

Countable dense homogeneous filters and the Menger covering property

by

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Abstract. We present a ZFC construction of a non-meager filter which fails to be countable dense homogeneous. This answers a question of Hernández-Gutiérrez and Hrušák. The method of the proof also allows us to obtain for any $n \in \omega \cup \{\infty\}$ an n -dimensional metrizable Baire topological group which is strongly locally homogeneous but not countable dense homogeneous.

1. Introduction. A topological space X has property *CDH* (abbreviated from *countable dense homogeneous*) if for arbitrary countable dense subsets D_0, D_1 of X there exists a homeomorphism $\phi : X \rightarrow X$ such that $\phi[D_0] = D_1$. The study of CDH filters on ω was initiated in [12] where the property CDH was used to find concrete examples of non-homeomorphic filters and ultrafilters, considered with the topology inherited from $\mathcal{P}(\omega)$.

The following theorem is the main result of this note. Let us stress that we do not use any additional set-theoretic assumptions in its proof.

THEOREM 1. *There exists a non-meager non-CDH filter.*

There are several constructions of non-CDH spaces by transfinite induction using some enumeration of all potential autohomeomorphisms, by adding points to the space under construction in such a way that these homeomorphisms are ruled out one by one (see, e.g., [12] and the references therein). However, this method often requires some equalities between cardinal characteristics and hence does not seem to lead to a construction of non-meager non-CDH filters outright in ZFC.

Instead of ruling out all potential autohomeomorphisms sending some countable dense subset D_0 onto some other countable dense subset D_1 one

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by one, we shall do this at once. Our idea is rather straightforward: if a space X admits two countable dense subsets D_0, D_1 such that $X \setminus D_0$ is not homeomorphic to $X \setminus D_1$, then there obviously is no autohomeomorphisms of X mapping D_0 onto D_1 . We shall prove Theorem 1 by constructing a non-meager filter \mathcal{F} on ω and two countable dense subsets $\mathcal{D}_0, \mathcal{D}_1$ of \mathcal{F} such that $\mathcal{F} \setminus \mathcal{D}_1$ has the Menger property whereas $\mathcal{F} \setminus \mathcal{D}_0$ does not; see the next section for its definition.

2. Covering properties of Menger and Hurewicz. We recall from [15] that a topological space X has

- *the Menger (covering) property* if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that each \mathcal{V}_n is a finite subfamily of \mathcal{U}_n and the collection $\{\bigcup \mathcal{V}_n : n \in \omega\}$ is a cover of X ;
- *the Hurewicz (covering) property* if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that each \mathcal{V}_n is a finite subfamily of \mathcal{U}_n and the collection $\{\bigcup \mathcal{V}_n : n \in \omega\}$ is a γ -cover of X (a family \mathcal{U} of subsets of a space X is called a γ -cover of X if every $x \in X$ belongs to all but finitely many $U \in \mathcal{U}$).

These properties were introduced by Hurewicz in [8] and [9], respectively. It is clear that every σ -compact space has the Hurewicz property, and the Hurewicz property implies the Menger one. It is known [6, 10, 16] that none of these implications can be reversed. The simplest example of a metrizable space without the Menger property is the Baire space ω^ω . Indeed $\langle \mathcal{U}_n : n \in \omega \rangle$, where $\mathcal{U}_n = \{\{x \in \omega^\omega : x(n) = k\} : k \in \omega\}$, is a sequence of open covers of ω^ω witnessing the failure of the Menger property.

In the proof of Theorem 1 we shall use without mention several basic facts about these properties summarized in the following proposition. Most likely these can be found somewhere in the literature. However, we did not try to locate them or to present their proofs as we believe that the proof of any of them should not take the reader more than a couple of minutes.

PROPOSITION 2.

- (i) *If a topological space X has the Menger [Hurewicz] property and Y is a continuous image of X , then Y is Menger [Hurewicz].*
- (ii) *If a topological space X has the Menger [Hurewicz] property and Y is a closed subspace of X , then Y is Menger [Hurewicz].*
- (iii) *If a topological space X has the Menger [Hurewicz] property and Y is compact, then $X \times Y$ is Menger [Hurewicz].*
- (iv) *If $\{Y_i : i \in \omega\}$ is a collection of Menger [Hurewicz] subspaces of a space X , then $\bigcup_{i \in \omega} Y_i$ is Menger [Hurewicz].*

- (v) If a topological space X has the Menger [Hurewicz] property and Y is an F_σ -subspace of X , then Y is Menger [Hurewicz].
- (vi) If a topological space X has the Menger [Hurewicz] property and Y is σ -compact, then $X \times Y$ is Menger [Hurewicz].

COROLLARY 3. *Let X be a metrizable space with the Menger property. Then $X \setminus A$ has the Menger property for all finite subsets A .*

Proof. $X \setminus A$ is an F_σ -subspace of X . ■

By a filter on a countable set C we mean a *free filter*, i.e., a filter containing all cofinite subsets of C . A family $\mathcal{B} \subset \mathcal{P}(C)$ is said to be *centered* if for any finite $\mathcal{B}' \subset \mathcal{B}$ the intersection $\bigcap \mathcal{B}'$ is infinite. Any centered family generates a filter in a natural way.

COROLLARY 4. *Suppose that a filter \mathcal{F} on a countable set C is generated by a centered family \mathcal{B} all of whose finite powers have the Menger property when \mathcal{B} is considered with the subspace topology inherited from $\mathcal{P}(C)$. Then \mathcal{F} is Menger.*

Proof. Consider the map $\phi_n : \mathcal{B}^n \times \mathcal{P}(C) \times [C]^{<\omega}, \phi : \langle B_0, \dots, B_{n-1}; X; x \rangle \mapsto (\bigcap_{i \in n} B_i \setminus x) \cup X$. It is clear that each ϕ_n is continuous and $\mathcal{F} = \bigcup_{n \in \omega} \phi_n[\mathcal{B}^n \times \mathcal{P}(C) \times [C]^{<\omega}]$. It suffices to use Proposition 2 several times. ■

3. Proof of Theorem 1. We divide the proof into two lemmas.

LEMMA 5. *Let \mathcal{F} be a filter on ω containing co-infinite sets. Then there exists a countable dense subset \mathcal{D} of \mathcal{F} such that $\mathcal{F} \setminus \mathcal{D}$ does not have the Menger property.*

Proof. Let us find $F \in \mathcal{F}$ such that $|\omega \setminus F| = \omega$ and consider the subspace $\mathcal{C} = \{F' \in \mathcal{F} : F \subset F'\}$ of \mathcal{F} . Let $\mathcal{D}', \mathcal{D}''$ be countable dense subspaces of \mathcal{C} and $\mathcal{F} \setminus \mathcal{C}$, respectively. Notice that \mathcal{C} is a copy of the Cantor set being a compact zero-dimensional space without isolated points. Therefore $\mathcal{C} \setminus \mathcal{D}'$ is homeomorphic to ω^ω . Thus $\mathcal{F} \setminus (\mathcal{D}' \cup \mathcal{D}'')$ has a closed subspace homeomorphic to ω^ω (namely $\mathcal{C} \setminus \mathcal{D}'$) and hence is not Menger. ■

A collection \mathcal{U} of subsets of X is called an ω -cover of X if $X \notin \mathcal{U}$ and for every $A \in [X]^{<\omega}$ there exists $U \in \mathcal{U}$ such that $A \subset U$. A collection \mathcal{U} is a *groupable ω -cover* of X if there is a partition $\{\mathcal{U}_n : n \in \omega\}$ of \mathcal{U} into pairwise disjoint finite sets such that, for each finite subset A of X and for all but finitely many n , there exists $U \in \mathcal{U}_n$ such that $A \subset U$.

The existence of spaces X as in the following lemma was first established in [6] (see also [16, Corollary 6.4]).

LEMMA 6. *Let X be a dense subspace of ω^ω such that all finite powers of X have the Menger property but X fails to have the Hurewicz property. Then there exists a clopen ω -cover \mathcal{V} of X which fails to be a groupable ω -cover*

of X , and is such that for any two disjoint finite subsets A, C of X the set $\{V \in \mathcal{V} : A \subset V \wedge V \cap C = \emptyset\}$ is infinite.

Proof. Applying [11, Theorem 16] we can find a sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of clopen ω -covers of X such that for any sequence $\langle \mathcal{V}_n : n \in \omega \rangle$, $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$, the union $\bigcup_{n \in \omega} \mathcal{V}_n$ fails to be a groupable ω -cover of X . Passing to finer ω -covers if necessary, we may additionally assume that \mathcal{U}_{n+1} is a refinement of \mathcal{U}_n for all n , and the projection of each element of \mathcal{U}_0 onto the 0th coordinate (recall that \mathcal{U}_0 is a family of subsets of ω^ω) is finite.

For every $s \in \omega^{<\omega}$ we shall denote the basic open subset $\{x \in \omega^\omega : x|s = s\}$ of ω^ω by $[s]$. Let $\mathcal{B} = \{B_k : k \in \omega\}$ be the family of all finite unions of the sets of the form $[s]$, $s \in \omega^{<\omega} \setminus \{\emptyset\}$. It follows from our restrictions on \mathcal{U}_n 's that $\mathcal{W}_{n,k} = \{U \setminus B_k : U \in \mathcal{U}_n\}$ is an ω -cover of $X \setminus B_k$ for all $n, k \in \omega$ (because no $U \in \mathcal{U}_n$ contains $X \setminus B_k$). Let us decompose ω into countably many disjoint infinite sets $\{I_k : k \in \omega\}$ and for every $k \in \omega$ consider the sequence $\langle \mathcal{W}_{n,k} : n \in I_k \rangle$ of clopen ω -covers of $X \setminus B_k$. Since $X \setminus B_k$ is a clopen subset of X , all of its finite powers have the Menger property, and hence there exists a sequence $\langle \mathcal{V}_{n,k} : n \in I_k \rangle$ such that $\mathcal{V}_{n,k} \in [\mathcal{W}_{n,k}]^{<\omega}$ and $\bigcup_{n \in I_k} \mathcal{V}_{n,k}$ is an ω -cover of $X \setminus B_k$ (see [2] or [10, Theorem 3.9]). Since $\{X \setminus B_k : k \in \omega\}$ is an ω -cover of X , we see that $\mathcal{V} = \bigcup_{k \in \omega, n \in I_k} \mathcal{V}_{n,k}$ is an ω -cover of X . Each element of $\mathcal{V}_{n,k}$ is included in some element of \mathcal{U}_n , and hence \mathcal{V} is not groupable.

Finally, let us fix some disjoint finite subsets $A, C \subset X$ and find $k \in \omega$ such that $C \subset B_k$ and $A \cap B_k = \emptyset$. Since $\bigcup_{n \in I_k} \mathcal{V}_{n,k}$ is an ω -cover of $X \setminus B_k$, there are infinitely many elements of $\bigcup_{n \in I_k} \mathcal{V}_{n,k}$ which contain A . By the construction, all of them are disjoint from C . ■

We are now in a position to complete the proof of Theorem 1.

In light of Lemma 5 it is enough to construct a non-meager filter \mathcal{F} and a countable dense $\mathcal{D} \subset \mathcal{F}$ such that $\mathcal{F} \setminus \mathcal{D}$ has the Menger property. Let X and \mathcal{V} be as in Lemma 6. Let us fix a bijective enumeration $\{V_n : n \in \omega\}$ of \mathcal{V} and for every $m \in \omega$ consider the mapping $f_m : X^m \rightarrow \mathcal{P}(\omega)$,

$$f_m \langle x_0, \dots, x_{m-1} \rangle = \{n : \{x_0, \dots, x_{m-1}\} \subset V_n\}.$$

Since \mathcal{V} is an ω -cover of X we have $f_m[X^m] \subset [\omega]^\omega$ for all m . Moreover, it is easy to see that $\mathcal{Y} = \bigcup_{m \in \omega} f_m[X^m]$ is closed under finite intersections of its elements and hence it generates the filter $\mathcal{F} = \{F \subset \omega : Y \subset^* F \text{ for some } Y \in \mathcal{Y}\}$. We claim that \mathcal{F} is as required. Indeed, by Talagrand and Jalali-Naini's characterization [4, Proposition 9.4] the non-meagerness of \mathcal{F} is a direct consequence of \mathcal{V} not being groupable.

Let us write \mathcal{F} in the form $\bigcup_{m \in \omega} \mathcal{F}_m$, where $\mathcal{F}_m = \{F \subset \omega : Y \subset^* F \text{ for some } Y \in f_m[X^m]\}$. Consider the map $g_m : X^m \times \mathcal{P}(\omega) \times \omega \rightarrow \mathcal{P}(\omega)$, $g_m(a, b, c) = (f_m(a) \setminus c) \cup b$. It is easy to see that g_m is continuous and

$\mathcal{F}_m = g_m[X^m \times \mathcal{P}(\omega) \times \omega]$. Since the Menger property is preserved under products with σ -compact spaces and continuous images, we conclude that \mathcal{F}_m has the Menger property for all $m \in \omega$.

Now let us fix any injective sequence $\langle x_m : m \in \omega \rangle$ of elements of X and set $F_m = f_m \langle x_0, \dots, x_{m-1} \rangle$ for all $m > 0$. Set also $F_0 = \omega$. The sequence $\langle F_m : m \in \omega \rangle \in \mathcal{F}^\omega$ has the property that $\{F_m : m \in \omega\} \cap \mathcal{F}_k = \{F_m : m \in k + 1\}$ for every $k \in \omega$. Indeed, otherwise there exists $\langle x'_0, \dots, x'_{k-1} \rangle \in X^k$ such that $\phi_k \langle x'_0, \dots, x'_{k-1} \rangle \subset^* F_{k+1}$ (because the sequence $\langle F_m : m \in \omega \rangle$ is decreasing). Since the sequence $\langle x_m : m \in \omega \rangle$ is injective, there exists $j \leq k$ such that $x_j \notin \{x'_0, \dots, x'_{k-1}\}$, and hence by our choice of \mathcal{V} there exist infinitely many $n \in \omega$ such that $\{x'_0, \dots, x'_{k-1}\} \subset V_n$ and $x_j \notin V_n$. However, all these n 's are in $\phi_k \langle x'_0, \dots, x'_{k-1} \rangle$ but not in $f_1(x_j) \supset f_{k+1} \langle x_0, \dots, x_k \rangle = F_{k+1}$, a contradiction.

Since each \mathcal{F}_k is closed under finite modifications of its elements, we can conclude that for any sequence $\langle F'_m : m \in \omega \rangle \in \mathcal{F}^\omega$, if $F_m =^* F'_m$ for all m , then $\{F'_m : m \in \omega\} \cap \mathcal{F}_k = \{F'_m : m \in k + 1\}$ for all k .

Let $\{s_m : m \in \omega\}$ be an enumeration of $2^{<\omega}$ and $F'_m = [F_m \setminus (\text{dom}(s_m))] \cup s_m^{-1}(1)$. It is clear that $\mathcal{D} = \{F'_m : m \in \omega\}$ is dense in \mathcal{F} . It follows from the above that

$$\mathcal{F} \setminus \mathcal{D} = \bigcup_{k \in \omega} \mathcal{F}_k \setminus \mathcal{D} = \bigcup_{k \in \omega} (\mathcal{F}_k \setminus \{F'_m : m \in k + 1\}).$$

Since each \mathcal{F}_k has the Menger property, by Corollary 3 we deduce that $\mathcal{F} \setminus \mathcal{D}$ is a countable union of its subspaces with the Menger property, and hence itself has the Menger property. This completes the proof of Theorem 1. ■

A space X is called *strongly locally homogeneous* if it has an open base \mathcal{B} such that, for each $U \in \mathcal{B}$ and points $x, y \in U$, there exists a homeomorphism $h : X \rightarrow X$ with $h(x) = y$ and $h|(X \setminus U)$ equal to the identity. It has been shown in [1] that every strongly locally homogeneous Polish space is CDH. This result is no more true for Baire spaces, even in the realm of separable metrizable spaces: for every $n \in \omega \cup \{\infty\}$ there exists an n -dimensional Baire space which is strongly locally homogeneous but not CDH (see [13, Remark 4.1]).

However, spaces constructed in [13] are not topological groups (see Theorem 3.5 there). Thus the filter \mathcal{F} constructed in the proof of Theorem 1 seems to be the first example of a metrizable separable Baire topological group which is strongly locally homogeneous but not CDH. It might also be worth noticing that it cannot be made CDH by products with metrizable compact spaces.

PROPOSITION 7. *Let \mathcal{F} be the filter constructed in the proof of Theorem 1. Then $\mathcal{F} \times Y$ is not CDH for any metrizable compact Y .*

Proof. We shall use the notation from the proof of Theorem 1. Let $\langle y_m : m \in \omega \rangle$ be a sequence of elements of Y such that $\mathcal{D}_p := \{ \langle F'_m, y_m \rangle : m \in \omega \}$ is dense in $\mathcal{F} \times Y$. Since $\mathcal{F}_k \cap \{ F'_m : m \in \omega \}$ is finite for all $k \in \omega$, we see that $(\mathcal{F}_k \times Y) \cap \mathcal{D}_p$ is finite for all $k \in \omega$. Since \mathcal{F}_k is Menger, so is $\mathcal{F}_k \times Y$, and hence $(\mathcal{F}_k \times Y) \setminus \mathcal{D}_p$ is Menger as well. Therefore $(\mathcal{F} \times Y) \setminus \mathcal{D}_p = \bigcup_{k \in \omega} (\mathcal{F}_k \times Y) \setminus \mathcal{D}_p$ is also Menger. Thus $\mathcal{F} \times Y$ has a countable dense subset with Menger complement.

On the other hand, the same argument as in the proof of Lemma 5 implies that $\mathcal{F} \times Y$ has a countable dense subset whose complement in $\mathcal{F} \times Y$ is not Menger. ■

We call a filter \mathcal{F} on ω a P^+ -filter if for every sequence $\langle A_n : n \in \omega \rangle$ of elements of $\mathcal{F}^+ = \{ X \subset \omega : \forall F \in \mathcal{F} (X \cap F \neq \emptyset) \}$ there is a sequence $\langle B_n : n \in \omega \rangle$ such that $B_n \in [A_n]^{<\omega}$ and $\bigcup_{n \in \omega} B_n \in \mathcal{F}^+$. Replacing \mathcal{F}^+ with \mathcal{F} in the definition above we get the classical notion of a P -filter. Every filter with the Menger property is a P^+ -filter. Indeed, the Menger property applied to the collection $\{ \mathcal{U}_A : A \in \mathcal{F}^+ \}$ of open covers of \mathcal{F} , where $\mathcal{U}_A = \{ \{ X \subset \omega : n \in X \} : n \in A \}$, gives nothing other than the definition of P^+ -filters. If \mathcal{F} is an ultrafilter then $\mathcal{F}^+ = \mathcal{F}$ and hence \mathcal{F} is a P -filter if and only if it is a P^+ -filter. Since the non-meager non-CDH filter constructed in the proof of Theorem 1 has the Menger property, we get the following

COROLLARY 8. *There exists a non-meager P^+ -filter which is not CDH.*

Corollary 8 follows directly from Theorem 1, and hence its proof does not require anything beyond ZFC. Corollary 8 implies that one of the main results of [7], which states that non-meager P -filters are CDH, is sharp in the sense that it cannot be extended to P^+ -filters, even under additional set-theoretic assumptions.

Following [3] we call filters \mathcal{F}_0 and \mathcal{F}_1 on ω *coherent* if there exists a monotone surjection $\psi : \omega \rightarrow \omega$ such that $\psi[\mathcal{F}_0] = \psi[\mathcal{F}_1]$. It is easy to see that coherence is an equivalence relation. It has been shown in [5] that, in the model constructed by Miller in [14], any two non-meager filters are coherent and there exists a P -point, i.e., an ultrafilter which is a P -filter. Together with Theorem 1 this proves the following

COROLLARY 9. *In the Miller model the collection of all CDH filters is not closed under the coherence relation.*

We do not know whether Corollary 9 is true in ZFC. Let us note that a ZFC proof of it would require at least a construction of a CDH filter without any additional set-theoretic assumption, which seems to be quite a difficult task.

We recall that \mathfrak{d} is, by definition, the minimal cardinality of a cover of ω^ω by its compact subspaces, and \mathfrak{u} is the minimal cardinality of a base of an ultrafilter on ω . We refer the reader to [4] for more information on \mathfrak{u} , \mathfrak{d} , and other cardinal characteristics of the continuum.

REMARK. In the proof of Theorem 1 we had to be rather careful with the choice of \mathcal{V} . This is because the same argument would not work if we started with \mathcal{V} which satisfies all requirements of Lemma 6 except for the last one, i.e., does not allow one to sufficiently distinguish between disjoint finite subsets of X . Indeed, assume $\mathfrak{u} < \mathfrak{d}$, which holds, e.g., in the aforementioned Miller model, and let \mathcal{U} be an ultrafilter with \mathfrak{u} -many generators. Then \mathcal{U} fails to have the Hurewicz property by [10, Theorem 4.3], being a non-meager subset of $[\omega]^\omega$. On the other hand, by [10, Theorem 4.4] all finite powers of \mathcal{U} have the Menger property because \mathcal{U} is a union of fewer than \mathfrak{d} compact spaces. Now set $\mathcal{V} = \{V_n : n \in \omega\}$, where $V_n = \{U \in \mathcal{U} : n \in U\}$. It is easy to check that \mathcal{V} is an ω -cover of \mathcal{U} which fails to be groupable. Letting $X = \mathcal{U}$ and defining ϕ_m 's and \mathcal{F} in the same way as in the proof of Theorem 1, one can easily check that $\mathcal{F} = \mathcal{U}$. However, \mathcal{U} is a P -point (see, e.g., [4, Theorem 9.25]). Therefore \mathcal{U} is CDH (see [7]) and hence by Lemma 5 it is impossible to select a countable dense $\mathcal{D} \subset \mathcal{U}$ such that $\mathcal{U} \setminus \mathcal{D}$ has the Menger property. ■

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