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Applications of some strong set-theoretic axioms to locally compact T_5 and hereditarily scwH spaces

by

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Abstract. Under some very strong set-theoretic hypotheses, hereditarily normal spaces (also referred to as T_5 spaces) that are locally compact and hereditarily collectionwise Hausdorff can have a highly simplified structure. This paper gives a structure theorem (Theorem 1) that applies to all such ω_1-compact spaces and another (Theorem 4) to all such spaces of Lindelöf number $\leq \aleph_1$. It also introduces an axiom (Axiom F) on crowding of functions, with consequences (Theorem 3) for the crowding of countably compact subspaces in certain continuous preimages of ω_1. It also exposes (Theorem 2) the fine structure of perfect preimages of ω_1 which are T_5 and hereditarily collectionwise Hausdorff. In these theorems, "T_5 and hereditarily collectionwise Hausdorff" is weakened to "hereditarily strongly collectionwise Hausdorff." Corollaries include the consistency, modulo large cardinals, of every hereditarily strongly collectionwise Hausdorff manifold of dimension > 1 being metrizable. The concept of an alignment plays an important role in formulating several of the structure theorems.

This is the second in a series of papers about some remarkably strong implications some axioms of set theory have for the structure of hereditarily normal (abbreviated T_5) locally compact spaces. The first, [Ny2], showed that under some strong axioms, all hereditarily collectionwise Hausdorff (abbreviated cwH) T_5 manifolds of dimension greater than 1 are metrizable. In this paper we will prove a structure theorem about locally compact spaces (Theorem 4) which has this result as a corollary. In fact, "hereditarily cwH and T_5" can be weakened to "hereditarily strongly cwH" (Definition 1.1). Future papers in this series will delve deeply into the theory of the locally connected case and will also continue the analysis of the general case, making liberal use of the theorems and lemmas of this paper.

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Some results en route to Theorem 4 are of independent interest. Section 1 shows how MA(ω_1) implies that locally compact, hereditarily strongly cwH spaces have a rather simple structure if they are either ω_1-compact or of Lindelöf degree $\aleph_1$. Section 2 utilizes the Proper Forcing Axiom (PFA), which is stronger than MA(ω_1), showing that it implies that hereditarily strongly cwH perfect preimages of ω_1 fall apart into finitely many copies of ω_1 and a paracompact subspace. This is an important ingredient in the proof of Theorem 4, and also in strengthening the main results of Section 1 under the PFA. Section 3 introduces a simple and natural sounding axiom with considerable large cardinal strength, and Theorem 3 there dovetails with a theorem in Section 4 to immediately give Theorem 4.

There is ongoing research into the possibility of dropping all mention of the cwH property in many of the results of this paper. At the end of Section 1 we give some specific ideas for substituting the T_5 property for the hereditarily scwH property in all results of that section.

I am indebted to the referee for calling my attention to the fact [H] that $(*_c)$ is a consequence of the PFA and thus reducing the set-theoretic assumptions in several of the theorems of this paper.

Throughout this paper, "space" will mean "Hausdorff topological space."

1. The global structure of locally compact, hereditarily scwH spaces under MA(ω_1)**.** In this section we will see some striking consequences of MA(ω_1) for locally compact, hereditarily strongly collectionwise Hausdorff spaces.

1.1. Definition. Given a subset D of a set X, an *expansion* of D is a family $\{U_d : d \in D\}$ of subsets of X such that $U_d \cap D = d$ for all $d \in D$. A space X is [*strongly*] *collectionwise Hausdorff* (abbreviated [*s*]*cwH*) if every closed discrete subspace has an expansion to a disjoint [resp. discrete] collection of open sets.

A well known, almost trivial fact is that every normal, cwH space is scwH: if D and $\{U_d : d \in D\}$ are as in 1.1, let V be an open set containing D whose closure is in $\bigcup\{U_d : d \in D\}$; then $\{U_d \cap V : d \in D\}$ is a discrete open expansion of D. Hence the class of hereditarily scwH spaces is somewhat broader than the class of T_5 hereditarily cwH spaces, and the only results which use even normality are Lemma 2.4 and Theorems 2.3 and 3; even there, the hereditary scwH property gives the same topological conclusions.

We begin with a general lemma of independent interest. Recall that a space is called ω_1-*compact* if every closed discrete subspace is countable. This class of spaces is a natural common generalization of countably compact spaces and Lindelöf spaces—the latter because every uncountable closed discrete subspace naturally gives rise to an uncountable open cover with no proper subcover.

1.2. LEMMA. [MA(ω_1)] *In a locally compact*, *hereditarily scwH space*, *every open Lindelöf* [*resp.* ω_1-*compact*] *subset has Lindelöf* [*resp.* ω_1-*compact*] *closure and hereditarily Lindelöf*, *hereditarily separable boundary.*

Proof. Let H be an open Lindelöf [resp. ω_1-compact] subset of a space X as described. It is enough to show that H has a boundary with countable spread, since MA(ω_1) implies every locally compact space of countable spread is hereditarily separable and hereditarily Lindelöf by Szentmiklóssy's Theorem (cf. [Ro]). Of course, any Lindelöf set with Lindelöf boundary has Lindelöf closure, and the same applies if "ω_1-compact" is substituted for "Lindelöf" everywhere.

So let D be a discrete subspace of the boundary of H; since H is open, its boundary is $\overline{H} \setminus H$. Let $C = \overline{D} \setminus D$. Then C is closed in X because D is discrete, and C is disjoint from H because $\overline{H} \setminus H$ is closed. Hence H is a subset of $W = \overline{H} \setminus C$. Also, D is closed in the relative topology of W. Using the fact that W is strongly cwH, let $\{U_d : d \in D\}$ be a discrete-in-W open expansion of D; then $\{U_d \cap H : d \in D\}$ is a discrete-in-H collection of open subsets of H, and is countable since H is ω_1-compact. (In fact, if we pick a point $d' \in U_d \cap H$ for each $d \in D$ and let D' be the resulting set, then D' is closed discrete in W.) Hence D is countable. ■

A similar but simpler argument shows that, under MA(ω_1), every compact hereditarily cwH space is sequentially compact: every countable subset has hereditarily Lindelöf closure, and every G_δ in any locally compact (Hausdorff) space is a point of first countability. If we strengthen the set-theoretic hypothesis to PFA, which implies that every countably compact (Hausdorff) space of countable spread is hereditarily Lindelöf, we can weaken "compact" to "countably compact." Similarly, PFA implies that "locally compact" can be weakened to "regular" in most (though perhaps not all) of Lemma 1.2:

1.3. COROLLARY. [PFA] *In a regular hereditarily scwH space*, *every open Lindelöf* [*resp.* ω_1-*compact*] *subset has Lindelöf* [*resp.* ω_1-*compact*] *closure and hereditarily Lindelöf boundary.*

Proof. The only place in the proof of Lemma 1.2 where local compactness was used was to get from the boundary of H being of countable spread to its being hereditarily Lindelöf; and the PFA is enough to do this for all regular spaces. ■

Lemma 1.2 is already strong enough to have some powerful global consequences, especially in conjunction with a general ZFC result which will be stated after some definitions.

1.4. Definitions. The *Lindelöf degree* or *Lindelöf number* of a space X, denoted by $L(X)$, is the least infinite cardinal number κ such that every open cover of X has a subcover of cardinality $\leq \kappa$. An *alignment* of a space X is a family $\langle X_\alpha : \alpha < \theta \rangle$ of open subspaces of X whose union is X, such that $\overline{X}_\alpha$ is a proper subset of X_β whenever $\alpha < \beta$. The ordinal θ is called the *length* of the alignment, while the *width at* α of the alignment equals the Lindelöf degree of $X_\alpha \setminus \bigcup_{\xi<\alpha} X_\xi$, and the *width* of the alignment is the supremum of the widths at all ordinals $< \theta$. An alignment is *continuous* if $X_\alpha = \bigcup_{\xi<\alpha} X_\xi$ whenever α is a limit ordinal.

The following well known concept is quite weak, being implied by each of the following: normality, the scwH property, countable paracompactness, and even realcompactness:

1.5. Definition. A space X satisfies *Property wD* if every infinite closed discrete subspace D has an infinite subspace D' that expands to a discrete collection of open sets.

Of course, it only takes countable D to verify Property wD.

1.6. Lemma. *Let X be a locally compact space satisfying wD hereditarily, and let $\langle X_\alpha : \alpha < \theta \rangle$ be a continuous alignment of X. For each limit ordinal γ of uncountable cofinality, the boundary of X_γ in X is a closed discrete subspace.*

Proof. Let p be a point on the boundary of X_γ, and let N_p be a compact neighborhood of p in X. Let κ be the cofinality of γ and inductively pick $x_\xi \in N_p \cap X_\gamma$ in such a way that if $\beta_\xi = \sup\{\eta : x_\xi \notin X_\eta\}$ then (1) $\beta_\xi < \beta_\eta$ if $\xi < \eta$, (2) $\{\beta_\xi : \xi < \kappa\}$ is cofinal in γ, and (3) $\beta_\nu = \sup\{\beta_\xi : \xi < \nu\}$ if ν is a limit ordinal. Conditions (1) and (2) are routine while (3) can be arranged using compactness of N_p: the net $\langle x_\xi : \xi < \nu \rangle$ has a convergent subnet, and any limit point is in N_p and also on the boundary of $X_{\beta_\nu} = \bigcup_{\xi<\nu} X_{\beta_\xi}$. Thus N_p meets every $\overline{X}_{\beta_\nu} \setminus X_{\beta_\nu}$ such that ν is a limit ordinal.

If p is not isolated on the boundary B of X_γ, let $\{p_n : n \in \omega\}$ be an infinite discrete subspace of $B \cap \operatorname{int}(N_p)$. It is possible to do this because every infinite (Hausdorff) space contains an infinite discrete subspace. Let $S = X_\gamma \cup \{p_n : n \in \omega\}$. Then the p_n form a closed discrete subspace of S.

Claim. *If $D' = \{p_n : n \in Q\}$ is an infinite subset of $\{p_n : n \in \omega\}$ then it is impossible to expand D' to a discrete collection of open subsets of S.*

This claim contradicts the hypothesis that X satisfies wD hereditarily, and this gives the conclusion that p is relatively isolated in B.

Proof of claim. Suppose $\{U_n : n \in Q\}$ is a disjoint open expansion of D' and let $V_n = U_n \cap \operatorname{int}(N_p)$. Let Lim denote the class of limit ordinals. Let

$$C_n = \{\nu \in \kappa \cap \mathrm{Lim} : \overline{V}_n \cap (\overline{X}_{\beta_\nu} \setminus X_{\beta_\nu}) \neq \emptyset\}.$$

An argument like that for constructing the β_ξ shows C_n is a closed unbounded ("club") subset of κ. Compactness of $\overline{V}_n \cap X_\alpha$ for all α makes it easy to show it is closed, while unboundedness follows by a standard argument: let $\nu(i)$ be defined for $i \in \omega$ by induction so that $\nu(i) < \nu(i+1)$ and $\overline{V}_n \cap (X_{\beta_{\nu(i+1)}} \setminus X_{\beta_{\nu(i)}}) \neq \emptyset$, starting with any desired $\nu = \nu(0) < \kappa$; then the supremum of the $\beta_{\nu(i)}$ is of the form β_ν, and compactness of $\overline{V}_n \cap \overline{X}_{\beta_\nu}$ ensures that $\overline{V}_n \cap (\overline{X}_{\beta_\nu} \setminus X_{\beta_\nu})$ is nonempty.

Now let $C = \bigcap_{n \in Q} C_n$. If $\nu \in C$ then $\overline{V}_n \cap (\overline{X}_{\beta_\nu} \setminus X_{\beta_\nu})$ is a nonempty closed subset of the compact set $N_p \cap (\overline{X}_{\beta_\nu} \setminus X_{\beta_\nu})$, and if we take q_n from $\overline{V}_n \cap (\overline{X}_{\beta_\nu} \setminus X_{\beta_\nu})$ then $\{q_n : n \in D'\}$ is not a closed discrete subspace of S. ∎

The following consequence of Lemmas 1.2 and 1.6 says that, informally speaking, every ω_1-compact, locally compact, hereditarily scwH space is narrow, and can be arbitrarily long depending on what $L(X)$ is.

THEOREM 1. [MA(ω_1)] *Let X be a locally compact, hereditarily scwH, ω_1-compact space. Then X has a continuous alignment $\{X_\alpha : \alpha < \theta\}$ of countable width such that each X_α is ω_1-compact and each $\overline{X}_\alpha \setminus X_\alpha$ is hereditarily Lindelöf. Moreover, if η is a limit ordinal of uncountable cofinality, then $\bigcup\{X_\alpha : \alpha < \eta\}$ has (countable, closed) discrete boundary.*

In a forthcoming paper, large cardinal axioms will be used to remove the cofinality restriction on η. Whether this can be done without using large cardinals is an open problem. It will also be shown in a future paper that ω_1-compactness cannot be dispensed with in Theorem 1.

Before proving Theorem 1, we prove a lemma which will be useful later on as well.

1.7. LEMMA. *In a locally compact space, every point has an open Lindelöf neighborhood.*

Proof. Let X be locally compact, let $x \in X$, and let N be a compact nbhd of x. Since X is Tikhonov, there is a continuous function $f : X \to [0,1]$ sending x to 0 and $\overline{N^{\mathrm{c}}}$ to 1, and then $f^{\leftarrow}[0,1)$ is (open, and) Lindelöf, being the union of the compact sets $f^{\leftarrow}[0, 1 - 1/n]$. ∎

A corollary of Lemma 1.7 is that H need not be open in Lemma 1.2 for its closure to be Lindelöf: any Lindelöf H can be covered by countably many open Lindelöf subsets, and their union will be Lindelöf, and the closure of the union contains that of H. On the other hand, openness of H still seems necessary for the conclusion that the boundary is hereditarily Lindelöf.

Proof of Theorem 1. The "moreover" part is immediate from Lemma 1.6, and we do not need MA(ω_1) to conclude the countability of the boundary of X_η, but only the fact that the boundary is a closed discrete subspace of an ω_1-compact space.

If X is empty, let $\theta = 0$. If X is nonempty and Lindelöf, let $\theta = 1$ and let $X_0 = X$; it is easy to see that this works. If X is not Lindelöf, let X_0 be any nonempty Lindelöf open subset of X. If X_α has been defined and is unequal to X for all $\alpha < \gamma$ and $\gamma = \xi + 1$ for some ξ, then we use ω_1-compactness and Lemmas 1.2 and 1.7 to cover the boundary of X_ξ with countably many open Lindelöf subsets, at least one of which is not a subset of $\overline{X}_\xi$, let U_ξ be their union, and let $X_\gamma = X_{\xi+1} = X_\xi \cup U_\xi$. Clearly X_γ is open and ω_1-compact, and so it has hereditarily Lindelöf boundary, by Lemma 1.2. Also, $X_\gamma \setminus X_\xi$ is Lindelöf, being relatively closed in the Lindelöf open set U_ξ.

If γ is a limit ordinal, then we define $X_\gamma = \bigcup\{X_\alpha : \alpha < \gamma\}$; then X_γ is obviously open. If γ is of countable cofinality, then X_γ is the union of countably many ω_1-compact spaces, hence ω_1-compact. If γ is a limit ordinal of cofinality ω_1, then we utilize local compactness of X to show ω_1-compactness of X_γ, as follows. Any uncountable relatively closed discrete subspace D of X_γ meets each X_α ($\alpha < \gamma$) in a countable set, but ω_1-compactness of X would then give D an accumulation point p outside X_γ. However, if N is a compact neighborhood of p, then $N \cap \overline{X}_\alpha$ is compact for each $\alpha < \gamma$, and one of these sets must contain an infinite subset of D, contradicting closed discreteness of D in X_γ. So now we can use Lemma 1.2 to conclude that the boundary of X is hereditarily Lindelöf.

Finally, if γ is a limit ordinal of cofinality $> \omega_1$, and D is an uncountable discrete subspace of X_γ, then an elementary cofinality argument gives $\alpha < \gamma$ such that $D \cap X_\alpha$ is uncountable, and so D has an accumulation point in X_α and thus cannot be closed in X_γ. Hence X_γ is ω_1-compact, and we deal with its boundary as before. ■

Another general class of locally compact, hereditarily scwH spaces for which Lemma 1.2 has strong consequences is those of Lindelöf degree $\aleph_1$: they have continuous alignments of width ω and length ω_1. Before showing this, we recall a closely related concept which was introduced in [Ny1]:

1.8. Definition. A space X is a *Type I space* if it is the union of an ω_1-sequence $\langle X_\alpha : \alpha < \omega_1 \rangle$ of open subspaces such that $\overline{X}_\alpha \subset X_\beta$ whenever $\alpha < \beta$ and such that $\overline{X}_\alpha$ is Lindelöf for all α. Such an ω_1-sequence will be called a *canonical sequence for* X if moreover $X_\gamma = \bigcup_{\beta<\gamma} X_\beta$ for all limit γ.

Examples of Type I spaces include all spaces with alignments of length ω_1 and width ω. This is because, in any space X with an alignment of countable width, each countable interval of the form $X_{\alpha+\xi} \setminus X_\alpha$ (that is, $\xi < \omega_1$) is Lindelöf, and each set of the form $X_\alpha \setminus \bigcup_{\xi<\alpha} X_\xi$ is Lindelöf, so that the boundary of $\bigcup_{\xi<\alpha} X_\xi$ is a closed, hence Lindelöf subset of X_α. If α is countable one then uses this to establish by induction that the closure of X_α is Lindelöf. For locally compact spaces, we also have:

1.9. Lemma. *Let X be a nonempty locally compact space. The following are equivalent*:

(1) *Every open Lindelöf subset of X has Lindelöf closure*, *and* $L(X) \leq \aleph_1$.
(2) *Every Lindelöf subset of X has Lindelöf closure*, *and* $L(X) \leq \aleph_1$.
(3) *X is of Type I.*
(4) *X has an alignment of width ω and length $\leq \omega_1$.*
(5) *X has a continuous alignment of width ω and length $\leq \omega_1$.*

Proof. (1) and (2) are equivalent by the remark following Lemma 1.7. The remark following 1.8 shows that (4) implies (3), while (5) obviously implies (4).

(3)⇒(2): If X is of Type I, then every Lindelöf subset is contained in some X_α ($\alpha < \omega_1$) and hence has Lindelöf closure; and any space X which is the union of $\leq \aleph_1$ Lindelöf subsets satisfies $L(X) \leq \aleph_1$.

(2)⇒(5): If X is Lindelöf, this is easy: see the beginning of the proof of Theorem 1. If X is not Lindelöf, cover X with nonempty open Lindelöf subsets $\{G_\alpha : \alpha < \omega_1\}$. Let $X_0 = G_0$. If $X_\xi \neq X$ is open and has Lindelöf closure, and $\alpha = \xi + 1$, cover the closure of $X_\xi \cup G_\xi$ with countably many open Lindelöf sets V_n, at least one of which is not a subset of X_ξ, and let X_α be the union of the V_n. Then X_α is also Lindelöf, being the union of countably many Lindelöf subsets, and $X_\alpha \setminus X_\xi$ is closed in X_α and so it is also Lindelöf.

If α is a countable limit ordinal and X_ξ has been defined for $\xi < \alpha$, let X_α be the union of the earlier X_ξ. By the obvious induction hypothesis, X_α is Lindelöf. Since the G_α ($\alpha < \omega_1$) cover X, it is clear that $\langle X_\alpha : \alpha < \omega_1 \rangle$ is a continuous alignment of width ω and length ω_1. ■

1.10. Corollary. [MA(ω_1)] *Let X be a locally compact*, *hereditarily scwH space such that $L(X) = \aleph_1$. Then X has a continuous alignment of length ω_1 and width ω such that each $\overline{X}_\alpha \setminus X_\alpha$ is hereditarily Lindelöf.*

Proof. Follow the construction of the X_α in the proof of Lemma 1.9. Then each X_α is an open Lindelöf set, and so Lemma 1.2 applies. ■

In Section 4 we will strengthen Corollary 1.10 by showing that some powerful axioms imply that every Type I locally compact, hereditarily scwH space can be given a canonical sequence such that every $\overline{X}_\alpha \setminus X_\alpha$ is *countable.*

We close this section by discussing possible alternatives to Lemma 1.2, the only place where MA(ω_1) was used directly. The other uses merely relied on Lemma 1.2., and did not use hereditary separability of the boundaries.

The two key ingredients in the proof of the rest of 1.2 were: 1) using the hereditary strong cwH property and local compactness to show that every open Lindelöf subset has a boundary of countable spread, and 2) using the fact that $\mathrm{MA}(\omega_1)$ implies every locally compact space of countable spread is hereditarily Lindelöf. Unfortunately, 1) does not go through if the hereditary strong cwH property is replaced by hereditary normality: $\mathrm{MA}(\omega_1)$ implies the existence of counterexamples. [On the other hand, if we add local connectedness, we come upon some interesting open problems discussed at the end of this paper.] So we need a different set-theoretic hypothesis, and the following conjecture suggests itself:

Conjecture 1. $2^{\aleph_0} < 2^{\aleph_1}$ implies that every open Lindelöf subset of a locally compact T_5 space has a boundary of countable spread.

Then, if first countability can be eliminated from the following recent result, we would be done:

Theorem A ([ENS]). $2^{\aleph_0} < 2^{\aleph_1}$ *is compatible with the statement that every first countable*, *locally compact space of countable spread is hereditarily Lindelöf.*

Conjecture 1 is especially attractive because a whole arsenal of techniques from topology, set theory, and combinatorics can potentially be brought to bear on it. For example, a negative solution to the following problem, for which we do not seem to have any consistency results, would establish Conjecture 1:

Problem 1. Can the product of more than $\mathfrak{c}$ copies of $[0,1]$ contain a dense subspace which is the union of countably many compact T_5 subsets?

The connection is established like in the second paragraph in the proof of Lemma 1.2, by taking a potential counterexample to Conjecture 1 and producing a locally compact subspace W with a dense open Lindelöf (hence σ-compact) subspace H whose complement is an uncountable closed discrete subspace D of H. Let f be a continuous map of D onto a dense subspace S of $[0,1]^{2^{\aleph_1}}$ such that $|S| = \aleph_1$, and use the normality of W and the Tietze extension theorem to extend the composition of f with each projection map to all of W. This induces a map of W onto a dense subspace of the product. The image of H will be both dense in the product and σ-compact. Now use the fact that the continuous image of a compact T_5 space is likewise T_5.

2. The fine structure of some perfect preimages of ω_1 under the PFA. In this section we will lay part of the foundation for the results of Section 4 by proving a structure theorem (Theorem 2) of independent interest about perfect preimages of ω_1.

2.1. Definition. A map $f : X \to Y$ is *perfect* if it is continuous and closed, and each point of Y has a compact preimage. A subset of ω_1 that is closed and uncountable ("unbounded") is called a *club*. Given a function $f : X \to \omega_1$, a subset of X will be said to be *unbounded* if its image in ω_1 is unbounded. A pair of unbounded subsets of X is *almost disjoint* if the image of their intersection is nonstationary.

2.2. Lemma. *If X is a space and $f : X \to \omega_1$ is continuous, and W is an unbounded copy of ω_1 in X, then W is closed in X.*

Proof. The following simple fact is the key: $W \cap f^{\leftarrow}[0, \alpha]$ is countable for each $\alpha < \omega_1$. The conclusion then follows from the fact that every countable subset of W has compact closure and every point of X has a neighborhood of the form $f^{\leftarrow}[0, \alpha]$ with $\alpha < \omega_1$.

The key fact follows thus: if $f^{\leftarrow}[0, \alpha]$ contains an uncountable subset of W, it contains a closed uncountable subset. Now if U is an open subset of ω_1 containing a club, then U is co-countable, so $W \setminus f^{\leftarrow}[0, \alpha + 1)$ is countable, hence is a subset of $f^{\leftarrow}[0, \gamma]$ for some $\gamma \geq \alpha,\ \gamma < \omega_1$. But this contradicts unboundedness of W. ∎

Theorem 2. [PFA] *Given a hereditarily scwH perfect preimage X of ω_1, there are finitely many disjoint unbounded copies $W_0, \ldots, W_n$ of ω_1 such that the subspace $X \setminus (W_0 \cup \ldots \cup W_n)$ is paracompact.*

Theorem 2 has the quick corollary that every unbounded copy of ω_1 in X must meet one of the W_k in a (closed) unbounded set. Indeed, if it met every one of them in a countable set, then $X \setminus (W_0 \cup \ldots \cup W_n)$ would contain an unbounded copy of ω_1; but every paracompact, countably compact space is compact, and Lemma 2.1 now gives a contradiction.

The proof of Theorem 2 will repeatedly utilize the fact that each perfect preimage of ω_1 is a Type I space (recall Definition 1.8), and one can let $X_\alpha = \pi^{\leftarrow}[0, \alpha)$. Another key fact is that if Y is a perfect preimage of ω_1 with hereditarily Lindelöf fibers, and W is a perfect preimage of ω_1 inside Y, then $Y \setminus W$ is also of Type I. Indeed, if Y_α and W_α are defined as the preimages in Y and W of $[0, \alpha)$, then Y_α is hereditarily Lindelöf and hence so is the closure of $Y_\beta \setminus W_\beta$ for all $\beta < \alpha$. We will be applying the following theorem of the late Zoltán Balogh to these preimages:

Theorem B. [PFA] *Every first countable closed preimage of ω_1 contains an unbounded copy of ω_1.*

Balogh's original proof of Theorem B has never been published, but a similar proof of Theorem B appears in [D]. A completely different proof will appear in [EN2].

Theorem B will be used along with another consequence of the PFA (Theorem C, below) to help generate infinitely many copies of ω_1 to which we can then apply the following ZFC theorem. Recall that a subset of ω_1 which meets every club is called *stationary*. This concept carries over to spaces homeomorphic to ω_1 without change.

2.3. THEOREM. *Let X be a space which is either T_5 or hereditarily scwH, for which there are a continuous $\pi : X \to \omega_1$ and a stationary subset S of ω_1 such that the fiber $\pi^{\leftarrow}\sigma$ is countably compact for all $\sigma \in S$. Then X cannot contain an infinite family of disjoint closed unbounded countably compact subspaces.*

This theorem is a corollary of:

2.4. LEMMA. *Let X be a space which is either hereditarily scwH or T_5 and let $\pi : X \to \omega_1$ be continuous. If $\mathcal{W}$ is a countable family of almost disjoint closed countably compact unbounded subsets of X, then there exists $\delta < \omega_1$ such that $\{W \setminus \pi^{\leftarrow}[0, \delta] : W \in \mathcal{W}\}$ is a discrete collection of closed sets.*

Proof. By making a preliminary choice of δ, we may assume $\mathcal{W}$ is actually disjoint. This is because the image of the intersection of any two members of $\mathcal{W}$ has countable image under π: the intersection, being closed, is also countably compact, and if it were unbounded its image would be a club.

Any finite collection of disjoint closed sets is discrete. So suppose $\{W_n : n \in \omega\}$ is a one-to-one listing of $\mathcal{W}$. First we show that there cannot be more than countably many points of W_0 in the closure of $\bigcup_{n=1}^{\infty} W_n$; then the same argument applies to the other W_n's and we obtain $\tau < \omega_1$ such that $\{W_n \setminus \pi^{\leftarrow}[0, \tau] : n \in \omega\}$ is relatively discrete, meaning that no point of any member of the collection is in the closure of the union of the other members. A slight modification of the argument will then take us to a discrete family.

Suppose that the points of W_0 in the closure of $\bigcup_{n=1}^{\infty} W_n$ have uncountable image under π. Let $\{x_\xi : \xi \in \omega_1\}$ list uncountably many of these points, chosen in such a way that $\pi(x_\alpha) > \sup\{\pi(x_\xi) : \xi < \alpha\}$ for all $\alpha < \omega_1$. Let C_0 be the derived set of $A = \{\pi(x_\xi) : \xi \in \omega_1\}$ in ω_1; then $C_0 (= \bar{A} \setminus A)$ is clearly a club. Let $Y = X \setminus (\pi^{\leftarrow} C_0 \cap W_0))$. Then $\{x_\xi : \xi \in \omega_1\}$ is a closed discrete subset of Y. Let $\{U_\xi : \xi \in \omega_1\}$ be an expansion of $\{x_\xi : \xi \in \omega_1\}$ to a family of disjoint open sets, chosen using continuity of π so that $U_\beta \subset \pi^{\leftarrow}(\gamma_\beta, \pi(x_\beta)]$ for all $\beta < \omega_1$, where $\gamma_\beta = \sup\{\pi(x_\xi) : \xi < \beta\}$. For some $n > 1$, W_n meets uncountably many U_β. Since W_n is countably compact, continuity of π implies $\bigcup_{\xi \in \omega_1} U_\xi \cap W_n$ has a limit point in $\pi^{\leftarrow}\{\gamma_\alpha\}$ for uncountably many α, and none of these γ_α are in $\{\pi(x_\xi) : \xi \in \omega_1\}$, and none of the limit points are in the union of the U_ξ. But if X is either scwH or T_5, then $\{U_\xi : \xi < \omega_1\}$ can be chosen so that this does not happen! To show this in the latter case, use the fact that $\{x_\xi : \xi \in \omega_1\}$ and $\pi^{\leftarrow} C_0 \cap Y$ are disjoint closed subsets of Y. Hence there exists τ_0 such that no point of $W_0 \setminus \pi^{\leftarrow}[0, \alpha_0]$ is in the closure of the union of the other W_n. Repeating this argument, obtain $\tau_n < \omega_1$ for all n and let $\tau = \sup_n \tau_n$; then τ is as desired.

Now suppose that $Z = \bigcup\{W_n \setminus \pi^{\leftarrow}[0,\tau] : n \in \omega\}$ is not closed, and moreover, that for each $\alpha < \omega_1$ there exists x in $\overline{Z} \setminus Z$ such that $\pi(x) > \alpha$. Relative discreteness of the sets $W_n \setminus \pi^{\leftarrow}[0,\tau]$ implies that $\overline{Z} \setminus Z$ is closed in X. So we can define $\{x_\xi : \xi \in \omega_1\}$ and C_0 as above, but with $\overline{Z} \setminus Z$ in place of W_0, this time letting Y be $X \setminus (\pi^{\leftarrow} C_0 \cap \overline{Z} \setminus Z)$. The rest of the argument is as before, yielding $\beta < \omega_1$ such that no point of $X \setminus \pi^{\leftarrow}[0,\beta]$ is in $\overline{Z} \setminus Z$. Now $\delta = \max\{\tau,\beta\}$ is as desired. ∎

Proof of Theorem 2.3. If there were such a family $\{W_n : n \in \omega\}$, then there would be one that is discrete, by 2.4. The image of each W_n is a club, hence so is $C = \bigcap_{n=0}^{\infty} \pi^{\rightarrow} W_n$. But $\{W_n \cap \pi^{\leftarrow}\{\sigma\} : n \in \omega\}$ is not a discrete collection if $\sigma \in C \cap S$, a contradiction. ∎

The example of $\omega_1 \times [0,1]$ and the projection to the first coordinate show how important the topological restrictions in 2.3 are: the projection map to the first coordinate has compact fibers, and the domain is normal and hereditarily collectionwise Hausdorff, yet contains a family of $\mathfrak{c}$ disjoint unbounded copies of ω_1.

To produce the needed copies of ω_1 for proving Theorem 2, we will utilize the following concepts and results. The concepts, including the CC_{ij} axioms we give below, are discussed in [EN1] along with a number of related axioms, and many topological applications of the various axioms are given. Two of the related axioms were introduced earlier in [AT] and [T], where several applications to combinatorial set theory were shown along with the compatibility of the respective axioms with CH.

2.5. Definition. An ideal $\mathcal{J}$ of subsets of a set S is *countable-covering* if for each countable subset Q of S, the ideal $\mathcal{J}{\restriction}Q$ is countably generated.

In other words, for each $Q \in [S]^\omega$ there is a countable subcollection $\{J_n^Q : n \in \omega\}$ of $\mathcal{J}$ such that every member J of $\mathcal{J}$ that is a subset of Q satisfies $J \subset J_n^Q$ for some n.

A dual concept is that of a *P-ideal* on S. This is an ideal $\mathcal{I}$ of countable subsets of S such that for each countable subcollection $\{I_n : n \in \omega\}$ of $\mathcal{I}$, there exists $I \in \mathcal{I}$ such that $I \setminus I_n$ is finite for all n.

The following axiom is a consequence of the PFA (see [H]):

($*_c$) Let $\mathcal{I}$ be a P-ideal on a cardinal κ of uncountable cofinality. Then either

(i) there is a closed uncountable subset A of κ with $[A]^\omega \subset \mathcal{I}$, or
(ii) there is a stationary subset B of κ such that $B \cap I$ is finite for all $I \in \mathcal{I}$.

The following axioms, the second of which was originally derived from the stronger axiom PFA$^+$ (see [Ny2]), are easy consequences of $(*_c)$:

2.6. Definition. *Axiom* CC_{21} [resp. *Axiom* CC_{22}] [resp. *Axiom* CC_{23}] is the axiom that for each countable-covering ideal $\mathcal{J}$ on a stationary subset S of ω_1, either

(i) there is a stationary $B \subset S$ such that $[B]^\omega \subset \mathcal{J}$, or

(ii) there is an uncountable [resp. stationary] [resp. closed uncountable] $A \subset S$ such that $A \cap J$ is finite for all $J \in \mathcal{J}$.

Substituting the weaker "uncountable" for "stationary" in (i) gives the axioms CC_{1j}, $j = 1, 2, 3$.

2.7. Theorem. *The PFA implies* $(*_c)$ *which implies* CC_{ij} *for* $i = 1, 2$ *and* $j = 1, 2, 3$.

Outline of proof. The first implication is shown in [H]. For the second, it is clearly enough to show that $(*_c)$ implies CC_{23}.

Let $\mathcal{J}'$ be the ideal of countable subsets of ω_1 which meet S in a subset of $\mathcal{J}$, let

$$\mathcal{I} = \{I \in [\omega_1]^{\le\omega} : I \cap J' \text{ is finite for all } J' \in \mathcal{J}'\}$$

and use the easy fact [EN1] that $\mathcal{I}$ is a P-ideal whenever $\mathcal{J}'$ is countable-covering. ■

Axiom CC_{12} is also an easy corollary of Axiom 0^* of [NyP] which in turn is a consequence of TOP + MA(ω_1), both of which follow from the PFA.

The proof of Theorem 2 makes use of the following result from [EN1].

Theorem C. *Assume* CC_{12} *and let* X *be a locally compact*, *cwH*, *Type I space. If* X *is either scwH or locally ccc*, *then either*:

(1) X *is paracompact*, *or*

(2) X *contains a perfect preimage of* ω_1 *that is closed in* X.

Of course, every perfect preimage of ω_1 is countably compact, and now we obtain:

2.8. Corollary. [PFA] *If* X *is a locally compact hereditarily scwH Type I space*, *then* X *is either paracompact or contains a copy of* ω_1.

Proof. Let $\langle X_\alpha : \alpha < \omega_1\rangle$ be a canonical sequence for X. Then for each limit ordinal γ, the subspace $\overline{X}_\gamma \setminus X_\gamma$ is locally compact, and is hereditarily Lindelöf by Lemma 1.2. These properties easily carry over to sets of the form $Y \cap X_\alpha$ $(\alpha < \omega_1)$, where

$$Y = \bigcup\{\overline{X}_\gamma \setminus X_\gamma : \gamma \text{ is a limit ordinal}\}.$$

Note that Y is closed in X and of Type I, itself. As is well known, every hereditarily Lindelöf, locally compact space is first countable. (Indeed, if one covers the complement of $\{y\} \subset Y$ with an ascending sequence of open sets, with the closure of each being a subset of the next open set in the sequence, then the complements of the closures form a local base at y.) By Theorems B and C, either Y contains a copy of ω_1, or else it is Lindelöf. In the latter case, $\overline{X}_\gamma \setminus X_\gamma$ is empty for all but countably many γ, and so (2) of Theorem C fails. ∎

If Fleissner's Axiom R is added to the set-theoretic hypotheses in Corollary 2.8, then one can eliminate "Type I" altogether and even weaken "scwH" to "ω_1-scwH." This is shown, in effect, in a posthumous paper by Balogh [B]. Balogh's actual theorem weakens "PFA" to "MA(ω_1)" and weakens the conclusion to "X is either paracompact or contains a perfect preimage of ω_1," but a combination of his argument and the proof of Corollary 2.8 readily gives the stronger conclusion under PFA. Both PFA$^+$ and Martin's Maximum are known to imply the combination of PFA + Axiom R.

Proof of Theorem 2. If X admits a perfect map $\pi : X \to \omega_1$ then each set of the form $U_\alpha = \pi^{\leftarrow}[0,\alpha]$ is compact since $[0,\alpha]$ is compact. Hence X is locally compact. For each limit ordinal γ, the set $V_\gamma = \pi^{\leftarrow}[0,\gamma)$ is open and Lindelöf, and so by Lemma 1.2, V_γ has a hereditarily separable, hereditarily Lindelöf boundary, $\overline{V}_\gamma \setminus V_\gamma \subset \pi^{\leftarrow}\{\gamma\}$. Let $Y = \bigcup\{\overline{V}_\gamma \setminus V_\gamma :$ γ is a limit ordinal$\}$. Then Y is a perfect preimage of ω_1 since it is clearly closed in X and has the set Λ of limit ordinals of ω_1 as its image. Moreover, $X \setminus Y$ falls apart into the relatively clopen sets $D_\xi \cup A_\xi$ where $D_\xi = \pi^{\leftarrow}(\lambda_\xi + 1, \lambda_\xi + \omega)$ and $A_\xi = \pi^{\leftarrow}\{\lambda_\xi\} \setminus Y$, and where λ_ξ is the ξth member of $\Lambda \cup \{0\}$. Since both D_ξ and A_ξ are Lindelöf, it follows that $X \setminus Y$ is an open paracompact subspace of X. We will now work on Y, breaking it apart into a finite union of copies of ω_1 and a subspace whose union with $X \setminus Y$ is paracompact.

Since Y is a perfect preimage of ω_1, it is Type I and is not paracompact, so Corollary 2.8 gives a copy W_0 of ω_1 in Y. The discussion preceding Theorem B shows that $Y \setminus W_0$ is of Type I. Hence it is either paracompact or itself contains a copy W_1 of ω_1. Proceeding in this way, we obtain a sequence of disjoint copies W_i of ω_1, which must terminate after finitely many steps because of Theorem 2.3, with Y playing the role of X. Thus $Z = Y \setminus (W_0 \cup \ldots \cup W_n)$ is paracompact for some n. A standard fact about locally compact paracompact spaces is that they are topological direct sums of Lindelöf subspaces [E, 5.1.27]. Using this, let $\{U_\psi : \psi \in \Psi\}$ be a partition of Z into clopen Lindelöf subspaces. Then each U_ψ is a subset of some $\pi^{\leftarrow}[0,\alpha]$ and there can be only countably many U_ψ contained in any of these subspaces. By a standard argument [see the proof that C_n is a club in 1.6], one easily shows that the following subset C of ω_1 is closed unbounded:

$$C = \{\gamma \in \Lambda : \forall \beta < \gamma, \text{ if } U_\psi \text{ meets } \pi^{\leftarrow}[0,\beta] \text{ then } U_\psi \subset f^{\leftarrow}[0,\gamma)\}.$$

This enables us to partition all of $X \setminus (W_0 \cup \ldots \cup W_n)$ into clopen Lindelöf subsets, as follows. For each $\gamma \in C \cup \{0\}$, let γ^* be the immediate successor of γ in C and let

$$\mathcal{U}_\gamma = \{U_\psi : U_\psi \cap \pi^{\leftarrow}[\gamma, \gamma^*) \neq \emptyset\}, \quad \mathcal{V}_\gamma = \{D_\xi \cup A_\xi : \lambda_\xi \leq \gamma \leq \gamma^*\}.$$

Then if $H_\gamma = (\bigcup \mathcal{U}_\gamma) \cup (\bigcup \mathcal{V}_\gamma)$ for all $\gamma \in C \cup \{0\}$, it is easy to see that the set of all these H_γ is as desired. Of course, any space that can be partitioned into clopen Lindelöf subsets is paracompact. ∎

3. An axiom on crowding of functions, and a topological application. Our applications of the PFA have thus far not required any large cardinal axioms. In this section, however, we introduce an axiom whose consistency implies that of measurable cardinals. It will be used along with the PFA in Section 4 to do for spaces of Lindelöf degree $\aleph_1$ something similar to what Theorem 1 did for ω_1-compact spaces at ordinals of uncountable cofinality.

Axiom F. Any family of $\aleph_2$ functions from ω_1 to ω has an infinite subfamily that is bounded on a stationary set.

Axiom F is an easy consequence of the axiom designated SSA in [Ny2]. The combination SSA + PFA is consistent if it is consistent that there is a cardinal κ with a stationary set of supercompact cardinals below it [S, p. 660]. Axiom SSA states that there is a stationary subset S of ω_1 such that the ideal of nonstationary subsets of S is $(\omega_2, \omega_2, \omega)$-saturated. This means that for each collection of ω_2 stationary subsets of S, there is a subcollection $\mathcal{E}$ of cardinality ω_2 such that every countable subfamily of $\mathcal{E}$ has stationary intersection. To see that SSA implies Axiom F, let $\{f_\alpha : \alpha < \omega_2\}$ be a family of functions from ω_1 to ω, and, for each α, let n_α be such that $f_\alpha^{\leftarrow}\{n_\alpha\} \cap S$ is stationary. Let n satisfy $n = n_\alpha$ for ω_2-many α. By SSA there is an infinite $\mathcal{A} \subset \omega_2$ and a stationary $E \subset S$ such that $f_\alpha(\xi) = n$ for all $\alpha \in A$ and all $\xi \in E$.

Further information on Axiom F will be provided in a forthcoming paper [Ny3]. The following application of Axiom F will be combined in Section 4 with a lemma that uses the PFA, to produce the main result of this paper (Theorem 4).

Theorem 3. [Axiom F] *Let X be a space which is either T_5 or hereditarily scwH, for which there is a continuous $\pi : X \to \omega_1$ with locally compact σ-compact fibers. Then X cannot contain an almost disjoint family of $\aleph_2$ closed unbounded countably compact subsets.*

Proof. Suppose $\mathcal{W} = \{W_\nu : \nu < \omega_2\}$ is an almost disjoint family of $\aleph_2$ closed unbounded countably compact subsets. By Theorem 2.3, it suffices to show that X has a subspace Y such that $Y \cap \pi^{\leftarrow}\{\sigma\}$ is compact for all σ

in some stationary subset S of ω_1, and to find an infinite subfamily $\mathcal{W}_0$ of $\mathcal{W}$ such that $W \subset Y$ for all $W \in \mathcal{W}_0$.

Let $\pi^{\leftarrow}\{\alpha\}$ be the ascending union of compact sets K^n_α ($n \in \omega$) for each $\alpha \in \omega_1$, with each K^n_α in the interior of K^{n+1}_α; if $\pi^{\leftarrow}\{\alpha\}$ is compact, let it equal K^n_α for all n. Let $C_\nu = \pi^{\rightarrow} W_\nu$ for all $\nu < \omega_2$.

Let $f_\nu : \omega_1 \to \omega$ be defined by letting $f_\nu(\xi)$ be the least n such that $\pi^{\leftarrow}\{\xi\} \cap W_\nu \subset K^n_\xi$ whenever $\xi \in C_\nu$. Such an n exists since $\{\xi\} \cap W_\nu$ is countably compact and so any countable open cover has a finite subcover. If $\xi \notin C_\nu$ let $f_\nu(\xi) = 0$. Axiom F now gives a stationary subset S of ω_1 and an infinite subset Z of ω_2 and $n \in \omega$ such that $f_\nu(\xi) < n$ for all $\nu \in Z$ and all $\xi \in S$. Let $C = \bigcap\{C_\nu : \nu \in Z\}$. Then C is a club subset of ω_1 and $\pi^{\leftarrow}\{\xi\} \cap W_\nu \subset K^n_\xi$ for all $\xi \in S \cap C$ and all $\nu \in Z$. Let $X_\alpha = \pi^{\leftarrow}[0, \alpha)$ for all α and let

$$Y = \bigcup\{K^n_\xi : \xi \in S \cap C\} \cup \bigcup\{\overline{X}_\alpha \setminus X_\alpha : \alpha \notin S \cap C\}.$$

Then Y is as claimed above, and we can let $\mathcal{W}_0 = \{W_\nu : \nu \in Z\}$. ∎

4. The structure theorem for the Lindelöf degree $\aleph_1$ case. In this section, we combine Theorem 3 with another theorem (4.4 below) to arrive at:

THEOREM 4. [PFA + Axiom F] *Every locally compact, hereditarily scwH space X satisfying $L(X) = \aleph_1$ is a Type I space for which there is a canonical sequence $\langle X_\alpha : \alpha < \omega_1\rangle$ such that the boundary of each X_α is countable.*

To appreciate the power of Theorem 4, we now show how to get from it to the main theorem of [Ny2] (Corollary A below), via:

4.1. COROLLARY. [PFA + Axiom F] *Every hereditarily scwH manifold of dimension greater than* 1 *is metrizable.*

Proof. The statement uses the definition of a manifold which states that it is a connected Hausdorff space in which every point has a neighborhood homeomorphic to $\mathbb{R}^n$ for some (unique) n. We will use the same axioms as for Theorem 4. In particular, since we are assuming MA(ω_1), any hereditarily scwH manifold X has the property that every open Lindelöf subset has Lindelöf closure.

CLAIM. *X is of Type I.*

Assuming this claim for the moment, we see that the Lindelöf degree of X is either $\aleph_0$ or $\aleph_1$, and if it is $\aleph_0$ then X has a countable open cover by copies of $\mathbb{R}^n$ and so it is second countable, hence metrizable. If it is $\aleph_1$ then we use Theorem 4 to conclude that X has a canonical sequence in which every $\overline{X}_\alpha \setminus X_\alpha$ is countable. But if $\dim(X) > 1$ this means $\overline{X}_\alpha \setminus X_\alpha$ is actually empty, because it disconnects X and hence it also disconnects any nbhd of any point of $\overline{X}_\alpha \setminus X_\alpha$; but no countable set disconnects $\mathbb{R}^n$ if $n > 0$. So $X_\alpha = \overline{X}_\alpha$ for all α, and any nonempty set of this form is all of X because X is connected. So X is Lindelöf after all.

Proof of claim. Let X_0 be any open subset of X that is homeomorphic to $\mathbb{R}^n$. Suppose X_α has been defined for all $\alpha < \beta$ to be open with Lindelöf closure, and that $\overline{X}_\xi \subset X_\alpha$ whenever $\xi < \alpha$. Let $Y_\beta = \bigcup\{X_\alpha : \alpha < \beta\}$; if β is a limit ordinal then Y_β is Lindelöf, while if β is not a limit ordinal then either $\beta = 0$ or $\beta = \alpha + 1$ for some α, and in each case, Y_β has Lindelöf closure. Cover the closure of Y_β with countably many open copies of the second countable space $\mathbb{R}^n$, each of which has compact closure, and let X_β be the union of this cover. Then X_β is second countable and open, and its closure is thus Lindelöf. By first countability of X, $\bigcup\{X_\alpha : \alpha < \omega_1\}$ is closed in X and it is clearly open, so it is all of X because X is connected. ∎

Since every T_5 hereditarily cwH space is hereditarily scwH, we get the main theorem of [Ny2]:

Corollary A. *It is consistent, modulo large cardinals, that every T_5 hereditarily cwH manifold of dimension greater than* 1 *is metrizable.* ∎

In a forthcoming paper, Theorem 4 will be applied to arbitrary locally compact, locally connected, hereditarily strongly cwH spaces, to show a wealth of strong properties for them, one of which is that each component is ω_1-compact—and then Theorem 1 already says a lot about such spaces. The following folklore lemma is the first step in the proof of Theorem 4.

4.2. Lemma. *Let $\pi : X \to \omega_1$ be continuous and let W be an unbounded copy of ω_1 in X. There is a club subset C of ω_1 and a club subset Ω of W such that the restriction of π to Ω is a bijection of Ω to C.*

Proof. Let $f : \omega_1 \to X$ be an embedding with range W, and let Ω be the image of f restricted to

$$C = \{\alpha : \pi(f(\xi)) < \alpha \text{ for all } \xi < \alpha \text{ and } \pi(f(\alpha)) = \alpha\}.$$

Clearly, the restriction of π to Ω is a bijection. It is easy to see that C is closed: if $\alpha_i \nearrow \alpha$ and $\alpha_i \in C$ for all $i \in \omega$, and $\xi < \alpha$, then $\xi < \alpha_i$ for some i and so $\pi(f(\xi)) < \alpha_i < \alpha$; and since $\langle f(\alpha_i) : n \in \omega\rangle$ converges to $f(\alpha)$, it follows from the continuity of π that $\langle \pi(f(\alpha_i)) : i \in \omega\rangle = \langle \alpha_i : i \in \omega\rangle$ converges to $\pi(f(\alpha)) = \alpha$. The proof that C is unbounded is done by a standard argument using unboundedness of W. To wit: given $\alpha \in \omega_1$, let $\alpha_0 = \alpha$; then, with α_i defined, let $\beta_i = \sup\{\pi(f(\xi)) : \xi < \alpha_i\}$ and let α_{i+1} be a countable ordinal γ such that both $\pi(f(\gamma))$ and γ are greater than both α_i and β_i. Such a γ exists since W meets each set of the form $\pi^{\leftarrow}[0, \eta]$ in a countable set: $W \cap \pi^{\leftarrow}[0, \eta]$ is relatively clopen in W and so it must be either countable or co-countable in W. Let $\delta = \sup\{\alpha_i : i \in \omega\}$. Then $\pi(f(\alpha_i))$ converges to $\pi(f(\delta))$ and its terms are sandwiched in between the α_i, so it converges to δ; and the other criterion for δ being in C is trivially satisfied. ■

The following lemma features progressively stronger hypotheses (see 2.7), the one for part (a) being simply ZFC.

4.3. Lemma. *Let $\langle X_\alpha : \alpha < \omega_1\rangle$ be a canonical sequence for a Type I space X. If X is locally compact and hereditarily strongly cwH, and E is a stationary subset of ω_1, and $x_\alpha \in \overline{X}_\alpha \setminus X_\alpha$ for all $\alpha \in E$, then*:

(a) *$N = \{\alpha \in E : x_\alpha \notin \mathrm{cl}\{x_\beta : \beta < \alpha\}$ is nonstationary.*

(b) *If CC_{22} holds, then there is a stationary $S \subset E$ and a perfect preimage Z of ω_1 such that $\{x_\alpha : \alpha \in S\} \subset Z$.*

(c) *If PFA holds, then there is a stationary $S \subset E$, a club $C \subset \omega_1$, and an embedding $g : C \to X$ such that $g(\alpha) \in \overline{X}_\alpha \setminus X_\alpha$ for all $\alpha \in C$ and $g(\alpha) = x_\alpha$ for all α in S.*

Proof of (a). Suppose N is stationary. Each x_η has a nbhd that meets $\{x_\alpha : \alpha \in S\}$ in $\{x_\alpha : \alpha \le \eta, \alpha \in S\}$, and so $\{x_\eta : \eta \in N\}$ is a discrete subspace of X. Arguing as in the proof of Lemma 1.2, and using the Pressing-Down Lemma (PDL), we get a contradiction to "hereditarily strongly cwH," as follows. Let $D = \{x_\eta : \eta \in N\}$; then D is closed discrete in the relative topology of the open subspace $V = X \setminus (\overline{D} \setminus D)$. Let $\{U_\eta : \eta \in N\}$ be a discrete-in-V open expansion of $\{x_\eta : \eta \in N\}$. Since each x_α is in the closure of X_α, it follows that U_α meets X_α; and, since the sequence is canonical, U_α actually meets X_β for some $\beta < \alpha$ if α is a limit ordinal. Since the set of limit points of N in N is itself stationary, we can use the PDL to obtain $\eta < \omega_1$ such that uncountably many U_α meet X_η; but this violates the claim that the U_α's are a discrete collection, since $\overline{X}_\alpha$ is Lindelöf. [Compare the proof of Lemma 1.6.]

Proof of (b). Let $\mathcal{J}$ be the ideal of all countable $J \subset E$ such that $\{x_\eta : \eta \in J\}$ has compact closure.

Claim 1. *$\mathcal{J}$ is countable-covering.*

Once Claim 1 is proved, apply CC_{22} to $\mathcal{J}$. The second alternative in the statement of CC_{22} is impossible since it would give an uncountable $D = \{x_\eta : \eta \in B\}$ such that B is stationary, and such that no infinite subset of D has compact closure; but in a locally compact space like X, this would mean D is closed discrete, and we obtain a contradiction like in the proof of (a). Thus there is a stationary $S \subset W$ such that $[S]^\omega \subset \mathcal{J}$. Let Z be the closure of $\{x_\eta : \eta \in S\}$. Let $g : Z \to \omega_1$ be defined by taking each point of $Z \cap \overline{X}_\alpha \setminus X_\alpha$ to α.

Claim 2. *This defines g on all of Z, and g is a perfect map onto its range.*

Once this claim is proved, part (b) follows from the well known fact that ω_1 is homeomorphic to each of its club subsets and from the fact that the range of g is closed in ω_1. This latter fact follows from the fact that if $g(z_n) \nearrow \gamma$, then any accumulation point of $\{z_n : n \in \omega\}$ [a set with compact closure] is in $\overline{X}_\gamma \setminus X_\gamma$.

Proof of Claim 2. Let $Y = \bigcup\{\overline{X}_\alpha \setminus X_\alpha : \alpha \in \omega_1\}$. Clearly, Y is closed in X and Z is a closed subset of Y. Let $h : Y \to \omega_1$ extend g to Y in the obvious way. By the nature of a canonical sequence, the preimage $h^{\leftarrow}(\beta, \alpha]$ is open in Y for each basic open set $(\beta, \alpha] = [\beta + 1, \alpha]$ and this is also true of $h^{\leftarrow}[0, \alpha]$. So h and g are continuous. To show g is perfect, we use the well known fact that if B is a locally compact space (more generally, a k-space), then a continuous function $f : A \to B$ is perfect if, and only if, the preimage of every compact subset of B is compact [E, 3.7.18]. Now, every compact subset of ω_1 is countable and is a subset of some $[0, \alpha]$. And every closed subspace of $Z \cap f^{\leftarrow}[0, \alpha]$ is compact, being in the compact closure of $\{x_\eta : \eta \le \alpha,\ \eta \in S\}$.

Proof of Claim 1. Let $Q \subset [E]^\omega$ and let $\{V_n : n \in \omega\}$ be an ascending cover of the Lindelöf closure of $\{x_\eta : \eta \in Q\}$ by open subsets of X, each V_n having compact closure. Let $J_n = \{\xi \in Q : x_\xi \in V_n\}$. Then $J_n \in \mathcal{J}$ for all n. Now suppose that $J \in \mathcal{J}$ and $J \subset Q$. Since the closure of $\{x_\eta : \eta \in J\}$ is a subset of the closure of $\{x_\eta : \eta \in Q\}$, it is is covered by $\{V_n : n \in \omega\}$. Let $\{V_n : n \in F\}$ be a finite subcover. Then $J \subset \bigcup\{J_n : n \in F\}$.

Proof of (c). We apply Theorem 1 to the space

$$X_0 = Z \cap g^{\leftarrow}(\mathrm{ran}(g) \cap \Lambda)$$

where Z is obtained as in (b) above. By Lemma 1.2, each $g^{\leftarrow}\{\alpha\} \cap X_0$ is hereditarily Lindelöf, hence sequentially compact. Theorem 2 gives copies $W_0, \ldots, W_n$ of ω_1 in X_0 such that $X_0 \setminus (W_0 \cup \ldots \cup W_n)$ is paracompact. By part (a), $\{x_\alpha : \alpha \in E\}$ is not paracompact and neither is any subspace indexed by a stationary set; in fact, a use of the PDL as in the proof of (a) shows that no open cover by countable sets can have even a point-countable open refinement. Hence $S_0 = \{\alpha : x_\alpha \in W_k\}$ must be stationary for some k. Let $f : \omega_1 \to X_0$ be an embedding with range W_k. By cutting down W_k as in Lemma 4.2 if necessary, we obtain a club subset C_0 of $\overline{E} \cap \Lambda$ such that the restriction of $g \circ f$ to C_0 is the identity on C_0. Let C_1 be the set of all limit points of $S_0 \cap C_0$ in C_0. Apply part (a) again to show that the set of all α in $S_0 \cap C_1$ such that $f(\alpha) \neq x_\alpha$ is nonstationary. Let C be a club subset of C_1 missing this nonstationary set and let $S = S_0 \cap C$. ■

Theorem 4 is now an immediate corollary of Theorem 3 and:

4.4. Lemma. [PFA] *If X is a locally compact, hereditarily scwH Type I space with a canonical sequence $\langle X_\alpha : \alpha < \omega_1\rangle$ such that $\overline{X}_\alpha \setminus X_\alpha$ is uncountable for a stationary collection of α's, then there is a subspace Y of X and a continuous $\pi : Y \to \omega_1$ for which there is an almost disjoint collection of $\aleph_2$ unbounded copies of ω_1 in Y, such that $\pi^{\leftarrow}\{\xi\}$ is locally compact and σ-compact for each $\xi \in \omega_1$.*

Proof. Let E be the stationary set described. If $\alpha \in E$ then $\overline{X}_\alpha \setminus X_\alpha$ is of cardinality $\geq \mathfrak{c} \geq \aleph_2$. [Actually, equality holds in both places, the latter because we are assuming PFA, but we will not be needing this.] This is because $\overline{X}_\alpha \setminus X_\alpha$ is the countable union of compact sets; because every uncountable, hereditarily Lindelöf space has a closed dense-in-itself subspace; and because every compact Hausdorff dense-in-itself space has at least $\mathfrak{c}$ points.

Define a family $\{h_\nu : \nu < \omega_2\}$ of embeddings of club subsets C_ν of ω_1 using Lemma 4.3(c) as follows. Let $x_\alpha \in \overline{X}_\alpha \setminus X_\alpha$ for all $\alpha \in E$, let g and C be defined exactly as in 4.3(c), and let $h_0 = g$, $C_0 = C$. If $\nu < \omega_2$, and h_μ has been defined for all $\mu < \nu$, let x^ν_α be taken from the part of $\overline{X}_\alpha \setminus X_\alpha$ (where $\alpha \in E$) that is outside the union of the ranges of the earlier h_μ. Let h_ν be again like g in 4.3(c) and let C_ν be the domain of h_ν. Then $h^{\rightarrow}_\nu C_\nu$ meets each earlier $h^{\rightarrow}_\mu C_\mu$ in at most a countable set. This is where CC_{22} (as opposed to just CC_{12}) comes in: unless two copies of ω_1 agree on a club, their intersection is only countable; and CC_{22} is being used to ensure that $h^{\rightarrow}_\nu C_\nu$ and $h^{\rightarrow}_\mu C_\mu$ do not agree on a club. On the other hand, two copies of ω_1 could agree on a club and yet have uncountable symmetric difference; and CC_{12} alone could conceivably be giving us $\aleph_2$ copies of ω_1 where any two agree on a club subset of each but also fail to share uncountable subsets of each.

Finally, let $Y = \bigcup\{\overline{X}_\xi \setminus X_\xi : \xi \in C\}$ and let π be the composition of the obvious projection to C with the unique order-preserving function from C to ω_1. The sets $h^{\rightarrow}_\nu (C_\nu \cap C)$ ($\nu < \omega_2$) are the desired family. ∎

Since the T_5 property is enough in Theorem 3, it would be most interesting to know the answer to:

Problem 2. Is there a model of set theory in which Axiom F holds along with the statement that every T_5 manifold of dimension > 1 is either metrizable or contains a family of $\aleph_2$ almost disjoint copies of ω_1?

In such a model, the argument for Corollary 4.1 would tell us that every T_5 manifold of dimension > 1 is metrizable. The model for Theorem 4 has not been ruled out; in fact, the following problem is still unsolved:

Problem 3. Is every T_5 manifold collectionwise Hausdorff?

If "normal" is substituted for T_5 we have a favorite problem of Mary Ellen Rudin, which is also unsolved, even if "locally compact, locally connected space" is substituted for "manifold". We know that $V = L$ implies the answer to all of these questions is Yes, because $V = L$ implies that every locally compact normal space is collectionwise Hausdorf; but it is not known whether a Yes answer to any of these questions is compatible with $\mathrm{MA}(\omega_1)$. Since every open subset of a manifold is a topological direct sum of manifolds of the same dimension, and since a space is T_5 iff every open subset is T_5, the statement that every T_5 manifold is cwH has the corollary that every T_5 manifold is hereditarily cwH, hence hereditarily scwH. So an affirmative answer to Problem 3 also gives one to Problem 2.

Finally, one might even ask whether "hereditarily strongly cwH" is a genuine weakening of "T_5 + hereditarily cwH" in the presence of local compactness:

PROBLEM 4. Is every locally compact, hereditarily scwH space (hereditarily) normal?

The point of the parenthetical "hereditarily" is that a space is hereditarily normal (i.e., T_5) iff every open subspace is normal, while every open subspace of a locally compact space is locally compact. Remarkably enough, we do not even seem to have any consistent answers to Problem 4, not even if "locally compact" is weakened to "regular".

Added in proof (January 2003). The author has found an example, using the axiom $\diamondsuit$, of a locally compact hereditarily scwH non-normal space. However, Problem 4 and its generalization to regular spaces remain open under the set-theoretic assumptions used in this article.

References

[AT] U. Abraham and S. Todorčević, *Partition properties of ω_1 compatible with CH*, Fund. Math. 152 (1997), 165–181.

[B] Z. Balogh, *Locally nice spaces and Axiom R*, Topology Appl. 125 (2002), 335–341.

[D] A. Dow, *Theory in topology*, in: Recent Progress in General Topology, M. Hušek and J. van Mill (eds.), Elsevier, 1992, 167–197.

[EN1] T. Eisworth and P. Nyikos, *Antidiamond principles and topological applications*, Trans. Amer. Math. Soc., submitted. Preprint available at http://www.math.uni.edu/~eisworth/research/papers/papers.html.

[EN2] —, *First countable, countably compact spaces and the continuum hypothesis*, Trans. Amer. Math. Soc., submitted. Preprint available at above website.

[ENS] T. Eisworth, P. Nyikos, and S. Shelah, *Gently killing S-spaces*, Israel J. Math., to appear. Preprint available at above website.

[E] R. Engelking, *General Topology*, Heldermann, Berlin, 1989.

[H] J. Hirschorn, *Random trees under CH*, Israel J. Math., submitted. Preprint available at http://mailbox.univie.ac.at/James.Hirschorn/research.html.

[Ny1] P. Nyikos, *The theory of nonmetrizable manifolds*, in: Handbook of Set-Theoretic Topology, K. Kunen and J. Vaughan (eds.), North-Holland, 1984, 633–684.

[Ny2] —, *Complete normality and metrization theory of manifolds*, Topology Appl. 123 (2002), 181–192.

[Ny3] —, *Crowding of functions, para-saturation of ideals, and topological applications*, Topology Proc., submitted.

[NyP] P. Nyikos and L. Piątkiewicz, *On the equivalence of certain consequences of the proper forcing axiom*, J. Symbolic Logic 60 (1995), 431–443.

[Ro] J. Roitman, *Basic S and L*, in: Handbook of Set-Theoretic Topology, K. Kunen and J. Vaughan (eds.), North-Holland, 1984, 295–326.

[S] S. Shelah, *Proper and Improper Forcing*, Perspectives in Mathematical Logic, Springer, 1998.

[T] S. Todorčević, *A dichotomy for P-ideals of countable sets*, Fund. Math. 166 (2000), 251–267.

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