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Equidecomposability of Jordan domains under groups of isometries

by

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Dedicated to Professor Jan Mycielski on the occasion of his 70th birthday

Abstract. Let G_d denote the isometry group of $\mathbb{R}^d$. We prove that if G is a paradoxical subgroup of G_d then there exist G-equidecomposable Jordan domains with piecewise smooth boundaries and having different volumes. On the other hand, we construct a system $\mathcal{F}_d$ of Jordan domains with differentiable boundaries and of the same volume such that $\mathcal{F}_d$ has the cardinality of the continuum, and for every amenable subgroup G of G_d, the elements of $\mathcal{F}_d$ are *not* G-equidecomposable; moreover, their interiors are not G-equidecomposable as geometric bodies. As a corollary, we obtain Jordan domains $A, B \subset \mathbb{R}^2$ with differentiable boundaries and of the same area such that A and B are not equidecomposable, and $\operatorname{int} A$ and $\operatorname{int} B$ are not equidecomposable as geometric bodies. This gives a partial solution to a problem of Jan Mycielski.

1. Introduction and main results. By a well known theorem of Tarski [12, Corollary 9.2] every discrete group is either paradoxical or amenable. A classical theorem of Tits [11] states that for linear groups this dichotomy takes the following sharper form: a linear group G either contains a free subgroup of rank two (and, *a fortiori*, is paradoxical), or G is almost solvable, that is, has a normal subgroup H such that H is solvable and G/H is finite (and, *a fortiori*, is amenable). Let G_d denote the group of all isometries of $\mathbb{R}^d$. Since G_d is isomorphic to a linear group (see [12, Appendix A]), it follows that the Tits alternative holds for each subgroup of G_d.

Let G be a subgroup of G_d. We shall say that the sets $A, B \subset \mathbb{R}^d$ are *G-equidecomposable* (and write $A \overset{G}{\sim} B$) if there are finite decompositions

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$A = A_1 \cup \ldots \cup A_n$, $B = B_1 \cup \ldots \cup B_n$ and transformations $g_1, \ldots, g_n \in G$ such that $B_i = g_i(A_i)$ $(i = 1, \ldots, n)$. If we want to indicate that we use n pieces in the decompositions then we shall write $A \overset{G}{\sim}_n B$. It is well known that if $d \geq 3$ then G_d itself contains a free subgroup of rank two. This fact is the basis of the so-called Banach–Tarski paradox stating that if $d \geq 3$ then there exist G_d-equidecomposable measurable subsets of $\mathbb{R}^d$ having different measures; in fact, whenever A and B are bounded subsets of $\mathbb{R}^d$ with nonempty interior then $A \overset{G_d}{\sim} B$ (see [12]). As an immediate corollary we deduce that if $d \geq 3$ then λ_d, the Lebesgue measure on $\mathbb{R}^d$, cannot be extended to all subsets of $\mathbb{R}^d$ as a finitely additive measure invariant under all isometries. This result was extended to all paradoxical subgroups of $\mathbb{R}^d$ as follows: if G is a paradoxical subgroup of G_d then λ_d cannot be extended to all subsets of $\mathbb{R}^d$ as a finitely additive measure invariant under G. See [12, Theorem 11.20] with a simple proof due to J. Mycielski.

We may ask whether or not the statement of the Banach–Tarski paradox itself can be generalized to all paradoxical subgroups of G_d. We cannot expect that the statement in its full strength generalizes. For example, let $G = O_d$, the group of all orthogonal linear transformations of $\mathbb{R}^d$ (that is, the group of all isometries that leave the origin fixed). If $d \geq 3$ then O_d is paradoxical. On the other hand, it is clear that, say, two balls of different size cannot be O_d-equidecomposable. However, we shall prove that whenever G is a paradoxical subgroup of G_d then there are G-equidecomposable measurable sets of different measure. Moreover, these sets can be chosen to be Jordan measurable (bounded sets with λ_d-negligible boundaries), or even Jordan domains (homeomorphic images of the closed ball) with piecewise smooth boundary.

Theorem 1. *For every paradoxical group $G \subset G_d$ there exist Jordan domains $A, B \subset \mathbb{R}^d$ with piecewise smooth boundary such that $A \overset{G}{\sim} B$, but $\lambda_d(A) \neq \lambda_d(B)$.*

For the proof we shall need the following result on groups of isometries. We shall say that a set $H \subset \mathbb{R}^d$ is a *K-net* if for every $x \in \mathbb{R}^d$ there exists a $y \in H$ with $|y - x| \leq K$. By a *flat* we shall mean a translated copy of a subspace of $\mathbb{R}^d$.

Theorem 2. *For every subgroup G of G_d exactly one of the following statements is true.*

(i) *There exists a flat E in $\mathbb{R}^d$ such that $0 \leq \dim E < d$, and every isometry $g \in G$ maps E onto itself.*

(ii) *There is a positive number K such that the set $\{g(x) : g \in G\}$ is a K-net for every $x \in \mathbb{R}^d$.*

The proof of Theorem 2 will be based on the following result: *For every convex set $C \subset \mathbb{R}^d$, $C \neq \emptyset$, $C \neq \mathbb{R}^d$, there exists a flat E such that* $0 \leq \dim E < d$, *and whenever an isometry $g \in G_d$ maps C onto itself then g also maps E onto itself* (Lemma 5). We shall prove these statements in the next section. The proof of Theorem 1 will be given in Section 3.

In the second part of the paper (Sections 4 and 5) we shall consider amenable subgroups of G_d. It follows from Mycielski's invariant measure extension theorem that if G is an amenable subgroup of $\mathbb{R}^d$ and if A and B are measurable and G-equidecomposable subsets of $\mathbb{R}^d$, then $\lambda_d(A) = \lambda_d(B)$. (See [12, Corollary 10.9].) In other words, if G is amenable then the condition $\lambda_d(A) = \lambda_d(B)$ is necessary for the G-equidecomposability of the measurable sets A and B. As we proved in [6], if the box dimensions of the boundaries of A and B are less than d, then $\lambda_d(A) = \lambda_d(B) > 0$ is sufficient for the equidecomposability of A and B under the group of all translations. In particular, if A and B are Jordan domains with Lipschitz boundaries and if $\lambda_d(A) = \lambda_d(B)$ holds, then A and B are equidecomposable under translations. Now the question we address is the following: what happens under other amenable subgroups of G_d? Suppose that A and B are Jordan domains with $\lambda_d(A) = \lambda_d(B)$. Is it possible that some weaker conditions on the boundaries of A and B imply $A \overset{G}{\sim} B$ for *some* amenable group G? The case $d = 2$ is particularly interesting since G_2 is amenable. We know that if $A, B \subset \mathbb{R}^2$ are Jordan domains of the same area and having rectifiable boundaries then they are equidecomposable under the group of translations. Suppose we impose a weaker condition on the boundaries. Assume, for example, that A and B have differentiable boundaries. Can we expect that A and B are equidecomposable using *arbitrary* plane isometries? In Theorem 3 below we shall prove that the answer to this question is negative.

In 1977 Jan Mycielski introduced two variants of the notion of equidecomposability using regular-open sets as pieces [10]. A set $H \subset \mathbb{R}^d$ is called *regular-open* if it equals the interior of its closure. The family of all bounded regular-open sets in $\mathbb{R}^d$ will be denoted by $\mathbf{B}_d^*$. For $A, B \in \mathbf{B}_d^*$ we shall denote by $A \vee B$ the interior of the closure of $A \cup B$. We say that $A, B \in \mathbf{B}_d^*$ are *equidecomposable in* $\mathbf{B}_d^*$ if there are pairwise disjoint sets $A_1, \ldots, A_k \in \mathbf{B}_d^*$ and isometries $g_1, \ldots, g_k$ such that $A = A_1 \vee \ldots \vee A_k$, the sets $g_1(A_1), \ldots, g_k(A_k)$ are pairwise disjoint, and $B = g_1(A_1) \vee \ldots \vee g_k(A_k)$.

A set is called a *geometric body* if it is bounded, regular-open and Jordan measurable. The family of geometric bodies in $\mathbb{R}^d$ will be denoted by $\mathbf{B}_d$. We shall say that $A, B \in \mathbf{B}_d$ are *equidecomposable in* $\mathbf{B}_d$ if they are equidecomposable in $\mathbf{B}_d^*$ in such a way that the pieces of the decompositions belong to $\mathbf{B}_d$. Clearly, if A and B are equidecomposable in $\mathbf{B}_d$ then they are also equidecomposable in $\mathbf{B}_d^*$, and $\lambda_d(A) = \lambda_d(B)$.

In [10] Mycielski proved that for $d \le 2$, if $A, B \in \mathbf{B}_d$ are equidecomposable in $\mathbf{B}_d^*$ then $\lambda_d(A) = \lambda_d(B)$. This is a surprising result, as $A = A_1 \vee \ldots \vee A_k$ does *not* imply $\lambda_d(A) = \lambda_d(A_1) + \ldots + \lambda_d(A_k)$ unless $A_1, \ldots, A_k \in \mathbf{B}_d$ (see also [12, pp. 117–119]). Actually, Mycielski's argument yields the following generalization. *If $A, B \in \mathbf{B}_d$ are equidecomposable in $\mathbf{B}_d^*$ under an amenable subgroup of $\mathbb{R}^d$ then $\lambda_d(A) = \lambda_d(B)$.* Indeed, it follows from Mycielski's invariant measure extension theorem [12, Theorem 10.8] that the Jordan measure can be extended to all subsets of $\mathbb{R}^d$ as a G_d-invariant finitely additive measure that vanishes on meager sets. It is easy to see that the existence of such a measure implies the statement above.

In [10] Jan Mycielski posed several problems concerning these notions. He asked whether or not all nonempty sets in $\mathbf{B}_d^*$ for $d \ge 3$ are pairwise equidecomposable in $\mathbf{B}_d^*$. He noted that this problem is equivalent to Marczewski's problem (see also [12, Theorem 9.5]). Now the solution of Marczewski's problem by Dougherty and Foreman ([2] and [3]) implies that the answer to the problem above is affirmative.

Mycielski also asked whether the conditions $A, B \in \mathbf{B}_d$ and $\lambda_d(A) = \lambda_d(B)$ are sufficient for the equidecomposability of A and B in $\mathbf{B}_d$. In the next theorem we give a partial answer: we construct sets $A, B \in \mathbf{B}_d$ with $\lambda_d(A) = \lambda_d(B)$ which are not equidecomposable in $\mathbf{B}_d$ under amenable groups of isometries. Since G_2 is amenable, our theorem provides a negative answer to Mycielski's problem for $d = 2$. (A similar statement for $d = 1$ was announced in [9, p. 180] with a sketch of proof based on the results of [8].) Mycielski's problem for $d \ge 3$ remains open.

Let ∂A denote the boundary of the set A. Let $A \subset \mathbb{R}^d$ be a Jordan domain. We shall say that ∂A is differentiable everywhere and infinitely differentiable everywhere except at one point if there is a homeomorphism between ∂A and the sphere $S = \{x \in \mathbb{R}^d : |x| = 1\}$ which is differentiable everywhere and infinitely differentiable everywhere except at one point. Our main result is the following.

THEOREM 3. *For every $d \ge 2$ there exists a family $\mathcal{F}_d$ of Jordan domains with the following properties.*

(i) $\lambda_d(D) = 1$ *for every* $D \in \mathcal{F}_d$.

(ii) *For each $D \in \mathcal{F}_d$ the boundary of D is differentiable everywhere and infinitely differentiable everywhere except at one point.*

(iii) *The elements of $\mathcal{F}_d$ are pairwise nonequidecomposable under any amenable subgroup of G_d.*

(iv) *The interiors of the elements of $\mathcal{F}_d$ are pairwise nonequidecomposable in $\mathbf{B}_d$ under any amenable subgroup of G_d.*

(v) *The cardinality of $\mathcal{F}_d$ is continuum.*

COROLLARY 4. *There are Jordan domains $A, B \subset \mathbb{R}^2$ with differentiable boundaries such that $\lambda_2(A) = \lambda_2(B)$, but A and B are not equidecomposable, and* $\operatorname{int} A$ *and* $\operatorname{int} B$ *are not equidecomposable in* $\mathbf{B}_2$.

The proof of Theorem 3 is based on the fact that the amenable subgroups of G_d are uniformly amenable, that is, they satisfy a uniform version of Følner's condition. In Section 4 we shall prove that all amenable subgroups of G_d satisfy *one single* condition of Følner type, and so they are, in a sense, *uniformly* uniformly amenable. Using this result, we shall give a necessary condition for the equidecomposability of sets under amenable groups of isometries (Theorem 9). The proof of Theorem 3 will be given in Section 5. We shall use the following additional notation.

- $B_d(x, r) = \{y \in \mathbb{R}^d : |y - x| < r\}$,
- $B_d(r) = \{y \in \mathbb{R}^d : |y| < r\}$,
- $U_d(r) = \{x \in \mathbb{R}^d : |x| \le r\}$ for every $r > 0$,
- $U_d(H, r) = \{x \in \mathbb{R}^d : \operatorname{dist}(x, H) \le r\}$ for every $H \subset \mathbb{R}^d$,
- χ_H is the characteristic function of the set H,
- $|H|$ is the cardinality of the set H,
- $\mathbb{N}$ is the set of positive integers.

2. Two results on groups of isometries

LEMMA 5. *For every convex set $C \subset \mathbb{R}^d$, $C \ne \emptyset$, $C \ne \mathbb{R}^d$, there exists a flat E in $\mathbb{R}^d$ such that $0 \le \dim E < d$, and whenever an isometry $g \in G_d$ maps C onto itself then g also maps E onto itself.*

Proof. We may assume that C is closed because if an isometry g maps C onto itself then g also maps the closure of C onto itself. Note that if C is bounded then every isometry mapping C onto itself fixes the center of gravity of C. Since every point is a flat (being a translate of the subspace $\{0\}$), the statement of the lemma is true for bounded sets.

First we shall prove the lemma in the case when C does not contain a line. Let V denote the set of vectors $v \in \mathbb{R}^d$ such that the set of real numbers $\{v \cdot x : x \in C\}$ is bounded from above. (Here $v \cdot x$ denotes the scalar product of v and x.) Then V is a cone, that is, if $v_i \in V$ and $\lambda_i \ge 0$ $(i = 1, 2)$ then $\lambda_1 v_1 + \lambda_2 v_2 \in V$. We claim that V is not contained in any subspace of dimension less than d. Suppose this is not true. Then there is a nonzero vector w perpendicular to every $v \in V$. We prove that if $x \in C$ then the whole line $x + tw$ $(t \in \mathbb{R})$ is in C. Indeed, C is the intersection of all half-spaces containing C. These half-spaces are of the form $\{x : v \cdot x \le b\}$, where $v \in V$. If $x \in C$ and $v \in V$ then, as $v \cdot w = 0$, we have $v \cdot (x + tw) = v \cdot x$ for every $t \in \mathbb{R}$. Therefore, if a half-space contains x then it also contains the line $x + tw$ $(t \in \mathbb{R})$. That is, C contains the line $x + tw$ $(t \in \mathbb{R})$. However, C does not contain any line by assumption, so that V cannot be contained in any subspace of dimension less than d. Since V is a cone, it follows that $\operatorname{int} V$, the interior of V, is nonempty. Now we shall distinguish between two cases.

Case I: $V = \mathbb{R}^d$. We claim that in this case C is bounded. Indeed, as $(1, 0, \dots, 0) \in V$, it follows from the definition of V that the set C_1 of the first coordinates of the elements of C is bounded from above. Since $(-1, 0, \dots, 0) \in V$, it follows that C_1 is also bounded from below. Similarly, the set of all coordinates of the elements of C is bounded; that is, C is bounded. Then, as we saw earlier, the statement of the lemma is true.

Case II: $V \neq \mathbb{R}^d$. Since V is a cone, it follows that there is a subspace E of dimension $d-1$ such that V lies in one of the half-spaces determined by E. The set $B_d(1) \cap \operatorname{int} V$ is nonempty, convex, open, and lies in the half-space described above. Therefore its center of gravity, c, belongs to $B_d(1) \cap \operatorname{int} V$, and is distinct from the origin. If an orthogonal transformation $O \in O_d$ maps V into itself then it also maps $B_d(1) \cap \operatorname{int} V$ into itself, and thus O fixes c.

Now let $g \in G_d$ be an isometry mapping C onto itself. Then there is an orthogonal transformation $O \in O_d$ and a vector d such that $g(x) = O(x) + d$ for every $x \in \mathbb{R}^d$. We prove that O^{-1} maps V into itself. Let $v \in V$ be arbitrary. Then there is a $b \in \mathbb{R}$ such that $v \cdot x \le b$ for every $x \in C$. If $x \in C$ then $g(x) = O(x) + d \in C$, therefore $v \cdot (O(x) + d) \le b$ and $O^{-1}(v) \cdot x = v \cdot O(x) \le b - v \cdot d$. Hence the set $\{O^{-1}(v) \cdot x : x \in C\}$ is bounded from above, that is, $O^{-1}(v) \in V$. This proves $O^{-1}(V) \subset V$. Therefore, as we showed above, $O^{-1}(c) = c$ and thus $O(c) = c$. Let $H = \{x \in \mathbb{R}^d : c \cdot x = 0\}$. Then H is a subspace of dimension $d - 1$, and O maps H onto itself.

Since $c \in \operatorname{int} V \subset V$, the set $B = \{c \cdot x : x \in C\}$ is bounded from above. Let $b_0 = \sup B$. Since $g(C) = \{O(x) + d : x \in C\} = C$, it follows that

$$\begin{aligned} b_0 &= \sup\{c \cdot O(x) + c \cdot d : x \in C\} = \sup\{O^{-1}(c) \cdot x + c \cdot d : x \in C\} \\ &= \sup\{c \cdot x : x \in C\} + c \cdot d = b_0 + c \cdot d, \end{aligned}$$

that is, $c \cdot d = 0$. Therefore $d \in H$ and thus $x \mapsto g(x) = O(x) + d$ maps the subspace H onto itself. This completes the proof of the lemma in the case when C does not contain a line.

We shall prove the lemma in the general case by induction on d. If $d = 1$ then either C is bounded (namely, is an interval), and then, as we saw above, the statement is true, or C is a half-line. In the latter case the only isometry that maps C onto itself is the identity, which fixes every point. Let $d > 1$, and suppose that the statement is true for every dimension less than d. Let $C \subset \mathbb{R}^d$ be convex such that $C \neq \emptyset$ and $C \neq \mathbb{R}^d$. If C does not contain a line then, as we proved already, the statement of the lemma is true. Therefore we may assume that C contains a line. Let F

be a flat of maximal dimension which is contained in C. By assumption, $\dim F \geq 1$ and, as $C \neq \mathbb{R}^d$, $\dim F < d$. Let $\dim F = k$. We may assume that $F = \{(x_1, \ldots, x_d) : x_1 = \ldots = x_{d-k} = 0\}$. If $x \in C$ then the closed convex hull of $F \cup \{x\}$ contains $F + x$. Since C is closed and convex, it follows that $F + x \subset C$ for every $x \in C$; that is, $C + F = C$. Therefore C is of the form $D \times \mathbb{R}^k$, where $D \subset \mathbb{R}^{d-k}$. It is clear that D is closed, convex, $D \neq \emptyset$ and $D \neq \mathbb{R}^{d-k}$. By the induction hypothesis, there is a subspace E of $\mathbb{R}^{d-k}$ and a vector $a \in \mathbb{R}^{d-k}$ such that $0 \leq \dim E < d - k$, and whenever an isometry $h \in G_{d-k}$ maps D onto itself then h also maps the flat $E + a$ onto itself. Let $E' = E \times \mathbb{R}^k$ and $a' = (a, 0, \ldots, 0) \in \mathbb{R}^d$. We shall prove that if $g \in G_d$ maps C onto itself then g maps $E' + a'$ onto itself.

Suppose $g \in G_d$ maps C onto itself. Let $g(F) = F' + b$, where F' is a subspace of $\mathbb{R}^d$ of dimension k. We show that $F' = F$. If this is not true then there is an element $x \in F' \setminus F$. Since C is convex and $tx + b \in F' + b = g(F) \subset C$ for every $t \in \mathbb{R}$, it follows that $\frac{1}{2}f + \frac{t}{2}x + \frac{b}{2} \in C$ for every $f \in F$ and $t \in \mathbb{R}$. Therefore, the set $F'' = \left\{\frac{1}{2}f + \frac{t}{2}x : f \in F,\ t \in \mathbb{R}\right\}$ is a subspace of dimension $k + 1$ such that C contains a translate of F''. Since k was maximal, this is impossible. Therefore $F' = F$ and thus $g(F)$ is a translate of F.

Let $(x_0, y_0) \in E' + a'$ be fixed, where $x_0 \in E + a$ and $y_0 \in \mathbb{R}^k$. We show that there is a $z \in \mathbb{R}^k$ and a map $h : \mathbb{R}^{d-k} \to \mathbb{R}^{d-k}$ such that $g(x, y_0) = (h(x), z)$ for every $x \in \mathbb{R}^{d-k}$. Indeed, if $x, x' \in \mathbb{R}^{d-k}$ then the vector $(x, y_0) - (x', y_0)$ is perpendicular to F. Since g is an isometry, it follows that $g(x, y_0) - g(x', y_0)$ is perpendicular to $g(F) = F + b$ or, what is the same, to F. Therefore the last k coordinates of $g(x, y_0)$ and $g(x', y_0)$ must coincide for every $x, x' \in \mathbb{R}^{d-k}$. This proves that for a suitable $z \in \mathbb{R}^k$, we have $g(x, y_0) = (h(x), z)$ for every $x \in \mathbb{R}^{d-k}$, where h is a suitable map from $\mathbb{R}^{d-k}$ into itself. It is obvious that h must be an isometry. Since $C = D \times \mathbb{R}^k$ and g maps C onto itself, it is also clear that $h(D) = D$. Then, by the choice of E and a, we have $h(E + a) = E + a$. Thus

$$g(x_0, y_0) = (h(x_0), z) \in (E + a) \times \mathbb{R}^k = E' + a'.$$

Since $(x_0, y_0) \in E' + a'$ was arbitrary, we have $g(E' + a') \subset E' + a'$, which completes the proof. ∎

Proof of Theorem 2. Suppose that (i) of the theorem holds. Then, for every $x \in E$, the set $\{g(x) : g \in G\}$ is a subset of E and thus it cannot be a K-net for any $K > 0$. That is, in this case, the statement (ii) is not true.

Next suppose that (i) does not hold. We prove (ii). First we show that the set $H = \{g(0) : g \in G\}$ is a K-net for a suitable $K > 0$. Suppose this is not true. Then there is a sequence of points $x_n \in \mathbb{R}^d$ such that $r_n = \operatorname{dist}(x_n, H) \to \infty$ as $n \to \infty$. Choose elements $y_n \in H$ such that $|y_n - x_n| < r_n + 1/n$ $(n = 1, 2, \ldots)$. Let $y_n = g_n(0)$, where $g_n \in G$ for every n, and put $z_n = g_n^{-1}(x_n)$. Then $\mathrm{dist}(z_n, H) = r_n$, and $0 \in B_d(z_n, r_n + 1/n)$. The set $U = \bigcup_{n=1}^{\infty} B_d(z_n, r_n)$ contains an open half-space, as $r_n \to \infty$ and $0 \in B_d(z_n, r_n + 1/n)$. Also, $U \cap H = \emptyset$. Let C denote the convex hull of H. Then $C \neq \emptyset$ and $C \neq \mathbb{R}^d$, as $C \cap H = \emptyset$. Also, every $g \in G$ maps C onto itself (as every $g \in G$ maps H onto itself). By Lemma 5, there exists a flat E such that $0 \le \dim E < d$, and whenever an isometry $g \in G_d$ maps C onto itself, then g maps E onto itself. Therefore, every $g \in G$ maps E onto itself. This, however, contradicts the assumption that (i) is false, and thus the set $H = \{g(0) : g \in G\}$ must be a K-net for a suitable $K > 0$.

Now we prove that for every $x \in \mathbb{R}^d$, the set $H_x = \{g(x) : g \in G\}$ is a $2K$-net. Indeed, let $y \in \mathbb{R}^d$ be arbitrary. Since $\{g(0) : g \in G\}$ is a K-net, there are isometries $g_1, g_2 \in G$ such that $|g_1(0) - x| \le K$ and $|g_2(0) - y| \le K$. Then $|g_2 g_1^{-1}(x) - y| \le |g_2 g_1^{-1}(x) - g_2(0)| + |g_2(0) - y| \le 2K$, and thus H_x is a $2K$-net. Therefore (ii) holds, and the proof is complete. ∎

3. Proof of Theorem 1. First we shall consider two special cases.

LEMMA 6. *For every paradoxical group $G \subset O_d$ there exist G-equidecomposable Jordan domains with piecewise smooth boundaries and different volumes.*

Proof. We shall argue by induction on d. For $d = 1$ the statement is true, since O_1 does not contain paradoxical subgroups. Suppose $d > 1$ and that the statement is true for every dimension less than d. Let G be a paradoxical subgroup of O_d. By the Tits theorem, H contains a free subgroup of rank two; we may clearly assume that G itself is such a group. Let S be the unit sphere of $\mathbb{R}^d$. We shall distinguish between two cases.

CASE I: *The action of G on S is not locally commutative.* Then we can choose noncommuting elements $g, h \in G$ having a common fixed point $x \in S$. Being a subgroup of a free group, the group H generated by g and h is also free and, as g and h do not commute, it contains a free subgroup H_1 of rank two. Let U denote the one-dimensional subspace generated by x; then the elements of U are fixed under H_1. Let V be the complementary subspace of U; that is, let V be perpendicular to U of dimension $d-1$. Then V is H_1-invariant. We may assume that $V = \mathbb{R}^{d-1}$. By the induction hypothesis, there are H_1-equidecomposable Jordan domains $A, B \subset V$ with piecewise smooth boundary having different $d-1$-dimensional volumes. Then $A \times [0,1]$ and $B \times [0,1]$ are also H_1-equidecomposable Jordan domains with piecewise smooth boundary and different d-dimensional volumes. Since $H_1 \subset G$, this completes the proof of Case I.

CASE II: *The action of G on S is locally commutative.* Then, by [12, Theorem 4.5], S is paradoxical under G. Let $g \in G$ be an element different from the identity, and let $x \in S$ be selected such that $g(x) \neq x$.

Let $\delta = |x - g(x)|/2$, and put $S_1 = S \setminus B_d(x, \delta)$, $S_2 = S \setminus B_d(g(x), \delta)$. We claim that S, S_1 and S_2 are pairwise G-equidecomposable. Let T denote the type semigroup of the action of G on S (see [12, Chapter 8]). Then to every set $A \subset S$ there corresponds a type $[A] \in T$ such that $[A] = [B]$ if and only if A and B are G-equidecomposable. Let $[S] = a$ and $[S_1] = [S_2] = b$. (Note that $g(S_1) = S_2$ and thus $[S_1] = [S_2]$.) Since S is paradoxical, we have $a = 2a$. On the other hand, $S = S_1 \cup S_2$ gives $b \le a \le 2b$. Now these relations imply $a = b$. Indeed, $a = 2a \le 2b$ gives $a \le b$ by the inequality version ($2a \le 2b \Rightarrow a \le b$) of the cancellation law [12, Theorem 8.7], and then $a \le b$, $b \le a$ give $a = b$ by the Banach–Schröder–Bernstein theorem [12, Theorem 3.4]. (We can also argue as follows: $2b = a + c \Rightarrow 2b + c = a + 2c = 2a + 2c = 2(a + c) = 4b$, and thus $4b = (a + c) + 2b = a + (2b + c) = a + 4b = 4a + 4b = 4(a + b)$, therefore $b = a + b$ by the cancellation law. Then $a \le b$ and we infer $a = b$ as above.)

Now $b = a$ means $S_1 \overset{G}{\sim} S$. Let $E^* = \{tx : 0 < t \le 1,\ x \in E\}$ for every $E \subset S$. Now we put $A = (S_1)^* \cup \{0\}$ and $B = U_d(1) = S^* \cup \{0\}$. Then A and B are G-equidecomposable. Indeed, If $S = E_1 \cup \ldots \cup E_n$ and $S_1 = F_1 \cup \ldots \cup F_n$ are decompositions such that $F_i = g_i(E_i)$ for some $g_i \in G$ $(i = 1, \ldots, n)$, then $A = E_1^* \cup \ldots \cup E_n^* \cup \{0\}$ and $B = F_1^* \cup \ldots \cup F_n^* \cup \{0\}$; and $F_i^* = g_i(E_i^*)$ for every i. Since A and B are Jordan domains with piecewise smooth boundaries and different volumes, this concludes the proof. ∎

Lemma 7. *Let G be a paradoxical subgroup of G_d such that* (i) *its action on $\mathbb{R}^d$ is locally commutative*, *and* (ii) *the set $\{g(x) : g \in G\}$ is a K-net for every $x \in \mathbb{R}^d$. Then there are disjoint balls of the same size*, *B_1 and B_2*, *such that $B_1 \overset{G}{\sim} B_1 \cup B_2$.*

Proof. By the Tits theorem, G contains a free subgroup of rank 2. Let $g_0, h_0 \in G$ be independent elements that generate such a subgroup. It is well known (and easy to check) that the elements $g_n = g_0^n h_0 g_0^n$ $(n = 1, 2, \ldots)$ are also independent; that is, they do not satisfy any nontrivial relation. We define

$$N = [(\sqrt{d} + 2)^d 4^d] + 1, \qquad M = \max_{1 \le n \le 6N} |g_n(0)|.$$

We put $B_0 = U_d(M+2K)$. Since the set $\{g(0) : g \in G\}$ is unbounded (in fact, a K-net), there are elements $h_n \in G$ such that the balls $B_n = h_n(B_0)$ $(n = 1, \ldots, 2N)$ are pairwise disjoint. Our aim is to show that $B_1 \overset{G}{\sim} B_1 \cup B_2$. We shall prove this in three steps. First we show that the set $X = B_1 \cup \ldots \cup B_{2N}$ is G-equidecomposable to a subset of the ball $U_d(2M + 2K)$. Next we shall prove that $U_d(2M+2K)$ is G-equidecomposable to a subset of $B_1 \cup \ldots \cup B_N$. Finally, we shall prove $B_1 \overset{G}{\sim} B_1 \cup B_2$ by using these two statements.

STEP I. $X = B_1 \cup \ldots \cup B_{2N}$ *is G-equidecomposable to a subset of* $U_d(2M+2K)$.

If $x \in B_0$ and $n \le 6N$ then $|g_n(x) - g_n(0)| = |x - 0| \le M + 2K$, and thus $|g_n(x)| \le |g_n(0)| + M + 2K \le 2M + 2K$. Therefore each of $g_1, \ldots, g_{6N}$ maps B_0 into $U_d(2M+2K)$. Let Γ_0 denote the set of pairs $(x, g_n(x))$ $(x \in B_0,\ n = 1, \ldots, 6N)$. We consider Γ_0 as a bipartite graph between the sets B_0 and $U_d(2M+2K)$ (multiple edges are allowed). The crucial property of Γ_0 that we shall exploit is that each component of Γ_0 contains at most one cycle. This follows from the fact that $g_1, \ldots, g_{6N}$ freely generate a group whose action is locally commutative on $\mathbb{R}^d$ by assumption (i) (see the proof of [7, Theorem 3]). Now let Γ_n denote the set of pairs (x, y) such that $x \in B_n$ and $y = g_{3(n-1)+i}(h_n^{-1}(x))$ for at least one $i = 1, 2, 3$. Then Γ_n is a bipartite graph between B_n and $U_d(2M+2K)$, and the degree of each vertex $x \in B_n$ equals three (counting the edges with multiplicities). In order to show that $X = B_1 \cup \ldots \cup B_{2N}$ is G-equidecomposable to a subset of $U_d(2M+2K)$ it is enough to prove that the graph $\Gamma = \bigcup_{n=1}^{2N} \Gamma_n$ contains a matching between X and a subset of $U_d(2M+2K)$, that is, a set of independent edges that covers X. Clearly, it is enough to show that every component C of Γ contains a set of independent edges that covers $X \cap V_C$, where V_C is the set of the vertices of the edges belonging to C.

Let C be an arbitrary component of Γ. Then the degree of each vertex $x \in X \cap V_C$ equals three. We claim that C contains at most one cycle. Indeed, Γ is obtained from Γ_0 by replacing the edge $(x, g_{3(n-1)+i}(x))$ by $(h_n(x), g_{3(n-1)+i}(x))$ for every $x \in B_0$, $n = 1, \ldots, 2N$ and $i = 1, 2, 3$. It is easy to check that this operation does not produce new cycles and, as each component of Γ_0 contains at most one cycle, the same is true for C. Therefore either C is a tree (that is, a connected graph containing no cycles), or C contains exactly one cycle. In the latter case we delete one of the edges of the cycle contained in C. The remaining graph C' is a tree in which the degree of each vertex $x \in X \cap V_C$ is at least two. If C is a tree then we put $C' = C$. Now we prove that C' contains a set of independent edges covering $X \cap V_C$. Let $x_0 \in X \cap V_C$ be a fixed vertex. For every $v \in V_C$ let $n(v)$ denote the distance between v and x_0, that is, the length of the unique path from x_0 to v. If $x \in X \cap V_C$ then the degree of x is at least two, and thus we can select a vertex $y_x \in U_d(2M+2K)$ such that $(x, y_x) \in C'$ and $n(y_x) = n(x) + 1$. Then the edges (x, y_x) $(x \in X \cap V_C)$ are independent. Indeed, suppose $x_1 \ne x_2$ and $y_{x_1} = y_{x_2} = y$. Then $n(x_1) = n(x_2) = n(y) - 1$, and thus the path P_1 from x_0 to x_1 does not contain x_2, and the path P_2 from x_0 to x_2 does not contain x_1. But then the union of the paths P_1 and P_2 together with the edges (x_1, y) and (x_2, y) contains a cycle, which contradicts the fact that C' is a tree. Therefore $\{(x, y_x) : x \in X \cap V_C\}$ is a set of independent edges covering $X \cap V_C$. This concludes the proof of Step I.

STEP II. *$U_d(2M+2K)$ is G-equidecomposable to a subset of $B_1 \cup \ldots \cup B_N$.*

First we show that if $0 < s < t$, then any set $H \subset \mathbb{R}^d$ of diameter t can be covered by at most $(\sqrt{d}+2)^d \cdot (t/s)^d$ sets of diameter s. Indeed, H can be covered by a cube Q of side length t. If we cover $\mathbb{R}^d$ by nonoverlapping cubes of side length $s/\sqrt{d}$, then at most $((t/(s/\sqrt{d}))+2)^d \le (\sqrt{d}+2)^d(t/s)^d$ of these cubes can intersect Q. Since these cubes have diameter s and cover H, the statement follows.

The diameter of the ball $U_d(2M+2K)$ is $4M+4K$. Therefore it can be covered by sets $H_1, \ldots, H_k$ of diameter $M+K$, where $k \le (\sqrt{d}+2)^d \cdot 4^d \le N$. Let a point $x_i \in H_i$ be selected for every $i = 1, \ldots, k$. Since $\{g(x_i) : g \in G\}$ is a K-net, there is an isometry $u_i \in G$ such that $|u_i(x_i) - h_i(0)| \le K$ $(i = 1, \ldots, k)$. Since the diameter of H_i is $M+K$, it follows that $u_i(H_i) \subset h_i(B_0) = B_i$, and thus $H_1 \cup \ldots \cup H_k$ is G-equidecomposable to a subset of $B_1 \cup \ldots \cup B_k \subset B_1 \cup \ldots \cup B_N$. As $U_d(2M+2K)$ is a subset of $H_1 \cup \ldots \cup H_k$, our statement is proved.

STEP III. $B_1 \overset{G}{\sim} B_1 \cup B_2$.

Let T denote the type semigroup of the action of G on $\mathbb{R}^d$. Let $a \in T$ denote the type of $U_d(2M+2K)$, and let $b \in T$ denote the type of B_1. Since the balls B_n $(n = 1, \ldots, 2N)$ are pairwise G-equidecomposable (in fact, G-congruent), the type of each B_n is b. By Step I, we have $2Nb \le a$, and by Step II, we have $a \le Nb$. Then $2Nb \le Nb$, and thus $2Nb = Nb$. Therefore, by the cancellation law, $2b = b$; that is, $B_1 \overset{G}{\sim} B_1 \cup B_2$. ∎

Now we turn to the proof of Theorem 1. First we note that if the statement of the theorem is true for a group G then it is also true for the conjugate group $G^t = tGt^{-1}$ for every $t \in G_d$. Indeed, the groups G and G^t are isomorphic, and thus if G^t is paradoxical then so is G. Also, if the sets A and B are G-equidecomposable then $t(A)$ and $t(B)$ are G^t-equidecomposable. Finally, if A and B are Jordan domains with piecewise smooth boundary and with different volumes then so are $t(A)$ and $t(B)$.

We shall prove the theorem by induction on d. If $d = 1$ then the statement is true, since G_1 does not contain paradoxical subgroups. Let $d > 1$, and suppose that the statement is true for every dimension less than d. Let G be a paradoxical subgroup of G_d. We may assume that G is a free group of rank two, since otherwise we replace G by a subgroup with this property.

First we suppose that the action of G on $\mathbb{R}^d$ is not locally commutative. Then we can choose noncommuting elements $g, h \in G$ having a common fixed point p. We may assume that p is the origin, since otherwise we replace G by the conjugate group tGt^{-1}, where t is the translation $x \mapsto x - p$. Let H

denote the group generated by g and h; then H is a subgroup of O_d. Being a subgroup of a free group, H is also free and, as g and h do not commute, it contains a free subgroup H_1 of rank two. In particular, H_1 is paradoxical. Summing up: H_1 is a paradoxical subgroup of O_d and thus, by Lemma 6, the statement of the theorem is true.

Therefore we may assume that the action of G on $\mathbb{R}^d$ is locally commutative. By Theorem 2, one of the following statements is true:

(i) there exists a flat $E \subset \mathbb{R}^d$ of dimension $k < d$ such that every element of G maps E onto itself;

(ii) for a suitable $K > 0$, the set $\{g(x) : g \in G\}$ is a K-net for every $x \in \mathbb{R}^d$.

By Lemma 7, if (ii) holds then the statement of the theorem is true. Therefore we may suppose that (i) holds. Replacing G by a suitable conjugate group, we may also assume that $E = \{(x_1, \ldots, x_d) : x_{k+1} = x_{k+2} = \ldots = x_d = 0\}$. Then each $g \in G$ maps E onto itself.

For every $g \in G$ let $\bar{g}$ denote the restriction of g to E, and put $\overline{G} = \{\bar{g} : g \in G\}$. Then $\overline{G}$ is a group of isometries mapping E into itself. We show that $\overline{G}$ is paradoxical. Since G is a free group of rank two, it also contains infinitely many independent elements (as we mentioned already in the proof of Lemma 7). Let $g_1, g_2, g_3, g_4 \in G$ be independent. Every word w formed by the letters g_i, g_i^{-1} $(i = 1, 2, 3, 4)$ defines an element of G also denoted by w. It is clear that if we replace g_i and g_i^{-1} by $\bar{g}_i$ and $\bar{g}_i^{-1}$ in the word w then the resulting map equals $\overline{w}$.

Let G_1 denote the group generated by $\bar{g}_1$ and $\bar{g}_2$, and let G_2 be the group generated by $\bar{g}_3$ and $\bar{g}_4$. We prove that at least one of G_1 and G_2 is paradoxical. Since G_1 and G_2 are both subgroups of $\overline{G}$, this will prove that $\overline{G}$ is paradoxical.

Suppose that G_1 is not paradoxical. Then, in particular, $\bar{g}_1$ and $\bar{g}_2$ are not independent. Consequently, there exists a word w_1 of the letters g_i, g_i^{-1} $(i = 1, 2)$ such that $\overline{w}_1$, as a map from E into itself, is the identity map. Similarly, if G_2 is not paradoxical then there exists a word w_2 of the letters g_i, g_i^{-1} $(i = 3, 4)$ such that $\overline{w}_2$ is the identity map. Therefore every point of E is a common fixed point of the elements $w_1, w_2 \in G$. Since G is locally commutative, it follows that w_1 and w_2 commute, that is, $w_1 w_2 = w_2 w_1$. However, both $w_1 w_2$ and $w_2 w_1$ are words formed by the letters g_i, g_i^{-1} $(i = 1, 2, 3, 4)$ in such a way that in the juxtapositions $w_1 w_2$ and $w_2 w_1$ no cancellation can occur between w_1 and w_2, since w_1 and w_2 do not contain common letters. It follows then that the words $w_1 w_2$ and $w_2 w_1$ are formally different and thus, as g_1, g_2, g_3, g_4 are independent, they cannot define the same map. This contradiction shows that at least one of G_1 and G_2 is paradoxical, and then so is $\overline{G}$.

If we identify E with $\mathbb{R}^k$ then we find that $\overline{G}$ is a paradoxical subgroup of G_k. Then, by the induction hypothesis, there are Jordan domains C, $D \subset \mathbb{R}^k$ with piecewise smooth boundaries such that $C \overset{\overline{G}}{\sim} D$, but $\lambda_k(C) \neq \lambda_k(D)$. Let U denote the closed unit ball of $\mathbb{R}^{d-k}$, and put $A = C \times U$ and $B = D \times U$. Then A, B are Jordan domains with piecewise smooth boundary, and $\lambda_d(A) \neq \lambda_d(B)$. We prove that $A \overset{G}{\sim} B$. Since $C \overset{\overline{G}}{\sim} D$, there are decompositions $C = C_1 \cup \ldots \cup C_n$, $D = D_1 \cup \ldots \cup D_n$, and there are elements $g_1, \ldots, g_n \in G$ such that $D_i = \bar{g}_i(C_i)$ $(i = 1, \ldots, n)$. It is clear that $\bigcup_{i=1}^n C_i \times U$ is a decomposition of A and $\bigcup_{i=1}^n D_i \times U$ is a decomposition of B. In order to show $A \overset{G}{\sim} B$ it is enough to prove that $g_i(C_i \times U) = D_i \times U$.

We shall prove the more general statement that whenever $C \subset E = \mathbb{R}^k$ and $g \in G$ then $g(C \times U) = \bar{g}(C) \times U$. Let $F_x = \{x\} \times \mathbb{R}^{d-k}$ for every $x \in \mathbb{R}^k$. Then F_x is a translated copy of the subspace $\{0\} \times \mathbb{R}^{d-k}$ and is perpendicular to E for every $x \in E$. Since g is an isometry and maps E onto itself, it follows that $g(F_x)$ is also perpendicular to E, and thus $g(F_x) = F_{g(x)}$. If $(x, y) \in C \times U$ then since $(x, y) \in F_x$ we obtain $g(x, y) = (g(x), z) = (\bar{g}(x), z)$, where $z \in \mathbb{R}^{d-k}$. Now $(x, y) \in C \times U \subset E \times U$ gives $|y| \leq 1$, and thus $\operatorname{dist}((x, y), E) = |y| \leq 1$. Therefore $|z| = \operatorname{dist}(g(x, y), E) \leq 1$, that is, $z \in U$, which proves that $g(C \times U) \subset \bar{g}(C) \times U$. The same argument shows that $g^{-1}(\bar{g}(C) \times U) \subset C \times U$, and thus $g(C \times U) = \bar{g}(C) \times U$, as we stated. This completes the proof of Theorem 1. ■

4. Uniformly amenable groups and a necessary condition for equidecomposability. A group G is called *uniformly amenable* if there is a function $c : \mathbb{N} \times (0, \infty) \to \mathbb{N}$ with the following property: for every nonempty finite subset $A \subset G$ and for every $\varepsilon > 0$ there exists a nonempty finite subset $U \subset G$ such that $|U| \leq c(|A|, \varepsilon)$ and $|(UA) \setminus U| < \varepsilon |U|$. If the condition above is satisfied then we shall say that c is a *uniform amenability function* (u.a.f.) of G.

It was proved by G. Keller [5] and M. Bożejko [1] that every solvable group is uniformly amenable; moreover, the class of uniformly amenable groups is closed under group extensions. As we mentioned in the introduction, every amenable subgroup G of G_d is almost solvable, that is, has a normal subgroup H such that H is solvable and G/H is finite. Since finite groups are obviously uniformly amenable, the next statement is a consequence of Keller's and Bożejko's theorem: *Every amenable subgroup of G_d is uniformly amenable.* Now we claim that the amenable subgroups of G_d are, in fact, *uniformly* uniformly amenable, in the following sense.

PROPOSITION 8. *There exists a function $c : \mathbb{N} \times (0, \infty) \to \mathbb{N}$ with the following property. For every almost solvable group G* (*in particular*, *for every amenable subgroup of G_d*) *there is a positive integer $k_0(G)$ such that* *for every nonempty finite subset* $A \subset G$ *with* $|A| \geq k_0(G)$ *and for every* $\varepsilon > 0$ *there exists a nonempty finite subset* $U \subset G$ *such that* $|U| \leq c(|A|, \varepsilon)$ *and* $|(UA) \setminus U| < \varepsilon|U|$.

Proof. We note first that there is a single function c_0 that is a u.a.f. of every Abelian group. Indeed, if k and $\varepsilon > 0$ are given, then let N be chosen such that $(1+\varepsilon)^{N-1} > N^k$, and define $c_0(k,\varepsilon) = N^k$. If $F = \{a_1, \dots, a_k\}$ is an arbitrary k-element subset of an Abelian group G, then put $K_n = \{a_1^{i_1} \dots a_k^{i_k} : 0 \leq i_1, \dots, i_k < n\}$. Then $|K_n| \leq n^k$ for every n. Therefore $|K_{n+1} \setminus K_n| < \varepsilon |K_n|$ for at least one $n = 1, \dots, N-1$, since otherwise $|K_{n+1}| \geq (1+\varepsilon)|K_n|$ would hold for every $n = 1, \dots, N-1$ implying $N^k \geq |K_N| \geq (1+\varepsilon)^{N-1}|K_1|$, which is impossible. As $FK_n \subset K_{n+1}$, we obtain $|(FK_n) \setminus K_n| \leq |K_{n+1} \setminus K_n| < \varepsilon |K_n|$ and $|K_n| = n^k \leq N^k$. In other words, c_0 is a u.a.f. of G.

Next we show that there exists a countable system S of functions such that every almost solvable group G has a u.a.f. belonging to S. Let $\mathcal{G}_n$ denote the class of groups G for which there is a sequence $\{e\} = G_0, G_1, \dots, G_n = G$ such that each G_{i-1} is a normal subgroup of G_i and the factor group G_i/G_{i-1} is either finite or Abelian. Since $\bigcup_{n=0}^{\infty} \mathcal{G}_n$ contains every almost solvable group, it is enough to show that for every n there exists a countable system S_n of functions such that every $G \in \mathcal{G}_n$ has a u.a.f. belonging to S_n.

We prove this statement by induction on n. The case $n = 0$ is trivial since $\mathcal{G}_0$ only consists of the one-element group with u.a.f. $c \equiv 1$. Suppose $n > 0$ and that there exists a countable system S_{n-1} such that every $G \in \mathcal{G}_{n-1}$ has a u.a.f. belonging to S_{n-1}. For every $G \in \mathcal{G}_n$ there is a normal subgroup H of G such that $H \in \mathcal{G}_{n-1}$ and G/H is either finite or Abelian. If G/H is finite and $|G/H| = k$ then the function $c \equiv k$ is a u.a.f. of G/H. On the other hand, if G/H is Abelian then, as we saw above, c_0 is a u.a.f. of G/H. By Bożejko's theorem, G is uniformly amenable. Moreover, what Bożejko actually proves in [1, Theorem 3] is that to every pair (d_1, d_2) of functions there corresponds a function d such that whenever H is a normal subgroup of G, d_1 is a u.a.f. of H and d_2 is a u.a.f. of G/H, then d is a u.a.f. of G. Since there is a countable set of functions containing u.a.f.'s of every group which is either finite, Abelian or belongs to $\mathcal{G}_{n-1}$, it is clear that there is a countable system S_n containing u.a.f.'s of every group $G \in \mathcal{G}_n$. This proves the existence of a countable system S with the required property.

Let $c_1, c_2, \dots$ be an enumeration of S. We claim that the function $c(k, \varepsilon) = \max\{c_n(k, \varepsilon) : n \leq k\}$ satisfies the requirements of the proposition. Indeed, let G be an arbitrary almost solvable group. Then there is an n such that c_n is a u.a.f. of G. We put $k_0(G) = n$. Let A be a nonempty finite subset of G with $|A| \geq k_0(G)$. Then there is a nonempty finite set $U \subset G$ such that $|(UA) \setminus U| < \varepsilon |U|$ and $|U| \leq c_n(|A|, \varepsilon)$. Since $c_n(|A|, \varepsilon) \leq c(|A|, \varepsilon)$ because $n \leq |A|$ and by the definition of c, the proof is complete. ∎

From now on we shall fix a pair of functions $c(k,\varepsilon)$ and $k_0(G)$ satisfying the requirements of Proposition 8.

THEOREM 9. *For every $d \ge 1$ there exists a function $N_d : \mathbb{N}\times(0,\infty) \to \mathbb{N}$ with the following properties.*

(i) *Whenever G is an amenable subgroup of G_d, $Q \subset \mathbb{R}^d$ is a closed cube, $A, B \subset Q$, $A \overset{G}{\sim}_k B$, $\varepsilon > 0$, and $N \ge N_d(\max(k, k_0(G)), \varepsilon)$, then there is a decomposition of Q into convex sets $C_1, \dots, C_N$ and there are isometries $a_{i,j}$ $(i,j = 1,\dots,N)$ such that $a_{i,j}(C_j) \subset Q$ for every i,j, and*

$$\left|\frac{1}{N}\sum_{i,j=1}^{N} \chi_{a_{i,j}(A\cap C_j)}(x) - \frac{1}{N}\sum_{i,j=1}^{N} \chi_{a_{i,j}(B\cap C_j)}(x)\right| < \varepsilon \tag{1}$$

everywhere on $\mathbb{R}^d$.

(ii) *Whenever G is an amenable subgroup of G_d, $Q \subset \mathbb{R}^d$ is a closed cube, $A, B \subset Q$ are equidecomposable in $\mathbf{B}_d$ under G using k pieces, $\varepsilon > 0$, and $N \ge N_d(\max(k, k_0(G)), \varepsilon)$, then there is a decomposition of Q into convex sets $C_1, \dots, C_N$ and there are isometries $a_{i,j}$ $(i,j = 1,\dots,N)$ such that $a_{i,j}(C_j) \subset Q$ for every i,j, and* (1) *holds everywhere on $\mathbb{R}^d$ except at the points of a nowhere dense set of measure zero.*

Proof. For $k \in \mathbb{N}$ and $0 < \varepsilon < 1$ we define $\eta = \varepsilon(4[\sqrt{d}+2]^d k)^{-1}$, $N_0 = c(k,\eta)$, $M = [\sqrt{d}+2]^{dN_0}$, and $N_d(k,\varepsilon) = [4N_0M/\varepsilon] + 1$. We shall prove that $N_d(k,\varepsilon)$ satisfies the requirements.

We may assume that $Q = [0,1]^d$, since otherwise we apply a similarity transformation γ mapping Q onto $[0,1]^d$. Then we apply the theorem with $\gamma(A)$ and $\gamma(B)$ instead of A and B, and obtain C_i and $a_{i,j}$. Clearly, the decomposition $Q = \bigcup_{i=1}^N \gamma^{-1}(C_i)$ and the isometries $\gamma^{-1}a_{i,j}\gamma$ will satisfy the requirements for A and B.

Let G, $Q = [0,1]^d$, A, B, k, ε be as in (i) of the theorem. Then there are decompositions $A = \bigcup_{n=1}^k A_n$ and $B = \bigcup_{n=1}^k B_n$ such that $B_n = a_n(A_n)$ $(n = 1,\dots,k)$, where $a_1,\dots,a_k \in G$. We may assume that $k \ge k_0(G)$, since otherwise we replace k by $k_0(G)$, and put $A_n = B_n = \emptyset$ and $a_n = \mathrm{id}$ for every $k < n \le k_0(G)$.

Put $F = \{a_1,\dots,a_k\}$. By Proposition 8, there is a nonempty finite set $K = \{c_1,\dots,c_s\} \subset G$ such that $s \le N_0$ and $|(KF)\setminus K| < \eta s$. We have

$$\begin{aligned}\sum_{i=1}^{s}(\chi_{c_iA} - \chi_{c_iB}) &= \sum_{i=1}^{s}\sum_{n=1}^{k}(\chi_{c_iA_n} - \chi_{c_iB_n}) \\ &= \sum_{n=1}^{k}\sum_{i=1}^{s}(\chi_{c_iA_n} - \chi_{c_ia_nA_n}) =: \sum_{n=1}^{k}\sigma_n.\end{aligned} \tag{2}$$

In the sum defining σ_n the terms $\chi_{c_iA_n}$ with $c_i \in Ka_n$ and $\chi_{c_ia_nA_n}$ with $c_ia_n \in K$ cancel out. Since

$$|K \setminus (Ka_n)| = |(Ka_n) \setminus K| \le |(KF) \setminus K| < \eta s,$$

it follows that $\sum_{n=1}^k \sigma_n = \sum_{\mu=1}^m \pm\chi_{D_\mu}$, where $m \le 2k\eta s = s\varepsilon(2[\sqrt{d}+2]^d)^{-1}$, and each D_μ is congruent to a subset of A.

Let $t_1, t_2, \ldots$ be an enumeration of all translations by vectors with integer coordinates, and let $Q_r = t_r([0,1)^d)$. Multiplying (2) by χ_{Q_r} we obtain

$$\sum_{i=1}^{s}(\chi_{(c_iA)\cap Q_r} - \chi_{(c_iB)\cap Q_r}) = \sum_{\mu=1}^{m} \pm\chi_{D_\mu\cap Q_r}. \tag{3}$$

Let T_r denote the operator $T_rf(x) = f(t_rx)$ ($f : \mathbb{R}^d \to \mathbb{R},\ x \in \mathbb{R}^d$). Then T_r is a linear operator defined on the functions $f : \mathbb{R}^d \to \mathbb{R}$, and $T_r(\chi_H) = \chi_{t_r^{-1}H}$ for every $H \subset \mathbb{R}^d$. Applying T_r to both sides of (3), and taking the sum over all r we obtain

$$\sum_{i=1}^{s}\sum_{r=1}^{\infty}(\chi_{t_r^{-1}((c_iA)\cap Q_r)} - \chi_{t_r^{-1}((c_iB)\cap Q_r)}) = \sum_{\mu=1}^{m}\sum_{r=1}^{\infty} \pm\chi_{t_r^{-1}(D_\mu\cap Q_r)}. \tag{4}$$

(Note that for every i and μ we have $(c_iA)\cap Q_r = (c_iB)\cap Q_r = D_\mu \cap Q_r = \emptyset$ for all but a finite number of indices r.)

For every i and r we define $P_r^i = (c_i^{-1}Q_r) \cap [0,1]^d$. Then, for every i, $[0,1]^d = \bigcup_{r=1}^\infty P_r^i$ is a decomposition of $[0,1]^d$ into convex sets of which at most $[\sqrt{d}+2]^d$ can be nonempty. Let $C_1, \ldots, C_L$ be an enumeration of all nonempty sets of the form $P_{r_1}^1 \cap \ldots \cap P_{r_s}^s$. Then $L \le [\sqrt{d}+2]^{ds} \le M$, and $C_1 \cup \ldots \cup C_L$ is a decomposition of $[0,1]^d$ into disjoint convex sets. For every i and r we have

$$t_r^{-1}((c_iA)\cap Q_r) = t_r^{-1}c_i(A \cap (c_i^{-1}Q_r)) = t_r^{-1}c_i(A \cap P_r^i). \tag{5}$$

Clearly, for each $1 \le i \le s$ and $1 \le j \le L$ we can select an r such that $C_j \subset P_r^i$. We define $a_{i,j} = t_r^{-1}c_i$; then

$$a_{i,j}(C_j) \subset a_{i,j}(P_r^i) = t_r^{-1}c_i(P_r^i) \subset t_r^{-1}c_i(c_i^{-1}Q_r) = [0,1)^d.$$

For every i and r,

$$t_r^{-1}c_i(A \cap P_r^i) = \bigcup a_{i,j}(A \cap C_j), \tag{6}$$

where the union is taken for all j's satisfying $C_j \subset P_r^i$. Let $\alpha_{i,j} = \chi_{a_{i,j}(A\cap C_j)}$ and $\beta_{i,j} = \chi_{a_{i,j}(B\cap C_j)}$. The union on the right hand side of (6) consists of disjoint sets, and hence by (5) we obtain

$$\sum_{r=1}^{\infty}\chi_{t_r^{-1}((c_iA)\cap Q_r)} = \sum_{j=1}^{L}\alpha_{i,j}. \tag{7}$$

A similar equation holds with B in place of A and thus, by (4),

$$\Big|\sum_{i=1}^{s}\sum_{j=1}^{L}(\alpha_{i,j}-\beta_{i,j})\Big| \le \sum_{\mu=1}^{m}\sum_{r=1}^{\infty}\chi_{t_r^{-1}(D_\mu\cap Q_r)} \le s\varepsilon/2. \tag{8}$$

The second inequality of (8) follows from the fact that for every μ, the set D_μ can be covered by a unit square, and hence the number of indices r with $D_\mu\cap Q_r\ne\emptyset$ is at most $[\sqrt{d}+2]^d$. Therefore the middle term of (8) is at most $[\sqrt{d}+2]^d m\le s\varepsilon/2$.

Now let $N\ge N_d(k,\varepsilon)$ be arbitrary, and define $a_{\nu s+q,j}=a_{q,j}$ for every $0\le\nu<[N/s],\ 1\le q\le s$ and $j=1,\dots,L$. Then, by (8), we have

$$\Big|\sum_{i=1}^{[N/s]s}\sum_{j=1}^{L}(\alpha_{i,j}-\beta_{i,j})\Big| \le [N/s]s\varepsilon/2\le N\varepsilon/2. \tag{9}$$

Finally, we put $C_j=\emptyset$ for every $L<j\le N$ and $a_{i,j}=\mathrm{id}$ whenever $[N/s]s<i\le N$ or $L<j\le N$. Since $L\le M,\ s\le N_0$ and $4N_0M/\varepsilon<N$, we deduce from (9) that

$$\begin{aligned}\Big|\sum_{i,j=1}^{N}(\alpha_{i,j}-\beta_{i,j})\Big| &= \Big|\sum_{i=1}^{N}\sum_{j=1}^{L}(\alpha_{i,j}-\beta_{i,j})\Big| \\ &\le \Big|\sum_{i=1}^{[N/s]s}\sum_{j=1}^{L}(\alpha_{i,j}-\beta_{i,j})\Big| + \Big|\sum_{i=[N/s]s+1}^{N}\sum_{j=1}^{L}(\alpha_{i,j}-\beta_{i,j})\Big| \\ &\le N\varepsilon/2+s\cdot 2L\le N\varepsilon/2+2N_0M<N\varepsilon/2+N\varepsilon/2=N\varepsilon.\end{aligned} \tag{10}$$

Dividing (10) by N, we obtain (1), and this completes the proof of (i).

In order to prove (ii), suppose that $A,B\in\mathbf{B}_d$ are equidecomposable in $\mathbf{B}_d$ under G using k pieces. Then there are disjoint sets $A_1,\dots,A_k\in\mathbf{B}_d$ and isometries $g_1,\dots,g_k\in G$ such that $A=A_1\vee\ldots\vee A_k$ and $B=g_1(A_1)\vee\ldots\vee g_1(A_k)$. It is easy to see that $\chi_A(x)=\sum_{i=1}^k\chi_{A_i}(x)$ everywhere except at the points of the boundaries of $A_1,\dots,A_k$. Since the sets A_i are geometric bodies, it follows that $\chi_A=\sum_{i=1}^k\chi_{A_i}$ holds everywhere except at the points of a nowhere dense set of measure zero. Therefore we can follow the proof of (i) step by step, using the convention that by the equality of functions we mean that the functions are equal at the points of an everywhere dense open set of full measure. ∎

5. Proof of Theorem 3

LEMMA 10. *For every $N \in \mathbb{N}$, $0 < a < b < 1$, and $\delta > 0$ there is a positive number s and a set $H \subset [a,b]^{d-1}$ with the following properties.*

(i) *H is the union of finitely many disjoint $d-1$-dimensional closed rectangular boxes.*

(ii) $\lambda_{d-1}(H) > (b-a)^{d-1}/2$.

(iii) *Whenever $A_1, \dots, A_N$ are congruent copies of the set*

$$([0,s]^{d-1} \cup H) \times [0,1]$$

then $A_1 \cup \ldots \cup A_N$ does not contain any d-dimensional ball of radius δ.

Proof. Let $P \subset (a,b)$ be a nowhere dense closed set with $\lambda(P) > (b-a)/\sqrt[d-1]{2}$. Let $[a,b] \supset A_1 \supset A_2 \supset \ldots$ be a sequence of sets such that $\bigcap_{n=1}^{\infty} A_n = P$, and each A_n is a finite union of closed intervals. Our aim is to prove that there exists an n such that $s = 1/n$ and $H = A_n^{d-1}$ satisfy the requirements.

It is clear that (i) and (ii) hold true for every n. Suppose that (iii) is false for every n, and let $C_n = ([0,1/n]^{d-1} \cup A_n^{d-1}) \times [0,1]$. Then for every n there are isometries $\alpha_1^n, \dots, \alpha_N^n \in G_d$ such that $D_n = \bigcup_{i=1}^N \alpha_i^n(C_n)$ contains a ball of radius δ. Clearly, we may assume that $B_d(\delta) \subset D_n$ for every n. We may also suppose that $|\alpha_i^n(0)| \le \sqrt{d} + \delta$ for every i and n. Indeed, otherwise $\alpha_i^n(C_n) \cap B_d(\delta) = \emptyset$, and we may replace α_i^n by the identity map.

Then, selecting a subsequence if necessary, we may suppose that for every $i = 1, \dots, N$, the sequence α_i^n $(n = 1, 2, \dots)$ converges to an isometry $\alpha_i \in G_d$ in the sense that $\alpha_i^n \to \alpha_i$ uniformly on every bounded subset of $\mathbb{R}^d$.

Now, it is easy to see, using $\bigcap_{n=1}^{\infty} A_n = P$ and $\lim_{n\to\infty} \alpha_i^n = \alpha_i$, that

$$\bigcup_{i=1}^{N} \alpha_i([\{0\} \cup P^{d-1}] \times [0,1]) \supset B_d(\delta). \tag{11}$$

Indeed, let $y \in B_d(\delta)$ be arbitrary. Then $y \in \bigcup_{i=1}^N \alpha_i^n(C_n)$ for every n, and thus there is an i such that $y \in \alpha_i^n(C_n)$ for infinitely many n. Since $\bigcap_{n=1}^{\infty} C_n = (\{0\} \cup P^{d-1}) \times [0,1]$, it is clear that $y \in \alpha_i([\{0\} \cup P^{d-1}] \times [0,1])$, which proves (11). However, the set $[\{0\} \cup P^{d-1}] \times [0,1]$ is nowhere dense in $\mathbb{R}^d$. Thus (11) is impossible, which concludes the proof. ∎

LEMMA 11. *If $C \subset [a,b]^d$ is convex then*

$$\lambda_d(U_d(\partial C, h)) \le 4d(b-a+2)^{d-1}h \quad \textit{for every } 0 \le h \le 1.$$

Proof. Let $f(x) = \operatorname{dist}(x, \partial C)$. By [4, Lemma 3.2.34],

$$\lambda_d(U_d(\partial C, h)) = \int_0^h \mu^{d-1}(f^{-1}(\{y\}))\, dy$$

for every $h > 0$, where μ^{d-1} denotes the $d-1$-dimensional Hausdorff measure. For every $y \le h$, $f^{-1}(\{y\})$ consists of at most two convex surfaces covered by $[a-h, b+h]^d$. Therefore $\mu^{d-1}(f^{-1}(\{y\})) \le 2\mu^{d-1}(\partial([a-h, b+h]^d)) = 4d(b-a+2h)^{d-1} \le 4d(b-a+2)^{d-1}$, from which the statement is clear. ∎

Now we turn to the proof of Theorem 3. First we fix a Jordan domain A with infinitely differentiable boundary such that A lies in the half-space $\{(x_1, \ldots, x_d) : x_d \le 0\}$, the boundary of A contains the $d-1$-dimensional unit cube $[0,1]^{d-1}$, and $\lambda_d(A) = 1/2$. We can construct such an A as follows. Let A_0 be a Jordan domain with infinitely differentiable boundary such that A_0 lies in the plane $\{(x_1, x_d) : x_d \le 0\}$, the boundary of A_0 contains the segment $[-\sqrt{d}, \sqrt{d}] \times \{0\}$, and A_0 is symmetric about the x_d axis. Then we rotate A_0 in $\mathbb{R}^d$ about the x_d axis, and apply a suitable affine transformation of the form $(x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_{d-1}, tx_d)$ in order to get volume 1/2. We may assume that $A \subset [-2\sqrt{d}, 2\sqrt{d}]^{d-1} \times [-1, 0]$.

We shall construct a sequence of positive numbers $1 = a_0 > a_1 > \ldots$ such that $a_n < (a_{n-1}/2)^2$ for every $n \ge 1$, and a sequence of functions $f_n \in C^\infty(\mathbb{R}^{d-1})$ $(n = 1, 2, \ldots)$ such that f_n vanishes outside the set

$$B_n := [a_{n-1}/2, a_{n-1}]^{d-1},$$

and $0 \le f_n \le a_{n-1}^2/2$ on B_n. We define

$$F_n = \{(x_1, \ldots, x_d) : (x_1, \ldots, x_{d-1}) \in B_n,\ 0 \le x_d \le f_n(x_1, \ldots, x_{d-1})\}.$$

Then, for every $I \subset \mathbb{N}$, we define $f_I = \sum_{\nu \in I} f_\nu$ and $F_I = \bigcup_{\nu \in I} F_\nu$. Finally, we put $B_I = F_I \cup (t_I A)$, where the number t_I is chosen such that $\lambda_d(B_I) = 1$.

It is clear that for every $I \subset \mathbb{N}$ the function f_I is infinitely differentiable everywhere on $\mathbb{R}^{d-1}$ except at the origin. Also, f_I is differentiable at the origin, since $x \in B_n$ implies $|f_I(x)| \le a_{n-1}^2/2 \le 2|x|^2$, and f_I vanishes outside $\bigcup_n B_n$. Therefore, for every I, the boundary of B_I is differentiable everywhere and infinitely differentiable everywhere except at one point. We shall prove that for a suitable set $\mathcal{I} \subset P(\mathbb{N})$ of cardinality continuum, the Jordan domains B_I $(I \in \mathcal{I})$ are pairwise nonequidecomposable under any amenable subgroup of G_d.

Let $Q = [-4\sqrt{d}, 4\sqrt{d}]^d$. Note that for every $I \subset \mathbb{N}$, we have $f_I \le 1/2$ everywhere and thus $\lambda_d(F_I) \le 1/2$. Since $\lambda_d(A) = 1/2$ and the number t_I is selected such that $\lambda_d(B_I) = \lambda_d(F_I \cup (t_I A)) = 1$, it follows that $1 \le t_I \le 2$. Therefore

$$B_I \subset (t_I A) \cup [0,1]^d \subset [-4\sqrt{d}, 4\sqrt{d}]^{d-1} \times [-2, 1] \subset Q.$$

Now we turn to the construction of the sequences (a_n) and (f_n). We put $a_0 = 1$. Let $n > 0$, and suppose that $a_0 > a_1 > \ldots > a_{n-1} > 0$ and

$f_1, \dots, f_{n-1}$ have been defined. Then we define

$$(12)\qquad \begin{aligned} \varepsilon_n &= a_{n-1}^{d+1}(2^{d+3}\lambda_d(Q))^{-1}, \qquad N = \max_{1\le i\le n} N_d(i,\varepsilon_n),\\ \eta_n &= \varepsilon_n(N^2 2^{2N^2})^{-1}, \end{aligned}$$

where N_d is the function defined in Theorem 9. The set $S_n = \partial A \cup \bigcup_{i<n} \partial F_i$ is closed, and $\lambda_d(S_n) = 0$. Therefore we can select a positive number

$$\delta_n < \eta_n(12d(8\sqrt{d}+2)^{d-1}N^2)^{-1}$$

such that

$$(13)\qquad \lambda_d(U_d(S_n,\delta_n)) < \frac{\eta_n}{3(2^d+1)N^2}.$$

According to Lemma 10, we can select a number $0 < a_n < (a_{n-1}/2)^2$ and a set $H_n \subset B_n = [a_{n-1}/2, a_{n-1}]^{d-1}$ such that H_n is the union of finitely many disjoint rectangular boxes, $\lambda_{d-1}(H_n) > \lambda_{d-1}(B_n)/2 = a_{n-1}^{d-1}\cdot 2^{-d}$, and no ball of radius δ_n can be covered by any N^2 congruent copies of the set $([0,a_n]^{d-1} \cup H_n) \times [0,1]$. Then we select a function $f_n \in C^\infty(\mathbb{R}^{d-1})$ such that f_n vanishes outside H_n, $0 \le f_n \le a_{n-1}^2/2$ in H_n, and $\int_{B_n} f_n\, d\lambda_{d-1} > a_{n-1}^{d+1}\cdot 2^{-d-1}$.

In this way we have defined the sequences (a_n) and (f_n). Then we define the sets B_I as described above. Then we have, for every $h > 0$,

$$\begin{aligned} \lambda_d(U_d(\partial(t_I A), h)) &\le \lambda_d(U_d(\partial(t_I A), t_I h)) = t_I^d \lambda_d(U_d(\partial A, h))\\ &\le 2^d \lambda_d(U_d(\partial A, h)), \end{aligned}$$

and thus, by (13),

$$(14)\qquad \lambda_d(U_d(\partial(t_I A),\delta_n)) \le 2^d\,\frac{\eta_n}{3(2^d+1)N^2}$$

for every n and I.

LEMMA 12. *If I and J are sets of positive integers such that $I\setminus J$ is infinite*, *then* (i) *B_I and B_J are not equidecomposable under any amenable subgroup of G_d*, *and* (ii) *$\operatorname{int} B_I$ and $\operatorname{int} B_J$ are not equidecomposable in $\mathbf{B}_d$ under any amenable subgroup of G_d*.

Proof. Suppose that $B_I \overset{G}{\sim}_k B_J$, where G is an amenable subgroup of G_d. Since $I\setminus J$ is infinite, there is an $n \in I\setminus J$ such that $n > \max(k, k_0(G))$. Note that $B_I \cup B_J \subset Q = [-4\sqrt{d}, 4\sqrt{d}]^d$. Then, by Theorem 9 and by the definition of N in (12), there is a decomposition of Q into convex sets $C_1,\dots,C_N$ and there are isometries $a_{i,j}$ $(i,j = 1,\dots,N)$ such that $a_{i,j}(C_j) \subset Q$ for every i,j, and

$$(15)\qquad \left|\frac{1}{N}\sum_{i,j=1}^N \chi_{a_{i,j}(B_I\cap C_j)} - \frac{1}{N}\sum_{i,j=1}^N \chi_{a_{i,j}(B_J\cap C_j)}\right| < \varepsilon_n$$

everywhere on $\mathbb{R}^d$. We put

$$U_1 = \bigcup_{\substack{\nu\in I\\ \nu\ge n}} F_\nu, \quad U_2 = (t_I A)\cup \bigcup_{\substack{\nu\in I\\ \nu< n}} F_\nu, \quad V_1 = \bigcup_{\substack{\nu\in J\\ \nu> n}} F_\nu, \quad V_2 = (t_J A)\cup \bigcup_{\substack{\nu\in J\\ \nu< n}} F_\nu.$$

Then $B_I = U_1\cup U_2$ and $B_J = V_1\cup V_2$. Since

$$\partial U_2 \subset \partial(t_I A)\cup \bigcup_{i<n}\partial F_i \quad \text{and} \quad \partial V_2 \subset \partial(t_J A)\cup \bigcup_{i<n}\partial F_i,$$

it follows from (13) and (14) that

$$\begin{aligned} &\lambda_d(U_d(\partial U_2,\delta_n)) < (2^d+1)\,\frac{\eta_n}{3(2^d+1)N^2} = \frac{\eta_n}{3N^2},\\ &\lambda_d(U_d(\partial V_2,\delta_n)) < \frac{\eta_n}{3N^2}. \end{aligned} \tag{16}$$

Since $C_j\subset Q$, it follows from Lemma 11 that

$$\lambda_d(U_d(\partial C_j,\delta_n)) \le 4d(8\sqrt{d}+2)^{d-1}\delta_n \le \frac{\eta_n}{3N^2} \tag{17}$$

for every $j=1,\dots,N$. Let

$$g = \sum_{i,j=1}^N \chi_{a_{i,j}(V_2\cap C_j)} - \sum_{i,j=1}^N \chi_{a_{i,j}(U_2\cap C_j)}.$$

Then, by (15), we have

$$\begin{aligned} \sum_{i,j=1}^N \chi_{a_{i,j}(U_1\cap C_j)} &\ge \sum_{i,j=1}^N \chi_{a_{i,j}(U_1\cap C_j)} - \sum_{i,j=1}^N \chi_{a_{i,j}(V_1\cap C_j)}\\ &> g - N\varepsilon_n. \end{aligned} \tag{18}$$

Let $D_1,\dots,D_P$ be an enumeration of the atoms of the algebra of sets generated by $a_{i,j}(U_2\cap C_j)$ and $a_{i,j}(V_2\cap C_j)$ $(i,j=1,\dots,N)$ in Q. Then $P\le 2^{2N^2}$, $D_1\cup\ldots\cup D_P$ is a disjoint decomposition of Q, and g is constant on each D_μ. Consequently, $g=\sum_{\mu=1}^P \alpha_\mu\chi_{D_\mu}$, where $|\alpha_\mu|\le N^2$ for every μ, since $|g|\le N^2$ everywhere. For every $\mu=1,\dots,P$, ∂D_μ is covered by

$$\bigcup_{i,j=1}^N [\partial(a_{i,j}(U_2))\cup\partial(a_{i,j}(V_2))\cup\partial(a_{i,j}(C_j))].$$

Therefore

$$\begin{aligned} &\lambda_d(U_d(\partial D_\mu,\delta_n))\\ &\quad\le N^2[\lambda_d(U_d(\partial U_2,\delta_n))+\lambda_d(U_d(\partial V_2,\delta_n))] + N\sum_{j=1}^N \lambda_d(U_d(\partial C_j,\delta_n))\\ &\quad\le N^2(\eta_n/N^2) = \eta_n \end{aligned}$$

by (16) and (17). Consequently, if $\lambda_d(D_\mu) > \eta_n$ then $U_d(\partial D_\mu, \delta_n)$ does not cover D_μ. If $x \in D_\mu \setminus U_d(\partial D_\mu, \delta_n)$ then $B_d(x, \delta_n) \subset D_\mu$, that is, D_μ contains a ball of radius δ_n. Since

$$U_1 = \bigcup_{\substack{\nu \in I \\ \nu \geq n}} F_\nu \subset \Big(\bigcup_{\nu > n} F_\nu \Big) \cup F_n \subset ([0, a_n]^{d-1} \cup H_n) \times [0,1],$$

it follows from the choice of H_n that N^2 congruent copies of U_1 cannot cover D_μ. That is, if $\lambda_d(D_\mu) > \eta_n$ then there is a point $y \in D_\mu$ such that $\sum_{i,j=1}^N \chi_{a_{i,j}(U_1)}(y) = 0$, and thus $g(y) = \alpha_\mu < N\varepsilon_n$ by (18). Therefore we have

$$\begin{aligned} (19) \quad \int_Q g\,dx &= \sum_{\mu=1}^P \alpha_\mu \lambda_d(D_\mu) = \sum_{\lambda_d(D_\mu) > \eta_n} \alpha_\mu \lambda_d(D_\mu) + \sum_{\lambda_d(D_\mu) \leq \eta_n} \alpha_\mu \lambda_d(D_\mu) \\ &\leq N\varepsilon_n \lambda_d(Q) + PN^2 \eta_n \\ &\leq 2N\varepsilon_n \lambda_d(Q) = N a_{n-1}^{d+1} \cdot 2^{-d-2} \end{aligned}$$

by (12). On the other hand,

$$\begin{aligned} \int_Q g\,dx &= N\lambda_d(V_2) - N\lambda_d(U_2) \\ &= N(1 - \lambda_d(V_1)) - N(1 - \lambda_d(U_1)) \\ &= N\lambda_d(U_1) - N\lambda_d(V_1) \\ &\geq N\lambda_d(F_n) - N\lambda_d([0, a_n]^d) \\ &> N a_{n-1}^{d+1} \cdot 2^{-d-1} - N a_n^d > N a_{n-1}^{d+1} \cdot 2^{-d-2}, \end{aligned}$$

which contradicts (19). This completes the proof of (i). The second statement can be proved in the same way, using (ii) of Theorem 9. ∎

In order to complete the proof of Theorem 3, we take a system $\mathcal{I}$ of infinite sets of positive integers such that $\mathcal{I}$ has the cardinality of the continuum, and either $I \setminus J$ or $J \setminus I$ is infinite for every $I, J \in \mathcal{I},\ I \neq J$. (We may take

$$\mathcal{I} = \{\phi(\{r \in \mathbb{Q} : r < c\}) : c \in \mathbb{R}\},$$

where ϕ is any injection from $\mathbb{Q}$ into $\mathbb{N}$.) It is clear that the system $\mathcal{F} = \{B_I : I \in \mathcal{I}\}$ satisfies the requirements of Theorem 3. ∎

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