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Bad Wadge-like reducibilities on the Baire space

by

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Abstract. We consider various collections of functions from the Baire space ${}^{\omega}\omega$ into itself naturally arising in (effective) descriptive set theory and general topology, including computable (equivalently, recursive) functions, contraction mappings, and functions which are nonexpansive or Lipschitz with respect to suitable complete ultrametrics on ${}^{\omega}\omega$ (compatible with its standard topology). We analyze the degree-structures induced by such sets of functions when used as reducibility notions between subsets of ${}^{\omega}\omega$, and we show that the resulting hierarchies of degrees are much more complicated than the classical Wadge hierarchy; in particular, they always contain large infinite antichains, and in most cases also infinite descending chains.

1. Introduction. We work in $\mathsf{ZF} + \mathsf{DC}(\mathbb{R})$, where $\mathsf{DC}(\mathbb{R})$ is the Axiom of Dependent Choice over the reals. Let ${}^{\omega}\omega$ denote the Baire space of ω-sequences of natural numbers (endowed with the product of the discrete topology on ω). Given a set of functions $\mathcal{F}$ from ${}^{\omega}\omega$ into itself and $A, B \subseteq {}^{\omega}\omega$, we set

$$A \leq_{\mathcal{F}} B \iff \text{there is some } f \in \mathcal{F} \text{ which reduces } A \text{ to } B,$$

where a function $f \colon {}^{\omega}\omega \to {}^{\omega}\omega$ *reduces A to B* if $A = f^{-1}(B)$. When $A \leq_{\mathcal{F}} B$, we say that A is *$\mathcal{F}$-reducible to B*. We also set $A \equiv_{\mathcal{F}} B \Leftrightarrow A \leq_{\mathcal{F}} B \leq_{\mathcal{F}} A$ and $A <_{\mathcal{F}} B \Leftrightarrow A \leq_{\mathcal{F}} B \wedge B \nleq_{\mathcal{F}} A$. If $\leq_{\mathcal{F}}$ is a preorder [1] (which is the case if e.g. $\mathcal{F}$ contains the identity function and is closed under composition), then $\equiv_{\mathcal{F}}$ is an equivalence relation, and hence we can consider the *$\mathcal{F}$-degree* of a set $A \subseteq {}^{\omega}\omega$ defined by

$$[A]_{\mathcal{F}} = \{B \subseteq {}^{\omega}\omega \mid B \equiv_{\mathcal{F}} A\}.$$

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[1] A binary relation is called a *preorder* if it is reflexive and transitive.

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A set $A \subseteq {}^\omega\omega$ (equivalently, its $\mathcal{F}$-degree $[A]_\mathcal{F}$) is called *$\mathcal{F}$-selfdual* (respectively, *$\mathcal{F}$-nonselfdual*) if $A \leq_\mathcal{F} {}^\omega\omega \setminus A$ (respectively, $A \nleq_\mathcal{F} {}^\omega\omega \setminus A$). The collection of all $\mathcal{F}$-degrees will be denoted by $\mathsf{Deg}(\mathcal{F})$, and for every $\Gamma \subseteq \mathscr{P}({}^\omega\omega)$ closed under $\equiv_\mathcal{F}$ we will denote by $\mathsf{Deg}_\Gamma(\mathcal{F})$ the collection of all $\mathcal{F}$-degrees of sets in Γ. The preorder $\leq_\mathcal{F}$ canonically induces the partial order $\leq$ on $\mathsf{Deg}(\mathcal{F})$ defined by

$$[A]_\mathcal{F} \leq [B]_\mathcal{F} \;\Leftrightarrow\; A \leq_\mathcal{F} B,$$

and $(\mathsf{Deg}(\mathcal{F}), \leq)$ (sometimes simply denoted by $\mathsf{Deg}(\mathcal{F})$ again) is called the *structure of $\mathcal{F}$-degrees* or *degree-structure induced by $\mathcal{F}$* or *$\mathcal{F}$-hierarchy* (*of degrees*). Similar terminology and notation will be used when considering the restriction of $\leq$ to $\mathsf{Deg}_\Gamma(\mathcal{F})$ for some Γ as above.

Several preorders of the form $\leq_\mathcal{F}$ have been fruitfully considered in the literature until now, including those where $\mathcal{F}$ is one of the following sets of functions (see Section 2 for the omitted definitions):

(1) the collection L of all *nonexpansive functions* and the collection W of all *continuous functions* (see e.g. the survey paper [VW78] or the more recent [And07]);
(2) the collection Lip of all *Lipschitz functions* and the collection UCont of all *uniformly continuous functions* (see [MR10a]);
(3) the collection Bor of all *Borel(-measurable) functions* (see [AM03]);
(4) for $1 \leq \alpha < \omega_1$, the collection D_α of all *$\mathbf{\Delta}^0_\alpha$-functions*, i.e. those $f \colon {}^\omega\omega \to {}^\omega\omega$ such that $f^{-1}(D) \in \mathbf{\Delta}^0_\alpha$ for all $D \in \mathbf{\Delta}^0_\alpha$ (see [And06] for the case $\alpha = 2$ and [MR09] for arbitrary α's);
(5) for $1 \leq \gamma < \omega_1$ an additively closed ordinal ([2]), the collection $\mathscr{B}_\gamma$ of all functions which are of Baire class $< \gamma$ (see [MR10a]);
(6) for $1 \leq n \in \omega$, the collection of all *$\mathbf{\Sigma}^1_{2n}$-measurable functions* (see [MR10b]).

Assuming the Axiom of Determinacy AD, the degree-structure induced by each of the above sets of functions $\mathcal{F}$ is extremely well-behaved: it is well-founded and almost linear, meaning that antichains have size at most 2 and are in fact *$\mathcal{F}$-nonselfdual pairs*, i.e. they are of the form $\{[A]_\mathcal{F}, [{}^\omega\omega \setminus A]_\mathcal{F}\}$ for some $\mathcal{F}$-nonselfdual $A \subseteq {}^\omega\omega$. Therefore all the above notions of reducibility can be reasonably used as tools for measuring the complexity of subsets of ${}^\omega\omega$.

Notice that, except for the case when $\mathcal{F}$ is the collection of all $\mathbf{\Sigma}^1_{2n}$-measurable functions, the axiom AD is always used only in a local way to determine the degree-structures described above, that is: if $\mathbf{\Gamma} \subseteq \mathscr{P}({}^\omega\omega)$ is closed under continuous preimages (i.e. $\mathbf{\Gamma}$ is a *boldface pointclass*), the

([2]) An ordinal γ is *additively closed* if $\alpha + \beta < \gamma$ for every $\alpha, \beta < \gamma$. This condition is required to ensure that $\mathscr{B}_\gamma$ be closed under composition.

structure $\mathsf{Deg}_{\boldsymbol{\Gamma}}(\mathcal{F}) = (\mathsf{Deg}_{\boldsymbol{\Gamma}}(\mathcal{F}), \leq)$ of $\mathcal{F}$-degrees of sets in $\boldsymbol{\Gamma}$ can be fully determined as soon as we assume the determinacy of games with payoff set in the closure under complements and finite intersections of $\boldsymbol{\Gamma}$. Therefore, if $\boldsymbol{\Gamma}$ is the collection of all Borel sets, we do not need to explicitly assume any determinacy axiom because of Martin's Borel determinacy (see e.g. [Kec95, Theorem 20.5]) (3). Similar considerations will apply to the results of this paper as well, so readers unfamiliar with determinacy axioms may simply restrict their attention to Borel subsets of ${}^{\omega}\omega$ throughout the paper.

In general, all classes of functions $\mathcal{F}$ used as reducibility notions (including the ones above) are required to contain at least all nonexpansive functions—in fact, the condition $\mathcal{F} \supseteq \mathsf{L}$ is part of the definition of the notion of *set of reductions* introduced in [MR09, Definition 1]. Why is it so? On the one hand, this condition already guarantees that the resulting $\mathcal{F}$-hierarchy of degrees is well-behaved (in fact, when $\mathcal{F} \supseteq \mathsf{L}$ only a few characteristics of the induced hierarchy of degrees really depend on the actual $\mathcal{F}$; see Theorem 2.6 below). On the other hand, the received opinion is that if $\mathcal{F}$ lacks such a condition then it is very likely that the resulting structure of degrees will not be well-behaved, i.e. it will contain infinite descending chains and/or infinite antichains (see Definition 2.7). However, besides the trivial example of constant functions briefly considered in [MR09, Section 3], to the best of our knowledge the problem of whether this opinion is correct has been overlooked in the literature: in particular, no "natural" example of an $\mathcal{F}$ inducing an ill-founded hierarchy of degrees (without further set-theoretical assumptions) has been presented so far.

In this note, we fill this gap and confirm the above-mentioned intuition by considering some concrete examples of sets of functions $\mathcal{F} \not\supseteq \mathsf{L}$ which naturally appear in (effective) descriptive set theory and in general topology. In particular, after recalling some basic notation and results in Section 2, in Section 3 we show that when considering the effective counterpart of the W-hierarchy one gets a considerably complicated structure of degrees (Theorem 3.2). In Section 4 we fully describe the hierarchy of degrees induced by the collection c of all *contractions* (see Figure 3), showing in particular that such a hierarchy contains infinite antichains (Corollary 4.8) but no descending chains (Corollary 4.9). The analysis of the c-hierarchy involves a characterization of the selfcontractible subsets of ${}^{\omega}\omega$ (see Definition 2.9 and Corollary 4.4) which may be of independent interest. Finally, in Section 5 we show that the behavior of the classical L-hierarchy heavily relies on the metric chosen: replacing in the definition of L (or of Lip) the "standard" metric d with another complete ultrametric (still compatible with the

(3) In fact, in the very special case of Borel sets one can even just work in second-order arithmetic (discharging our original assumption $\mathsf{ZF} + \mathsf{DC}(\mathbb{R})$) by [LSR88].

topology of ${}^\omega\omega$) may in fact lead to extremely wild hierarchies of degrees (Theorems 5.4 and 5.11).

2. Definitions and preliminaries

Basic notation. The power set of X is denoted by $\mathscr{P}(X)$. The identity function on X is denoted by id_X, with the reference to X dropped when this is not a source of confusion. When $A \subseteq X$, we will write $\neg A$ for $X \setminus A$ whenever the space X is clear from the context. The reals are denoted by $\mathbb{R}$, and we set $\mathbb{R}^+ = \{r \in \mathbb{R} \mid r \geq 0\}$. The set of natural numbers is denoted by ω, and ${}^\omega\omega$ and ${}^{<\omega}\omega$ denote the collections of, respectively, all ω-sequences and all finite sequences of natural numbers. For $s \in {}^{<\omega}\omega$, $\mathrm{lh}(s)$ denotes the length of s, and if $x \in {}^{<\omega}\omega \cup {}^\omega\omega$ then $s^\frown x$ denotes the concatenation of s with x. To simplify the notation, when $s = \langle n \rangle$ for some $n \in \omega$ we will write e.g. $n^\frown x$ in place of the formally more correct $\langle n \rangle^\frown x$ (similar simplifications will be applied also to the other notation below). If $A \subseteq {}^\omega\omega$ and $s \in {}^{<\omega}\omega$, we set $s^\frown A = \{s^\frown x \mid x \in A\}$ and $A_{\lfloor s \rfloor} = \{x \in {}^\omega\omega \mid s^\frown x \in A\}$. Given $A_n \subseteq {}^\omega\omega$ for $n \in \omega$, we set $\bigoplus_{n\in\omega} A_n = \bigcup_{n\in\omega} n^\frown A_n$. When $A, B \subseteq {}^\omega\omega$, we also set $A \oplus B = \bigoplus_{n\in\omega} C_n$, where $C_{2i} = A$ and $C_{2i+1} = B$ for every $i \in \omega$. Given $n, i \in \omega$, the symbol $n^{(i)}$ denotes the unique sequence of length i which is constantly equal to n, and similarly $\vec{n}$ denotes the ω-sequence with constant value n.

The Baire space. When ${}^\omega\omega$ is endowed with the product of the discrete topology on ω, the resulting topological space is called the *Baire space*. It is a zero-dimensional Polish space (i.e. a completely metrizable second-countable topological space admitting a basis of clopen sets). A compatible complete metric $d\colon ({}^\omega\omega)^2 \to \mathbb{R}^+$ for ${}^\omega\omega$ is given by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 2^{-n} & \text{if } x \neq y \text{ and } n \in \omega \text{ is smallest such that } x(n) \neq y(n). \end{cases}$$

In fact, d is an *ultrametric* (that is, $d(x,y) \leq \max\{d(x,z), d(y,z)\}$ for every $x, y, z \in {}^\omega\omega$), and it will be referred to as the *standard metric* on ${}^\omega\omega$.

For $s \in {}^{<\omega}\omega$ we set

$$\mathbf{N}_s = \{x \in {}^\omega\omega \mid s \subseteq x\}.$$

The collection $\{\mathbf{N}_s \mid s \in {}^{<\omega}\omega\}$ is a countable clopen basis for the topology of ${}^\omega\omega$, and in fact it is the collection of all open balls with respect to d.

Classes of functions. Fix a metric space $X = (X, d)$.

DEFINITION 2.1. A function $f\colon X \to X$ is called:

- *Lipschitz with constant* $L \in \mathbb{R}^+$ if $d(f(x), f(y)) \leq L \cdot d(x,y)$ for every $x, y \in X$;

- a *contraction* if it is Lipschitz with constant $L < 1$;
- *nonexpansive* if it is Lipschitz with constant $L \leq 1$;
- *Lipschitz* if it is Lipschitz with constant L for some $L \in \mathbb{R}^+$;
- *uniformly continuous* if for every $\varepsilon > 0$ there is $\delta > 0$ such that $d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$ for every $x, y \in {}^\omega\omega$;
- *continuous* if for every $x \in {}^\omega\omega$ and every $\varepsilon > 0$ there is $\delta > 0$ such that $d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$ for every $y \in {}^\omega\omega$.

The collection of all contractions (respectively, nonexpansive functions, Lipschitz functions, uniformly continuous functions, continuous functions) from the metric space ${}^\omega\omega = ({}^\omega\omega, d)$ into itself will be denoted by c (respectively, L, Lip, UCont, W).

When we want to stress the dependence of the corresponding definitions on the standard metric d, we will write $\mathsf{c}(d)$ (respectively, $\mathsf{L}(d)$, $\mathsf{Lip}(d)$, $\mathsf{UCont}(d)$) in place of c (respectively, L, Lip, UCont) ([4]). In Section 5, we will also consider different metrics d' on ${}^\omega\omega$, and therefore we will denote by $\mathsf{c}(d')$ (respectively, $\mathsf{L}(d')$, $\mathsf{Lip}(d')$, $\mathsf{UCont}(d')$) the class of all contraction (respectively, nonexpansive, Lipschitz, uniformly continuous) mappings from the metric space $({}^\omega\omega, d')$ into itself.

REMARK 2.2. (i) Nonexpansive functions from ${}^\omega\omega$ into itself are often called “Lipschitz functions” in papers dealing with Wadge theory (see e.g. the survey papers [VW78, And07]); this is why their collection is usually denoted by L. However, since in this paper we will also consider the collection Lip of all Lipschitz functions (with arbitrary constant), we had to disambiguate the terminology.

(ii) The class of all continuous functions from ${}^\omega\omega$ into itself is usually denoted by W in honor of W. W. Wadge, who initiated a systematic analysis of the associated reducibility preorder $\leq_{\mathsf{W}}$.

(iii) Clearly we have

$$\mathsf{c} \subsetneq \mathsf{L} \subsetneq \mathsf{Lip} \subsetneq \mathsf{UCont} \subsetneq \mathsf{W},$$

and c is closed under both left and right composition with nonexpansive functions.

All classes of functions $\mathcal{F}$ from ${}^\omega\omega$ into itself considered in Definition 2.1 or in the comment following it are closed under composition, and (except for c and its variants) they contain $\mathrm{id} = \mathrm{id}_{{}^\omega\omega}$. Therefore, all such $\mathcal{F} \neq \mathsf{c}$ induce a reducibility *preorder* $\leq_{\mathcal{F}}$, and consequently we can analyze their induced degree-structures $\mathsf{Deg}(\mathcal{F}) = (\mathsf{Deg}(\mathcal{F}), \leq)$. As for $\mathcal{F} = \mathsf{c}$, the relation $\leq_{\mathsf{c}}$ is transitive but in general not reflexive (see Lemma 2.11). Nevertheless it can

([4]) As is well-known, the collection W does not really depend on d but only on its induced topology.

be naturally extended to a preorder, which will be denoted by $\leq_{\mathsf{c}}$ again, by setting, for $A, B \subseteq {}^{\omega}\omega$,

$$A \leq_{\mathsf{c}} B \;\Leftrightarrow\; \text{either } A = B \text{ or } A = f^{-1}(B) \text{ for some } f \in \mathsf{c}.$$

(Equivalently, $\leq_{\mathsf{c}}$ is the preorder induced by considering as reducing functions those in the collection $\mathcal{F} = \mathsf{c} \cup \{\mathrm{id}\}$; notice that such a set of functions remains closed under composition.) We will see in Lemma 2.11 that all sets $A \subseteq {}^{\omega}\omega$ are c-nonselfdual, i.e. that $A \not\leq_{\mathsf{c}} \neg A$. Notice also that for every $A, A', B, B' \subseteq {}^{\omega}\omega$, if $A = f^{-1}(B)$ for some $f \in \mathsf{c}$, then

$$A' \leq_{\mathsf{L}} A \wedge B \leq_{\mathsf{L}} B' \;\Rightarrow\; A' \leq_{\mathsf{c}} B' \tag{2.1}$$

by Remark 2.2(iii).

Boldface pointclasses. A *boldface pointclass* $\boldsymbol{\Gamma}$ is a nonempty collection of subsets of ${}^{\omega}\omega$ which is closed under continuous preimages, that is, $B \in \boldsymbol{\Gamma}$ whenever $B \leq_{\mathsf{W}} A$ for some $A \in \boldsymbol{\Gamma}$. The *dual* of $\boldsymbol{\Gamma}$ is the boldface pointclass $\check{\boldsymbol{\Gamma}} = \{\neg A \mid A \in \boldsymbol{\Gamma}\}$, and the associated *ambiguous pointclass* is the boldface pointclass $\boldsymbol{\Delta}_{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$. A boldface pointclass $\boldsymbol{\Gamma}$ is *nonselfdual* if $\boldsymbol{\Gamma} \neq \check{\boldsymbol{\Gamma}}$, and *selfdual* otherwise. A set $A \subseteq {}^{\omega}\omega$ is *properly in* $\boldsymbol{\Gamma}$ or is a *proper* $\boldsymbol{\Gamma}$ *set* if $A \in \boldsymbol{\Gamma} \setminus \check{\boldsymbol{\Gamma}}$. Given a boldface pointclass $\boldsymbol{\Gamma}$ and a collection of functions $\mathcal{F}$ from ${}^{\omega}\omega$ into itself, we say that $A \subseteq {}^{\omega}\omega$ is $\mathcal{F}$*-complete for* $\boldsymbol{\Gamma}$ if $A \in \boldsymbol{\Gamma}$ and $B \leq_{\mathcal{F}} A$ for every $B \in \boldsymbol{\Gamma}$. When $\mathcal{F} \subseteq \mathsf{W}$, A is $\mathcal{F}$-complete for $\boldsymbol{\Gamma}$ if and only if $\boldsymbol{\Gamma} = \{B \subseteq {}^{\omega}\omega \mid B \leq_{\mathcal{F}} A\}$; moreover, in this case $\boldsymbol{\Gamma}$ is nonselfdual if and only if A is $\mathcal{F}$-nonselfdual. Examples of nonselfdual boldface pointclasses are the levels $\boldsymbol{\Sigma}^0_{\xi}$ and $\boldsymbol{\Pi}^0_{\xi}$ (for $1 \leq \xi < \omega_1$) of the classical stratification of the Borel subsets of ${}^{\omega}\omega$.

Lipschitz games and determinacy axioms. Given $A, B \subseteq {}^{\omega}\omega$, the so-called *Lipschitz game* $G_{\mathsf{L}}(A, B)$ (with payoff sets A and B) is the two-player zero-sum infinite game in which the two players I and II take turns in playing natural numbers, so that after ω-many turns I will have enumerated a sequence $a \in {}^{\omega}\omega$ and II will have enumerated a sequence $b \in {}^{\omega}\omega$; the winning condition for II is then $a \in A \Leftrightarrow b \in B$.

A *strategy for player* I is simply a function $\sigma \colon {}^{<\omega}\omega \to \omega$, and for every $y \in {}^{\omega}\omega$ we denote by $\sigma * y$ the ω-sequence enumerated by I in a play of G_{L} in which II enumerates y and I follows σ, i.e. $\sigma * y = \langle \sigma(y \restriction n) \mid n \in \omega \rangle$. Similarly, a *strategy for* II is a function $\tau \colon {}^{<\omega}\omega \setminus \{\emptyset\} \to \omega$, and for every $x \in {}^{\omega}\omega$ we set $x * \tau = \langle \tau(x \restriction (n+1)) \mid n \in \omega \rangle$.

Let $A, B \subseteq {}^{\omega}\omega$. Then a *winning strategy for* II *in the game* $G_{\mathsf{L}}(A, B)$ is a strategy τ for II such that $x \in A \Leftrightarrow x * \tau \in B$ for every $x \in {}^{\omega}\omega$, and similarly we can define *winning strategies for* I *in* $G_{\mathsf{L}}(A, B)$. Notice that given $A, B \subseteq {}^{\omega}\omega$, at most one of I and II has a winning strategy in $G_{\mathsf{L}}(A, B)$. We say that $G_{\mathsf{L}}(A, B)$ is *determined* if at least one (and, by the comment above, only one) of the players I and II has a winning strategy in $G_{\mathsf{L}}(A, B)$.

The next folklore result shows the relationship between the Lipschitz game and the reducibility preorders $\leq_{\mathsf{c}}$ and $\leq_{\mathsf{L}}$. We fully reprove it here for the reader's convenience.

PROPOSITION 2.3 (Folklore). *Let $A, B \subseteq {}^{\omega}\omega$.*

(1) $A \leq_{\mathsf{c}} B \Leftrightarrow A = B \vee \mathrm{I}$ *wins* $G_{\mathsf{L}}(\neg B, A)$. *In fact, if* I *wins* $G_{\mathsf{L}}(\neg B, A)$ *then* $A = f^{-1}(B)$ *for some* $f \in \mathsf{c}$.
(2) $A \leq_{\mathsf{L}} B \Leftrightarrow \mathrm{II}$ *wins* $G_{\mathsf{L}}(A, B)$.

Proof. (1) Assume first that $A \leq_{\mathsf{c}} B$ and $A \neq B$, so that there is $f \in \mathsf{c}$ such that $A = f^{-1}(B)$. Notice that since $d(f(x), f(y)) \leq \frac{1}{2} d(x, y)$ for all $x, y \in {}^{\omega}\omega$ (because all nonzero distances used by d are of the form 2^{-n} for some $n \in \omega$), for every $s \in {}^{<\omega}\omega$ there is a unique $t_s \in {}^{<\omega}\omega$ such that $\mathrm{lh}(t_s) = \mathrm{lh}(s) + 1$ and $f(\mathbf{N}_s) \subseteq \mathbf{N}_{t_s}$. Define the strategy σ for I by setting $\sigma(s) = t_s(\mathrm{lh}(s))$ for every $s \in {}^{<\omega}\omega$; it is easy to check that such a strategy is winning in $G_{\mathsf{L}}(\neg B, A)$.

Conversely, if σ is a winning strategy for I, then the map $f \colon {}^{\omega}\omega \to {}^{\omega}\omega$, $y \mapsto \sigma * y$, is easily seen to be a contraction, and $y \in A \Leftrightarrow f(y) \notin \neg B \Leftrightarrow f(y) \in B$, whence $A = f^{-1}(B)$.

(2) The proof is similar to that of (1). If $f \in \mathsf{L}$ witnesses $A \leq_{\mathsf{L}} B$ then for every $s \in {}^{<\omega}\omega$ there is a unique $t_s \in {}^{<\omega}\omega$ such that $\mathrm{lh}(t_s) = \mathrm{lh}(s)$ and $f(\mathbf{N}_s) \subseteq \mathbf{N}_{t_s}$; setting $\tau(s) = t_s(\mathrm{lh}(s) - 1)$ for each $s \in {}^{<\omega}\omega \setminus \{\emptyset\}$, we see that τ is a winning strategy for II in $G_{\mathsf{L}}(A, B)$. Conversely, if τ is a winning strategy for II in $G_{\mathsf{L}}(A, B)$, then the map $x \mapsto x * \tau$ is nonexpansive and witnesses $A \leq_{\mathsf{L}} B$. ∎

Since Lipschitz games can straightforwardly be coded as classical Gale–Stewart games on ω, the full AD implies ([5]) the *Axiom of Determinacy for Lipschitz Games*

$$(\mathsf{AD}^{\mathsf{L}}) \qquad \forall A, B \subseteq {}^{\omega}\omega\ (G_{\mathsf{L}}(A, B) \text{ is determined}).$$

It immediately follows from Proposition 2.3 that AD^{L} is equivalent to the following *Strong Semilinear Ordering Principle for* L:

$$(\mathsf{SSLO}^{\mathsf{L}}) \qquad \forall A, B \subseteq {}^{\omega}\omega\ (A \leq_{\mathsf{L}} B \vee \neg B \leq_{\mathsf{c}} A).$$

In particular, since $\mathsf{c} \subseteq \mathsf{L}$ the principle $\mathsf{SSLO}^{\mathsf{L}}$ implies the so-called *Semilinear Ordering Principle for* L, i.e. the statement

$$(\mathsf{SLO}^{\mathsf{L}}) \qquad \forall A, B \subseteq {}^{\omega}\omega\ (A \leq_{\mathsf{L}} B \vee \neg B \leq_{\mathsf{L}} A).$$

([5]) Actually, it was conjectured by Solovay that if $\mathsf{ZF} + \mathsf{V} = \mathsf{L}(\mathbb{R})$ then $\mathsf{AD}^{\mathsf{L}} \Rightarrow \mathsf{AD}$; this is still a major open problem in this area.

Actually, by [And03, Theorem 1] we find that $\mathsf{SSLO}^{\mathsf{L}}$ is equivalent to $\mathsf{SLO}^{\mathsf{L}}$ when assuming $\mathsf{ZF}+\mathsf{DC}(\mathbb{R})+\mathsf{BP}$, where BP is the statement "every $A \subseteq {}^{\omega}\omega$ has the Baire property".

It is a consequence of $\mathsf{SLO}^{\mathsf{L}}$ that if $\boldsymbol{\Gamma}$ is a nonselfdual boldface pointclass, then every proper $\boldsymbol{\Gamma}$ set is $\leq_{\mathsf{L}}$-complete for $\boldsymbol{\Gamma}$ (the converse is always true). Using Proposition 2.3, this can be strengthened by replacing nonexpansive functions with contractions.

Corollary 2.4. (AD^{L}) *Let $\boldsymbol{\Gamma}$ be a nonselfdual boldface pointclass. Then for every proper $\boldsymbol{\Gamma}$ set A and every $B \in \boldsymbol{\Gamma}$ we have $B = f^{-1}(A)$ for some* contraction f. *In particular, a set $A \subseteq {}^{\omega}\omega$ is $\leq_{\mathsf{c}}$-complete for $\boldsymbol{\Gamma}$ if and only if $A \in \boldsymbol{\Gamma} \setminus \check{\boldsymbol{\Gamma}}$.*

Proof. Let $A, B \subseteq {}^{\omega}\omega$ be distinct sets in $\boldsymbol{\Gamma}$. If there is no contraction f such that $B = f^{-1}(A)$, then $A \leq_{\mathsf{L}} \neg B$ by $\mathsf{SSLO}^{\mathsf{L}}$ (which is equivalent to AD^{L}). Since nonexpansive functions are continuous and $\check{\boldsymbol{\Gamma}}$ is a boldface pointclass, this implies that A is not a proper $\boldsymbol{\Gamma}$ set. ∎

Classical degree-hierarchies. The L-hierarchy and the W-hierarchy are the prototypes for the degree-structures induced by each of the $\mathcal{F}$'s mentioned in the introduction. They can be described as follows (for a full proof of Theorem 2.5, see e.g. [And07]).

Theorem 2.5. ($\mathsf{AD}^{\mathsf{L}}+\mathsf{BP}$)

(1) $\leq_{\mathsf{L}}$ (*and hence also the induced partial order on* L-*degrees*) *is well-founded;*
(2) $\mathsf{SLO}^{\mathsf{L}}$ *holds, and thus each level of the* L-*hierarchy contains either a single* L-*selfdual degree or an* L-*nonselfdual pair;*
(3) *at the bottom of the* L-*hierarchy there is the* L-*nonselfdual pair consisting of* $[{}^{\omega}\omega]_{\mathsf{L}} = \{{}^{\omega}\omega\}$ *and* $[\emptyset]_{\mathsf{L}} = \{\emptyset\}$*;*
(4) *successor levels and limit levels of countable cofinality are occupied by a single* L-*selfdual degree;*
(5) *at limit levels of uncountable cofinality there is an* L-*nonselfdual pair.*

Therefore in the L-hierarchy we have an alternation of L-nonselfdual pairs with ω_1-blocks of consecutive L-selfdual degrees, with L-selfdual degrees (followed by an ω_1-block as above) at limit levels of countable cofinality and L-nonselfdual pairs at limit levels of uncountable cofinality (see Figure 1).

By the Steel–Van Wesep theorem [VW78, Theorem 3.1], under $\mathsf{AD}^{\mathsf{L}}+\mathsf{BP}$ the W-hierarchy is obtained from the L-hierarchy by gluing together each ω_1-block of consecutive L-selfdual degrees into a single W-selfdual degree, so that: $\leq_{\mathsf{W}}$ is well-founded, at each level of the W-hierarchy there is either a single W-selfdual degree or a W-nonselfdual pair, W-selfdual degrees coincide exactly with the collapses of maximal ω_1-blocks of consecutive L-selfdual degrees, and W-nonselfdual pairs coincide exactly with L-nonselfdual pairs.

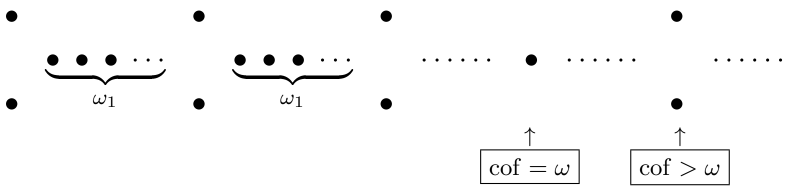

Fig. 1. The L-hierarchy (bullets represent L-degrees)

It follows that W-nonselfdual pairs alternate with single W-selfdual degrees, with W-selfdual degrees (followed by a W-nonselfdual pair) at limit levels of countable cofinality and W-nonseldfual pairs at limit levels of uncountable cofinality. The first W-selfdual degree consists of all nontrivial clopen sets, while the first nontrivial W-nonselfdual pair consists of all proper open and proper closed sets (Figure 2).

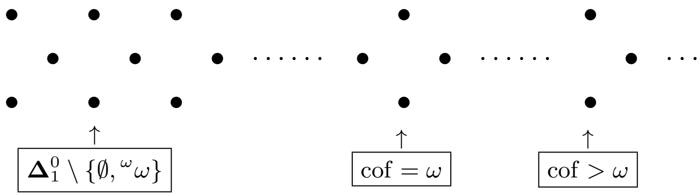

Fig. 2. The W-hierarchy (bullets represent W-degrees)

Gathering various easy observations, it is possible to show that every collection of functions $\mathcal{F}$ closed under composition and containing id induces a degree-structure very close to the L-hierarchy and the W-hierarchy as long as $\mathcal{F} \supseteq \mathsf{L}$; in fact, the next theorem ([6]) leaves open only the problem of determining what happens after an $\mathcal{F}$-selfdual degree, and what happens at limit levels of uncountable cofinality (for a proof of Theorem 2.6, see [MR09, Theorem 3.1]).

THEOREM 2.6. ($\mathsf{AD}^{\mathsf{L}} + \mathsf{BP}$) *Let $\mathcal{F}$ be a set of functions from ${}^\omega\omega$ into itself which is closed under composition and contains* id. *If $\mathcal{F} \supseteq \mathsf{L}$, then*

(1) $\leq_{\mathcal{F}}$ (*and hence also the partial order induced on the $\mathcal{F}$-degrees*) *is well-founded;*

(2) *the* Semilinear Ordering Principle for $\mathcal{F}$*:*

$$(\mathsf{SLO}^{\mathcal{F}}) \qquad \forall A, B \subseteq {}^\omega\omega\, (A \leq_{\mathcal{F}} B \vee \neg B \leq_{\mathcal{F}} A)$$

is satisfied, and thus each level of the $\mathcal{F}$-hierarchy contains either a single $\mathcal{F}$-selfdual degree or an $\mathcal{F}$-nonselfdual pair;

([6]) In this generality, Theorem 2.6 first appeared in [MR09]. However, restricted forms of it (in which only the so-called *amenable* collections of functions $\mathcal{F}$ were considered) already appeared in [AM03, And06].

(3) *the first level of the $\mathcal{F}$-hierarchy is occupied by the $\mathcal{F}$-nonselfdual pair consisting of $[{}^{\omega}\omega]_{\mathcal{F}} = \{{}^{\omega}\omega\}$ and $[\emptyset]_{\mathcal{F}} = \{\emptyset\}$;*
(4) *after an $\mathcal{F}$-nonselfdual pair and at limit levels of countable cofinality there is always a single $\mathcal{F}$-selfdual degree;*
(5) *if $A \subseteq {}^{\omega}\omega$ is $\mathcal{F}$-nonselfdual, then $[A]_{\mathcal{F}} = [A]_{\mathsf{L}}$ (in particular, A is also L-nonselfdual).*

As recalled above, all the reducibilities $\mathcal{F}$ mentioned in the introduction satisfy the condition $\mathcal{F} \supseteq \mathsf{L}$, and in fact, except for the case of the $\mathbf{\Sigma}^1_{2n}$-measurable functions considered in [MR10b], under suitable determinacy assumptions their induced degree-structure is isomorphic either to the L-hierarchy or to the W-hierarchy. More precisely:

(1) $(\mathsf{Deg}(\mathsf{Lip}), \leq)$, $(\mathsf{Deg}(\mathsf{UCont}), \leq)$, and $(\mathsf{Deg}(\mathscr{B}_\gamma), \leq)$ (for $1 \leq \gamma < \omega_1$ an additively closed ordinal) are all isomorphic to the L-hierarchy (Figure 1). Moreover, $(\mathsf{Deg}(\mathsf{Lip}), \leq) = (\mathsf{Deg}(\mathsf{UCont}), \leq)$.
(2) $(\mathsf{Deg}(\mathsf{Bor}), \leq)$ and $(\mathsf{Deg}(\mathsf{D}_\alpha), \leq)$ (for every $1 \leq \alpha < \omega_1$) are all isomorphic to the W-hierarchy (Figure 2).

A classification of the $\mathcal{F}$-hierarchies of degrees. In [MRSS12] a rough classification of $\mathcal{F}$-hierarchies was proposed according to whether they provide an acceptable measure of "complexity" for subsets of ${}^{\omega}\omega$ ([7]).

DEFINITION 2.7. Let $\mathcal{F}$ be a collection of functions from ${}^{\omega}\omega$ to itself which is closed under composition and contains id. The structure of the $\mathcal{F}$-degrees is called:

- *very good* if it is semi-well-ordered, i.e. it is well-founded and $\mathsf{SLO}^{\mathcal{F}}$ holds ([8]);
- *good* if it is a well-quasi-order, i.e. it contains neither infinite descending chains nor infinite antichains;
- *bad* if it contains infinite antichains;
- *very bad* if it contains both infinite descending chains and infinite antichains.

According to this classification, under $\mathsf{AD}^{\mathsf{L}} + \mathsf{BP}$ all the $\mathcal{F}$-hierarchies considered above are very good, and in fact by Theorem 2.6 we find that $\mathcal{F} \supseteq \mathsf{L}$ is a sufficient condition for the structure of the $\mathcal{F}$-degrees to be very good. Albeit this is literally not a necessary condition (see the discussion

([7]) The two guiding principles for such a classification are the following: (1) the $\mathcal{F}$-hierarchy must be at least *well-founded*, so that one can associate a rank function to it which measures how much complicated is a given $\mathcal{F}$-degree, and (2) the shorter are the antichains, the better is the classification given by the $\mathcal{F}$-hierarchy (this is because it is arguably preferable to have as few as possible distinct $\mathcal{F}$-degrees on each of the levels).

([8]) Of course when we are interested in the restriction of the $\mathcal{F}$-hierarchy to some $\Gamma \subseteq \mathscr{P}({}^{\omega}\omega)$, then we just require that $\mathsf{SLO}^{\mathcal{F}}$ holds for $A, B \in \Gamma$.

in Section 6), in Sections 3–5 we will show that in many relevant cases if $\mathcal{F} \not\supseteq \mathsf{L}$ then one gets a bad degree-structure, or even a very bad one.

REMARK 2.8. Bad and very bad hierarchies of degrees have been considered in several papers [Her93, Her96, MRSS12, IST13, Sch13]. However, all these examples were obtained by considering Wadge-like reducibilites on topological spaces different from ${}^{\omega}\omega$. To the best of our knowledge, the ones reported in the present paper are the first "natural" examples of hierarchies of degrees *defined on the classical Baire space* which can be proven to be (very) bad, without any further set-theoretical assumption beyond our basic theory $\mathsf{ZF} + \mathsf{DC}(\mathbb{R})$.

Selfcontractible sets. Fix a metric space $X = (X, d)$.

DEFINITION 2.9. A set $A \subseteq X$ is called *selfcontractible* if there is a contraction $f \colon X \to X$ such that $f^{-1}(A) = A$.

Notice that, in particular, if $A \subseteq {}^{\omega}\omega$ is selfcontractible and $B \in [A]_{\mathsf{L}}$, then B is selfcontractible as well by Remark 2.2(iii).

REMARK 2.10. In Definition 2.9 we could further require that the Lipschitz constant of f be bounded by some $0 < r < 1$. More precisely, given $0 < r < 1$ we could say that a set $A \subseteq X$ is *r-selfcontractible* if $A = f^{-1}(A)$ for some $f \colon X \to X$ such that $d(f(x), f(y)) \leq r \cdot d(x, y)$ for every $x, y \in X$. However, it is easy to check that $A \subseteq X$ is selfcontractible if and only if it is r-selfcontractible *for some* $0 < r < 1$, if and only if it is r-selfcontractible *for all* $0 < r < 1$. (For the nontrivial direction, notice that if f witnesses that $A \subseteq X$ is selfcontractible, then for every $0 < r < 1$ there is $n(r) \in \omega$ large enough so that $f^{n(r)} = \underbrace{f \circ \cdots \circ f}_{n(r)}$ witnesses that A is r-selfcontractible.)

By the Banach fixed-point theorem, if (X, d) is a nonempty complete metric space and $f \colon X \to X$ is a contraction, then there is a (unique) fixed point $x_f \in X$ for f. From this classical result and (2.1), it easily follows that:

LEMMA 2.11. *Let $X = (X, d)$ be a complete metric space. For every $A \subseteq X$ there is no contraction $f \colon X \to X$ such that $f^{-1}(\neg A) = A$. In particular, all sets $A \subseteq {}^{\omega}\omega$ are c-nonselfdual (that is, $A \not\leq_{\mathsf{c}} \neg A$), and if A is L-selfdual then A is not selfcontractible.*

Lemma 2.11 shows that L-nonselfduality is a necessary condition for $A \subseteq {}^{\omega}\omega$ to be selfcontractible; in Corollary 4.4 we will obtain a full characterization of selfcontractible subsets of ${}^{\omega}\omega$ by showing that such a condition is also sufficient (under suitable determinacy assumptions).

The next simple observation will not be used for the main results of this paper, but it may be interesting as it shows that each selfcontractible set can be shrunk to arbitrarily small subsets which maintain the same topological complexity.

PROPOSITION 2.12. *Let $X = (X, d)$ be a complete metric space, $A \subseteq X$ be selfcontractible, and f be a witness of this fact. Then for every open neighborhood U of x_f there is a contraction $g\colon X \to X$ such that $A = g^{-1}(A \cap U)$. Moreover, if U is clopen and $A \neq X$, then $A \equiv_{\mathsf{W}} A \cap U$.*

Proof. It is enough to notice that for every $\varepsilon > 0$ there is $n(\varepsilon) \in \omega$ such that the range of the $n(\varepsilon)$-told composition $f^{n(\varepsilon)}$ has diameter $< \varepsilon$. Since x_f, being the fixed point of f, is always in that range, we deduce that if ε is such that $B(x_f, \varepsilon) = \{x \in X \mid d(x_f, x) < \varepsilon\} \subseteq U$, the range of $f^{n(\varepsilon)}$ is totally contained in $B(x_f, \varepsilon)$, and hence also in U. Finally, $f^{n(\varepsilon)}$ is clearly a contraction and $(f^{n(\varepsilon)})^{-1}(A) = A$ because $f^{-1}(A) = A$ by assumption.

For the last part, since contractions are continuous functions we just need to show $A \cap U \leq_{\mathsf{W}} A$; but it is easy to see that this is witnessed by the continuous function $(\mathrm{id}_X \restriction U) \cup f_{\bar{y}}$, where $f_{\bar{y}}$ is the constant function with value $\bar{y} \in \neg A$. ∎

3. Computable functions. Throughout this section, we assume a certain familiarity with the basic concepts and terminology of recursion theory and effective descriptive set theory, in particular with the notions of recursive/recursively enumerable subset of ω (and its Cartesian products), and with the Kleene pointclasses Σ^0_n and Σ^1_n (for $n \in \omega$). A good reference for these topics, containing all necessary definitions, is [Mos80].

As explained in [Kec95, Proposition 2.6], every continuous function from ${}^{\omega}\omega$ into itself can be represented by a monotone and length-increasing $\varphi\colon {}^{<\omega}\omega \to {}^{<\omega}\omega$. More precisely, $f\colon {}^{\omega}\omega \to {}^{\omega}\omega$ is continuous if and only if there is $\varphi\colon {}^{<\omega}\omega \to {}^{<\omega}\omega$ such that

(a) $s \subseteq t \Rightarrow \varphi(s) \subseteq \varphi(t)$ for all $s, t \in {}^{<\omega}\omega$;
(b) $\lim_{n \to \infty} \mathrm{lh}(\varphi(x \restriction n)) = \infty$ for all $x \in {}^{\omega}\omega$;
(c) $f(x) = \bigcup_{n \in \omega} \varphi(x \restriction n)$ for all $x \in {}^{\omega}\omega$.

When (a)–(c) above are satisfied by some φ, we say that φ is an *approximating function for f*.

If we require that a φ as above be computable, then f itself may be dubbed computable. To be more precise, let $\mathfrak{G}\colon {}^{<\omega}\omega \to \omega$ be the Gödel bijection, and call a function $\varphi\colon {}^{<\omega}\omega \to {}^{<\omega}\omega$ *computable* if $\mathfrak{G} \circ \varphi \circ \mathfrak{G}^{-1}\colon \omega \to \omega$ is computable. Then we may introduce the following definition.

DEFINITION 3.1. A function $f\colon {}^{\omega}\omega \to {}^{\omega}\omega$ is *computable* if there is a computable approximating function φ for f.

It is easy to check that Definition 3.1 is actually equivalent to the definition of a *recursive function* given in [Mos80, Section 3D] ([9]).

Since $\mathrm{id} = \mathrm{id}_{^\omega\omega}$ is computable and the composition of computable functions is computable, setting $\mathsf{Comp} = \{f : {}^\omega\omega \to {}^\omega\omega \mid f \text{ is computable}\}$ we find that $\leq_{\mathsf{Comp}}$ is a preorder, and thus it induces a degree-structure on $\mathscr{P}({}^\omega\omega)$. Of course, a meaningful use of that preorder should be confined to subsets of ${}^\omega\omega$ which can be defined in a "recursive fashion", e.g. to the Kleene pointclasses Σ^0_n and Σ^1_n. However, we are now going to show that even when restricted to Π^0_1 or to Σ^0_2, the preorder $\leq_{\mathsf{Comp}}$ induces quite a complicated hierarchy of degrees. This will provide a first example of a widely considered ([10]) and reasonably complex class of functions lacking the crucial condition ([11]) $\mathcal{F} \supseteq \mathsf{L}$ and whose induced degree-structure is very bad. In particular, Theorem 3.2 gives a precise mathematical formulation to the common opinion that the effective counterpart of the Wadge hierarchy cannot be used as a tool for getting a reasonable classification of subsets of ${}^\omega\omega$.

THEOREM 3.2.

(1) *The structure of* recursive *subsets of* ω *under inclusion can be embedded into* $\mathsf{Deg}_{\Pi^0_1}(\mathsf{Comp})$.
(2) *The structure of* recursively enumerable *subsets of* ω *under inclusion can be embedded into* $\mathsf{Deg}_{\Sigma^0_2}(\mathsf{Comp})$.

Proof. Part (1) is somehow implicit in [FFT10, Theorem 9]. By [FFT10, Theorem 6], there exists a uniform sequence $\langle A_n \mid n \in \omega\rangle$ of nonempty Π^0_1 sets such that for every $n \in \omega$ there is no computable (in fact, no hyperarithmetical) function g such that $g(A_n) \subseteq \bigcup_{m\neq n} A_m$. In particular, none of the A_n's can contain a recursive element $x \in {}^\omega\omega$, as otherwise the constant function with value x would contradict the choice of the A_n's.

Given a recursive $X \subseteq \omega$, set
$$\psi_0(X) = \bigcup_{n\in X} n^\frown A_n.$$
Then $\psi_0(X) \in \Pi^0_1$ because for every $n \in \omega$ both $\mathbf{N}_{\langle n\rangle} = n^\frown{}^\omega\omega$ and $n^\frown(\neg A_n)$ are in Σ^0_1, the sequence of the A_n's is uniform, and under our assumption $\omega \setminus X$ is recursively enumerable. We claim that ψ_0 is the desired embedding.

([9]) A function $f : {}^\omega\omega \to {}^\omega\omega$ is called *recursive* if the set $G^f = \{(x,n) \in {}^\omega\omega\times\omega \mid f(x) \in \mathbf{N}_{\mathfrak{G}^{-1}(n)}\}$ is Σ^0_1, i.e. of the form $\bigcup_{(l,k)\in A} \mathbf{N}_{\mathfrak{G}^{-1}(l)} \times \{k\}$ for some recursively enumerable $A \subseteq \omega\times\omega$.

([10]) For example the class of computable functions is used to define the Weihrauch reducibility and its induced lattice of degrees. These notions are central in computable analysis, and allow us to e.g. classify the computational content of some classical theorems—see e.g. [BGM12] and the references contained therein.

([11]) It is easy to see that Comp does not even contain e.g. constant functions whose unique value is not recursive (as a function from ω into itself).

Let $X, Y \subseteq \omega$ be two recursive sets. If $X \subseteq Y$, then the map $f\colon {}^\omega\omega \to {}^\omega\omega$ defined by

$$f(x) = \begin{cases} \vec{0} & \text{if } x(0) \notin X, \\ x & \text{otherwise,} \end{cases}$$

is computable and clearly reduces $\psi_0(X)$ to $\psi_0(Y)$, since $\vec{0}$, being a recursive point of ${}^\omega\omega$, does not belong to $\psi_0(Y)$.

Conversely, let f witness $\psi_0(X) \leq_{\mathsf{Comp}} \psi_0(Y)$ and assume towards a contradiction that there is $n \in X \setminus Y$. Then since $\psi_0(X) \cap \mathbf{N}_{\langle n \rangle} = n^\frown A_n$, the map $g\colon {}^\omega\omega \to {}^\omega\omega$ defined by

$$g(x) = \langle f(n^\frown x)(k+1) \mid k \in \omega \rangle$$

would be computable and such that $g(A_n) \subseteq \bigcup_{m \in Y} A_m \subseteq \bigcup_{m \neq n} A_m$, contradicting the choice of the A_n's. Therefore $X \subseteq Y$.

(2) We slightly modify the construction of (1). For every recursively enumerable $X \subseteq \omega$, set

$$\psi_1(X) = \bigcup_{n \in X} \bigcup_{k,i \in \omega} n^\frown 0^{(k)\frown}(i+1)^\frown A_n.$$

Then $\psi_1(X)$ is clearly a Σ^0_2 set, and we claim that it is the desired embedding.

Let $X, Y \subseteq \omega$ be recursively enumerable sets, and let T_X be a Turing machine enumerating X. If $X \subseteq Y$, then let $\varphi\colon {}^{<\omega}\omega \to {}^{<\omega}\omega$ be defined by setting $\varphi(\emptyset) = \emptyset$, $\varphi(n^\frown s) = n^\frown 0^{(\mathrm{lh}(s))}$ if n is *not* enumerated by T_X in $\leq \mathrm{lh}(s)$ steps, and $\varphi(n^\frown s) = n^\frown 0^{(k)\frown} s{\restriction}(\mathrm{lh}(s) - k)$ if n is enumerated by T_X in k steps for some $k \leq \mathrm{lh}(s)$ (for every $n \in \omega$ and $s \in {}^{<\omega}\omega$). Then it is easy to check that φ is computable and satisfies conditions (a)–(b) above, so that φ is an approximating function for the computable function $f\colon {}^\omega\omega \to {}^\omega\omega\colon x \mapsto \bigcup_{n \in \omega} \varphi(x{\restriction}n)$. Moreover, $f(n^\frown x) \neq n^\frown \vec{0}$ if and only if $n \in X$ and $x \neq \vec{0}$, and in that case $f(n^\frown x) = n^\frown 0^{(k)\frown} x$ for some $k \in \omega$. This easily implies that f reduces $\psi_1(X)$ to $\psi_1(Y)$ (since we assumed $X \subseteq Y$).

Conversely, let $f \in \mathsf{Comp}$ be a witness of $\psi_1(X) \leq_{\mathsf{Comp}} \psi_1(Y)$, and assume towards a contradiction that there is $n \in X \setminus Y$. Let $\varphi\colon {}^{<\omega}\omega \to {}^{<\omega}\omega$ be a computable approximating function for f, and let $T \subseteq {}^{<\omega}\omega$ be a computable tree with $A_n = [T]$, where $[T] = \{x \in {}^\omega\omega \mid \forall n \in \omega\ (x{\restriction}n \in T)\}$. Define $\varphi'\colon {}^{<\omega}\omega \to {}^{<\omega}\omega$ by setting

(i) $\varphi'(s) = \emptyset$ if $s \in T$ and $\varphi(n^\frown 1^\frown s)$ is of the form $m^\frown 0^{(k)}$ for some $m, k \in \omega$;
(ii) $\varphi'(s) = t$ if $s \in T$ and $\varphi(n^\frown 1^\frown s)$ is of the form $m^\frown 0^{(k)\frown}(i+1)^\frown t$ for some $m, k, i \in \omega$ and $t \in {}^{<\omega}\omega$;
(iii) $\varphi'(s) = \varphi'(s{\restriction}l)^\frown 0^{(\mathrm{lh}(s))}$ if $s \notin T$ and $l < \mathrm{lh}(s)$ is largest such that $s{\restriction}l \in T$.

The map φ' clearly satisfies condition (a) by the fact that T is closed under subsequences and that φ satisfies (a) as well. To see that φ' also satisfies condition (b), notice that if $x \in A_n = [T]$ then $\varphi(x{\restriction}l)$ must be of the form $m^\frown 0^{(k)\frown}(i+1)^\frown t$ for all large enough $l \in \omega$ because $n^\frown 1^\frown x \in \psi_1(X)$ and φ is an approximating function for the reduction f of $\psi_1(X)$ to $\psi_1(Y)$, while if $x \notin A_n$ then for all large enough $l \in \omega$ one has $x{\restriction}l \notin T$, and hence $\mathrm{lh}(\varphi'(x{\restriction}l)) \geq l$. Since φ' is clearly computable, this implies that φ' is an approximating function for the computable map $g\colon {}^\omega\omega \to {}^\omega\omega\colon x \mapsto \bigcup_{i\in\omega} \varphi'(x{\restriction}i)$. Moreover, by the choice of f and φ one easily sees that ([12]) $g(x) \in \bigcup_{m\in Y} A_m \subseteq \bigcup_{m\neq n} A_n$ for every $x \in A_n$, contradicting the choice of the A_n's. Therefore $X \subseteq Y$, as required. ∎

Obviously, Theorem 3.2 can be relativized to any oracle $z \in {}^\omega\omega$. Moreover, using the same methods one can easily see that similar results hold when Comp is replaced with other larger classes of functions which are defined in an "effective way": for example, one can show that the structure of hyperarithmetical subsets of ω under inclusion can be embedded into the degree-structure $\mathsf{Deg}_{\Delta^1_1}(\mathsf{Hyp})$, where Hyp is the collection of all hyperarithmetical functions from ${}^\omega\omega$ into itself.

4. Contractions. Many of the following results will be stated assuming either AD^L or $\mathsf{AD}^\mathsf{L} + \mathsf{BP}$. As recalled in Section 2, both these assumptions are (seemingly weaker) consequences of AD, so the reader unfamiliar with these special determinacy axioms may safely assume the full AD throughout the section. Moreover, we remark that all the above-mentioned determinacy axioms are always used only in a local way (in the sense explained in the introduction); therefore, the restriction of each of the results below to the Borel realm is true *without any further assumption beyond* $\mathsf{ZF}+\mathsf{DC}(\mathbb{R})$—this feature will be tacitly used several times (see e.g. Corollary 4.6).

PROPOSITION 4.1. (AD^L) *Let $A, B \subseteq {}^\omega\omega$. If A and B belong to different* L*-degrees (i.e. $A \not\equiv_\mathsf{L} B$), then*

$$A \leq_\mathsf{c} B \;\Leftrightarrow\; A \leq_\mathsf{L} B.$$

Proof. One implication is obvious because $\mathsf{c} \subseteq \mathsf{L}$. For the other direction, if $A \leq_\mathsf{L} B$ then $\neg B \not\leq_\mathsf{L} A$, since if $\neg B \leq_\mathsf{L} A$ then we would get $\neg B \leq_\mathsf{L} B$, and hence also $A \leq_\mathsf{L} B \leq_\mathsf{L} \neg B \leq_\mathsf{L} A$ (contradicting our assumption $A \not\equiv_\mathsf{L} B$). Therefore I wins $G_\mathsf{L}(\neg B, A)$ by Proposition 2.3(2) and AD^L, whence $A \leq_\mathsf{c} B$ by Proposition 2.3(1). ∎

PROPOSITION 4.2. (AD^L) *Let $A, B \subseteq {}^\omega\omega$ be distinct sets such that $A \equiv_\mathsf{L} B$. Then*

$$A \leq_\mathsf{c} B \;\Leftrightarrow\; A \not\leq_\mathsf{L} \neg A.$$

([12]) In fact, the function g would witness that $A_n \leq_{\mathsf{Comp}} \bigcup_{m\in Z} A_m$ for every $Z \supseteq Y$.

Proof. For the forward direction, assume towards a contradiction that $A \leq_{\mathsf{c}} B$ but $A \leq_{\mathsf{L}} \neg A$. Then $B \equiv_{\mathsf{L}} A \leq_{\mathsf{L}} \neg A$ by assumption, and since $A \neq B$ implies that $A \leq_{\mathsf{c}} B$ can be witnessed by a function in c, we would get $A \leq_{\mathsf{c}} \neg A$ by (2.1), contradicting Lemma 2.11.

If instead $A \not\leq_{\mathsf{L}} \neg A$, then I wins $G_{\mathsf{L}}(\neg A, A)$ by Proposition 2.3(2) and AD^{L}, and therefore A is selfcontractible by Proposition 2.3(1), that is, $A \leq_{\mathsf{c}} A$ can be witnessed by a function in c. Since $A \equiv_{\mathsf{L}} B$, we get $A \leq_{\mathsf{c}} B$ by (2.1) again. ∎

Remark 4.3. Notice that we actually did not use any determinacy axiom to show $A \leq_{\mathsf{c}} B \Rightarrow A \not\leq_{\mathsf{L}} \neg A$ (for A, B distinct subsets of ${}^{\omega}\omega$ such that $A \equiv_{\mathsf{L}} B$). This fact will be used later in Corollary 4.8.

Despite their simplicity, Propositions 4.1 and 4.2 have many interesting consequences. First of all, they provide a characterization of all selfcontractible subsets of ${}^{\omega}\omega$.

Corollary 4.4. (AD^{L}) *For every $A \subseteq {}^{\omega}\omega$, A is selfcontractible if and only if it is L-nonselfdual.*

By the Steel–Van Wesep theorem [VW78, Theorem 3.1], under $\mathsf{AD}^{\mathsf{L}} + \mathsf{BP}$ we know that $A \subseteq {}^{\omega}\omega$ is L-selfdual if and only if it is W-selfdual. Therefore we also get the following variant of Corollary 4.4.

Corollary 4.5. ($\mathsf{AD}^{\mathsf{L}} + \mathsf{BP}$) *For every $A \subseteq {}^{\omega}\omega$, A is selfcontractible if and only if A is W-nonselfdual.*

In particular, all sets lying properly in some level of the Baire stratification of the Borel sets are selfcontractible (in fact, this result can be obtained working in $\mathsf{ZF} + \mathsf{DC}(\mathbb{R})$ alone by Borel determinacy).

Corollary 4.6. *Every proper $\boldsymbol{\Sigma}^0_\xi$ or $\boldsymbol{\Pi}^0_\xi$ subset of ${}^{\omega}\omega$ is selfcontractible.*

Proof. Proper $\boldsymbol{\Sigma}^0_\xi$ (respectively, $\boldsymbol{\Pi}^0_\xi$) sets are always L-nonselfdual (because both $\boldsymbol{\Sigma}^0_\xi$ and $\boldsymbol{\Pi}^0_\xi$ are nonselfdual boldface pointclasses and $\mathsf{L} \subseteq \mathsf{W}$). ∎

Corollary 4.6 can clearly be extended to arbitrary nonselfdual boldface pointclasses $\boldsymbol{\Gamma}$ if sufficiently strong determinacy axioms are assumed. Moreover, by Remark 2.10 one easily sees that Corollaries 4.4–4.6 can be restated using r-contractibility (for an arbitrary $0 < r < 1$) instead of contractibility.

Concerning the structure of the c-degrees, we already observed in Lemma 2.11 that there are no c-selfdual degrees. The next corollary of Proposition 4.2 shows how the c- and the L-degree of a set $A \subseteq {}^{\omega}\omega$ are related to each other with respect to inclusion.

Corollary 4.7. (AD^{L}) *Let $A \subseteq {}^{\omega}\omega$. If A is L-nonselfdual then $[A]_{\mathsf{c}} = [A]_{\mathsf{L}}$, while if A is L-selfdual then $[A]_{\mathsf{c}} = \{A\} \subsetneq [A]_{\mathsf{L}}$.*

Proof. The inclusion $[A]_{\mathsf{c}} \subseteq [A]_{\mathsf{L}}$ (for an arbitrary $A \subseteq {}^\omega\omega$) follows from $\mathsf{c} \subseteq \mathsf{L}$.

Assume first that A is L-nonselfdual and that $B \in [A]_{\mathsf{L}}$ is distinct from A. Then $A \leq_{\mathsf{c}} B$ by Proposition 4.2. Moreover, B is L-nonselfdual as well by $B \equiv_{\mathsf{L}} A$, so switching the roles of A and B we also get $B \leq_{\mathsf{c}} A$, and hence $B \in [A]_{\mathsf{c}}$. This shows that $[A]_{\mathsf{L}} \subseteq [A]_{\mathsf{c}}$, and hence $[A]_{\mathsf{c}} = [A]_{\mathsf{L}}$.

Assume now that A is L-selfdual, and that there is $B \neq A$ such that $B \in [A]_{\mathsf{c}}$. Then $B \equiv_{\mathsf{L}} A$ (since $[A]_{\mathsf{c}} \subseteq [A]_{\mathsf{L}}$), and hence $A \nleq_{\mathsf{c}} B$ by Proposition 4.2 again. ∎

Proposition 4.2 can also be used to show (in $\mathsf{ZF} + \mathsf{DC}(\mathbb{R})$ alone) that the degree-structure induced by c is bad, as it contains very large antichains.

COROLLARY 4.8. *The preorder $\leq_{\mathsf{c}}$ contains antichains of size ${}^\omega 2$, i.e. there is an injection $\psi\colon {}^\omega 2 \to \mathscr{P}({}^\omega\omega)$ such that $\psi(x)$ and $\psi(y)$ are $\leq_{\mathsf{c}}$-incomparable whenever $x \neq y$. In fact, a $\leq_{\mathsf{c}}$-antichain of size ${}^\omega 2$ can be found inside every L-selfdual degree.*

Proof. Set

$$\psi(x) = \mathbf{N}_{\langle 0\rangle} \cup \bigcup\{\mathbf{N}_{\langle n+2\rangle} \mid x(n) = 1\}.$$

Then $\psi\colon {}^\omega 2 \to \mathscr{P}({}^\omega\omega)$ is injective and $\psi(x) \equiv_{\mathsf{L}} \psi(y) \equiv_{\mathsf{L}} \mathbf{N}_{\langle 0\rangle}$ for all $x, y \in {}^\omega 2$. Since $\mathbf{N}_{\langle 0\rangle}$ is clearly L-selfdual, the result follows from (the forward direction of) Proposition 4.2 (together with Remark 4.3).

The additional part is obtained in the same way, using the following general claim.

CLAIM 4.8.1. *For every $A \neq {}^\omega\omega, \emptyset$ there is an injection $\psi\colon {}^\omega 2 \to [A]_{\mathsf{L}}$.*

Proof. For $x \in {}^\omega 2$, set

$$\psi(x) = \bigcup_{n\in\omega} (2n)^\frown A_{\lfloor n\rfloor} \cup \bigcup\{\mathbf{N}_{\langle 2n+1\rangle} \mid x(n) = 1\}.$$

Fix $y_0 \notin A$ and $y_1 \in A$. It is then easy to check that the maps $f, g\colon {}^\omega\omega \to {}^\omega\omega$ defined by

$$f(n^\frown y) = (2n)^\frown y \quad \text{and} \quad g(n^\frown y) = \begin{cases} i^\frown y & \text{if } n = 2i, \\ y_0 & \text{if } n = 2i+1 \text{ and } x(i) = 0, \\ y_1 & \text{if } n = 2i+1 \text{ and } x(i) = 1, \end{cases}$$

witness $A \equiv_{\mathsf{L}} \psi(x)$ for every $x \in {}^\omega\omega$. ∎

This finishes the proof of Corollary 4.8. ∎

On the other hand, assuming sufficiently strong determinacy axioms one can show that the c-hierarchy is not very bad, i.e. that it is at least well-founded.

Corollary 4.9. (AD^{L}) *For every* $A, B \subseteq {}^{\omega}\omega$,

$$A <_{\mathsf{c}} B \;\Leftrightarrow\; A <_{\mathsf{L}} B.$$

In particular, further assuming BP, *the preorder* $\leq_{\mathsf{c}}$ *is well-founded.*

Proof. If $A <_{\mathsf{L}} B$ then $A <_{\mathsf{c}} B$ by Proposition 4.1. Conversely, assume $A <_{\mathsf{c}} B$, so that, in particular, $A \neq B$. Then $A \leq_{\mathsf{L}} B$ because $\mathsf{c} \subseteq \mathsf{L}$. Assume towards a contradiction that $A \equiv_{\mathsf{L}} B$; then $A \not\leq_{\mathsf{L}} \neg A$ by Proposition 4.2 and since $A \leq_{\mathsf{c}} B$, whence $[A]_{\mathsf{L}} = [A]_{\mathsf{c}}$ by Corollary 4.7. But then $B \equiv_{\mathsf{c}} A$, contradicting our choice of A and B.

In particular, every infinite strictly $\leq_{\mathsf{c}}$-decreasing chain is also strictly $\leq_{\mathsf{L}}$-decreasing, and so by Theorem 2.5(1), $\leq_{\mathsf{c}}$ is well-founded. ∎

More generally, combining Corollary 4.7 with Proposition 4.1, we get a full description of the degree-structure induced by c. In fact, the relation $\leq_{\mathsf{c}}$ is simply the refinement of $\leq_{\mathsf{L}}$ in which all sets belonging to the same L-selfdual degree are made pairwise $\leq_{\mathsf{c}}$-incomparable. Therefore the c-hierarchy of degrees is obtained from the L-hierarchy by splitting each L-selfdual degree into the singletons of its elements. Figure 3 summarizes the situation (compare it with Figure 1): bullets represent c-degrees, while the boxes around them represent the L-degrees they come from.

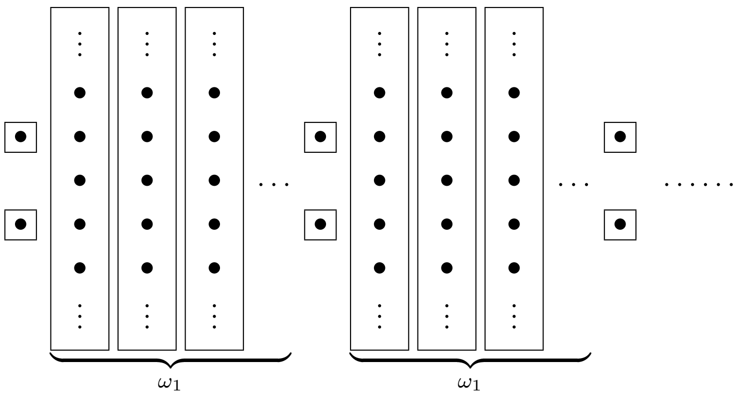

Fig. 3. The c-hierarchy

One may wonder what happens if we further restrict our attention to the collection of all contractions admitting a Lipschitz constant smaller than or equal to a fixed $0 < r < 1$. More precisely, given $0 < r < 1$ let $\mathsf{c}(r)$ be the collection of all functions $f\colon {}^{\omega}\omega \to {}^{\omega}\omega$ such that $d(f(x), f(y)) \leq r \cdot d(x, y)$ for every $x, y \in {}^{\omega}\omega$. Then $\mathsf{c}(r)$ is closed under composition, and hence (with a little abuse of notation) we can define the preorder

$$A \leq_{\mathsf{c}(r)} B \;\Leftrightarrow\; \text{either } A = B \text{ or } A = f^{-1}(B) \text{ for some } f \in \mathsf{c}(r).$$

In particular, $\leq_{\mathsf{c}(r)} = \leq_{\mathcal{F}}$ for $\mathcal{F} = \mathsf{c}(r) \cup \{\mathrm{id}\}$.

Given $0 < r < 1$, let $n(r)$ be the smallest $n \in \omega$ such that $2^{-(n+1)} \leq r$. Then $\mathsf{c}(r) = \mathsf{c}(2^{-(n(r)+1)})$, and the relation $\leq_{\mathsf{c}(r)}$ admits a characterization via winning strategies for I in suitable reduction games similar to the one we obtained in Proposition 2.3 for $\leq_{\mathsf{c}}$ (which corresponds to the case $\frac{1}{2} \leq r < 1$). In fact, it is enough to replace the Lipschitz game G_{L} with the *$n(r)$-Lipschitz game* $G_{n(r)\text{-}\mathsf{Lip}}$ introduced in [MR11, Section 3] to get:

(1) $A \leq_{\mathsf{c}(r)} B \Leftrightarrow A = B \vee \mathrm{I}$ wins $G_{n(r)\text{-}\mathsf{Lip}}(\neg B, A)$. In fact, if I wins $G_{n(r)\text{-}\mathsf{Lip}}(\neg B, A)$ then $A = f^{-1}(B)$ for some $f \in \mathsf{c}(r)$.

(2) $A = f^{-1}(B)$ for some Lipschitz function with constant $2^{n(r)} \Leftrightarrow$ II wins $G_{n(r)\text{-}\mathsf{Lip}}(A, B)$.

Using this characterization of $\leq_{\mathsf{c}(r)}$, one can reprove suitable variants of most of the results needed to determine the corresponding degree-structure $\mathsf{Deg}(\mathsf{c}(r))$ ([13]). In particular, the analogue of Proposition 4.2 in which $\leq_{\mathsf{c}}$ is replaced by $\leq_{\mathsf{c}(r)}$ (for an arbitrary $0 < r < 1$) is true. (For the forward direction use $\mathsf{c}(r) \subseteq \mathsf{c}$, while for the backward direction use the fact that under AD^{L}, a set $A \subseteq {}^{\omega}\omega$ is L-nonselfdual if and only if it is Lip-nonselfdual—see [MR10a].) Therefore also the analogues of Corollaries 4.7 and 4.8 remain true when c is replaced with $\mathsf{c}(r)$.

However, not all the results of this section can be generalized to arbitrary preorders of the form $\leq_{\mathsf{c}(r)}$. For example, Proposition 4.1 fails if $r < 1/2$, because if $A \subseteq {}^{\omega}\omega$ is L-selfdual, then $A <_{\mathsf{L}} 0^\frown A$ but $A \not\leq_{\mathsf{c}(r)} 0^\frown A$ (in particular, this counterexample also shows that the first part of Corollary 4.9 fails for such r's as well); nevertheless, we can still prove that $\leq_{\mathsf{c}(r)}$ is well-founded (under $\mathsf{AD}^{\mathsf{L}} + \mathsf{BP}$) using a slightly different argument.

Corollary 4.10. ($\mathsf{AD}^{\mathsf{L}} + \mathsf{BP}$) *Let $0 < r < 1$. Then the preorder $\leq_{\mathsf{c}(r)}$ is well-founded.*

Proof. Assume for a contradiction that there is a sequence $\langle A_n \mid n \in \omega \rangle$ of subsets of ${}^{\omega}\omega$ such that $A_{n+1} <_{\mathsf{c}(r)} A_n$ for every $n \in \omega$.

Assume first that there is $N \in \omega$ such that $A_n \equiv_{\mathsf{L}} A_m$ for every $n, m \geq N$. If A_N is L-nonselfdual (so that all the A_m's with $m \geq N$ are L-nonselfdual as well), then $[A_N]_{\mathsf{L}} = [A_N]_{\mathsf{c}(r)}$ by (the analogue of) Corollary 4.7, and hence $A_n \equiv_{\mathsf{c}(r)} A_m$ for every $n, m \geq N$; if instead A_N is L-selfdual, then $A_n \not\leq_{\mathsf{c}(r)} A_m$ for all distinct $n, m \geq N$ by (the analogue of) Proposition 4.2. Thus in both cases we reach a contradiction to our choice of the A_n's.

Therefore, passing to a subsequence if necessary, we can assume without loss of generality that $A_n \not\equiv_{\mathsf{L}} A_m$ for all distinct $n, m \in \omega$. But then

([13]) In fact, using the game-theoretic characterization mentioned above (together with the fact that $\mathsf{Lip} \subseteq \mathsf{W}$), one can also show that we can replace c with $\mathsf{c}(r)$ (for an arbitrary $0 < r < 1$) in the completeness result of Corollary 2.4.

the sequence of A_n's is also $\leq_{\mathsf{L}}$-descending because $\mathsf{c}(r) \subseteq \mathsf{L}$, contradicting Theorem 2.5(1). ∎

This shows that the degree-structure induced by $\mathsf{c}(r)$ is, under suitable determinacy assumptions, another example of a bad degree-structure which is not very bad. However, when $0 < r < 1/2$ (i.e. when $\mathsf{c}(r) \neq \mathsf{c}$), the $\mathsf{c}(r)$-hierarchy is much more difficult to describe: this is mainly due to the counterexample presented before Corollary 4.10. However, using the above game-theoretic characterization of $\leq_{\mathsf{c}(r)}$ (together with the fact that, under our set-theoretical assumptions, L-nonselfduality and Lip-nonselfduality coincide) we can still give a full description of the $\leq_{\mathsf{c}(r)}$-preorders in terms of L-selfduality and $\leq_{\mathsf{L}}$-reducibility, from which a full description of the $\mathsf{c}(r)$-hierarchy can easily be recovered.

Proposition 4.11. ($\mathsf{AD}^{\mathsf{L}} + \mathsf{BP}$) *For every* $0<r<1$ *and every* $A, B \subseteq {}^{\omega}\omega$,

$$\begin{aligned} A \leq_{\mathsf{c}(r)} B \ \Leftrightarrow\ & A = B \vee (A \not\leq_{\mathsf{L}} \neg A \wedge A \leq_{\mathsf{L}} B) \\ & \vee (A \leq_{\mathsf{L}} \neg A \wedge 0^{(n(r)+1)}{}^\frown A \leq_{\mathsf{L}} B). \end{aligned}$$

Sketch of the proof. For the forward direction, assume that $A \leq_{\mathsf{c}(r)} B$. Since $A \leq_{\mathsf{c}(r)} B \Rightarrow A \leq_{\mathsf{L}} B$ by $\mathsf{c}(r) \subseteq \mathsf{L}$, the only nontrivial case that needs to be considered is when $A \leq_{\mathsf{L}} \neg A$ with $A \neq B$. Assume towards a contradiction that $0^{(n(r)+1)}{}^\frown A \not\leq_{\mathsf{L}} B$. Then

$$\neg A \leq_{\mathsf{L}} A \leq_{\mathsf{L}} B \leq_{\mathsf{L}} 0^{(n(r))}{}^\frown A \equiv_{\mathsf{L}} 0^{(n(r))}{}^\frown(\neg A),$$

so that $B = f^{-1}(\neg A)$ via some Lipschitz function f with constant $2^{n(r)}$. Therefore II would win $G_{n(r)\text{-}\mathsf{Lip}}(\neg B, A)$, and hence $A \not\leq_{\mathsf{c}(r)} B$ because I could not win such a game, a contradiction.

For the backward direction, assume first that $A \not\leq_{\mathsf{L}} \neg A$ and $A \neq B$. If $A \equiv_{\mathsf{L}} B$, then $A \leq_{\mathsf{c}(r)} B$ by (the analogue of) Proposition 4.2. If instead $A <_{\mathsf{L}} B$, then $A <_{\mathsf{Lip}} B, \neg B$ as well; hence II cannot win $G_{n(r)\text{-}\mathsf{Lip}}(\neg B, A)$, and since such a game is determined by our assumptions ([14]), we get $A \leq_{\mathsf{c}(r)} B$. Finally, assume that $A \leq_{\mathsf{L}} \neg A$ and $0^{(n(r)+1)}{}^\frown A \leq_{\mathsf{L}} B$ (which in particular implies $A \neq B$). If $A \not\leq_{\mathsf{c}(r)} B$, then I could not win $G_{n(r)\text{-}\mathsf{Lip}}(\neg B, A)$. Since such a game is determined and $A \leq_{\mathsf{L}} \neg A$, we would then have $B = f^{-1}(A)$ via some Lipschitz function f with constant $2^{n(r)}$, which in turn would imply $B \leq_{\mathsf{L}} 0^{(n(r))}{}^\frown A$, a contradiction. ∎

5. Changing the metric. As long as reducibility preorders $\leq_{\mathcal{F}}$ between subsets of ${}^{\omega}\omega$ are concerned, there are three kinds of sets of functions $\mathcal{F}$ that have been considered in the literature whose definition actually depends on the standard metric d on ${}^{\omega}\omega$ (rather than on its topology):

([14]) Recall that by [MR11, Lemma 6.1] the principle AD^{L} implies that all games of the form $G_{k\text{-}\mathsf{Lip}}(A, B)$ (for an arbitrary $k \in \omega$) are determined.

(1) the collection $\mathsf{L} = \mathsf{L}(d)$ of nonexpansive functions;
(2) the collection $\mathsf{Lip} = \mathsf{Lip}(d)$ of all Lipschitz functions (with arbitrary constant);
(3) the collection $\mathsf{UCont} = \mathsf{UCont}(d)$ of all uniformly continuous functions.

As recalled in Section 2, under suitable determinacy assumptions all three degree-structures induced by these notions of reducibility are very good and isomorphic to one another (see Figure 1); in fact the degree-structures $(\mathsf{Deg}(\mathsf{Lip}), \leq)$ and $(\mathsf{Deg}(\mathsf{UCont}), \leq)$ coincide despite the fact that $\mathsf{Lip} \subsetneq \mathsf{UCont}$. A natural question is then the following:

QUESTION 5.1. *What happens if we replace d with another complete* (*ultra*)*metric d' compatible with the topology of ${}^{\omega}\omega$? Are the degree-structures induced by $\mathsf{L}(d')$, $\mathsf{Lip}(d')$, and $\mathsf{UCont}(d')$ still well-behaved* (*i.e. good or very good*)*?*

Of course trivial modifications of d, such as replacing the distances $\langle 2^{-n} \mid n \in \omega\rangle$ used in the definition of d with any strictly decreasing sequence of reals converging to 0, yield exactly the same classes of functions (and hence the same induced degree-structures). However, slightly more elaborated variants can heavily modify the resulting hierarchies of degrees.

DEFINITION 5.2. Let $d_0 \colon ({}^{\omega}\omega)^2 \to \mathbb{R}^+$ be the metric on ${}^{\omega}\omega$ defined by

$$d_0(x,y) = \begin{cases} 0 & \text{if } x = y, \\ d(x,y) & \text{if } x(0) = y(0), \\ \max\{x(0), y(0)\} & \text{if } x(0) \neq y(0). \end{cases}$$

Thus $({}^{\omega}\omega, d_0)$ is essentially obtained by "gluing" together the subspaces $\mathbf{N}_{\langle n\rangle}$ of $({}^{\omega}\omega, d)$ by letting all the points in $\mathbf{N}_{\langle n\rangle}$ have distance $\max\{n,m\}$ from all the points in $\mathbf{N}_{\langle m\rangle}$ (for distinct $n,m \in \omega$).

The trivial but crucial observation is that, for $x,y \in {}^{\omega}\omega$,

(†) $\quad d(x,y) < 1 \;\Leftrightarrow\; d_0(x,y) < 1$, and in that case $d(x,y) = d_0(x,y)$.

Moreover, $d(x,y) \leq d_0(x,y)$ for every $x,y \in {}^{\omega}\omega$.

PROPOSITION 5.3. *The metric d_0 is a complete ultrametric compatible with the product topology on ${}^{\omega}\omega$.*

Proof. Since clearly $d_0(x,y) = d_0(y,x)$ and $d_0(x,y) = 0 \Leftrightarrow x = y$ (for all $x,y \in {}^{\omega}\omega$), to see that d_0 is an ultrametric it is enough to fix $x,y,z \in {}^{\omega}\omega$ and show that $d_0(x,y) \leq \max\{d_0(x,z), d_0(y,z)\}$; the latter can be straightforwardly checked by considering various cases, depending on whether the values of $x(0)$, $y(0)$, and $z(0)$ coincide or are distinct. Finally, the fact that d_0 is complete and compatible with the product topology on ${}^{\omega}\omega$ easily follows from (†) above. ∎

Denote by $\subseteq^*$ the relation of inclusion modulo finite sets on $\mathscr{P}(\omega)$, that is, for $X, Y \subseteq \omega$ set

$$X \subseteq^* Y \iff \exists \bar{k} \in \omega \; \forall k \geq \bar{k} \; (k \in X \Rightarrow k \in Y).$$

THEOREM 5.4. *Let $A \subseteq {}^{\omega}\omega$ be a* W*-selfdual set. Then there is a map $\psi \colon \mathscr{P}(\omega) \to [A]_{\mathsf{W}}$ such that, for every $X, Y \subseteq \omega$,*

(1) *if $X \subseteq^* Y$, then $\psi(X) \leq_{\mathsf{L}(d_0)} \psi(Y)$;*
(2) *if $\psi(X) \leq_{\mathsf{Lip}(d_0)} \psi(Y)$, then $X \subseteq^* Y$.*

Proof. Without loss of generality, we may assume that $A \leq_{\mathsf{L}(d)} \neg A$ (otherwise we replace A with $A \oplus \neg A$). Recursively define a sequence $\langle A_m \mid m \in \omega \rangle$ of subsets of ${}^{\omega}\omega$ by setting

$$A_0 = A, \qquad A_{m+1} = \bigoplus_{n \in \omega} 0^{(n)} {}^\frown A_m.$$

Arguing as in [MR10a], it is easy to check that:

(a) $A_m \leq_{\mathsf{L}(d)} \neg A_m$ (hence, in particular, $A_m \neq {}^{\omega}\omega$) for every $m \in \omega$;
(b) $A_n \leq_{\mathsf{L}(d)} A_m$ for every $n \leq m \in \omega$;
(c) $A_m \not\leq_{\mathsf{Lip}(d)} A_n$ for every $n < m \in \omega$;
(d) $s^\frown A_m \equiv_{\mathsf{Lip}(d)} A_m$ for every $m \in \omega$ and $s \in {}^{<\omega}\omega$;
(e) $A_m \equiv_{\mathsf{W}} A$ for every $m \in \omega$.

Recursively define the sequence $\langle n_k \mid k \in \omega \rangle$ by

$$n_0 = 0, \qquad n_{k+1} = n_k \cdot n_k + 1,$$

and for $i \in \omega$ let $\# i$ be the unique $k \in \omega$ such that $n_k \leq i < n_{k+1}$ (so that, in particular, $\# n_k = k$). Finally, for $X \subseteq \omega$ set

$$\psi(X) = \bigoplus_{i \in \omega} A_{3\# i + \rho_X(\# i)},$$

where $\rho_X \colon \omega \to 2$ is the characteristic function of the set X, i.e. $\rho_X(j) = 1 \iff j \in X$. It is trivial to check that (e) implies $\psi(X) \equiv_{\mathsf{W}} A$ for every $X \subseteq \omega$. We claim that ψ is as desired.

First we show that if $X, Y \subseteq \omega$ are such that $X \subseteq^* Y$, then $\psi(X) \leq_{\mathsf{L}(d_0)} \psi(Y)$. Fix $\bar{k} \in \omega$ such that $\forall k \geq \bar{k} \, (k \in X \Rightarrow k \in Y)$. For $k < \bar{k}$, let g_k be a witness of $A_{3k + \rho_X(k)} \leq_{\mathsf{L}(d)} A_{3\bar{k} + \rho_Y(\bar{k})}$, which exists by property (b) above; for $k \geq \bar{k}$, let g_k be a witness of $A_{3k+\rho_X(k)} \leq_{\mathsf{L}(d)} A_{3k + \rho_Y(k)}$, which exists by our choice of $\bar{k}$ and (b) again. Then define $f \colon {}^{\omega}\omega \to {}^{\omega}\omega$ by setting (for every $i \in \omega$ and $x \in {}^{\omega}\omega$)

$$f(i^\frown x) = \max\{i, n_{\bar{k}}\}^\frown g_{\# i}(x).$$

It is straightforward to check that f reduces $\psi(X)$ to $\psi(Y)$, so it remains to verify that $f \in \mathsf{L}(d_0)$. Fix $x, y \in {}^{\omega}\omega$. If $x(0) = y(0)$, then $f(x)(0) = f(y)(0)$, so that both $d_0(x, y) = d(x, y)$ and $d_0(f(x), f(y)) = d(f(x), f(y))$; therefore the inequality $d_0(f(x), f(y)) \leq d_0(x, y)$ follows from the fact that all the g_k's are in $\mathsf{L}(d)$ together with the observation that the definition of f on $\mathbf{N}_{\langle x(0)\rangle} = \mathbf{N}_{\langle y(0)\rangle}$ involves only $g_{\#x(0)}$. Now assume that $x(0) \neq y(0)$. If at least one of $x(0)$ and $y(0)$ is strictly above $n_{\bar{k}}$, then $f(x)(0) \neq f(y)(0)$, and so by definition of f and case assumption,

$$d_0(f(x), f(y)) = \max\{f(x)(0), f(y)(0)\} = \max\{x(0), y(0)\} = d_0(x, y).$$

If instead $x(0), y(0) \leq n_{\bar{k}}$, then $f(x)(0) = f(y)(0) = n_{\bar{k}}$, so that

$$d_0(f(x), f(y)) \leq \tfrac{1}{2} < d_0(x, y)$$

because we assumed $x(0) \neq y(0)$. Thus, in all cases $d_0(f(x), f(y)) \leq d_0(x, y)$, and hence we are done.

Assume now that $X, Y \subseteq \omega$ are such that $\psi(X) \leq_{\mathsf{Lip}(d_0)} \psi(Y)$, let $f \colon {}^\omega\omega \to {}^\omega\omega$ be a witness of this, and let $0 \neq l \in \omega$ be such that

$$d_0(f(x), f(y)) \leq 2^l \cdot d_0(x, y)$$

for every $x, y \in {}^\omega\omega$.

Claim 5.4.1. *Fix an arbitrary $i \in \omega$. If there is $2^{l-1} < j \in \omega$ such that $f(\mathbf{N}_{\langle i\rangle}) \cap \mathbf{N}_{\langle j\rangle} \neq \emptyset$, then $f(\mathbf{N}_{\langle i\rangle}) \subseteq \mathbf{N}_{\langle j\rangle}$.*

Proof. Let $x \in {}^\omega\omega$ satisfy $f(i^\frown x)(0) = j$, and suppose towards a contradiction that there is $y \in {}^\omega\omega$ such that $f(i^\frown y)(0) \neq j$. Then

$$d_0(f(i^\frown x), f(i^\frown y)) = \max\{f(i^\frown x)(0), f(i^\frown y)(0)\} \geq j > 2^{l-1}.$$

But since $d_0(i^\frown x, i^\frown y) \leq \frac{1}{2}$, by our choice of l we get

$$d_0(f(i^\frown x), f(i^\frown y)) \leq 2^l \cdot d_0(i^\frown x, i^\frown y) \leq 2^l \cdot \tfrac{1}{2} = 2^{l-1} < d_0(f(i^\frown x), f(i^\frown y)),$$

a contradiction. ∎

Claim 5.4.2. *For every $i \in \omega$, if $n_{\#i} > 2^{l-1}$ then there is $j \in \omega$ such that $\#j \geq \#i$ and $f(\mathbf{N}_{\langle i\rangle}) \subseteq \mathbf{N}_{\langle j\rangle}$.*

Proof. Set $s = i^\frown(l-1)^\frown 0^{(l-1)}$, so that $\mathrm{lh}(s) = l+1$. Then $d_0(x, y) = d(x, y) \leq 2^{-(l+1)}$ for every $x, y \in \mathbf{N}_s$. By our choice of l, it follows that $d_0(f(x), f(y)) \leq 1/2$, and hence $f(x)(0) = f(y)(0)$ by definition of d_0. This shows that $f(\mathbf{N}_s) \subseteq \mathbf{N}_{\langle j\rangle}$ for some $j \in \omega$.

Let $g \colon {}^\omega\omega \to {}^\omega\omega$ be defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \in \mathbf{N}_s, \\ (j+1)^\frown\vec{0} & \text{otherwise.} \end{cases}$$

Then g reduces $\psi(X) \cap \mathbf{N}_s$ to $\psi(Y) \cap \mathbf{N}_{\langle j\rangle}$ because f is a reduction of $\psi(X)$ to $\psi(Y)$; we claim that $g \in \mathsf{Lip}(d)$. Fix $x, y \in {}^\omega\omega$. If $x, y \in \mathbf{N}_s$, then since we showed that $f(x)(0) = f(y)(0)$, and moreover $x(0) = y(0)$ because $\mathrm{lh}(s) = l+1 > 0$, we get

$$d(g(x), g(y)) = d(f(x), f(y)) = d_0(f(x), f(y)) \leq 2^l \cdot d_0(x, y) = 2^l \cdot d(x, y).$$

If $x, y \notin \mathbf{N}_s$, then $g(x) = g(y)$ and hence $d(g(x), g(y)) = 0 \leq d(x, y)$. If $x \in \mathbf{N}_s$ and $y \notin \mathbf{N}_s$, then $d(x, y) \geq 2^{-l}$, and hence

$$d(g(x), g(y)) = 1 = 2^l \cdot 2^{-l} \leq 2^l \cdot d(x, y).$$

The case $x \notin \mathbf{N}_s$ and $y \in \mathbf{N}_s$ is treated similarly. So in all cases $d(g(x), g(y)) \leq 2^l \cdot d(x, y)$, and hence $g \in \mathsf{Lip}(d)$.

Since

$$\begin{aligned}\psi(X) \cap \mathbf{N}_s &= (i^\frown A_{3\#i+\rho_X(\#i)}) \cap \mathbf{N}_{i^\frown(l-1)^\frown 0^{(l-1)}} \\ &= \Big(i^\frown \bigoplus_{n\in\omega} 0^{(n)\frown} A_{3\#i+\rho_X(\#i)-1}\Big) \cap \mathbf{N}_{i^\frown(l-1)^\frown 0^{(l-1)}} \\ &= s^\frown A_{3\#i+\rho_X(\#i)-1}\end{aligned}$$

and $\psi(Y) \cap \mathbf{N}_{\langle j\rangle} = j^\frown A_{3\#j+\rho_Y(\#j)}$, it follows from (d) and the fact that g witnesses $\psi(X) \cap \mathbf{N}_s \leq_{\mathsf{Lip}(d)} \psi(Y) \cap \mathbf{N}_{\langle j\rangle}$ that $A_{3\#i+\rho_X(\#i)-1} \leq_{\mathsf{Lip}(d)} A_{3\#j+\rho_Y(\#j)}$. By (c),

$$3\#i + \rho_X(\#i) - 1 \leq 3\#j + \rho_Y(\#j),$$

which implies $\#i \leq \#j$.

Finally, since $\#i \leq \#j$ obviously implies $n_{\#i} \leq j$, we get $f(\mathbf{N}_{\langle i\rangle}) \subseteq \mathbf{N}_{\langle j\rangle}$ from $f(\mathbf{N}_s) \subseteq \mathbf{N}_{\langle j\rangle}$, $n_{\#i} > 2^{l-1}$, and Claim 5.4.1. ∎

By Claim 5.4.1, either $f(\mathbf{N}_{\langle 0\rangle}) \subseteq \bigcup_{i\leq 2^{l-1}} \mathbf{N}_{\langle i\rangle}$, or else $f(\mathbf{N}_{\langle 0\rangle}) \subseteq \mathbf{N}_{\langle j\rangle}$ for some $j > 2^{l-1}$. Therefore, in both cases there is $\bar{k} \in \omega$ such that $n_{\bar{k}} \geq 2^l$ (hence, in particular, also $n_{\bar{k}} > 2^{l-1}$) and $f(\mathbf{N}_{\langle 0\rangle}) \subseteq \bigcup_{i\leq n_{\bar{k}}} \mathbf{N}_{\langle i\rangle}$; we claim that $\forall k \geq \bar{k}$ $(k \in X \Rightarrow k \in Y)$, so that $X \subseteq^* Y$.

Fix $k \geq \bar{k}$. Since $n_k \geq n_{\bar{k}} > 2^{l-1}$ and clearly $n_{\#n_k} = n_k$, by Claim 5.4.2 applied to $i = n_k$ there is $j \in \omega$ such that $\#j \geq \#i = k$ (which implies $n_k \leq j$) and $f(\mathbf{N}_{\langle n_k\rangle}) \subseteq \mathbf{N}_{\langle j\rangle}$. Assume towards a contradiction that $j \geq n_{k+1}$. Then since $f(\mathbf{N}_{\langle 0\rangle}) \subseteq \bigcup_{i\leq n_{\bar{k}}} \mathbf{N}_{\langle i\rangle}$ and $j \geq n_{k+1} > n_k \geq n_{\bar{k}} \geq 2^l$, we would get

$$d_0(f(\vec{0}), f(n_k{}^\frown\vec{0})) = j \geq n_{k+1} > n_k \cdot n_k \geq 2^l \cdot n_k = 2^l \cdot d_0(\vec{0}, n_k{}^\frown\vec{0}),$$

contradicting the choice of l. Therefore $j < n_{k+1}$, and hence $\#j = k = \#n_k$. Since $f(\mathbf{N}_{\langle n_k\rangle}) \subseteq \mathbf{N}_{\langle j\rangle}$, arguing as in the proof of Claim 5.4.2 one can show that $\psi(X) \cap \mathbf{N}_{\langle n_k\rangle} \leq_{\mathsf{Lip}(d)} \psi(Y) \cap \mathbf{N}_{\langle j\rangle}$. Since, by (d),

$$\begin{aligned}\psi(X) \cap \mathbf{N}_{\langle n_k\rangle} &= n_k{}^\frown A_{3\#n_k+\rho_X(\#n_k)} \equiv_{\mathsf{Lip}(d)} A_{3\#n_k+\rho_X(\#n_k)}, \\ \psi(Y) \cap \mathbf{N}_{\langle j\rangle} &= j^\frown A_{3\#j+\rho_Y(\#j)} \equiv_{\mathsf{Lip}(d)} A_{3\#j+\rho_Y(\#j)},\end{aligned}$$

this implies that

$$3k + \rho_X(k) = 3\#n_k + \rho_X(\#n_k) \leq 3\#j + \rho_Y(\#j) = 3k + \rho_Y(k)$$

by (c). Therefore $\rho_X(k) \leq \rho_Y(k)$, and hence we are done. ∎

Corollary 5.5. *The partial order* $(\mathscr{P}(\omega), \subseteq^*)$ *can be embedded into both* [15] $\mathsf{Deg}_{\mathbf{\Delta}^0_1}(\mathsf{L}(d_0))$ *and* $\mathsf{Deg}_{\mathbf{\Delta}^0_1}(\mathsf{Lip}(d_0))$. *In particular, both degree-structures contain antichains of size* ${}^{\omega}2$ (*in the sense of Corollary* 4.8) *and infinite descending chains.*

Using Parovichenko's result [Par63] that (under AC) all partial orders of size $\aleph_1$ embed into $(\mathscr{P}(\omega), \subseteq^*)$, we get the following corollary.

Corollary 5.6. *Assume* AC. *Then every partial order of size* $\aleph_1$ *can be embedded into both* $\mathsf{Deg}_{\mathbf{\Delta}^0_1}(\mathsf{L}(d_0))$ *and* $\mathsf{Deg}_{\mathbf{\Delta}^0_1}(\mathsf{Lip}(d_0))$.

For what concerns the mutual relationships with respect to inclusion of the classes of functions related to d_0 and d considered above, we have the following full description.

Proposition 5.7.

(1) $\mathsf{c}(d) \subsetneq \mathsf{L}(d_0) \subsetneq \mathsf{L}(d)$.
(2) $\mathsf{L}(d) \not\subseteq \mathsf{Lip}(d_0) \subsetneq \mathsf{Lip}(d) \subsetneq \mathsf{UCont}(d_0) = \mathsf{UCont}(d)$.

Proof. (1) Fix $f \in \mathsf{c}(d) \subseteq \mathsf{L}(d)$. Since $d(f(x), f(y)) < 1$ for every $x, y \in {}^{\omega}\omega$, it follows from (†) that $d(f(x), f(y)) = d_0(f(x), f(y))$. Therefore, for every $x, y \in {}^{\omega}\omega$ we have

$$d_0(f(x), f(y)) = d(f(x), f(y)) \leq d(x, y) \leq d_0(x, y),$$

whence $f \in \mathsf{L}(d_0)$.

To show $\mathsf{L}(d_0) \subseteq \mathsf{L}(d)$, let $f \in \mathsf{L}(d_0)$ and let $x, y \in {}^{\omega}\omega$. If $x(0) \neq y(0)$, then $d(x, y) = 1$ and trivially $d(f(x), f(y)) \leq d(x, y)$. If instead $x(0) = y(0)$, then $d(x, y) = d_0(x, y)$, whence

$$d(f(x), f(y)) \leq d_0(f(x), f(y)) \leq d_0(x, y) = d(x, y).$$

Finally, both inclusions are proper because the functions $f = \mathrm{id}_{{}^{\omega}\omega}$ and $g \colon {}^{\omega}\omega \to {}^{\omega}\omega \colon n^\frown x \mapsto (n+1)^\frown x$ are in $\mathsf{L}(d_0) \setminus \mathsf{c}(d)$ and $\mathsf{L}(d) \setminus \mathsf{L}(d_0)$, respectively.

(2) Consider the function $f \colon {}^{\omega}\omega \to {}^{\omega}\omega \colon n^\frown x \mapsto (n^2+1)^\frown x$. Clearly $f \in \mathsf{L}(d)$, and since, for every $0 \neq n \in \omega$,

$$d_0(f(\vec{0}), f(n^\frown\vec{0})) = n^2 + 1 > n \cdot n = n \cdot d_0(\vec{0}, n^\frown\vec{0}),$$

we get $f \notin \mathsf{Lip}(d_0)$. This shows $\mathsf{L}(d) \not\subseteq \mathsf{Lip}(d_0)$, and hence also $\mathsf{Lip}(d) \not\subseteq \mathsf{Lip}(d_0)$ because $\mathsf{L}(d) \subseteq \mathsf{Lip}(d)$.

The inclusion $\mathsf{Lip}(d_0) \subseteq \mathsf{Lip}(d)$ can be proved similarly to $\mathsf{L}(d_0) \subseteq \mathsf{L}(d)$ above, while $\mathsf{UCont}(d_0) = \mathsf{UCont}(d)$ follows directly from (†). Since clearly $\mathsf{Lip}(d) \subsetneq \mathsf{UCont}(d)$, we are done. ∎

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(15) The closure of $\mathbf{\Delta}^0_1$ under both $\equiv_{\mathsf{L}(d_0)}$ and $\equiv_{\mathsf{Lip}(d_0)}$ follows from the fact that the metric d_0 is compatible with the topology on ${}^{\omega}\omega$ by Proposition 5.3, and hence $\mathsf{L}(d_0), \mathsf{Lip}(d_0) \subseteq \mathsf{W}$; see also Proposition 5.7.

Corollary 5.8. ($\mathsf{AD}^{\mathsf{L}}+\mathsf{BP}$) *The* $\mathsf{UCont}(d_0)$*-hierarchy is isomorphic to the* $\mathsf{L}(d)$*-hierarchy (see Theorem* 2.5 *and Figure* 1)*, and in fact it coincides with both the* $\mathsf{UCont}(d)$*-hierarchy and the* $\mathsf{Lip}(d)$*-hierarchy.*

Since obviously $\mathsf{L}(d_0) \subsetneq \mathsf{Lip}(d_0)$, from Proposition 5.7 and Corollary 5.5 we deduce that both $\mathsf{L}(d_0)$ and $\mathsf{Lip}(d_0)$ are examples of classes of functions which are much larger than the set of contractions $\mathsf{c} = \mathsf{c}(d)$ considered in Section 4, but which still miss the crucial condition of containing $\mathsf{L} = \mathsf{L}(d)$, and in fact they induce very bad degree-structures.

Slightly modifying the definition of the metric d_0, we can get a closely related metric d_1 which is much closer to the standard metric d, in the sense that in this case $\mathsf{Lip}(d_1) = \mathsf{Lip}(d)$ (while keeping the conditions $\mathsf{L}(d_1) \subsetneq \mathsf{L}(d)$ and $\mathsf{UCont}(d_1) = \mathsf{UCont}(d)$). We will see that also in this case the degree-structure $\mathsf{Deg}(\mathsf{L}(d_1))$ is very bad, while the degree-structures $\mathsf{Deg}(\mathsf{Lip}(d_1))$ and $\mathsf{Deg}(\mathsf{UCont}(d_1))$ both coincide with $\mathsf{Deg}(\mathsf{Lip}(d)) = \mathsf{Deg}(\mathsf{UCont}(d))$ (and are therefore isomorphic to the classical $\mathsf{L}(d)$-hierarchy described in Theorem 2.5, see Figure 1) when $\mathsf{AD}^{\mathsf{L}} + \mathsf{BP}$ is assumed.

Definition 5.9. Let $d_1 \colon ({}^{\omega}\omega)^2 \to \mathbb{R}^+$ be the metric on ${}^{\omega}\omega$ defined by

$$d_1(x,y) = \begin{cases} 0 & \text{if } x = y, \\ d(x,y) & \text{if } x(0) = y(0), \\ 2 - 2^{-(\max\{x(0),y(0)\}-1)} & \text{if } x(0) \neq y(0). \end{cases}$$

So the metric space $({}^{\omega}\omega, d_1)$ is constructed exactly as the space $({}^{\omega}\omega, d_0)$ except for the fact that we modify the distances used to "glue" together the subspaces $\mathbf{N}_{\langle n\rangle}$ of $({}^{\omega}\omega, d)$ in such a way that they form a bounded set.

Clearly, (†) remains true also after replacing d_0 with d_1, it is still the case that $d(x,y) \leq d_1(x,y)$ for every $x, y \in {}^{\omega}\omega$, and arguing as in Proposition 5.3 one sees that d_1 is a complete ultrametric compatible with the topology of ${}^{\omega}\omega$.

Proposition 5.10.

(1) $\mathsf{L}(d_1) = \mathsf{L}(d_0)$. *Therefore* $\mathsf{c}(d) \subsetneq \mathsf{L}(d_1) \subsetneq \mathsf{L}(d)$.
(2) $\mathsf{Lip}(d_1) = \mathsf{Lip}(d)$ *and* $\mathsf{UCont}(d_1) = \mathsf{UCont}(d)$.

Proof. (1) Use the fact that the map i defined by $i(0) = 0$, $i(2^{-(n+1)}) = 2^{-(n+1)}$, and $i(n+1) = 2 - 2^{-n}$ is an order-preserving map such that $d_1(x,y) = i(d_0(x,y))$ for every $x, y \in {}^{\omega}\omega$.

(2) The equality $\mathsf{UCont}(d_1) = \mathsf{UCont}(d)$ follows again from the analog of (†) with d_0 replaced by d_1, so it remains only to show that $\mathsf{Lip}(d_1) = \mathsf{Lip}(d)$.

Assume first that $f \in \mathsf{Lip}(d)$ and let $l \in \omega$ be such that $d(f(x), f(y)) \leq 2^l \cdot d(x,y)$ for every $x, y \in {}^{\omega}\omega$. We consider two cases: if $x \restriction (l+1) = y \restriction (l+1)$, then $d(x,y) \leq 2^{-(l+1)}$, and hence $d(f(x), f(y)) \leq 2^l \cdot 2^{-(l+1)} = 1/2$.

By the analog of (†) for d_1 and by the definition of d_1, this means that both $d_1(x,y) = d(x,y)$ and $d_1(f(x),f(y)) = d(f(x),f(y))$, whence $d_1(f(x),f(y)) \leq 2^l \cdot d_1(x,y)$. If instead $x\restriction(l+1) \neq y\restriction(l+1)$, then $d_1(x,y) \geq 2^{-l}$, and since all distances realized by d_1 are bounded by 2 we get

$$d_1(f(x),f(y)) \leq 2 = 2^{l+1} \cdot 2^{-l} \leq 2^{l+1} \cdot d_1(x,y).$$

Therefore $d_1(f(x),f(y)) \leq 2^{l+1} \cdot d_1(x,y)$ for every $x,y \in {}^\omega\omega$, and hence $f \in \mathsf{Lip}(d_1)$. This shows $\mathsf{Lip}(d) \subseteq \mathsf{Lip}(d_1)$. The inclusion $\mathsf{Lip}(d_1) \subseteq \mathsf{Lip}(d)$ can be proved in a similar way (or, alternatively, using the argument contained in the proof of $\mathsf{Lip}(d_0) \subseteq \mathsf{Lip}(d)$ in Proposition 5.7), hence we are done. ∎

From Proposition 5.10 and Theorem 5.4 we immediately get the following result.

THEOREM 5.11. *Let $A \subseteq {}^\omega\omega$ be a W-selfdual set. Then there is a map $\psi\colon \mathscr{P}(\omega) \to [A]_{\mathsf{W}}$ such that, for every $X, Y \subseteq \omega$,*

$$X \subseteq^* Y \iff \psi(X) \leq_{\mathsf{L}(d_1)} \psi(Y).$$

In particular, $(\mathscr{P}(\omega), \subseteq^*)$ embeds into $\mathsf{Deg}_{\mathbf{\Delta}^0_1}(\mathsf{L}(d_1))$, and so $\mathsf{Deg}_{\mathbf{\Delta}^0_1}(\mathsf{L}(d_1))$ contains both antichains of size ${}^\omega 2$ (in the sense of Corollary 4.8) and infinite descending chains.

6. Questions and open problems. By Theorem 2.6, the inclusion $\mathcal{F} \supseteq \mathsf{L}$ is a sufficient condition for $\mathsf{Deg}(\mathcal{F})$ being very good (under suitable determinacy assumptions). However, literally this is not a necessary condition: in fact, letting $\mathcal{F} = \{f\colon {}^\omega\omega \to {}^\omega\omega \mid f \text{ is two-valued}\} \cup \{\mathrm{id}\}$, one sees that $\mathcal{F} \not\supseteq \mathsf{L}$, but $\mathsf{Deg}(\mathcal{F})$ consists of the $\mathcal{F}$-nonselfdual pair $\{[{}^\omega\omega]_{\mathcal{F}}, [\emptyset]_{\mathcal{F}}\}$ plus a unique $\mathcal{F}$-degree above it containing all sets $\emptyset, {}^\omega\omega \neq A \subseteq {}^\omega\omega$, and is thus (trivially) very good. This phenomenon is apparently due to the fact that such an $\mathcal{F}$ is too large (that is, it is in bijection with $\mathscr{P}({}^\omega\omega)$), and this makes its induced degree-structure collapse to an extremely simple finite structure. In contrast, to the best of our knowledge the problem of whether any degree-structure induced by a *not too large* $\mathcal{F} \not\supseteq \mathsf{L}$ must be (very) bad remains open.

QUESTION 6.1. *Work in $\mathsf{ZF} + \mathsf{DC}(\mathbb{R})$ (or in $\mathsf{ZF} + \mathsf{DC}(\mathbb{R}) + \mathsf{AD}$), and let $\mathcal{F}$ be a collection of functions from ${}^\omega\omega$ into itself closed under composition and containing* id. *Assume that $\mathcal{F}$ is a surjective image of ${}^\omega\omega$. Is it true that if $\mathcal{F} \not\supseteq \mathsf{L}$ then $\mathsf{Deg}(\mathcal{F})$ is (very) bad?*

Another open problem related to the results in Section 5 is the following.

QUESTION 6.2. *Is there a complete ultrametric d' on the Baire space which is compatible with its topology and such that $\mathsf{L}(d) \not\subseteq \mathsf{L}(d'), \mathsf{Lip}(d'), \mathsf{UCont}(d')$? Can d' be chosen so that all the hierarchies $\mathsf{Deg}(\mathsf{L}(d'))$, $\mathsf{Deg}(\mathsf{Lip}(d'))$, and $\mathsf{Deg}(\mathsf{UCont}(d'))$ are (very) bad?*

All the degree-structures considered in this paper were either very good, or else (very) bad. Thus it seems natural to ask the following:

QUESTION 6.3. *Is there any "natural" collection of functions from* ${}^{\omega}\omega$ *into itself* (*closed under composition and containing* id) *such that* $\mathsf{Deg}(\mathcal{F})$ *is good but not very good?*

Finally, it could be interesting to further investigate the notion of self-contractible subsets of metric spaces considered in Sections 2 and 4.

QUESTION 6.4. *Given* $1 \leq \xi < \omega_1$, *is it true that every proper* $\boldsymbol{\Sigma}^0_\xi$ *or proper* $\boldsymbol{\Pi}^0_\xi$ *subset of* $\mathbb{R}$ *is selfcontractible? What if* $\mathbb{R}$ *is replaced by an arbitrary uncountable Polish space? Is it possible to characterize the collection of all selfcontractible subsets of* $\mathbb{R}$ (*or, more generally, of an arbitrary uncountable Polish space*) *similarly to Corollaries* 4.4 *and* 4.5*?*

Added in proof. Recently, some of the above-mentioned questions have been answered in [MRs14]. In particular, it is shown in [MRs14, Theorem 3.10] that the hierarchy $\mathsf{Deg}(\mathsf{UCont}(d'))$ is very good for any complete ultrametric d' on ${}^{\omega}\omega$. Moreover, by [MRs14, Proposition 3.4] there is such a d' which is compatible with the topology of ${}^{\omega}\omega$ and such that $\mathsf{L}(d) \not\subseteq \mathsf{UCont}(d')$ (hence also $\mathsf{L}(d) \not\subseteq \mathsf{L}(d'), \mathsf{Lip}(d')$). Since for such a d' the collection $\mathsf{UCont}(d')$, being a subcollection of the continuous functions, is clearly a surjective image of ${}^{\omega}\omega$, this fully answers Questions 6.1 and 6.2. To the best of our knowledge, Questions 6.3 and 6.4 are still open.

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