eta > \xi$ and $x = x_\eta$ or $x \in 2^\omega \setminus \mathcal{X}$, then there are $k \in \mathbb{N}$ and $j \in [N]$ such that $V_{x|k} \subseteq A_{\xi,j}$.

Definition 2.2. Given an N-splitting family $(A_{\xi,i} : \xi < \omega_1,\ i \in [N])$, we call the space $(K_N, \mathcal{T})$ an *unordered N-split Cantor set* if the topology $\mathcal{T}$ on K_N is defined by indicating neighbourhood bases $\mathcal{B}_x$ at x for every $x \in K_N$ in the following way: if $x \in 2^\omega \setminus \mathcal{X}$, then

$$\mathcal{B}_x = \{V_s : s \subseteq x\},$$

and if $x = (x_\xi, j) \in K_N$, then

$$\mathcal{B}_x = \{V_s \cap A_{\xi,j} : s \subseteq x_\xi\}.$$

The intuitive meaning of the above definitions is the following: each point x_ξ of 2^ω is split into N points $(x_\xi, 1), \dots, (x_\xi, N)$. If we view K_N as constructed inductively, when at step $\xi < \omega_1$ we construct the splitting clopen neighbourhoods $A_{\xi,1},\ldots,A_{\xi,N}$ of the points $(x_\xi,1),\ldots,(x_\xi,N)$ and these neighbourhoods split only x_ξ and no other previously constructed (x_η,i) for $\eta<\xi$ (condition 2.1(4)) nor x_η for $\eta>\xi$ nor $x\in 2^\omega\setminus\mathcal{X}$ (condition 2.1(5)). On the other hand, note that $A_{\xi,i}$'s may split x_η for $\eta<\xi$, and in this case, by condition 2.1(4), they do it "the same way" as the $A_{\eta,j}$'s.

Proposition 2.3. *Let $N\in\mathbb{N}$. If $(A_{\xi,i}:\xi<\omega_1,\, i\in[N])$ is an N-splitting family, then the corresponding unordered N-split Cantor set is a compact, Hausdorff, totally disconnected topological space.*

Proof. Since $V_\emptyset=K_N$, conditions (1)–(3) of Definition 2.1 imply that $A_{\xi,i}$'s are clopen sets. Now using Proposition 1.2.3 of [5], we will prove that the above families satisfy the axioms BP1–BP3 for neighbourhood bases from [5]. The only nontrivial part is to prove that given $x\in V\in\mathcal{B}_y$, there is $U\in\mathcal{B}_x$ such that $x\in U\subseteq V$.

Suppose $x\in 2^\omega\setminus\mathcal{X}$ and $x\in V_s\in\mathcal{B}_y$. Then $s\subseteq x$ and so V_s itself is in $\mathcal{B}_x$. If $x\in V_s\cap A_{\xi,i}$, we also have $s\subseteq x$ and by (5) of Definition 2.1 there is $k\in\mathbb{N}$ such that $V_{x|k}\subseteq A_{\xi,j}$ for some $j\in N$. Put $t=s\cup x|k$ and note that $V_t\subseteq A_{\xi,j}$, so by disjointness (condition 2.1(2)) we have $j=i$ with $x\in V_t\in\mathcal{B}_x$ and $V_t\subseteq V_s\cap A_{\xi,i}$.

Now suppose that $x=(x_\eta,i)$ and $x\in V_s\in\mathcal{B}_y$, hence $s\subseteq x$ and so $V_s\cap A_{\eta,i}\in\mathcal{B}_x$ and $x\in V_s\cap A_{\eta,i}\subseteq V_s$.

Finally, let $x=(x_\eta,i)$ and $x\in V_s\cap A_{\xi,j}\in\mathcal{B}_{(x_\xi,j)}$, then $s\subseteq x_\eta$.

First consider $\eta<\xi$. Then by (5) of Definition 2.1 there are $k\in\mathbb{N}$ and j' such that $A_{\eta,i}\cap V_{x_\eta|k}\subseteq A_{\xi,j'}\cap V_{x_\eta|k}$ and by disjointness we get $j'=j$. So, if we put $t=s\cup x_\eta|k$, then $A_{\eta,i}\cap V_t\subseteq A_{\xi,j}\cap V_t\subseteq A_{\xi,j}\cap V_s$ and of course $A_{\eta,i}\cap V_t\in\mathcal{B}_{(x_\eta,i)}$.

Secondly, if $\eta\ge\xi$ and $(x_\eta,i)\in V_s\cap A_{\xi,j}$, we also have $s\subseteq x_\eta$ and by 2.1(4) there are $k\in\mathbb{N}$ and j' such that $V_{x_\eta|k}\subseteq A_{\xi,j'}$ for some j'. By disjointness we have $j=j'$. If $t=s\cup x_\eta|k$ we have $V_t\subseteq A_{\xi,j}$, so $x\in V_t\in\mathcal{B}_x$ and $V_t\subseteq V_s\cap A_{\xi,i}$. This completes the proof that $\mathcal{B}_x$'s form a local neighbourhood base.

The Hausdorff property is easy since basic sets are clopen.

To prove the compactness, suppose $\mathcal{U}$ is an open cover of K_N. We may assume that it consists of basic open sets. For each $x\in 2^\omega\setminus\mathcal{X}$ define $s_x\in 2^{<\omega}$ such that $x\in V_{s_x}\subseteq U\in\mathcal{U}$ for some U, and for each $\xi<\omega_1$ define $s_\xi\in 2^{<\omega}$ such that $(x_\xi,i)\in V_{s_\xi}\cap A_{\xi,i}\subseteq U\in\mathcal{U}$ for some U, and for each $1\le i\le N$. This actually implies by (3) of Definition 2.1 that V_{s_ξ} is covered by finitely many $U\in\mathcal{U}$.

Now $\{\langle s_x\rangle,\langle s_\xi\rangle : x\in 2^\omega\setminus\mathcal{X},\ \xi<\omega_1\}$ forms an open cover of 2^ω which is compact and so it has a finite subcover, which easily yields a finite subcover of $\mathcal{U}$. ■

Definition 2.4. Suppose $N \in \mathbb{N}$ and K_N is an unordered N-split Cantor set. Under the above notation, we define the following:

- $R_\xi = \{(x_\xi, 1), \ldots, (x_\xi, N)\}$.
- $\mathcal{A}_\alpha$ is the subalgebra of $\mathrm{Clop}(K_N)$ generated by $(V_s : s \in 2^{<\omega})$ and $\{A_{\xi,i} : \xi < \alpha,\ i \in [N]\}$ for $\alpha \leq \omega_1$.
- C_α is the closure (in the norm) of the set of finite linear combinations of characteristic functions of elements of $\mathcal{A}_\alpha$ inside $C(K)$.

Note that C_0 can be naturally identified with $C(2^\omega)$ inside $C(K)$.

Lemma 2.5. *Let $N \in \mathbb{N}$ and let K_N be an unordered N-split Cantor set. For every $n \in \mathbb{N}$ and for every $\alpha \in \omega_1$ and every $i \in [N]$ we have*

$$A_{\alpha,i} \setminus V_{x_\alpha|n} \in \mathcal{A}_\alpha.$$

Proof. By the properties 2.1(4)&(5) of $A_{\xi,i}$'s any point of $K_N \setminus R_\alpha$ has a neighbourhood V such that for every $i \in [N]$ it is included in $A_{\alpha,i}$ or disjoint from $A_{\alpha,i}$ and moreover $V \in \mathcal{A}_\alpha$.

Since $A_{\alpha,i} \setminus V_{x_\alpha|n}$ is a compact subspace of $K_N \setminus R_\alpha$, we have a finite subcover consisting of subsets, i.e. $A_{\alpha,i} \setminus V_{x_\alpha|n}$ is the supremum of a finite family of elements of $\mathcal{A}_\alpha$ as required. ∎

Let us see the general form of continuous rational simple functions on an unordered N-split Cantor set. By a *rational simple function* we mean a function assuming only finitely many rational values.

Lemma 2.6. *Suppose that $N \in \mathbb{N}$ and that K_N is an unordered N-split Cantor set, $\varepsilon > 0$, μ is a (regular) Radon measure on K_N and f is a continuous rational simple function on K_N. Then there is a simple rational function $g \in C(2^\omega)$, distinct $\xi_1, \ldots, \xi_k < \omega_1$ and rationals $q_{i,l}$, non-negative integers m_i and $s_i \in 2^{m_i}$ with $s_i = x_{\xi_i}|m_i$, for $1 \leq i \leq k \in \omega$ and $1 \leq l < N$, such that*

$$f = g + \sum_{1 \leq i \leq k} \sum_{1 \leq l < N} q_{i,l} \chi_{A_{\xi_{i,l}} \cap V_{s_i}}$$

and

$$\sum_{1 \leq i \leq k} \max_{1 \leq l < N} (|q_{i,l}|) |\mu| (V_{s_i} \setminus R_{\xi_i}) \leq \varepsilon.$$

Proof. By induction on ξ we prove that any continuous simple rational function in C_ξ can be written in the form as in the lemma. The Stone–Weierstrass theorem and the uncountable cofinality of ω_1 imply that the union of C_ξ's is the entire $C(K_N)$.

The limit stage is trivial. So, suppose we have proved the conclusion for C_ξ and we are given a continuous simple rational function f in $C_{\xi+1}$. Note that

$$\bigcap_{m \in \mathbb{N}} V_{x_\xi|m} = R_\xi.$$

Hence, the regularity of the Radon measures implies that $|\mu|(V_{x_\xi|m}\setminus R_\xi)$'s converge to 0. Let m_1 be such that

$$|\mu|(V_{x_\xi|m}\setminus R_\xi)\le \frac{\varepsilon}{4\|f\|}$$

for $m\ge m_1$.

Note also that a simple function is a linear combination of characteristic functions of clopen sets, hence there are $\xi_1,\dots,\xi_{k-1}<\xi<\omega_1$ and m_2 such that the preimages under f of each of its finite rational values belong to the subalgebra of $\mathcal{A}_{\xi+1}$ generated by V_t's for $|t|<m_2$ and $A_{\xi_1,j},\dots,A_{\xi_{k-1},j},A_{\xi,j}$ for $1\le j\le N$. Now let $n\ge m_1,m_2$ be such that for every $1\le i<k$ there is $1\le j\le N$ such that $V_{x_\xi|m}\subseteq A_{\xi_i,j}$, which can be obtained by the property (5) (of Definition 2.1) of A_ξ's and $\eta=\xi_i$.

It follows that f is constant on $A_{\xi,j}\cap V_{x_\xi|m}$ for every $1\le j\le N$. Let $q'_1,\dots,q'_N\in\mathbb{Q}$ be the corresponding values and note that $|q'_l-q'_N|\le 2\|f\|$ for any $1\le l\le N$. So, by conditions (2) and (3) (of Definition 2.1) of $A_{\xi,j}$'s we have

$$f=[f|(K\setminus V_{x_\xi|m})+q'_N\chi_{V_{x_\xi|m}}]+\sum_{1\le l<N}(q'_l-q'_N)\chi_{A_{\xi,l}\cap V_{x_\xi|m}}.$$

Note that $f|(K\setminus V_{x_\xi|m})$ belongs to C_ξ by Lemma 2.5, and so

$$f=h+\sum_{1\le l<N}q_l\chi_{A_{\xi,l}\cap V_{x_\xi|m}},\qquad \max_{1\le l<N}|q_l|\,|\mu|(V_{x_\xi|m}\setminus R_\xi)\le\frac{\varepsilon}{2}$$

where $q_l=q'_l-q'_N$ and $h\in C_\xi$. Hence the inductive assumption for $\varepsilon/2$ can be used, which completes the proof of the lemma. ∎

Definition 2.7. We say that an N-splitting family $(A_{\xi,i}:\xi<\omega_1,\ i\in[N])$ is *balanced* if it satisfies the following additional condition:

(6) for all distinct $\xi,\eta\in\omega_1$ and all $j\in[2n]$,

$$|\{i\in\{1,3,\dots,2n-1\}:(x_\eta,i)\in A_{\xi,j}\}| = |\{i\in\{2,4,\dots,2n\}:(x_\eta,i)\in A_{\xi,j}\}|.$$

Lemma 2.8. *Suppose that $n\in\mathbb{N}$ and K_{2n} is an unordered $2n$-split Cantor set, where the N-splitting family $(A_{\xi,i}:\xi<\omega_1,i\in[2n])$ is balanced. Then:*

(a) *K_{2n}^{n+1} contains an uncountable discrete subspace;*
(b) *there is an uncountable biorthogonal system in $C(K_{2n})$ with $2n$-supported functionals.*

Proof. To prove (a), let us show that the subset $\{((x_\xi,1),(x_\xi,2),(x_\xi,4),\dots,(x_\xi,2n)):\xi<\omega_1\}$ of K_{2n}^{n+1} is relatively discrete.

Let $U_\xi=A_{\xi,1}\times A_{\xi,2}\times A_{\xi,4}\times\dots\times A_{\xi,2n}$, which is clearly an open neighbourhood of $((x_\xi,1),(x_\xi,2),(x_\xi,4),\dots,(x_\xi,2n))$. Now, fix distinct $\xi,\eta<\omega_1$ and let us prove that $((x_\eta,1),(x_\eta,2),(x_\eta,4),\dots,(x_\eta,2n))\notin U_\xi$.

For contradiction, suppose $((x_\eta,1),(x_\eta,2),(x_\eta,4),\ldots,(x_\eta,2n)) \in U_\xi$, that is, $(x_\eta,j) \in A_{\xi,j}$ for each $j = 1,2,4,\ldots,2n$. By condition 2.7(6), we see that for each $j \in [2n]$,

$$|\{i \in \{1,3,\ldots,2n-1\} : (x_\eta,i) \in A_{\xi,j}\}| \\ = |\{i \in \{2,4,\ldots,2n\} : (x_\eta,i) \in A_{\xi,j}\}|.$$

Hence, each set $A_{\xi,2}, A_{\xi,4},\ldots,A_{\xi,2n}$ must contain at least one of the $(x_\eta,1)$, $(x_\eta,3),\ldots,(x_\eta,2n-1)$. By the disjointness of the $A_{\xi,j}$'s (property (2) of Definition 2.1), $(x_\eta,1)$ has to be in one of the sets $A_{\xi,2},A_{\xi,4},\ldots,A_{\xi,2n}$. But by our assumption, $(x_\eta,1) \in A_{\xi,1}$ and again by the disjointness of the $A_{\xi,j}$'s, this is a contradiction.

To show (b), for each $\xi < \omega_1$, let $f_\xi = \chi_{A_{\xi,2n}}$ and

$$\mu_\xi = \sum_{k=1}^{n} (\delta_{(x_\xi,2i)} - \delta_{(x_\xi,2i-1)})$$

and note that $(f_\xi,\mu_\xi)_{\xi<\omega_1} \subseteq C(K_{2n}) \times M(K_{2n})$. Let us prove that this is a biorthogonal system.

For each $\xi < \omega_1$, since $(x_\xi,i) \in A_{\xi,i}$ and these sets are disjoint (property (2) of Definition 2.1), we get

$$\begin{aligned}\mu_\xi(f_\xi) &= \sum_{k=1}^{n} (\delta_{(x_\xi,2k)} - \delta_{(x_\xi,2k-1)})(\chi_{A_{\xi,2n}}) \\ &= \sum_{k=1}^{n} (\chi_{A_{\xi,2n}}((x_\xi,2k)) - \chi_{A_{\xi,2n}}((x_\xi,2k-1))) = \chi_{A_{\xi,2n}}((x_\xi,2n)) = 1.\end{aligned}$$

On the other hand, for distinct $\xi,\eta < \omega_1$, by property (6), we know that for all $j \in [2n]$,

$$|\{i \in \{1,3,\ldots,2n-1\} : (x_\eta,i) \in A_{\xi,j}\}| = |\{i \in \{2,4,\ldots,2n\} : (x_\eta,i) \in A_{\xi,j}\}|.$$

Hence,

$$\begin{aligned}\mu_\xi(f_\eta) &= \sum_{k=1}^{n} (\delta_{(x_\xi,2k)} - \delta_{(x_\xi,2k-1)})(\chi_{A_{\eta,2n}}) \\ &= \sum_{k=1}^{n} (\chi_{A_{\eta,2n}}((x_\xi,2k)) - \chi_{A_{\eta,2n}}((x_\xi,2k-1))) \\ &= \sum_{k=1}^{n} \chi_{A_{\eta,2n}}((x_\xi,2k)) - \sum_{k=1}^{n} \chi_{A_{\eta,2n}}((x_\xi,2k-1)) \\ &= |\{i \in \{2,4,\ldots,2n\} : (x_\xi,i) \in A_{\eta,2n}\}| \\ &\quad - |\{i \in \{1,3,\ldots,2n-1\} : (x_\xi,i) \in A_{\eta,2n}\}| = 0,\end{aligned}$$

showing that $(f_\xi,\mu_\xi)_{\xi<\omega_1} \subseteq C(K_{2n}) \times M(K_{2n})$ is a biorthogonal system. ∎

3. The generic construction. This section is devoted to a generic construction of an unordered $2n$-split Cantor set which exhibits quite random features. This type of uncountable structures was first investigated systematically in [19]. One can describe this random behaviour as: in any uncountable sequence of finite substructures there are two which are related as we wish (up to constraints). We fix an uncountable sequence $(x_\xi : \xi < \omega_1) \subseteq 2^\omega$ consisting of distinct elements.

DEFINITION 3.1. Let $\mathbb{P}$ be the forcing formed by the conditions

$$p = (F_p, n_p, (f_\xi^p : \xi \in F_p)),$$

where:

1. $F_p \in [\omega_1]^{<\omega}$;
2. $n_p \in \omega$ is such that for all $\xi \neq \eta$ in F_p, $x_\xi|n_p \neq x_\eta|n_p$;
3. for all $\xi \in F_p$,

$$f_\xi^p : 2^{n_p} \setminus \{x_\xi|n_p\} \to [2n]^{[2n]} \times [F_p \cap (\xi+1)]$$

is such that

(a) if $f_\xi^p(s) = (\varphi, \xi)$, then φ is a constant function;
(b) if $f_\xi^p(s) = (\varphi, \eta)$ for some $\eta < \xi$, then

$$\forall j \in [2n] \quad |\varphi^{-1}(j) \cap \{1, 3, 5, \ldots, 2n-1\}| = |\varphi^{-1}(j) \cap \{2, 4, \ldots, 2n\}|.$$

We put $q \leq p$ if $F_q \supseteq F_p$, $n_q \geq n_p$ and for all $\xi \in F_p$, all $s \in 2^{n_q} \setminus \{x_\xi|n_q\}$ and all $t \in 2^{n_p} \setminus \{x_\xi|n_p\}$,

$$t \subseteq s \ \Rightarrow \ f_\xi^p(t) = f_\xi^q(s).$$

Intuitively, we are of course trying to build a $2n$-split Cantor set which is determined by the choice of the balanced $2n$-splitting family formed by $A_{\xi,i}$'s. Thus the coordinate $f_\xi^p(s)$ describes the behaviour of $A_{\xi,i}$'s on V_s. The formal description is given in Definition 3.3. The value $f_\xi^p(s) = (\varphi, \xi)$, where φ has to be a constant function, say equal to i, means that the entire V_s is included in $A_{\xi,i}$. The value $f_\xi^p = (\varphi, \eta)$ for some $\eta < \xi$ means that $A_{\xi,i}$'s divide V_s as coded by φ, i.e. $A_{\eta,j} \cap V_s \subseteq A_{\xi,\varphi(j)}$ for each $j \in [N]$. Note that a condition $p \in \mathbb{P}$ carries no information about the behaviour of $A_{\xi,i}$'s on $V_{x_\xi|n_p}$, other than $(x_\xi, i) \in A_{\xi,i}$. This is the degree of freedom we have and which can be controlled by passing to an appropriate extension $q \leq p$. Condition (b) is to guarantee that the family of $A_{\xi,i}$'s is balanced, that is, satisfies property (6) of Definition 2.7.

LEMMA 3.2. *The following subsets of* $\mathbb{P}$ *are dense in* $\mathbb{P}$*:*

(i) $\{p \in \mathbb{P} : n_p \geq k\}$ *for each fixed* $k \in \mathbb{N}$*;*
(ii) $\{p \in \mathbb{P} : \xi \in F_p\}$ *for each fixed* $\xi < \omega_1$*.*

Proof. For (i), fix $k \in \mathbb{N}$ and let $p = (F_p, n_p, (f^p_\xi : \xi \in F_p)) \in \mathbb{P}$. If $n_p < k$, define $q = (F_q, n_q, (f^q_\xi : \xi \in F_q))$ by putting $F_q = F_p$, $n_q = k$ and for each $\xi \in F_q = F_p$, f^q_ξ is any function satisfying condition 3 of the definition of the forcing such that $f^q_\xi(t) = f^p_\xi(t|n_p)$ if $t|n_p \in 2^{n_p} \setminus \{x_\xi|n_p\}$; for example, let

$$f^q_\xi(t) = \begin{cases} f^p_\xi(t|n_p) & \text{if } t|n_p \in 2^{n_p} \setminus \{x_\xi|n_p\}, \\ (\varphi, \xi) & \text{otherwise,} \end{cases}$$

where φ is the constant function equal to 1. It is easy to see that $q \in \mathbb{P}$ and $q \le p$.

For (ii), fix $\xi < \omega_1$ and let $p = (F_p, n_p, (f^p_\xi : \xi \in F_p)) \in \mathbb{P}$. By (i), we may assume that n_p is such that $x_\eta|n_p \neq x_\xi|n_p$ for all $\eta \in F_p$. Define $q = (F_q, n_q, (f^q_\xi : \xi \in F_q))$ by putting $F_q = F_p \cup \{\xi\}$, $n_q = n_p$, $f^q_\eta = f^p_\eta$ for each $\eta \in F_p$, and f^q_ξ is any function satisfying condition 3 of the definition of the forcing; for example, let $f^q_\xi(t) = (\varphi, \xi)$, where φ is the constant function equal to 1. It is easy to see that $q \in \mathbb{P}$ and $q \le p$. ∎

Definition 3.3. Given a $\mathbb{P}$-generic filter G over a model V, we define the family $\{A_{\xi,j} : \xi \in \omega_1,\ j \in [2n]\}$ as follows: for each $\xi \in \omega_1$ and each $j \in [2n]$, let

$$\begin{aligned} A_{\xi,j} = {} & \bigcup\{V_s \cap A_{\eta,i} : \exists p \in G,\ f^p_\xi(s) = (\varphi, \eta) \text{ for some } \eta \neq \xi \text{ and } \varphi(i) = j\} \\ & \cup \bigcup\{V_s : \exists p \in G, f^p_\xi(s) = (\varphi, \xi) \text{ and } \varphi \text{ is the constant function equal to } j\} \\ & \cup \{(x_\xi, j)\}. \end{aligned}$$

The following lemma follows directly from the above definition.

Lemma 3.4. *Given $p \in G$, $\xi \in F_p$ and $s \in 2^{n_p} \setminus \{x_\xi|n_p\}$, we have:*

(a) *if $f^p_\xi(s) = (\varphi, \xi)$, then $V_s \subseteq A_{\xi,j}$ for $j = \varphi(1)$;*

(b) *if $f^p_\xi(s) = (\varphi, \eta)$ for some $\eta < \xi$, then $\forall i \in [2n]$, $V_s \cap A_{\eta,i} \subseteq A_{\xi,\varphi(i)}$.* ∎

Notice that in case $f^p_\xi(s) = (\varphi, \xi)$, φ is the constant function equal to j, so that we could have taken $j = \varphi(i)$ for any $i \in [2n]$.

Let us now check that the family $\{A_{\xi,j} : \xi \in \omega_1,\ j \in [2n]\}$ has the desired properties.

Theorem 3.5. *The family $\{A_{\xi,j} : \xi \in \omega_1,\ j \in [2n]\}$ is a balanced $2n$-splitting family.*

Proof. Let us prove that the family satisfies conditions 2.1(1)–(5) and 2.7(6).

(1) follows directly from the definition of $A_{\xi,j}$.

(2) This is proved by induction on ξ. First notice that by the definition of the forcing $\mathbb{P}$,

$$\forall p \in \mathbb{P}\ \forall \xi \in F_p\ \forall s \in \operatorname{dom} f^p_\xi \qquad R_\xi \cap V_s = \emptyset,$$

since $x_\xi|n_p \notin \operatorname{dom} f^p_\xi$. Thus, $(x_\xi, j_1) \in A_{\xi,j_2}$ iff $j_1 = j_2$.

Now, fix $\xi < \omega_1$ and suppose $A_{\eta,i}$ are pairwise disjoint for each fixed $\eta < \xi$. Suppose there is $x \in A_{\xi,j_1} \cap A_{\xi,j_2}$ for some distinct $j_1, j_2 \in [2n]$. By the above observation, $x \neq (x_\xi, j)$ for any $j \in [2n]$.

By the definition of A_{ξ,j_k}, for each $k \in \{1,2\}$ there are $p_k \in G$ and $s_k \in \operatorname{dom} f_\xi^{p_k}$ such that $x \in V_{s_k}$ and either

- $f_\xi^{p_k}(s_k) = (\varphi_k, \xi)$ and φ_k is the constant function equal to j_k, or
- $f_\xi^{p_k}(s_k) = (\varphi_k, \eta_k)$ for some $\eta_k < \xi$ and $x \in A_{\eta_k,i}$ for some $i \in \varphi_k^{-1}(j_k)$.

Let $p \in G$ be such that $p \leq p_1, p_2$ and let $t \in 2^{n_p} \setminus \{x_\xi | n_p\}$ be such that $x \in V_t$. Then $t \supseteq s_k$ since $x \in V_{s_k}$, and hence, by the definition of extension in $\mathbb{P}$, $f_\xi^{p_1}(s_1) = f_\xi^{p}(t) = f_\xi^{p_2}(s_2)$, so that $\varphi_1 = \varphi_2$.

Now, if $f_\xi^p(t) = (\varphi, \xi)$, this would mean that φ_1 and φ_2 are both constant equal to j_1 and j_2, contradicting the hypothesis that $j_1 \neq j_2$. Otherwise, if $f_\xi^p(t) = (\varphi, \eta)$, for some $\eta < \xi$, we would get $x \in A_{\eta,i_k}$ for some $i_k \in \varphi^{-1}(j_k)$. By the inductive hypothesis $i_1 = i_2 \in \varphi^{-1}(j_1) \cap \varphi^{-1}(j_2)$, which implies that $j_1 = j_2$, again contradicting the hypothesis.

This concludes the proof that the family satisfies condition (2) of Definition 2.1.

(3) is again proved by induction on ξ. So, let $\xi < \omega_1$, suppose $K = A_{\eta,1} \cup \cdots \cup A_{\eta,2n}$ for any $\eta < \xi$ and let $x \in K$.

If $x = (x_\xi, i)$ for some $i \in [2n]$, then $x \in A_{\xi,i}$ by definition.

By Lemma 3.2, let $p \in G$ be such that $x \in V_s$ for some $s \in 2^{n_p} \setminus \{x_\xi | n_p\}$.

If $f_\xi^p(s) = (\varphi, \xi)$, by Lemma 3.4(a) we get $V_s \subseteq A_{\xi,\varphi(1)}$, which guarantees that $x \in A_{\xi,\varphi(1)}$.

Otherwise, if $f_\xi^p(s) = (\varphi, \eta)$ for some $\eta < \xi$, by the inductive hypothesis, let $i \in [2n]$ be such that $x \in A_{\eta,i}$. Then, by Lemma 3.4(b), $V_s \cap A_{\eta,i} \subseteq A_{\xi,\varphi(i)}$, which implies that $x \in A_{\xi,\varphi(i)}$ and concludes the proof of condition (3) of Definition 2.1.

To prove (4), fix $\eta < \xi < \omega_1$ and $i \in [2n]$. By Lemma 3.2, let $p \in G$ be such that $\xi, \eta \in F_p$ and $x_\eta | n_p \neq x_\xi | n_p$.

If $f_\xi^p(x_\eta | n_p) = (\varphi, \xi)$, by Lemma 3.4(a) we get $V_{x_\eta | n_p} \subseteq A_{\xi,\varphi(1)}$ (and in particular $V_{x_\eta | n_p} \cap A_{\eta,i} \subseteq V_{x_\eta | n_p} \cap A_{\xi,\varphi(1)}$).

If $f_\xi^p(x_\eta | n_p) = (\varphi, \eta)$ for some $\eta < \xi$, then, by Lemma 3.4(b), we have $V_{x_\eta | n_p} \cap A_{\eta,i} \subseteq A_{\xi,\varphi(i)}$ (and in particular $V_{x_\eta | n_p} \cap A_{\eta,i} \subseteq V_{x_\eta | n_p} \cap A_{\xi,\varphi(i)}$), and we are done by condition (4) of Definition 2.1.

(5) is proved by induction on $\xi < \omega_1$. Let $\xi < \omega_1$ and $x \in 2^\omega \setminus \{x_\eta : \eta \leq \xi\}$.

If $x = x_\eta$ for some $\eta > \xi$, by Lemma 3.2 there is $p \in G$ such that $\xi, \eta \in F_p$. Otherwise, if $x \in 2^\omega \setminus \{x_\eta : \eta < \omega_1\}$, by Lemma 3.2 there is $p \in G$ such that $\xi \in F_p$ and $x | n_p \neq x_\xi | n_p$. In both cases, put $s = x | n_p$.

If $f_\xi^p(s) = (\varphi, \xi)$, then, by Lemma 3.4(a), $V_s \subseteq A_{\xi,\varphi(1)}$.

If $f^p_\xi(s) = (\varphi, \eta')$ for some $\eta' \in F_p \cap \xi$, by the inductive hypothesis, there are $k \in \mathbb{N}$ and $i \in [2n]$ such that $V_{x|k} \subseteq A_{\eta',i}$. By Lemma 3.2, let $q \in G$ be such that $q \le p$ and $n_q \ge k$. Putting $t = x|n_q$, we get $V_t \subseteq V_{x|k} \subseteq A_{\eta',i}$ and $f^q_\xi(t) = f^p_\xi(s) = (\varphi, \eta')$, since $t \supseteq s$. This implies by Lemma 3.4(b) that $V_t = V_t \cap A_{\eta',i} \subseteq A_{\xi,\varphi(i)}$, which concludes the proof of condition (5) of Definition 2.1.

Hence, the family formed by the $A_{\xi,i}$'s is a $2n$-splitting family.

(6) is proved by induction on $\xi < \omega_1$. So, fix $\xi < \omega_1$ and suppose we know that for all $\zeta < \xi$, all $\eta \neq \zeta$ and all $j \in [2n]$,

$$|\{i \in \{1,3,\ldots,2n-1\} : (x_\eta, i) \in A_{\zeta,j}\}| = |\{i \in \{2,4,\ldots,2n\} : (x_\eta, i) \in A_{\zeta,j}\}|.$$

Now, fix $\eta \neq \xi$. Let $p \in G$ be such that $\xi, \eta \in F_p$, so that $x_\eta|n_p \in \operatorname{dom} f^p_\xi$.

If $f^p_\xi(x_\eta|n_p) = (\varphi, \xi)$, then, by Lemma 3.4(a), $V_{x_\eta|n_p} \subseteq A_{\xi,\varphi(1)}$, which implies that $(x_\eta, i) \in A_{\xi,\varphi(1)}$ for all $i \in [2n]$. By the disjointness of the $A_{\xi,i}$'s, (3) and condition (6) of Definition 2.7 hold both for $A_{\xi,\varphi(1)}$ (which contains all (x_η, i)) and for $A_{\xi,j}$, $j \neq \varphi(1)$ (which contain no (x_η, i)).

If $f^p_\xi(x_\eta|n_p) = (\varphi, \zeta)$ for some $\zeta < \xi$ in F_p, then for all $i \in [2n]$, we have $V_{x_\eta|n_p} \cap A_{\zeta,i} \subseteq A_{\xi,\varphi(i)}$. This means that each $A_{\xi,j}$ contains exactly those (x_η, k) which are in $A_{\zeta,i}$ for some $i \in \varphi^{-1}(j)$. In particular,

$$\begin{aligned}
&\{k \in \{1,3,\ldots,2n-1\} : (x_\eta, k) \in A_{\xi,j}\} \\
&\qquad = \{k \in \{1,3,\ldots,2n-1\} : (x_\eta, k) \in A_{\zeta,i} \text{ for some } i \in \varphi^{-1}(j)\} \\
&\qquad = \bigcup_{i \in \varphi^{-1}(j)} \{k \in \{1,3,\ldots,2n-1\} : (x_\eta, k) \in A_{\zeta,i}\}
\end{aligned}$$

and

$$\begin{aligned}
&\{k \in \{2,4,\ldots,2n\} : (x_\eta, k) \in A_{\xi,j}\} \\
&\qquad = \{k \in \{2,4,\ldots,2n\} : (x_\eta, k) \in A_{\zeta,i} \text{ for some } i \in \varphi^{-1}(j)\} \\
&\qquad = \bigcup_{i \in \varphi^{-1}(j)} \{k \in \{2,4,\ldots,2n\} : (x_\eta, k) \in A_{\zeta,i}\}.
\end{aligned}$$

Let us now consider two cases:

If $\eta = \zeta$, since $(x_\eta, k) \in A_{\eta,k}$, we get

$$\{k \in \{1,3,\ldots,2n-1\} : (x_\eta, k) \in A_{\xi,j}\} = \{k \in \{1,3,\ldots,2n-1\} : k \in \varphi^{-1}(j)\}$$

and

$$\{k \in \{2,4,\ldots,2n\} : (x_\eta, k) \in A_{\xi,j}\} = \{k \in \{2,4,\ldots,2n\} : k \in \varphi^{-1}(j)\}.$$

By property 3(b) of the definition of the partial ordering, the sets on the right-hand side of these two equalities have the same size, which guarantees

$$|\{k \in \{1,3,\ldots,2n-1\} : (x_\eta, k) \in A_{\xi,j}\}| = |\{k \in \{2,4,\ldots,2n\} : (x_\eta, k) \in A_{\xi,j}\}|,$$

concluding the proof in this case.

If $\eta \neq \zeta$, by the inductive hypothesis we know that for all $i \in [2n]$, $|\{k \in \{1,3,\dots,2n-1\} : (x_\eta, k) \in A_{\zeta,i}\}| = |\{k \in \{2,4,\dots,2n\} : (x_\eta, k) \in A_{\zeta,i}\}|$. Hence,

$$\begin{aligned}
&|\{k \in \{1,3,\dots,2n-1\} : (x_\eta, k) \in A_{\xi,j}\}| \\
&\qquad = \Big|\bigcup_{i \in \varphi^{-1}(j)} \{k \in \{1,3,\dots,2n-1\} : (x_\eta, k) \in A_{\zeta,i}\}\Big| \\
&\qquad = \Big|\bigcup_{i \in \varphi^{-1}(j)} \{k \in \{2,4,\dots,2n\} : (x_\eta, k) \in A_{\zeta,i}\}\Big| \\
&\qquad = |\{k \in \{2,4,\dots,2n\} : (x_\eta, k) \in A_{\xi,j}\}|,
\end{aligned}$$

which concludes the proof of condition (6) of Definition 2.7, that is, the family of $A_{\xi,i}$'s is a balanced $2n$-splitting family. ∎

PROPOSITION 3.6. *Let* $p_1 = (F_1, n_1, (f^1_\xi : \xi \in F_1))$ *and* $p_2 = (F_2, n_2, (f^2_\xi : \xi \in F_2))$ *be conditions of* $\mathbb{P}$ *such that:*

- $F_1 \cap F_2 < F_1 \setminus F_2 < F_2 \setminus F_1$*;*
- $n_1 = n_2 = n$*;*
- *there is an order-preserving bijection* $e : F_1 \to F_2$ *such that*
  - *for all* $\xi \in F_1$, $x_\xi|n = x_{e(\xi)}|n$*;*
  - *for all* $\xi \in F_1$ *and all* $s \in 2^{n_1} \setminus \{x_\xi|n_1\}$ $(= 2^{n_2} \setminus \{x_{e(\xi)}|n_2\})$,

$$f^2_{e(\xi)}(s) = (\varphi, e(\eta)) \quad \textit{where} \quad f^1_\xi(s) = (\varphi, \eta).$$

Then, given $(\epsilon_\xi : \xi \in F_1 \setminus F_2) \subseteq [2n]^{[2n]}$ *such that for all* $\xi \in F_1 \setminus F_2$,

$$\forall j \in [2n] \quad |\epsilon_\xi^{-1}(j) \cap \{1,3,5,\dots,2n-1\}| = |\epsilon_\xi^{-1}(j) \cap \{2,4,6,\dots,2n\}|,$$

and given constant functions $(\delta_\xi : \xi \in F_1 \setminus F_2) \subseteq [2n]^{[2n]}$, *there is* $q \le p_1, p_2$, $q \in \mathbb{P}$, *such that*

$$\forall \xi \in F_1 \setminus F_2 \quad f^q_\xi(x_{e(\xi)}|n_q) = (\delta_\xi, \xi) \quad \textit{and} \quad f^q_{e(\xi)}(x_\xi|n_q) = (\epsilon_\xi, \xi). \tag{3.1}$$

Proof. Define $q = (F_q, n_q, (f^q_\xi : \xi \in F_q))$ as follows: let $F_q = F_1 \cup F_2$; let $n_q \in \mathbb{N}$ be such $n_p \le n_q$ and for all $\xi < \eta \in F_q$, $x_\xi|n_q \neq x_\eta|n_q$; for each $\xi \in F_q$ and $t \in 2^{n_q} \setminus \{x_\xi|n_q\}$, let

$$f^q_\xi(t) = \begin{cases} f^1_\xi(t|n) & \text{if } \xi \in F_1 \text{ and } t|n \neq x_\xi|n & \text{(Case 1)}, \\ f^2_\xi(t|n) & \text{if } \xi \in F_2 \text{ and } t|n \neq x_\xi|n & \text{(Case 2)}, \\ (\delta_\xi, \xi) & \text{if } \xi \in F_1 \text{ and } t|n = x_\xi|n & \text{(Case 3)}, \\ (\epsilon_{e^{-1}(\xi)}, e^{-1}(\xi)) & \text{if } \xi \in F_2 \setminus F_1 \text{ and } t|n = x_\xi|n & \text{(Case 4)}. \end{cases}$$

f^q_ξ is well-defined since $e(\xi) = \xi$ whenever $\xi \in F_1 \cap F_2$, so that $f^1_\xi(s) = f^2_{e(\xi)}(s) = f^2_\xi(s)$ for $s \in 2^n \setminus \{x_\xi|n\}$.

Let us now prove that $q \in \mathbb{P}$. Conditions 1 and 2 of Definition 3.1 follow directly from the definition of F_q and n_q.

To prove that q satisfies condition 3, fix $\xi \in F_q$ and $t \in 2^{n_q} \setminus \{x_\xi|n_q\}$. In Case 1 (resp. Case 2), both conditions 3(a) and 3(b) follow from the fact that p_1 (resp. p_2) is in $\mathbb{P}$.

In Case 3, we only have to check condition 3(a), which is guaranteed by the fact that $(\delta_\xi : \xi \in F_1 \setminus F_2) \subseteq [2n]^{[2n]}$ are assumed to be constant.

Similarly, in Case 4, we only have to check condition 3(b), which is guaranteed by the fact that $(\epsilon_\xi : \xi \in F_1 \setminus F_2) \subseteq [2n]^{[2n]}$ are assumed to be as needed.

Let us now prove that $q \leq p_1, p_2$. Trivially, $F_1, F_2 \subseteq F_q$ and $n_1, n_2 \leq n_q$.

Given $\xi \in F_q$, $s \in 2^n \setminus \{x_\xi|n\}$ and $t \in 2^{n_q} \setminus \{x_\xi|n_q\}$ such that $s \subseteq t$, let $k \in \{1,2\}$ be such that $\xi \in F_k$ and notice that we are in Cases 1 or 2, since $t|n = s$. Therefore, $f^q_\xi(t) = f^k_\xi(t|n) = f^k_\xi(s)$, which implies that $q \leq p_1, p_2$.

Finally, notice that the definition of $f^q_\xi(t)$ in Cases 1 or 2 implies (3.1). ∎

Theorem 3.7. *$\mathbb{P}$ is c.c.c.*

Proof. For each $\alpha < \omega_1$, let $p_\alpha = (F_\alpha, n_\alpha, (f^\alpha_\eta)_{\eta \in F_\alpha}) \in \mathbb{P}$.

By the Δ-system lemma, we can assume that $(F_\alpha)_{\alpha<\omega_1}$ forms a Δ-system with root Δ such that for every $\alpha < \beta < \omega_1$,

- $\Delta < F_\alpha \setminus \Delta < F_\beta \setminus \Delta$ and $|F_\alpha| = |F_\beta|$.

Since each $n_\alpha \in \mathbb{N}$, we can suppose that for every $\alpha < \beta < \omega_1$,

- $n_\alpha = n_\beta = n$.

Also, we may assume that if $e_{\alpha\beta} : F_\alpha \to F_\beta$ is the order-preserving bijective function, then

- for all $\xi \in F_\alpha$, $x_\xi|n = x_{e_{\alpha\beta}(\xi)}|n$ (since both belong to 2^n);
- for all $\xi \in F_\alpha$ and all $s \in 2^n \setminus \{x_\xi|n\}$,

$$f^\beta_{e_{\alpha\beta}(\xi)}(s) = (\varphi, e_{\alpha\beta}(\eta)), \quad \text{where} \quad f^\alpha_\xi(s) = (\varphi, \eta).$$

Now, fix $\alpha < \beta < \omega_1$. Note that p_α and p_β satisfy the hypothesis of Proposition 3.6. Let, for $\xi \in F_\beta \setminus \Delta$, ϵ_ξ be any function satisfying condition 3 of Definition 3.1 (for example, ϵ_ξ constant equal to 1); and for $\xi \in F_\alpha \setminus \Delta$, let $\delta_\xi \in [2n]^{[2n]}$ be any constant function. Then, by Proposition 3.6, there is $q \leq p_\alpha, p_\beta$ in $\mathbb{P}$, which concludes the proof. ∎

Theorem 3.8. *Let $n \geq 1$ be a natural number. It is consistent that there is a compact Hausdorff totally disconnected space K which is an unordered $2n$-split Cantor set corresponding to a balanced $2n$-splitting family $(A_{\xi,i} : \xi < \omega_1,\ i \in [2n])$ such that given any collection of pairwise disjoint sets $E_\alpha = \{\xi^1_\alpha, \ldots, \xi^k_\alpha\} \subseteq \omega_1$ for $\alpha < \omega_1$, given $\epsilon : [k] \times [2n] \to [2n]$ such that*

$$|\{l \in \{1,3,5,\ldots,2n-1\} : \epsilon(i,l) = j\}| = |\{l \in \{2,4,6,\ldots,2n\} : \epsilon(i,l) = j\}|,$$

and given $\delta : [k] \to [n]$, there are $\alpha < \beta$ such that for all $1 \le i \le k$,

$$R_{\xi^i_\beta} \subseteq A_{\xi^i_\alpha, \delta(i)} \quad \textit{and} \quad (x_{\xi^i_\alpha}, l) \in A_{\xi^i_\beta, \epsilon(i,l)}.$$

Proof. By Theorem 3.5, $\mathbb{P}$ forces that $(A_{\xi,i} : \xi < \omega_1,\ i \in [2n])$ as in Definition 3.3 is a balanced $2n$-splitting family. By Proposition 2.3, the corresponding unordered $2n$-split Cantor set is a compact, Hausdorff, totally disconnected space. Let us now prove the remaining desired property.

In V, suppose $(\dot{E}_\alpha)_{\alpha<\omega_1}$ and $(\dot{\xi}^i_\alpha)_{\alpha<\omega_1, 1\le i\le k}$ are sequences of names such that $\mathbb{P}$ forces that $\dot{E}_\alpha = \{\dot{\xi}^1_\alpha < \cdots < \dot{\xi}^k_\alpha\}$ and $(\dot{E}_\alpha)_{\alpha<\omega_1}$ is pairwise disjoint.

For each $\alpha < \omega_1$, let $p_\alpha = (F_\alpha, n_\alpha, (f^\alpha_\eta)_{\eta\in F_\alpha}) \in \mathbb{P}$, $\xi^1_\alpha, \ldots, \xi^k_\alpha \in \omega_1$ and $E_\alpha, \ldots, E_\alpha \subseteq \omega_1$ be finite such that

$$p_\alpha \Vdash \forall 1 \le i \le k \quad \dot{\xi}^i_\alpha = \check{\xi}^i_\alpha \text{ and } \dot{E}_\alpha = \check{E}_\alpha.$$

By Lemma 3.2, we can assume without loss of generality that for all $\alpha < \omega_1$, $E^\alpha_i \subseteq F_\alpha$.

By the Δ-system lemma, we can assume as well that $(F_\alpha)_{\alpha<\omega_1}$ forms a Δ-system with root Δ such that for every $\alpha < \beta < \omega_1$,

- $\Delta < F_\alpha \setminus \Delta < F_\beta \setminus \Delta$ and $|F_\alpha| = |F_\beta|$.

Since each $n_\alpha \in \mathbb{N}$, we can suppose that for every $\alpha < \beta < \omega_1$,

- $n_\alpha = n_\beta = n$.

Also, we may assume that if $e_{\alpha\beta} : F_\alpha \to F_\beta$ is the order-preserving bijective function, then

- for all $\xi \in F_\alpha$, $x_\xi|n = x_{e_{\alpha\beta}(\xi)}|n$ (since both belong to 2^n);
- for all $\xi \in F_\alpha$ and all $s \in 2^n \setminus \{x_\xi|n\}$,
$$f^\beta_{e_{\alpha\beta}(\xi)}(s) = (\varphi, e_{\alpha\beta}(\eta)), \quad \text{where} \quad f^\alpha_\xi(s) = (\varphi, \eta).$$
- for all $1 \le i \le k$, $e_{\alpha\beta}(\xi^i_\alpha) = \xi^i_\beta$.

Finally, we may assume that for all $1 \le i \le k$ we have: either $\xi^i_\alpha = \xi^i_\beta$ for all $\alpha < \beta < \omega_1$; or $\xi^i_\alpha \notin \Delta$ for all $\alpha < \omega_1$, and actually the second case holds by the assumption that E_α's are pairwise disjoint.

Now, fix $\alpha < \beta < \omega_1$. Note that p_α and p_β satisfy the hypothesis of Proposition 3.6. Taking $\epsilon_{\xi^i_\beta} = \epsilon(i, \cdot)$ and $\delta_{\xi^i_\alpha} = \delta(i)$ (and for $\xi \in F_\beta \setminus (\Delta \cup E_\beta)$, any function ϵ_ξ satisfying condition 3 of Definition 3.1, while for $\xi \in F_\alpha \setminus (\Delta \cup E_\alpha)$, any constant function $\delta_\xi \in [2n]^{[2n]}$), by Proposition 3.6, there is $q \le p_\alpha, p_\beta$ in $\mathbb{P}$ such that

$$\forall \xi \in F_\alpha \setminus \Delta \quad f^q_\xi(x_{e_{\alpha\beta}(\xi)}|n_q) = (\delta_\xi, \xi) \quad \text{and} \quad f^q_{e_{\alpha\beta}(\xi)}(x_\xi|n_q) = (\epsilon_{e_{\alpha\beta}(\xi)}, \xi).$$

In particular, for all $1 \le i \le k$,

$$f^q_{\xi^i_\alpha}(x_{\xi^i_\beta}|n_q) = (\delta(i), \xi^i_\alpha) \quad \text{and} \quad f^q_{\xi^i_\beta}(x_{\xi^i_\alpha}|n_q) = (\epsilon(i, \cdot), \xi^i_\alpha).$$

By the definition of $A_{\xi,j}$, we see that for all $1 \le i \le k$,

$$R_{\xi^i_\beta} \subseteq A_{\xi^i_\alpha,\delta(i)} \quad \text{and} \quad (x_{\xi^i_\alpha}, l) \in A_{\xi^i_\beta,\epsilon(i,l)},$$

which concludes the proof. ∎

The fact that $2n$ is even is exploited in the above proof. It turns out that there cannot be an analogue of an unordered N-split Cantor set for $N = 3$ which behaves as in Theorem 3.8, since we have the following:

LEMMA 3.9. *Let $N \ge 3$ be a natural number. Suppose that K is an unordered N-split Cantor set corresponding to an N-splitting family $(A_{\xi,i} : \xi < \omega_1,\ i \in [N])$ such that given any sequence $(\xi_\alpha : \alpha < \omega_1)$ of distinct ordinals and $j \in [N]$, there are $\alpha < \beta$ such that*

$$R_{\xi_\beta} \subseteq A_{\xi_\alpha,j}.$$

Suppose that $(f_\alpha, \mu_\alpha)_{\alpha<\omega_1}$ is a biorthogonal system such that $f_\alpha = \chi_{A_\alpha}$ for some clopen subset $A_\alpha \subseteq K$ and $\mu_\alpha = r_\alpha \delta_{(x_{\eta_\alpha},1)} + s_\alpha \delta_{(x_{\eta_\alpha},2)} + t_\alpha \delta_{(x_{\eta_\alpha},3)}$ for all $\alpha < \omega_1$, for some reals $r_\alpha, s_\alpha, t_\alpha$ and some sequence $(\eta_\alpha : \alpha < \omega_1)$. Then there is an uncountable nice biorthogonal system in $C(K)$.

Proof. If there is a biorthogonal system of the form $(\chi_{A_\alpha}, r_\alpha \delta_{y_\alpha})$ for $\alpha < \omega_1$ and $y_\alpha \in K$, then $r_\alpha = 1$ for all $\alpha < \omega_1$ and $y_\alpha \notin A_\beta$ for any $\beta \ne \alpha$ and $y_\alpha \in A_\alpha$. So $(\chi_{A_{\alpha+1}}, \delta_{y_{\alpha+1}} - \delta_{y_\alpha})$, say, for all limit ordinals α is a nice biorthogonal system.

If there is a biorthogonal system of the form $(\chi_{A_\alpha}, r_\alpha \delta_{y_\alpha} + s_\alpha \delta_{z_\alpha})$ for $\alpha < \omega_1$ and $y_\alpha, z_\alpha \in K$, and $r_\alpha, s_\alpha, r_\alpha + s_\alpha \ne 0$, then $r_\alpha, s_\alpha \notin A_\beta$ for any $\alpha \ne \beta$ and a similar argument to the one above gives a nice biorthogonal system. If $r_\alpha + s_\alpha = 0$ and $r_\alpha, s_\alpha \ne 0$, we may assume that $r_\alpha > 0$ and so $s_\alpha = -r_\alpha$. It follows from the fact that $(r_\alpha \delta_{y_\alpha} + s_\alpha \delta_{z_\alpha})(\chi_{A_\alpha}) = 1$ that $r_\alpha = 1$ and $s_\alpha = -1$, and so we have a nice biorthogonal system.

Hence, without loss of generality, we may assume that $r_\alpha, s_\alpha, t_\alpha \ne 0$ for all $\alpha < \omega_1$. First let us see that there is an uncountable $X \subseteq \omega_1$ such that $r_\alpha + s_\alpha + t_\alpha = 0$ for all $\alpha \in X$. If not, then there is an uncountable $X \subseteq \omega_1$ and an $\varepsilon > 0$ such that $|r_\alpha + s_\alpha + t_\alpha| > \varepsilon$ for each $\alpha \in X$.

Now note that as $\mu_\alpha(\chi_{A_\alpha}) = 1 \ne 0$, we have $j \in \{1, 2, 3\}$ such that $(x_{\eta_\alpha}, j) \in A_\alpha$. We may assume that it is the same j for all $\alpha \in X$. By the form of the basic neighbourhoods of points (x_{η_α}, j) we have $s \in 2^m$ for some $m \in \mathbb{N}$ such that $(x_{\eta_\alpha}, j) \in V_s \cap A_{\eta_\alpha,j} \subseteq A_\alpha$. We may assume that it is the same s for all $\alpha \in X$. It follows that for some $n \in \mathbb{N}$ we have $s = x_{\eta_\alpha}|n$ for all $\alpha \in X$ and so $R_{\eta_\alpha} \subseteq V_s$ for all $\alpha \in X$. Apply the hypothesis of the lemma and obtain $\alpha < \beta$, both in X, such that $R_{\eta_\beta} \subseteq A_{\eta_\alpha,j}$; thus we get $R_{\eta_\beta} \subseteq V_s \cap A_{\eta_\alpha,j} \subseteq A_\alpha$. This means that $0 = \mu_\beta(\chi_{A_\alpha}) = r_\beta + s_\beta + t_\beta$, contradicting the choice of $\beta \in X$. So we may assume that $r_\alpha + s_\alpha + t_\alpha = 0$ for all $\alpha < \omega_1$.

For three nonzero numbers whose sum is zero, there cannot be any subsum which is zero. This means that, for $\alpha \neq \beta$, as $\mu_\alpha(A_\beta) = 0$, we have either $\{x_\alpha, y_\alpha, z_\alpha\} \cap A_\beta = \emptyset$ or $\{x_\alpha, y_\alpha, z_\alpha\} \subseteq A_\beta$. So, to make an uncountable nice biorthogonal system out of points $\{x_\alpha, y_\alpha, z_\alpha\}$ and functions χ_{A_α}, we need to find any fixed pair of them which is separated by A_α for uncountably many α's.

But A_α must separate some pair as $\mu_\alpha(A_\alpha) = 1$, so choose an uncountable subset Y of ω_1 on which the same pair is separated, say $x_\alpha \in A_\alpha$ and $z_\alpha \notin A_\alpha$.

Define $\nu_\alpha = \delta_{x_\alpha} - \delta_{z_\alpha}$ and note that $(\chi_{A_\alpha}, \nu_\alpha)_{\alpha \in Y}$ is an uncountable nice biorthogonal system. ■

4. Biorthogonal and semibiorthogonal systems in $C(K_{2n})$'s

LEMMA 4.1. *Suppose that $\theta > \rho > 0$, $n \in \mathbb{N}$, $n \geq 2$, and $r_1, \ldots, r_{2n}$ are reals such that*

(1) $|\sum_{1 \leq i \leq 2n} r_i| < \rho$,
(2) *there is* $1 \leq i_0 \leq 2n$ *such that* $r_{i_0} > \theta$,
(3) *there is* $1 \leq i_1 \leq 2n$ *such that* $r_{i_1} = 0$.

Then there are $1 \leq i, j \leq 2n$ such that $(-1)^{i+j} = -1$ and
$$r_i + r_j < \frac{2n\rho - \theta}{n(2n-2)}.$$

Proof. By (1) and (2), since $\theta > \rho$, there must be an $i_2 \in \{1, \ldots, 2n\} \setminus \{i_0, i_1\}$ such that
$$r_{i_2} < -\frac{\theta - \rho}{2n-2} = \frac{\rho - \theta}{2n-2} < \frac{2n\rho - \theta}{n(2n-2)}.$$
So, if there is i_3 such that $(-1)^{i_2+i_3} = -1$ and $r_{i_3} \leq 0$, then we are done. Otherwise, there are at least n positive numbers r_i (at least for all i of parity other than i_2), and so, by (3), at most $n-1$ negative numbers r_i. Let r_{i_4} be the smallest number among r_i's with i of different parity than i_2, in particular $r_{i_4} > 0$. Let r_{i_5} be the smallest number among r_i's for i of the same parity as i_2, in particular $r_{i_5} \leq -\frac{\theta-\rho}{2n-2}$. So we have
$$nr_{i_4} + (n-1)r_{i_5} \leq \sum\{r_i : (-1)^{i+i_2} = -1\} + \sum\{r_i : (-1)^{i+i_2} = 1\} < \rho.$$
Hence,
$$n(r_{i_4} + r_{i_5}) < \rho + r_{i_5} \leq \rho - \frac{\theta - \rho}{2n-2},$$
so
$$r_{i_4} + r_{i_5} < \frac{1}{n}\left(\rho - \frac{\theta - \rho}{2n-2}\right) = \frac{(2n-1)\rho - \theta}{n(2n-2)},$$
as required. ■

Lemma 4.2. *Let $n \geq 2$. Suppose that $(f_\alpha)_{\alpha<\omega_1}$ is a sequence of continuous rational simple functions on K_{2n} as in Theorem 3.8 and $(\mu_\alpha)_{\alpha<\omega_1}$ is a sequence of $(2n-1)$-supported atomic Radon measures on K_{2n}. Then either there are $\alpha < \beta < \omega_1$ such that*

(a) $$\Big|\int f_\alpha\, d\mu_\beta\Big| > \frac{0.01}{2n^2(2n-2)},$$

or there is $\alpha \in \omega_1$ such that

(b) $$\int f_\alpha\, d\mu_\alpha < 0.99,$$

or there are $\alpha < \beta < \omega_1$ such that

(c) $$\int f_\beta\, d\mu_\alpha < -\frac{0.89}{2n^2(2n-2)}.$$

Proof. By the separability of $C_0 \equiv C(2^\omega)$ (see Definition 2.4), Lemma 2.6 and thinning out the sequence, we may assume that for all $\alpha < \omega_1$ we have

$$f_\alpha = g + \sum_{1\leq i\leq k} \sum_{1\leq l\leq 2n-1} q_{i,l}\chi_{A_{\xi^i_\alpha,l}\cap V_{s_i}}$$

for some simple rational function $g \in C_0$, $F_\alpha = \{\xi^1_\alpha, \dots, \xi^k_\alpha\} \subseteq \omega_1$, some $s_i \in 2^{m_i}$, $m_i \in N$ and some rationals $q_{i,l}$, $1 \leq i \leq k$ and $1 \leq l \leq 2n$, such that $s_i = r_{\xi^i_\alpha}|m_i$ and

$$\sum_{1\leq i\leq k} (\max_{1\leq l\leq 2n} |q_{i,l}|)|\mu_\alpha|(V_{s_i} \setminus R_{\xi^i_\alpha}) \leq \frac{0.01}{2n^2(2n-2)}.$$

By thinning out the sequence (applying the Δ-system lemma, see [Ku]) and moving some identical parts to g we may assume that F_α's are pairwise disjoint and g (no longer in C_0) is fixed. So, we will be allowed to use the following decompositions:

Claim 0. *For each $\alpha, \beta < \omega_1$ we have*

$$\begin{aligned}\int f_\alpha\, d\mu_\beta = \int g\, d\mu_\beta &+ \sum_{1\leq i\leq k} \sum_{1\leq l\leq 2n-1} q_{i,l}\mu_\beta(A_{\xi^i_\alpha,l} \cap R_{\xi^i_\beta} \cap V_{s_i}) \\ &+ \sum_{1\leq i\leq k} \sum_{1\leq l\leq 2n-1} q_{i,l}\mu_\beta(A_{\xi^i_\alpha,l} \cap V_{s_i} \setminus R_{\xi^i_\beta}).\end{aligned}$$

Here, the last term is small by the above application of Lemma 2.6, the first term will be shown to be small by the claim below, and so the value of the integral will depend on the relation of the points from $R_{\xi^i_\beta,l}$ to the sets $A_{\xi^i_\alpha,l}$ which is "as we wish" on any uncountable set by Theorem 3.8.

Claim 1. *Either* (a) *holds or for all but countably many α's in ω_1 we have*

$$\Big|\int g\,d\mu_\alpha\Big| \le \frac{0.02}{2n^2(2n-2)}.$$

Proof of the claim. If the inequality does not hold for uncountably many α's, then by Theorem 3.8 we can find among them $\alpha<\beta<\omega_1$ such that $R_{\xi^i_\beta} \subseteq A_{\xi^i_\alpha,2n}$ for all $1\le i\le k$. By Claim 0 we get

$$\Big|\int f_\alpha\,d\mu_\beta\Big| \ge \Big|\int g\,d\mu_\beta\Big| - \sum_{1\le i\le k}\sum_{1\le l\le 2n-1} |q_{i,l}|\,|\mu_\beta(A_{\xi^i_\alpha,l}\cap V_{s_i})|$$

$$\ge \Big|\int g\,d\mu_\beta\Big| - \sum_{1\le i\le k}(\max_{1\le l\le m}|q_{i,l}|)|\mu_\beta|(V_{s_i}\setminus R_{\xi^i_\beta}) > \frac{0.02-0.01}{2n^2(2n-2)} = \frac{0.01}{2n^2(2n-2)},$$

proving (a) of the lemma and Claim 1.

Claim 2. *Either* (a) *holds or for all but countably many α's in ω_1 we have, for each* $1\le l_0\le 2n-1$,

$$\Big|\sum_{1\le i\le k} q_{i,l_0}\mu_\alpha(R_{\xi^i_\alpha})\Big| \le \frac{0.04}{2n^2(2n-2)}.$$

Proof of the claim. Without loss of generality we may assume that the inequality of Claim 1 holds for all $\alpha<\beta<\omega_1$. Fix l_0 as above. Suppose that the inequality above does not hold for uncountably many α's; then by Theorem 3.8 we obtain among them $\alpha<\beta$ such that for all $1\le i\le k$,

$$R_{\xi^i_\beta} \subseteq A_{\xi^i_\alpha,l_0}.$$

So by Claim 0 we have

$$\Big|\int f_\alpha\,d\mu_\beta\Big| \ge \Big|\sum_{1\le i\le k}\sum_{1\le l\le 2n-1} q_{i,l}\mu_\beta(R_{\xi^i_\beta}\cap A_{\xi^i_\alpha,l}\cap V_{s_i})\Big| - \Big|\int g\,d\mu_\beta\Big|$$

$$- \sum_{1\le i\le k}(\max_{1\le l\le 2n-1}|q_{i,l}|)|\mu_\beta|(V_{s_i}\setminus R_{\xi^i_\beta})$$

$$\ge \Big|\sum_{1\le i\le k} q_{i,l_0}\mu_\beta(R_{\xi^i_\beta})\Big| - \Big|\int g\,d\mu_\beta\Big| - \sum_{1\le i\le k}(\max_{1\le l\le 2n-1}|q_{i,l}|)|\mu_\beta|(V_{s_i}\setminus R_{\xi^i_\beta})$$

$$> \frac{0.04-0.02-0.01}{2n^2(2n-2)} = \frac{0.01}{2n^2(2n-2)},$$

proving (a) and Claim 2.

Claim 3. *Either* (a) *or* (b) *holds or there is $l_0\in\{1,\dots,2n\}$ such that for uncountably many α's in ω_1 we have*

$$\sum_{1\le i\le k} q_{i,l_0}\mu_\alpha(\{(x_{\xi^i_\alpha},l_0)\}) > \frac{0.96}{2n}.$$

Proof of the claim. Assume that (a) does not hold, i.e., the inequalities of Claims 1 and 2 hold for all $\alpha < \omega_1$. Now, suppose also that the inequality above does not hold for any $l_0 \in \{1, \ldots, 2n\}$. By Claim 0 for $\alpha = \beta$ we have

$$\int f_\alpha \, d\mu_\alpha \le \sum_{1\le i\le k} \sum_{1\le l\le 2n-1} q_{i,l}\mu_\alpha(\{(x_{\xi_\alpha^i}, l)\}) - \frac{0.02}{2n^2(2n-2)} - \frac{0.01}{2n^2(2n-2)} \le \frac{(2n-1)0.96}{2n} - 0.03 < 0.99,$$

that is, we obtain (b), which concludes the proof of Claim 3.

To finish the proof of the lemma, we assume that (a) and (b) fail, i.e., the inequalities of all the above claims hold, and we need to get (c). Fix $\alpha < \omega_1$; we will apply Lemma 4.1 for

$$r_{l,\alpha} = \sum_{1\le i\le k} q_{i,l_0}\mu_\alpha(\{(x_{\xi_\alpha^i}, l)\})$$

and $l \in \{1, \ldots, 2n\}$. Since the supports of the measures μ_α have at most $2n-1$ elements, one of $r_{l,\alpha}$'s must be zero. By Claim 3 we have $r_{l_0,\alpha} > \theta = 0.96/(2n)$ and by Claim 2, $\sum_{1\le l\le 2n} r_{l,\alpha} < \rho = 0.04/(2n)^2$. So by Lemma 4.1 we find $1 \le l_{1,\alpha}, l_{2,\alpha} \le 2n$ of different parities such that

$$\sum_{1\le i\le k} q_{i,l_0}\mu_\alpha(\{(x_{\xi_\alpha^i}, l_{1,\alpha}), (x_{\xi_\alpha^i}, l_{2,\alpha})\}) < \frac{2n\rho - \theta}{n(2n-2)} = \frac{2n(0.04/(2n)^2) - 0.96/(2n)}{n(2n-2)} = -\frac{0.92}{2n^2(2n-2)}.$$

We may assume that $l_{1,\alpha} = l_1$ and $l_{2,\alpha} = l_2$ for all $\alpha < \omega_1$. Note that by Theorem 3.8 we can find $\alpha < \beta < \omega_1$ such that

$$\{(x_{\xi_\alpha^i}, l_1), (x_{\xi_\alpha^i}, l_2)\} \subseteq A_{\xi_\beta^i, l_0}$$

and

$$R_{\xi_\alpha^i} \setminus \{(x_{\xi_\alpha^i}, l_1), (x_{\xi_\alpha^i}, l_2)\} \subseteq A_{\xi_\beta^i, 2n}$$

for all $1 \le i \le k$. Together with Claim 0 with α and β switched, this implies that

$$\begin{aligned}\int f_\beta \, d\mu_\alpha &\le \sum_{1\le i\le k} q_{i,l_0}\mu_\alpha(\{(x_{\xi_\alpha^i}, l_1), (x_{\xi_\alpha^i}, l_2)\}) \\ &\quad + \sum_{1\le i\le k} (\max_{1\le l\le 2n-1} |q_{i,l}|)|\mu_\alpha|(V_{s_i} \setminus R_{\xi_\alpha^i}) + \left|\int g \, d\mu_\alpha\right| \\ &\le \frac{-0.92 + 0 + 0.01 + 0.02}{(2n)^2(2n-2)} = -\frac{0.89}{(2n)^2(2n-2)},\end{aligned}$$

which completes the proof of the lemma. ∎

THEOREM 4.3. *Let $n \geq 2$. If K_{2n} is an unordered $2n$-split Cantor set as in Theorem* 3.8, *then there are no uncountable semibiorthogonal sequences in $C(K_{2n})$ whose functionals are $(2n-1)$-supported but there is an uncountable biorthogonal system whose functionals are $2n$-supported.*

Proof. Suppose $(f_\alpha, \mu_\alpha)_{\alpha<\omega_1} \subseteq C(K_{2n}) \times M(K_{2n})$ is a sequence whose functionals are $2n-1$-supported and that $\int f_\alpha \, d\mu_\beta = 0$ for all $\alpha < \beta < \omega_1$ as well as $\int f_\alpha \, d\mu_\alpha = 1$ for all $\alpha < \omega_1$.

We may assume without loss of generality that $\|\mu_\alpha\| \leq M$ for some positive M. By the Stone–Weierstrass theorem we can choose $f'_\alpha \in C(K)$ which is a rational simple function and

$$\|f'_\alpha - f_\alpha\| < \frac{0.01}{2Mn^2(2n-2)}.$$

This means that (a) and (b) of Lemma 14 do not hold for f'_α's in place of f_α's, i.e. (c) holds, which implies that $(f_\alpha, \mu_\alpha)_{\alpha<\omega_1}$ is not semibiorthogonal. ∎

THEOREM 4.4. *If K_{2n} is an unordered $2n$-split Cantor set as in Theorem* 3.8, *then $hd(K_{2n}^n) = \omega$.*

Proof. We will be using the well-known fact that a regular space is hereditarily separable if and only if it has no uncountable left-separated sequence (see Theorem 3.1 of [16]).

Suppose $(y_\alpha)_{\alpha<\omega_1}$ is a left-separated sequence in K_{2n}^n of cardinality $\aleph_1$. Hence, for each $\alpha < \omega_1$, $y_\alpha = (y_\alpha^1, \ldots, y_\alpha^n)$, where each $y_\alpha^m \in K_{2n}$ and, by the definition of a left-separated sequence, for each $\alpha < \omega_1$ and each $m \in [N]$, there is an open basic neighbourhood U_α^m of y_α^m such that

$$\forall \alpha < \omega_1 \; \forall m \in [n] \quad y_\alpha^m \in U_\alpha^m$$

and

$$\forall \alpha < \beta < \omega_1 \; \exists m \in [n] \quad y_\alpha^m \notin U_\beta^m.$$

We may assume without loss of generality that

$$\{m \in [n] : y_\alpha^m \in 2^\omega \setminus \{x_\xi : \xi < \omega_1\}\} = \{m \in [n] : y_\beta^m \in 2^\omega \setminus \{x_\xi : \xi < \omega_1\}\}$$

for every $\alpha < \beta < \omega_1$ and let us call this set I.

For each $m \in [n] \setminus I$, let ξ_α^m be a countable ordinal and j_α^m be an element of $[n]$ such that $y_\alpha^m = (x_{\xi_\alpha^m}, j_\alpha^m)$.

Now, for each $m \in [n]$, let $s_\alpha^m \in 2^{<\omega}$ be such that

$$U_\alpha^m = \begin{cases} V_{s_\alpha^m} & \text{if } m \in I, \\ V_{s_\alpha^m} \cap A_{\xi_\alpha^m, j_\alpha^m} & \text{if } m \notin I. \end{cases}$$

Put $E_\alpha = \{\xi_\alpha^m : m \in [n] \setminus I\}$.

Without loss of generality, we may assume that:

- there is $j_m \in [n]$ such that $j_\alpha^m = j_m$ for all $\alpha < \omega_1$;

- there is $s_m \in 2^{<\omega}$ such that $s^m_\alpha = s_m$ for all $\alpha < \omega_1$ (this already guarantees that each $y^m_\alpha \in V_{s_m}$);
- for all $m \in [n] \setminus I$, either

$$\forall \alpha < \beta < \omega_1 \quad \xi^m_\alpha = \xi^m_\beta,$$

or

$$\forall \alpha < \beta < \omega_1 \quad \xi^m_\alpha < \xi^m_\beta.$$

- $(E_\alpha)_{\alpha<\omega_1}$ is a Δ-system with root Δ such that for every $\alpha < \beta < \omega_1$, $\Delta < E_\alpha \setminus \Delta < E_\beta \setminus \Delta$ and $|E_\alpha| = |E_\beta|$.

If $E_\alpha \setminus \Delta = \emptyset$, the left-separated sequence in K^n_{2n} would lead to a left-separated sequence in a finite power of 2^ω, which is not possible since 2^ω is hereditarily separable in all finite powers. Therefore, each $E_\alpha \setminus \Delta \neq \emptyset$ and they are pairwise disjoint.

For each $\alpha < \omega_1$, enumerate $E_\alpha \setminus \Delta = \{\eta^1_\alpha < \cdots < \eta^k_\alpha\}$. We may assume that $\xi^m_\alpha = \eta^i_\alpha$ if and only if $\xi^m_\beta = \eta^i_\beta$.

Claim. *For each $1 \leq i \leq k$, one can find $I_i \subseteq [2n]$ of cardinality N and a bijection $\sigma_i : I_i \to [2n] \setminus I_i$ such that $\sigma_i(l)$ and l have opposite parity and*

$$\{j \in [2n] : \exists m \in [n] \text{ such that } j = j_m \text{ and } \xi^m_\alpha = \eta^i_\alpha\} \subseteq I_i.$$

Proof of the claim. The claim follows easily from the fact that the set

$$\{j \in [2n] : \exists m \in [n] \text{ such that } j = j_m \text{ and } \xi^m_\alpha = \eta^i_\alpha\}$$

has cardinality at most n so that we can find I_i containing it, and that whenever we have a partition of $[2n]$ into two sets A and B, both of size n, then A has as many odds as B has evens, and vice versa.

Now, let $\epsilon : [k] \times [2n] \to [2n]$ be defined by

$$\epsilon(i,l) = \begin{cases} l & \text{if } l \in I_i, \\ \sigma_i^{-1}(l) & \text{if } l \in [2n] \setminus I_i. \end{cases}$$

Notice that for each $i \in [k]$, $l \in I_i$ and $j \in [2n]$, $\epsilon(i,l) = j$ if and only if $\epsilon(i,\sigma(l)) = j$. Since $\sigma(l)$ and l have opposite parities, we see that ϵ has the desired property, that is,

$$|\{l \in \{1,3,5,\ldots,2n-1\} : \epsilon(i,l) = j\}| = |\{l \in \{2,4,6,\ldots,2n\} : \epsilon(i,l) = j\}|.$$

By Theorem 3.8, there are $\alpha < \beta$ such that for all $i \in [k]$,

$$(x_{\eta^i_\alpha}, l) \in A_{\eta^i_\beta, \epsilon(i,l)}.$$

Fix $m \in [n]$ and let us prove that $y^m_\alpha \in U^m_\beta$, contradicting the assumption. If $m \notin I$, then $y^m_\alpha \in V_{s_m} = U^m_\beta$. If $m \in I$ and $\xi^m_\alpha \in \Delta$, then $\xi^m_\alpha = \xi^m_\beta \in U^m_\beta$. Finally, if $m \in I$ and $\xi^m_\alpha \notin \Delta$, then there is $i \in [k]$ such that $\xi^m_\alpha = \eta^i_\alpha$

and $\xi^m_\beta = \eta^i_\beta$. In this case we have $j_m \in I_i$, and so $\epsilon(i, j_m) = j_m$, which guarantees that

$$y^m_\alpha = (x_{\xi^m_\alpha}, j_m) = (x_{\eta^i_\alpha}, j_m) \in A_{\eta^i_\beta, j_m} = A_{\xi^m_\beta, j_m}.$$

Since also $y^m_\alpha \in V_{s_m}$, we get $y^m_\alpha \in U^m_\beta$, which concludes the proof. ∎

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