

## On the complexity of subspaces of $S_\omega$

by

Carlos Uzcátegui (Mérida)

**Abstract.** Let  $(X, \tau)$  be a countable topological space. We say that  $\tau$  is an analytic (resp. Borel) topology if  $\tau$  as a subset of the Cantor set  $2^X$  (via characteristic functions) is an analytic (resp. Borel) set. For example, the topology of the Arkhangel'skiĭ–Franklin space  $S_\omega$  is  $F_{\sigma\delta}$ . In this paper we study the complexity, in the sense of the Borel hierarchy, of subspaces of  $S_\omega$ . We show that  $S_\omega$  has subspaces with topologies of arbitrarily high Borel rank and it also has subspaces with a non-Borel topology. Moreover, a closed subset of  $S_\omega$  has this property iff it contains a copy of  $S_\omega$ .

**1. Introduction.** Let  $(X, \tau)$  be a countable topological space. We say that  $\tau$  is an *analytic* (resp. Borel) topology if  $\tau$  as a subset of the Cantor set  $2^X$  (identifying a subset of  $X$  with its characteristic function) is an analytic (resp. Borel) set. Most of the examples of countable topological spaces found in the literature are analytic. For example, every second countable topology is  $F_{\sigma\delta}$ , in particular, the topology of the rationals is (in fact a complete)  $F_{\sigma\delta}$  subset of  $2^{\mathbb{Q}}$ . Other examples of  $F_{\sigma\delta}$  topologies are Arens space [1] or its more general version, the Arkhangel'skiĭ–Franklin space  $S_\omega$  (see [2]). A systematic study of analytic topologies was initiated in [13, 12] where an explicit connection was shown between descriptive set theoretic properties and pure topological properties of a given space. For example, analytic topologies are closely related to spaces of continuous functions: a  $T_2$  regular countable space has an analytic topology iff it is homeomorphic to a countable subspace of  $C_{\mathbb{p}}(\mathbb{N}^{\mathbb{N}})$  (the space of real-valued continuous functions on the Baire space  $\mathbb{N}^{\mathbb{N}}$  with the pointwise topology) [13, Theorem 6.1].

In this note we are interested in the complexity of subspace topologies of a given countable space. It is clear that any subspace  $Y$  of a space  $X$  with an analytic topology also has an analytic topology. However, the complexity of the subspace topology of  $Y$  (measured in terms of the Borel hierarchy) might vary considerably depending on  $X$  and  $Y$ . On the one hand, if  $X$  is

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second countable or more generally has an  $F_\sigma$  basis (see Section 3.2 for the definition), then the topology of every subspace of  $X$  is  $F_{\sigma\delta}$ . On the other hand, we will show that the Arkhangel'skiĭ–Franklin space  $S_\omega$  (which has an  $F_{\sigma\delta}$  topology) has subspaces with arbitrarily high Borel rank and also has non-Borel subspaces (see §2 for the definition of  $S_\omega$  and some general information about it).

Our main result is the following

**THEOREM 1.1.** *Let  $X$  be a closed subset of  $S_\omega$ . The following are equivalent:*

- (i)  $X$  has a subspace whose topology is not Borel.
- (ii)  $X$  has subspaces with Borel topology of arbitrarily high Borel rank.
- (iii)  $X$  contains a copy of  $S_\omega$ .

The proof uses the fact that  $S_\omega$  is a sequential space, thus with every closed subspace  $X$  there is associated an ordinal  $\varrho(X)$  called the sequential rank (see the definition in §2). We will show the following

**THEOREM 1.2.** *Let  $X$  be a closed subset of  $S_\omega$ .*

- (i) *If  $\varrho(X) < \omega_1$ , then the subspace topology of every subset of  $X$  is Borel.*
- (ii) *If  $\varrho(X) = \omega_1$ , then  $X$  contains a closed copy of  $S_\omega$  and a subspace whose topology is not Borel.*

Examples of subspaces of  $S_\omega$  with Borel topology of arbitrarily high rank are essentially given by the terminal nodes of well founded trees. These subspaces will have only one non-isolated point and therefore their topology is given by a filter. Thus in §4 we will construct Borel filters of arbitrarily high rank which in fact are the nbhd filter of a point in a subspace of  $S_\omega$ . Parts (i) and (ii) of Theorem 1.2 are shown in §5 and §6 respectively.

A very natural question is to determine which countable spaces satisfy the conclusion of Theorem 1.1. In particular, we would like to know when a countable space contains a copy of  $S_\omega$  (a similar question was asked in [2]). In §3 we show that if in a countable space  $X$  with Borel topology the nbhd filter of every point is Borel, then every subspace of  $X$  also has a Borel topology. In particular, for the homogeneous space  $S_\omega$ , Theorem 1.1 implies that the nbhd filter of every point in  $S_\omega$  is not Borel. Moreover, we will see that these nbhd filters are complete analytic sets. This is in contrast with the fact that  $S_\omega$  has an  $F_{\sigma\delta}$  topology.

We end this introduction by making some comments about the connection between analytic topologies and the descriptive complexity of  $C_p(X)$ , the space of real-valued continuous functions on a non-discrete completely regular countable topological space  $X$  with the topology of pointwise convergence. There have been a lot of work on the classification of  $C_p(X)$

(see [4, 5, 6] and the references therein). One of the main results is that  $C_p(X)$  is homeomorphic to  $\sigma^\omega$  (the countable product of the space of sequences eventually equal to zero) whenever  $C_p(X)$  is  $F_{\sigma\delta}$  as a subset of  $\mathbb{R}^X$ . It can be shown that a regular topology on  $X$  is analytic iff  $C_p(X)$  is analytic. However the exact relationship between the complexity of the topology on  $X$  and that of  $C_p(X)$  has not been fully investigated. For the case of spaces with only one non-isolated point, i.e. spaces associated to filters, this has been done ([6, Lemma 4.2] and references therein). We have not pursued this issue here but we think it is worth studying.

The papers [11, 5] dealt with the problem of classifying  $C_D(X)$ , the set of continuous functions on  $X$  with the topology of pointwise convergence on  $D$ , where  $D$  is a countable dense subset of  $X$ , i.e,  $C_D(X) = \{f|D : f \in C(X)\} \subseteq \mathbb{R}^D$ . It was shown that the Borel complexity of  $C_D(X)$  might vary considerably depending on  $D$  and  $X$ . For instance, for every countable ordinal  $\alpha$  there is a space  $X_\alpha$  and a dense subset  $D_\alpha$  of  $X_\alpha$  such that  $C_p(X_\alpha)$  is  $F_{\sigma\delta}$  and  $C_{D_\alpha}(X_\alpha)$  has Borel rank larger than  $\alpha$  (see [5, Prop. 2.6]). Our results show that a similar phenomenon happens within the single space  $S_\omega$ .

**2. Preliminaries.** We will use the standard notions and terminology of descriptive set theory (see for instance [10]).  $\omega^{<\omega}$  denotes the collection of finite sequences of natural numbers. For  $s \in \omega^{<\omega}$ ,  $|s|$  denotes its length. For  $n \in \mathbb{N}$ ,  $s\hat{\ }n$  is the concatenation of  $s$  with  $n$ . For  $\alpha \in \mathbb{N}^\mathbb{N}$ , we denote by  $\alpha|n$  the restriction of  $\alpha$  to  $\{0, 1, \dots, n-1\}$ . The Borel sets of rank  $\alpha$  will be denoted by  $\Sigma_\alpha^0$  and  $\Pi_\alpha^0$ , where for instance  $\Sigma_1^0$  and  $\Pi_1^0$  are respectively the open and closed sets,  $\Sigma_2^0$  and  $\Pi_2^0$  the  $F_\sigma$  and  $G_\delta$  sets and so on. A subset of a Polish space is *analytic* (or  $\Sigma_1^1$ ) if it is a continuous image of the Baire space  $\mathbb{N}^\mathbb{N}$ . A well known result of Suslin is that a subset of a Polish space is Borel iff it is analytic and co-analytic (see for instance [10, Theorem 14.11]).

Let  $X, Y$  be Polish spaces and  $A \subseteq X$ ,  $B \subseteq Y$ . The set  $A$  is said to be *Wadge reducible* to  $B$ , denoted by  $A \leq_w B$ , if there is a continuous function  $f : X \rightarrow Y$  such that  $x \in A$  iff  $f(x) \in B$  (see [10, §21.E]). Notice that if  $A \leq_w B$  and  $A$  is Borel (resp. projective), then the Borel (resp. projective) type of  $B$  is at least that of  $A$ . Let  $\Gamma$  be a class of sets in Polish spaces. If  $Y$  is a Polish space, a set  $A \subseteq Y$  is called  $\Gamma$ -*complete* if  $A \in \Gamma(Y)$  and  $B \leq_w A$  for all  $B \in \Gamma$  (see [10, §22.B]). The archetypical  $\Sigma_1^1$ -complete set is the collection of ill founded trees on  $\mathbb{N}$ , i.e. trees with at least one infinite branch (see [10, 27.1]). Any  $\Sigma_1^1$ -complete set is not Borel. Thus to show that an analytic subset  $A$  of a Polish space  $Z$  is not Borel it suffices to show that the set of ill founded trees is Wadge reducible to  $A$ .

Let  $A$  be a subset of a topological space  $X$ . The sequential closure of  $A$  is defined by transfinite recursion as follows [2]. Let  $A^{(0)} = A$  and  $A^{(1)}$  be the set of all limits of convergent sequences in  $A$ ,  $A^{(\alpha+1)} = [A^{(\alpha)}]^{(1)}$  and

$A^{(\beta)} = \bigcup_{\alpha < \beta} A^{(\alpha)}$  for  $\beta$  a limit ordinal. The *sequential closure* of  $A$ , denoted by  $[A]_{\text{seq}}$ , is the set  $A^{(\omega_1)}$ . The space  $X$  is called *sequential* if for every  $A \subseteq X$  the closure of  $A$  is equal to its sequential closure, i.e.  $\bar{A} = [A]_{\text{seq}}$ . A subset  $O \subseteq X$  is said to be *sequentially open* if for all  $x \in O$  and any sequence  $x_n$  converging to  $x$  there is  $N$  such that  $x_n \in O$  for all  $n > N$ . A space is sequential iff every sequentially open set is in fact open. A closed subspace of a sequential space is sequential.

DEFINITION 2.1. Let  $X$  be a sequential space and  $A \subseteq X$ . The *sequential rank* of  $A$  in  $X$ , denoted by  $\sigma(A, X)$ , is defined by

$$\sigma(A, X) = \min\{\alpha : A^{(\alpha)} = A^{(\alpha+1)}\}.$$

The *sequential rank* of  $X$  is defined by

$$\varrho(X) = \sup\{\sigma(A, X) : A \subseteq X\}$$

The local versions of these ordinals are defined as follows. Given  $A \subseteq X$  and  $s \in X$  define

$$\begin{aligned}\sigma(s, A) &= \min\{\alpha : s \in A^{(\alpha)}\} \quad \text{for } s \in \bar{A}, \\ \varrho(s, X) &= \sup\{\sigma(s, A) : s \in \bar{A} \text{ \& } A \subseteq X\}.\end{aligned}$$

The following elementary facts about these ordinals are stated for later reference.

PROPOSITION 2.2. *Let  $X$  be a sequential space,  $A \subseteq X$  and  $s \in X$ .*

- (i)  $A^{(\sigma(A, X))} = \bar{A}$ .
- (ii)  $\varrho(s, X) = 0$  iff  $s$  is isolated in  $X$ .
- (iii)  $\sigma(A, X) = \sup_{s \in \bar{A}} \sigma(s, A)$ .
- (iv)  $\varrho(X) = \sup_{s \in X} \varrho(s, X)$ . ■

Now we recall the definition of  $S_\omega$  and some basic facts about it. Define a topology  $\tau$  on  $\omega^{<\omega}$  by

$$U \in \tau \Leftrightarrow \{n \in \mathbb{N} : s \hat{n} \notin U\} \text{ is finite for all } s \in U.$$

Let  $S_\omega$  be the space  $(\omega^{<\omega}, \tau)$ . It is clear that  $S_\omega$  is  $T_2$ , zero-dimensional and has no isolated points. Notice that a set  $U$  is  $\tau$ -open iff there is  $f : \omega^{<\omega} \rightarrow \mathbb{N}$  such that if  $s \in U$ , then  $s \hat{n} \in U$  for all  $n \geq f(s)$ . A sequence  $\{x_i\}_i$  in  $S_\omega$  converges to  $s$  iff  $\{x_i\}_i$  is eventually of the form  $s \hat{n}_i$  for some increasing sequence  $\{n_i\}$  of integers. From this it follows that  $S_\omega$  is sequential. For each  $t \in \omega^{<\omega}$  we define

$$N_t = \{s \in \omega^{<\omega} : t \preceq s\}.$$

Notice that  $N_t$  is a clopen set in  $S_\omega$ . If we consider  $\tau$  as a subset of  $2^{\omega^{<\omega}}$  (the latter with the product topology is homeomorphic to the Cantor set), then it is clear that  $\tau$  is  $F_{\sigma\delta}$ .

$S_\omega$  has showed up in many different contexts. The first occurrence was as an example of a sequential homogeneous space of sequential rank  $\omega_1$  (see [2]). A very interesting description of  $S_\omega$  as a translation invariant topology over  $\mathbb{Z}$  is given in [7].  $S_\omega$  has been implicitly used to study sequential convergence in  $C_p(X)$  (see [9]). For instance, if  $Z$  is a topological space such that there is a continuous surjection from  $Z$  onto a non-meager subset of  $\mathbb{R}$ , then  $C_p(Z)$  contains a copy of  $S_\omega$ . Another occurrence of  $S_\omega$  is in the following result about spaces with the Schur property. Let  $E$  be a linear normed space. Then every weakly convergent sequence in  $E$  is norm-convergent iff  $E$  with the weak topology contains no copy of  $S_\omega$  ([13, Theorem 5.3] and [9, Theorem 17]). Another interesting property of  $S_\omega$  appears in [3, Example 3.8].

**3. Complexity of neighborhood filters.** In this section we make some comments about the problem of when every subspace topology of a Borel topology is also Borel. Let us start by analyzing the case of a (Hausdorff) space with only one non-isolated point. Let  $\mathcal{F}$  be a filter over  $\omega$  and  $X$  be  $\omega + 1$  with the topology where every  $n \in \omega$  is isolated and the nbhds of  $\omega$  are the elements of  $\mathcal{F}$ . Let  $Y \subseteq X$ . Then the restriction of  $\mathcal{F}$  to  $Y$ , denoted by  $\mathcal{F}_Y$ , is easily seen to have the property that  $A \in \mathcal{F}_Y$  iff  $A \subseteq Y$  and  $A \cup (X \setminus Y) \in \mathcal{F}$ . This shows that  $\mathcal{F}_Y \leq_w \mathcal{F}$ . Therefore, if  $\mathcal{F}$  is Borel, then  $\mathcal{F}_Y$  is also Borel and thus the subspace topology of  $Y$  is Borel for every  $Y \subseteq X$ .

Recall that the nbhd filter  $\mathcal{F}_x$  of a point  $x \in X$  is the filter over  $X \setminus \{x\}$  defined by  $A \in \mathcal{F}_x$  if there is an open set  $V$  such that  $x \in V \subseteq A \cup \{x\}$ . Notice that if  $\tau$  is analytic, then every  $\mathcal{F}_x$  is also analytic. It is elementary to show that  $V$  is open iff  $V \setminus \{x\} \in \mathcal{F}_x$  for all  $x \in V$ . In particular, this says that if every  $\mathcal{F}_x$  is Borel, then  $\tau$  is also Borel. The converse is not true: as we will see in §6,  $S_\omega$  has a Borel topology but in fact all its nbhd filters are non-Borel.

A basis  $\mathcal{B}$  for a countable topological space  $X$  is said to be  $F_\sigma$  if  $\mathcal{B}$  is an  $F_\sigma$  set as a subset of  $2^X$ . Every space with an  $F_\sigma$  basis has an  $F_{\sigma\delta}$  topology [13, Proposition 3.2]. The converse is not true, since  $S_\omega$  has an  $F_{\sigma\delta}$  topology but it does not admit an  $F_\sigma$  basis [13, Proposition 5.2] (this will also be deduced from one of the results presented in this paper). A countable  $T_2$  regular space  $X$  has an  $F_\sigma$  basis iff  $X$  has a closed subbasis iff  $X$  is homeomorphic to a countable subspace of  $C_p(2^{\mathbb{N}})$  [13, Theorems 3.2, 3.4 and 6.1]. It is easy to check that having an  $F_\sigma$  basis is a hereditary property. Moreover, in this case, every nbhd filter is  $F_\sigma$ . In fact, let  $\{F_n\}_n$  be closed subsets of  $2^X$  such that  $\mathcal{B} = \bigcup_n F_n$  is a basis for  $\tau$ . Then

$$A \in \mathcal{F}_x \Leftrightarrow \exists n \in \mathbb{N} \exists V [V \in F_n \ \& \ x \in V \subseteq A \cup \{x\}].$$

The set of all  $(V, A) \in 2^X \times 2^X$  such that  $V \in F_n \ \& \ x \in V \subseteq A \cup \{x\}$  is

compact for any  $x \in X$  and  $n \in \mathbb{N}$ . So  $\mathcal{F}_x$  is a countable union of projections of compact sets, therefore it is  $F_\sigma$  for all  $x$ . We state this result in the following

**PROPOSITION 3.1.** *If  $\tau$  has an  $F_\sigma$  basis, then  $\mathcal{F}_x$  is  $F_\sigma$  for all  $x$ . ■*

We do not know if the converse holds. The Arens space  $S_2$  (which can be defined as  $\omega^{\leq 2}$  with the topology inherited from  $S_\omega$  (see [8, Example 1.6.19]) is an example of a space whose topology does not admit an  $F_\sigma$  basis [13] but all its nbhd filters are Borel (in fact, they are  $\Sigma_4^0$ , see Lemma 4.5).

We will denote the closure operator of a topological space  $(X, \tau)$  by  $\text{cl}_X$  or  $\text{cl}_\tau$ . The following result characterizes when every  $\mathcal{F}_x$  is Borel for an analytic (and therefore Borel) topology.

**THEOREM 3.2.** *Let  $\tau$  be an analytic topology over a countable set  $X$ . The following are equivalent:*

- (i)  $\mathcal{F}_x$  is Borel for every  $x \in X$ .
- (ii) For each  $x \in X$ , the set  $C_x = \{A \subseteq X : x \in \bar{A}\}$  is Borel.
- (iii)  $\text{cl}_\tau$  is a Borel function from  $2^X$  into  $2^X$ .
- (iv) The relation  $R(A, Y)$  given by “ $A$  is closed in  $Y$ ” is Borel (in  $2^X \times 2^X$ ).

*Proof.* Since  $\tau$  is analytic,  $R$  is analytic and each  $C_x$  is co-analytic. The following equivalences are straightforward:

$$\begin{aligned} A \in \mathcal{F}_x &\Leftrightarrow x \notin A \ \& \ X \setminus (A \cup \{x\}) \notin C_x, \\ A \in C_x &\Leftrightarrow X \setminus (A \cup \{x\}) \notin \mathcal{F}_x, \\ R(A, Y) &\Leftrightarrow \forall B [\text{cl}_\tau(A) = B \rightarrow B \cap Y \subseteq A], \\ A \in C_x &\Leftrightarrow x \in A \text{ or } [x \notin A \ \& \ \neg R(A, A \cup \{x\})], \\ \text{cl}_\tau(A) = B &\Leftrightarrow A \subseteq B \ \& \ \forall x (x \in B \rightarrow A \in C_x) \ \& \ X \setminus B \in \tau. \end{aligned}$$

Notice that the complementation mapping is a homeomorphism of the Cantor set and thus the function  $A \mapsto X \setminus (A \cup \{x\})$  is continuous for every  $x \in X$ . To finish the proof we notice that the first two equivalences above show that  $\mathcal{F}_x$  is Borel iff  $C_x$  is Borel. The third one shows that if  $\text{cl}_\tau$  is Borel, then  $R$  is co-analytic and, being analytic, it is then Borel by Suslin’s theorem [10, Theorem 14.11]. The fourth one shows that if  $R$  is Borel, then  $C_x$  is Borel for all  $x$ . And the last equivalence shows that if  $C_x$  is Borel for all  $x$ , then  $\text{cl}_\tau$  has an analytic graph and thus it is a Borel function [10, Theorem 14.12]. ■

In view of the previous result it is natural to introduce the following notion. Let us say that a topology on a countable set  $X$  is *hereditarily Borel* if the subspace topology of every  $Y \subseteq X$  is Borel. Thus by Theorem 3.2 we have the following result.

**COROLLARY 3.3.** *Let  $\tau$  be an analytic topology over  $X$ . If every nbhd filter  $\mathcal{F}_x$  of  $X$  is Borel, then the topology of  $X$  is hereditarily Borel. Moreover, the Borel rank of the subspace topologies is uniformly bounded. ■*

**REMARK 3.4.** We do not know whether the converse of 3.3 holds, that is to say, whether it is true that if  $X$  has an analytic topology such that the nbhd filter of some point is not Borel, then  $X$  has a subspace with a non-Borel topology.

We end this section by showing a general fact about the Baire measurability of  $\text{cl}_\tau$ .

**PROPOSITION 3.5.** *Let  $\tau$  be a meager (as a subset of  $2^X$ )  $T_1$  topology with infinitely many limit points. Then  $\text{cl}_\tau$  is not of Baire class 1.*

*Proof.* Since  $\tau$  is  $T_1$ , it is a dense subset of  $2^X$ . Thus the collection of  $\tau$ -closed sets is also dense and meager. Given any non- $\tau$ -closed set  $B$ , there is a sequence of finite sets  $F_n$  such that  $B = \lim_n F_n$  (in the product topology of  $2^X$ ). Since  $\tau$  is  $T_1$ , it follows that  $\bar{F}_n = F_n$  and therefore  $\text{cl}_\tau$  is not continuous at  $B$ . This shows that the collection of discontinuity points of  $\text{cl}_\tau$  is a co-meager set, and therefore  $\text{cl}_\tau$  cannot be of Baire class 1. ■

**REMARK 3.6.** In particular, by [13, Corollary 2.6]),  $\text{cl}_\tau$  is not of Baire class 1 when  $\tau$  is an analytic  $T_1$  topology with infinitely many limit points.

#### 4. Subspaces of $S_\omega$ with topology of arbitrarily high Borel rank.

In this section we will show the following

**THEOREM 4.1.** *For any countable ordinal  $\alpha$  there is  $X \subseteq S_\omega$  such that subspace topology of  $X$  is a Borel set of rank  $\geq \alpha$ .*

The idea of the proof of 4.1 is to associate to a well founded tree  $T$  on  $\mathbb{N}$  a subspace  $X_T$  of  $S_\omega$  in such a way that the Borel rank of the topology of  $X_T$  will be, roughly speaking, equal to the rank of  $T$ . Let  $E(T)$  be the terminal nodes of  $T$ . The subspaces we will construct are of the form  $\{\emptyset\} \cup E(T)$ . Let us observe that any antichain  $D$  (i.e. no element in  $D$  extends another one) is discrete as a subset of  $S_\omega$ , so in particular  $E(T)$  is a discrete set. Therefore, we will actually construct filters of arbitrarily high Borel rank. Our filters are similar to those constructed in [4]. It is interesting to realize that these filters correspond to nbhd filters of points in a subspace of  $S_\omega$ .

**DEFINITION 4.2.** For any well founded tree  $T$  on  $\mathbb{N}$ , let  $\mathcal{F}_T$  be the nbhd filter of  $\emptyset$  in the subspace  $\{\emptyset\} \cup E(T)$  of  $S_\omega$ .

We will construct by recursion an  $\omega_1$ -sequence of trees  $T_\alpha$  such that  $\mathcal{F}_{T_\alpha}$  is  $\Sigma_\alpha^0$ -complete, that is to say, they will satisfy the following two conditions:

- (i)  $\mathcal{F}_{T_\alpha}$  is  $\Sigma_\alpha^0$ .  
(ii) For every  $A$  in  $\Sigma_\alpha^0$  there is a continuous function  $V : 2^\mathbb{N} \rightarrow 2^{E(T_\alpha)}$  such that  $x \in A$  iff  $V(x) \in \mathcal{F}_{T_\alpha}$ .

Recall that the exact Borel rank of a  $\Sigma_\alpha^0$ -complete set is precisely  $\alpha$ .

Before stating the preliminary lemmas needed for the proof of Theorem 4.1 we make a general observation which shows that the subspaces we will construct cannot be sequential.

**PROPOSITION 4.3.** *Let  $X \subseteq S_\omega$ . If  $X$  is a sequential subspace of  $S_\omega$ , then the topology of  $X$  is  $\Pi_3^0$ .*

*Proof.* Since  $X$  is sequential,  $V \subseteq X$  is open in  $X$  iff  $V$  is sequentially open. Therefore  $V$  is open in  $X$  iff for all  $s \in V$  the following holds:

If  $\{n : \widehat{s}n \in X\}$  is infinite, then  $\exists N \forall m \geq N [\widehat{s}m \in X \rightarrow \widehat{s}m \in V]$ , and from this it follows that the topology of  $X$  is  $\Pi_3^0$ . ■

We will use the following result [4, Lemma 8.2] (see also [10, 23.5]).

**LEMMA 4.4.** *Let  $A \subseteq 2^\mathbb{N}$  and  $\alpha > 1$  a countable ordinal.*

(i)  *$A$  belongs to  $\Pi_{\alpha+1}^0$  iff there are sets  $A_m$  in  $\Pi_{\beta_m}^0$  for some  $\beta_m < \alpha$  such that*

$$x \in A \Leftrightarrow \forall n \exists m \geq n [x \in A_m].$$

(ii)  *$A$  is in  $\Sigma_{\alpha+1}^0$  iff there are sets  $A_m$  in  $\Sigma_{\beta_m}^0$  for some  $\beta_m < \alpha$  such that*

$$x \in A \Leftrightarrow \exists n \forall m \geq n [x \in A_m].$$

For  $\alpha = 1$ , the sets  $A_m$  can be chosen to be clopen. ■

The base for the induction is given in the following

**LEMMA 4.5.** *Let  $T = \omega^{\leq 2}$ . Then  $\mathcal{F}_T$  is  $\Sigma_4^0$ -complete.*

*Proof.* Notice that  $E(T) = \omega^2$ . Let  $V \subseteq E(T)$ . It is easy to check that

$$(1) \quad V \in \mathcal{F}_T \quad \text{iff} \quad \exists N \forall n \geq N \exists M \forall m \geq M [\langle n, m \rangle \in V].$$

From this it follows that  $\mathcal{F}_T$  is  $\Sigma_4^0$ . To see that  $\mathcal{F}_T$  is  $\Sigma_4^0$ -complete fix a  $\Sigma_4^0$  set  $A \subseteq 2^\mathbb{N}$ . By Lemma 4.4 there are clopen sets  $F(n, m)$  such that

$$(2) \quad x \in A \Leftrightarrow \exists N \forall n \geq N \exists M \forall m \geq M [x \in F(n, m)].$$

Let  $V : 2^\mathbb{N} \rightarrow 2^{E(T)}$  be given by  $V(x) = \{\langle n, m \rangle : x \in F(n, m)\}$ . Since the  $F(n, m)$ 's are clopen,  $V$  is continuous. From (1) and (2) we conclude that  $x \in A$  iff  $V(x) \in \mathcal{F}_T$ . ■

**REMARK 4.6.** Recall that the Arens space  $S_2$  is the subspace  $\omega^{\leq 2}$  of  $S_\omega$ . Thus the previous lemma might be known (see [4, Remark 8.11]) but we have included the proof for the sake of completeness.

LEMMA 4.7. Let  $T_n$  be well founded trees such that  $\mathcal{F}_{T_n}$  is  $\Sigma_{\alpha_n}^0$ -complete. Let  $T$  be the tree

$$T = \bigcup_n \{\langle n \rangle \hat{\ } s : s \in T_n\} \cup \{\emptyset\}.$$

Then  $T$  is well founded and  $\mathcal{F}_T$  is  $\Sigma_{\alpha+1}^0$ -complete for  $\alpha = \sup\{\alpha_n + 1 : n \in \mathbb{N}\}$ .

*Proof.* It is clear that  $T$  is well founded. Notice that  $E(T)$  is the union of  $\{\langle n \rangle \hat{\ } s : s \in E(T_n)\}$ . Let  $\Phi_n : S_\omega \rightarrow N_{\langle n \rangle}$  be defined by  $\Phi_n(s) = \langle n \rangle \hat{\ } s$ . It is clear that  $\Phi_n$  is a homeomorphism. Let  $\mathcal{G}_n = \Phi_n[\mathcal{F}_{T_n}]$ . Then  $\mathcal{G}_n$  is the nbhd filter of  $\langle n \rangle$  in the subspace  $\Phi_n[E(T_n) \cup \{\emptyset\}] = \{\langle n \rangle \hat{\ } s : s \in E(T_n)\} \cup \{\langle n \rangle\}$  and moreover  $\mathcal{G}_n$  is  $\Sigma_{\alpha_n}^0$ .

Let  $A \subseteq E(T)$ . We claim that

$$(3) \quad A \in \mathcal{F}_T \quad \text{iff} \quad \exists N \forall n \geq N [A \cap N_{\langle n \rangle} \in \mathcal{G}_n].$$

From this and Lemma 4.4 it follows that  $\mathcal{F}_T$  is  $\Sigma_{\alpha+1}^0$ .

To show (3), suppose  $A \in \mathcal{F}_T$  and let  $W$  be an open set in  $S_\omega$  such that  $\emptyset \in W$  and  $W \cap E(T) = A$ . There is  $N$  such that  $\langle n \rangle \in W$  for all  $n \geq N$ . Notice that  $W_n = W \cap N_{\langle n \rangle}$  is an open set in  $N_{\langle n \rangle}$ ,  $\langle n \rangle \in W_n$  and  $W_n \cap \Phi_n[E(T_n)] \subseteq A \cap N_{\langle n \rangle}$ . Conversely, suppose the right hand side of (3) holds and let  $W_n$  be an open set in  $N_{\langle n \rangle}$  such that  $\langle n \rangle \in W_n$  and  $W_n \cap \Phi_n[E(T_n)] = A \cap N_{\langle n \rangle}$  for all  $n \geq N$ . Let  $W$  be the union of the  $W_n$ 's together with  $\emptyset$ . Then  $W$  is an open nbhd of  $\emptyset$ . It is routine to check that  $W \cap E(T) \subseteq A$ .

Now we will show that  $\mathcal{F}_T$  is  $\Sigma_{\alpha+1}^0$ -complete. Let  $A \subseteq 2^\mathbb{N}$  be a  $\Sigma_{\alpha+1}^0$  set. By Lemma 4.4 there are  $\Sigma_{\beta_n}^0$  sets  $A_n$  with  $\beta_n < \alpha$  such that

$$x \in A \Leftrightarrow \exists N \forall n \geq N [x \in A_n].$$

We can assume that  $\beta_n \leq \alpha_n$ . (In fact, suppose  $\alpha_0 < \beta_0$ . Find the least  $n$  such that  $\beta_0 \leq \alpha_n$ . Replace the original sequence  $\{A_k\}$  by  $\{A'_k\}$  which now starts with  $n$  copies of  $2^\mathbb{N}$  followed by the original sequence  $\{A_k\}$ . Now  $\beta'_0 \leq \alpha_0$ . Repeat this procedure as many times as necessary.)

Since  $\mathcal{F}_{T_n}$  is  $\Sigma_{\alpha_n}^0$ -complete, there are continuous functions  $V_n : 2^\mathbb{N} \rightarrow 2^{E(T_n)}$  such that

$$(4) \quad x \in A_n \Leftrightarrow V_n(x) \in \mathcal{F}_{T_n}.$$

Let  $V(x) = \bigcup_n \Phi_n[V_n(x)]$ . Notice that  $V : 2^\mathbb{N} \rightarrow 2^{E(T)}$  is continuous and  $V(x) \cap N_{\langle n \rangle} = \Phi_n[V_n(x)]$ . From this, (3), (4) and the definition of  $\mathcal{G}_n$  we have  $V(x) \in \mathcal{F}_T$  iff  $\exists N \forall n \geq N [\Phi_n[V_n(x)] \in \mathcal{G}_n]$  iff  $\exists N \forall n \geq N [V_n(x) \in \mathcal{F}_{T_n}]$  iff  $x \in A$ . ■

REMARK 4.8. The definition of the filter  $\mathcal{F}_T$  occurring in the proof of the previous result could be stated in terms of the Hausdorff operation (see [10, Exercise 23.5]) and the Fréchet product (see [4, Section 8]). Thus the

$\Sigma_{\alpha+1}^0$ -completeness of  $\mathcal{F}_T$  can be proved basing on some general results about these operations. Our filters are similar to the filters  $F_\alpha$ 's constructed in [4, Section 8]. For instance,  $\mathcal{F}_T$  with  $T = \omega^{\leq 2}$  corresponds to  $F_2$ .

*Proof of 4.1.* We will define by recursion a sequence  $U_\alpha$  of well founded trees such that  $\mathcal{F}_{U_\alpha}$  is  $\Sigma_\alpha^0$ -complete where  $\alpha$  is either an even integer greater than 2 or an odd infinite ordinal.

We start with  $U_4 = \omega^{\leq 2}$  which works by Lemma 4.5. Now taking  $T_n$  equal to  $U_{2k}$  for all  $n$  and applying Lemma 4.7 we obtain  $U_{2k+2}$ . For infinite ordinals we start by taking  $T_n = U_{2n}$  in Lemma 4.7 and obtain  $U_{\omega+1}$ . Now for the inductive step the pattern should be clear. ■

It is quite easy to define topologies on the  $\Pi$  side of the Borel hierarchy once we have available topologies on the  $\Sigma$  side.

**PROPOSITION 4.9.** *Let  $(X, \tau)$  be a countable topological space. Suppose  $X = \bigcup_n U_n$  where  $U_n$  is a pairwise disjoint family of non-empty open sets. Suppose that  $\tau$  restricted to  $U_n$  is  $\Sigma_{\alpha_n}^0$ -complete and  $\alpha_n$  is an increasing sequence of countable ordinals. Then  $\tau$  is  $\Pi_\lambda^0$ -complete, where  $\lambda$  is  $\sup_n(\alpha_n + 1)$ .*

*Proof.* For  $V \subseteq X$ , it is clear that  $V \in \tau$  iff  $V \cap U_n$  is open in  $U_n$  for all  $n$ . Thus  $\tau$  is  $\Pi_\lambda^0$ . Fix a  $\Pi_\lambda^0$  subset  $A$  of a zero-dimensional Polish space  $Y$ . Let  $B_n$  be a  $\Sigma_{\beta_n}^0$  set with  $\beta_n < \lambda$  such that  $A = \bigcap_n B_n$ . We can suppose that  $\beta_n \leq \alpha_n$ . Then as  $\tau$  restricted to  $U_n$  is  $\Sigma_{\alpha_n}^0$ -hard, there are continuous functions  $f_n : Y \rightarrow 2^{U_n}$  such that  $y \in B_n$  iff  $f_n(y)$  is open in  $U_n$ . Define  $f : Y \rightarrow 2^X$  by  $f(y) = \bigcup_n f_n(y)$ . Since the  $U_n$ 's are pairwise disjoint,  $f$  is easily seen to be continuous and  $y \in A$  iff  $f(y) \in \tau$ . ■

**REMARK 4.10.** The method of constructing subspaces in the proof of 4.1 and 4.9 does not provide examples of topologies of any possible Borel type. For instance, it will be interesting to determine whether  $S_\omega$  has subspaces with topology of type  $\Pi_{2n}^0$ ,  $\Sigma_{2n+1}^0$  ( $n \geq 2$ ) and  $\Sigma_\omega^0$ .

**REMARK 4.11.** Notice also that from 4.1 and 3.1–3.3 it follows that  $S_\omega$  does not have an  $F_\sigma$  basis, a fact proved in [13, Proposition 5.2] by a different method.

**5. Subspaces of  $S_\omega$  whose topology is hereditarily Borel.** In this section we will show the following

**THEOREM 5.1.** *Let  $X \subseteq S_\omega$  be a closed subspace with  $\varrho(X) < \omega_1$ . Then the closure operator  $\text{cl}_X$  for the subspace topology of  $X$  is Borel. In particular, every subspace of  $X$  has a Borel topology and, moreover, the Borel rank of the topologies of subspaces of  $X$  is uniformly bounded.*

Let  $X$  be a closed subspace of  $S_\omega$  with  $\varrho(X) < \omega_1$ . In order to use 3.2 and 3.3 we need to show that the sets

$$C_s = \{A \subseteq X : s \in \bar{A}\}$$

are Borel for all  $s \in X$ . We will also allow  $s \notin X$ , since in this case  $C_s$  is empty.

LEMMA 5.2. *Let  $X$  be a closed subspace of  $S_\omega$ . Then for all  $N$ ,*

$$C_s = \{A \subseteq X : s \in A\} \cup \bigcap_{n \geq N} \bigcup_{m \geq n} C_{s \hat{\ } m}.$$

*Proof.* Let  $A \subseteq S_\omega$  and  $s \in \bar{A} \setminus A$ . A straightforward induction on  $\sigma(s, A)$  shows that there is an increasing sequence  $\{n_i\}_i$  of integers such that  $s \hat{\ } n_i \in \bar{A}$  for all  $i$ . From this the inclusion  $\subseteq$  follows. For the other one, just observe that  $s \hat{\ } n_i$  converges to  $s$ . ■

LEMMA 5.3. *Let  $X$  be a closed subspace of  $S_\omega$  and  $s \in X$ . If  $\varrho(s, X) < \omega_1$ , then there is  $N$  such that  $\varrho(s \hat{\ } m, X) < \varrho(s, X)$  for all  $m \geq N$  such that  $s \hat{\ } m \in X$ .*

*Proof.* Let  $\alpha = \varrho(s, X)$  and let  $B$  be the set of all  $m$  such that  $\varrho(s \hat{\ } m, X) \geq \alpha$  and  $s \hat{\ } m \in X$ . Suppose, towards a contradiction, that  $B$  is infinite. Notice that  $\alpha > 0$ , otherwise  $s$  would be isolated in  $X$  and there would be only finitely many  $m$  such that  $s \hat{\ } m \in X$ . We will only analyze the case when  $\alpha$  is a limit ordinal; the successor case can be treated similarly.

Since  $\alpha < \omega_1$ , we can fix an increasing sequence of ordinals  $\alpha_n < \omega_1$  converging to  $\alpha$ . For each  $n \in B$ , there is  $A_n$  such that  $s \hat{\ } n \in \bar{A}_n$  and  $\sigma(s \hat{\ } n, A_n) \geq \alpha_n$ . We can assume that  $A_n \subseteq N_{s \hat{\ } n}$ . Let

$$A = \bigcup_{n \in B} A_n.$$

Notice that  $s \in \bar{A}$ . We claim that  $\sigma(s, A) > \alpha$ , which is a contradiction. In fact, suppose  $s \in A^{(\alpha)}$ ; then there is  $m$  such that  $s \in A^{(\alpha_m+1)}$ . Therefore there is an increasing sequence of integers  $n_i$  such that  $s \hat{\ } n_i \in A^{(\alpha_m)}$  for all  $i$ . Since  $A_n \subseteq N_{s \hat{\ } n}$  and the  $N_{s \hat{\ } n}$ 's are disjoint, it follows that  $s \hat{\ } n_i \in A_{n_i}^{(\alpha_m)}$  for all  $i$ . Thus  $\alpha_{n_i} \leq \sigma(s \hat{\ } n_i, A_{n_i}) \leq \alpha_m$ , which is impossible if  $m < n_i$ . ■

*Proof of 5.1.* By Propositions 3.2 and 3.3 it suffices to show that  $C_s$  is a Borel subset of  $2^X$  for all  $s \in X$ . We will show this by induction on  $\varrho(s, X)$ .

If  $\varrho(s, X) = 0$ , then  $s$  is isolated in  $X$ , therefore  $C_s$  consists of all  $A \subseteq X$  such that  $s \in A$ , and thus  $C_s$  is a closed subset of  $2^X$ . Suppose that  $C_t$  is Borel for all  $t \in X$  with  $\varrho(t, X) < \alpha$  and let  $s \in X$  with  $\varrho(s, X) = \alpha$ . By Lemma 5.3 there is  $N$  such that  $\varrho(s \hat{\ } m, X) < \alpha$  for all  $m \geq N$  such that  $s \hat{\ } m \in X$ . By the inductive hypothesis,  $C_{s \hat{\ } m}$  is Borel for all  $m \geq N$ . Now from Lemma 5.2 it follows that  $C_s$  is also Borel. ■

REMARK 5.4. Let  $T$  be a tree on  $\mathbb{N}$ . Then  $T$  is closed as a subset of  $S_\omega$ , and thus a sequential space. By 4.3 the topology of  $T$  is  $F_{\sigma\delta}$ . If  $T$  is well founded, it has an associated rank as a tree, which we will denote by  $\text{rk}(T)$  (see [10, §2.E]). It is routine to check that  $\varrho(T) \leq \text{rk}(T)$ . For the trees  $U_\alpha$  constructed in the proof of 4.1, it can be easily verified by induction that  $\varrho(U_\alpha) = \text{rk}(U_\alpha) \leq \alpha$ . It can also be verified that every subspace of  $U_\alpha$  has a Borel topology of rank at most  $\alpha$  and there is one (namely  $E(U_\alpha) \cup \{\emptyset\}$ ) whose topology is Borel of rank exactly  $\alpha$ . So for this example, the sequential rank  $\varrho(X)$  gives a good (uniform) bound for the Borel complexity of the topology of subspaces of  $X$ .

**6. Subspaces of  $S_\omega$  with an analytic non-Borel topology.** In this section we will show the following

THEOREM 6.1. *Let  $X \subseteq S_\omega$  be a closed subspace with  $\varrho(X) = \omega_1$ . Then there is  $Y \subseteq X$  such that the subspace topology of  $Y$  is not Borel. Moreover, there is a closed copy of  $S_\omega$  inside  $X$ .*

The key lemma is

LEMMA 6.2. *Let  $D \subseteq \omega^{<\omega}$  be an antichain and  $s \in \bar{D}$  with  $\varrho(s, \bar{D}) = \omega_1$ . Then the topology of  $D \cup \{s\}$  is a complete analytic set, in particular, it is not Borel.*

Since any antichain is a discrete subset of  $S_\omega$ , it follows that  $s$  is the only non-isolated point of  $D \cup \{s\}$ . So the topology of  $D \cup \{s\}$  is given by the nbhd filter of  $s$  in  $D \cup \{s\}$ .

A concrete example of an antichain  $D$  such that  $\varrho(\emptyset, \bar{D}) = \omega_1$  is the following:

$$(5) \quad D = \{s \hat{=} 2n : s(i) \text{ is odd for all } i < |s| \text{ and } n \in \mathbb{N}\}.$$

Notice that the collection of all finite sequences of odd integers is a subset of  $\bar{D}$  and  $\emptyset \in \bar{D}$ . Thus  $\bar{D}$  contains a closed copy of  $S_\omega$ . Therefore  $\varrho(\emptyset, \bar{D}) = \omega_1$  and from Lemma 6.2 we conclude that the subspace topology of  $D \cup \{\emptyset\}$  is analytic and non-Borel.

From 6.1, 3.2 and 3.3 we know that there must be some  $s \in S_\omega$  such that the nbhd filter  $\mathcal{F}_s$  of  $s$  is not Borel. Since  $S_\omega$  is homogeneous, every  $\mathcal{F}_t$  is non-Borel. Even more, we will show that  $\mathcal{F}_t$  is a complete analytic set for every  $t \in S_\omega$ . It suffices to show this for  $t = \emptyset$ . In fact, let  $D$  be an antichain such that  $\emptyset \in \bar{D}$  and the topology of  $D \cup \{\emptyset\}$  is a complete analytic set (for instance, the one given by (5)). By 6.2 the nbhd filter  $\mathcal{G}$  of  $\emptyset$  in  $D \cup \{\emptyset\}$  is a complete analytic set. It is easy to check that  $A \in \mathcal{G}$  iff  $A \subseteq D$  and  $A \cup (S_\omega \setminus D) \in \mathcal{F}_\emptyset$ . Thus  $\mathcal{G} \leq_w \mathcal{F}_\emptyset$ . So we have shown the following

PROPOSITION 6.3. *Let  $\mathcal{F}_s$  be the nbhd filter of  $s$  in  $S_\omega$ . Then  $\mathcal{F}_s$  is a complete analytic set.*

This proposition also follows from a result of [14]. In fact, for every tree  $T$  let

$$F(T) = \{r \in S_\omega : \exists t \in T [|t| = |r| \ \& \ t(i) \leq r(i) \text{ for all } i < |r|]\}.$$

Let  $T_\alpha$  be the set of all initial segments of  $\alpha$ , where  $\alpha \in \mathbb{N}^{\mathbb{N}}$ . Let  $\mathcal{F}$  be the filter given by  $S \in \mathcal{F}$  iff there is  $\alpha \in \mathbb{N}^{\mathbb{N}}$  such that  $F(T_\alpha) \subseteq S$ . Notice that  $\mathcal{F} \subseteq \mathcal{F}_\emptyset$ . The proof of the main result in [14] shows that if  $T$  is not well founded, then  $F(T) \in \mathcal{F}$ , and when  $T$  is well founded, then  $F(T) \notin \mathcal{F}_\emptyset$ . Thus  $\mathcal{F}$  and  $\mathcal{F}_\emptyset$  are both complete analytic sets. The proof of 6.2 uses a similar argument and the fact that  $S_\omega$  is a sequential space. However, we do not know how to prove the existence of a subspace of  $S_\omega$  whose topology is not Borel just from the fact that the nbhd filters of  $S_\omega$  are not Borel. This is precisely an instance of the general question stated in 3.4.

Now we start the proof of Theorem 6.1. We will need another property of the ordinal  $\varrho$  defined in §2.

LEMMA 6.4. *Let  $X \subseteq S_\omega$  be a closed subspace and  $s \in \omega^{<\omega}$ . If  $\varrho(s, X) = \omega_1$ , then  $\varrho(\widehat{s}m, X) = \omega_1$  for infinitely many  $m$ 's.*

*Proof.* Suppose that  $\varrho(\widehat{s}m, X) < \omega_1$  for all  $m \geq N$  with  $\widehat{s}m \in X$ . Let  $\alpha = \sup\{\varrho(\widehat{s}m, X) : \widehat{s}m \in X, m \geq N\}$ . Let  $A \subseteq X$  be such that  $s \in \bar{A}$ . It suffices to show that  $\sigma(s, A) \leq \alpha + 1$ . We can assume that  $s \notin A$ . Then there is an increasing sequence  $\{n_i\}_i$  of integers such that  $\widehat{s}n_i \in \bar{A}$ . By hypothesis  $\varrho(\widehat{s}n_i, X) \leq \alpha$ . Therefore  $\sigma(\widehat{s}n_i, A) \leq \alpha$ , thus  $\sigma(s, A) \leq \alpha + 1$ . ■

*Proof of 6.1.* From 2.2(iv) we know that there is  $s \in X$  such that  $\varrho(s, X) = \omega_1$ . We will construct an antichain  $D \subseteq X$  such that  $s \in \bar{D}$  and  $\varrho(s, \bar{D}) = \omega_1$ . Thus  $Y = D \cup \{s\}$  will be the required subspace of  $X$ . By Lemma 6.4,  $\varrho(\widehat{s}m, X) = \omega_1$  for infinitely many  $m$ . The idea is to put in  $D$  “half” of these sequences  $\widehat{s}m$  and repeat this process with the other “half”. More formally, for each sequence  $t$  such that  $\varrho(t, X) = \omega_1$  put

$$B_t = \{m : \varrho(t\widehat{m}, X) = \omega_1\}$$

and let  $B_t^0, B_t^1$  be a partition of  $B_t$  into two infinite pieces. We define by recursion two sequences of sets  $D_n$  and  $E_n$  as follows:

$$\begin{aligned} D_1 &= \{\widehat{s}m : m \in B_s^1\}, \\ E_1 &= \{\widehat{s}m : m \in B_s^0\}, \\ D_{n+1} &= \{t\widehat{m} : t \in E_n \ \& \ m \in B_t^1\}, \\ E_{n+1} &= \{t\widehat{m} : t \in E_n \ \& \ m \in B_t^0\}. \end{aligned}$$

Let

$$D = \bigcup_{n \geq 1} D_n, \quad E = \bigcup_{n \geq 1} E_n.$$

It is not hard to verify by induction on  $n$  that  $E_n \subseteq \bar{D}$ . It is clear that  $D$  is an antichain and  $s \in \bar{D}$ . To see that  $\varrho(s, \bar{D}) = \omega_1$  it suffices to verify that  $E \cup \{s\}$  is a closed copy of  $S_\omega$ . It is clear that  $E \cup \{s\}$  is a copy of  $S_\omega$ . To check that it is closed, notice that if  $t \in E$ ,  $t' \prec t$  and  $|s| < |t'|$ , then  $t' \in E$ . ■

*Proof of 6.2.* Since  $S_\omega$  is a homogeneous space, we can assume that  $s = \emptyset$ . Consider the following function  $F$  that maps a tree  $T$  on  $\mathbb{N}$  to a subset of  $D$ :

$$F(T) = \{r \in D : \exists t \in T \ [|t| = |r| \ \& \ t(i) \leq r(i) \text{ for all } i < |r|]\}.$$

For a given  $r$  there are only finitely many sequences  $t$  such that  $|t| = |r|$  and  $t(i) \leq r(i)$  for all  $i < |r|$ , thus  $F$  is continuous.

We claim that  $T$  is ill founded iff  $F(T) \cup \{\emptyset\}$  is open in  $D \cup \{\emptyset\}$ . In fact, suppose first that  $T$  is ill founded. Let  $\alpha$  be an infinite branch of  $T$ . Define

$$W = \{t \in \omega^{<\omega} : \alpha(i) \leq t(i) \text{ for all } i < |t|\}.$$

It is clear that  $W$  is an open set of  $S_\omega$  and  $\emptyset \in W$ . Let

$$O = \bigcup_{t \in F(T)} N_t \cup W.$$

Then  $O$  is an open set of  $S_\omega$ . We will show that

$$F(T) \cup \{\emptyset\} = (D \cup \{\emptyset\}) \cap O.$$

It is clear that  $F(T) \cup \{\emptyset\} \subseteq (D \cup \{\emptyset\}) \cap O$ . Conversely, let  $r \in D \cap O$ . There are two cases: (i) If  $r \in N_t \cap O$  with  $t \in F(T)$ , then  $t = r$  as  $D$  is an antichain. (ii) If  $r \in W \cap D$ , then  $\alpha(i) \leq r(i)$  all  $i < |r|$ . Since  $\alpha$  is a branch of  $T$ , we have  $r \in F(T)$  by the definition of  $F(T)$ .

Suppose now that  $F(T) \cup \{\emptyset\}$  is open in  $D \cup \{\emptyset\}$  and let  $O$  be an open subset of  $S_\omega$  such that  $F(T) \cup \{\emptyset\} = (D \cup \{\emptyset\}) \cap O$ . By recursion we will define  $\alpha \in \mathbb{N}^{\mathbb{N}}$  and a sequence  $r_n \in \omega^{<\omega}$  such that:

- (1)  $r_n \in O \cap D$  for all  $n$ .
- (2)  $\alpha|j \in O \cap \bar{D}$  for all  $j$ .
- (3)  $r_n(i) \leq \alpha(i)$  for all  $i < |r_n|$ .
- (4)  $\varrho(\alpha|j, \bar{D}) = \omega_1$  for all  $j$ .
- (5)  $|r_n| < |r_{n+1}|$  for all  $n$ .

Granting this has been done we finish the proof. To show that  $T$  is not well founded, let

$$T_0 = \{t \in T : t(i) \leq \alpha(i) \text{ for all } i < |t|\}.$$

It is clear that  $T_0$  is a finitely branching subtree of  $T$ . So it suffices to show that  $T_0$  is infinite. In fact, by (1),  $r_n \in O \cap D \subseteq F(T)$ , thus there is  $t_n \in T$

such that  $t_n(i) \leq r_n(i)$  for all  $i < |r_n| = |t_n|$ . From (5) we conclude that the  $t_n$ 's are all different and from (3) we have  $t_n \in T_0$  for all  $n$ .

So it remains to show that such  $\alpha \in \mathbb{N}^{\mathbb{N}}$  and  $r_n \in \omega^{<\omega}$  exist. Since  $\emptyset \in O \cap \bar{D}$ , there is  $r_0 \in O \cap D$ . By Lemma 6.4 there are infinitely many  $n$  such that  $\varrho(\langle n \rangle, \bar{D}) = \omega_1$ . Thus let  $\alpha(0) \geq r_0(0)$  be such that  $\langle \alpha(0) \rangle \in O$  and  $\varrho(\langle \alpha(0) \rangle, \bar{D}) = \omega_1$ . We can continue this way and define  $\alpha(i)$  for all  $i < |r_0|$ . Thus (1) and (3) are satisfied for  $n = 0$ , and (2) and (4) for  $j < |r_0|$ .

Suppose we have defined  $r_n$  and  $\alpha(i)$  for all  $i < |r_n| = k$ . Let

$$s = \langle \alpha(0), \alpha(1), \dots, \alpha(k-1) \rangle.$$

By (2),  $s \in \bar{D} \cap O$ , thus there is  $r_{n+1} \in D \cap O$  extending  $s$ . By (4),  $\varrho(s, \bar{D}) = \omega_1$ , therefore  $r_{n+1}$  extends properly  $s$ . Hence  $|r_n| < |r_{n+1}|$  and (5) holds. Now we repeat the same argument as for the case  $n = 0$  and define  $\alpha$  up to  $|r_{n+1}|$  so that (2) and (4) hold for every  $j < |r_{n+1}|$ . ■

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Departamento de Matemáticas  
Facultad de Ciencias  
Universidad de Los Andes  
Mérida 5101, Venezuela  
E-mail: uzca@ciens.ula.ve

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