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Borel–Wadge degrees

by

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Abstract. Two sets of reals are Borel equivalent if one is the Borel pre-image of the other, and a Borel–Wadge degree is a collection of pairwise Borel equivalent subsets of $\mathbb{R}$. In this note we investigate the structure of Borel–Wadge degrees under the assumption of the Axiom of Determinacy.

1. Introduction and statements of the results. Let X be a Polish (i.e., separable, completely metrizable) space, and let $A, B \subseteq X$. We say that A is *Borel reducible to* B, in symbols $A \leq_{\mathbf{B}} B$, if $A = f^{-1}“B$ for some Borel $f : X \to X$. (The function f is called a *reduction* of A to B.) Since $\leq_{\mathbf{B}}$ is a pre-order, we can consider the associated equivalence relation $A \equiv_{\mathbf{B}} B \Leftrightarrow A \leq_{\mathbf{B}} B \;\&\; B \leq_{\mathbf{B}} A$. The equivalence classes under $\equiv_{\mathbf{B}}$ are called the *Borel–Wadge degrees*, and $[A]_{\mathbf{B}}$ is the degree of A. This is analogous to the usual notion of Wadge reducibility $\leq_{\mathrm{W}}$ and Wadge degree, where the reduction f is taken to be continuous. (For more on the Wadge hierarchy, see e.g. [Kec95, §21.E], [And03] and the references therein.) Since all uncountable Polish spaces are Borel isomorphic, the Borel–Wadge hierarchy does not depend on X, and therefore we can restrict ourselves to the Baire space ${}^{\omega}\omega$. (If X is countable, then the structure of the Borel–Wadge hierarchy becomes trivial.) This should be contrasted with the Wadge and Lipschitz hierarchies which are quite sensitive to the topological structure of the underlying Polish space.

The work of Wadge and others has shown that the Axiom of Determinacy, AD from now on, imposes a rich and detailed structure on the Wadge degrees. In this paper it is shown that the Borel–Wadge degrees exhibit a similar behavior, namely: the relation $\leq_{\mathbf{B}}$ is well-founded; self-dual degrees and non-self-dual pairs alternate, with a self-dual degree at limit levels of countable cofinality and non-self-dual pairs at limit levels of uncountable co-

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finality. In §4 we go further and characterize the non-self-dual Borel–Wadge degrees.

Instead of using the full AD as our hypothesis, we shall use one or the other of $\mathsf{SLO}^{\mathbf{B}}$ and $\mathsf{SLO}^{\mathrm{W}}$, which are both consequences of AD closely related to reducibility. $\mathsf{SLO}^{\mathbf{B}}$, the *semi-linear ordering principle* for Borel reducibility, is the statement: if $A, B \subseteq {}^{\omega}\omega$ then $A \leq_{\mathbf{B}} B$ or $\neg B \leq_{\mathbf{B}} A$. $\mathsf{SLO}^{\mathrm{W}}$ is the analogous statement with "$\leq_{\mathrm{W}}$" replacing "$\leq_{\mathbf{B}}$". Since $\mathsf{SLO}^{\mathbf{B}}$ follows at once from $\mathsf{SLO}^{\mathrm{W}}$ and since AD implies $\mathsf{SLO}^{\mathrm{W}}$ (Wadge's Lemma), $\mathsf{SLO}^{\mathbf{B}}$ follows from AD. It has been conjectured that AD follows from $\mathsf{SLO}^{\mathrm{W}}$, and a similar conjecture can be made for $\mathsf{SLO}^{\mathbf{B}}$, or even for more generous notions of reducibilities. There are essentially two reasons to adopt $\mathsf{SLO}^{\mathbf{B}}$ or $\mathsf{SLO}^{\mathrm{W}}$ rather than AD. The first is that it is interesting to develop the theory of the Borel–Wadge degrees from purely order-theoretic—and possibly weaker—assumptions. For example, the well-foundedness of $\leq_{\mathbf{B}}$ is a trivial consequence of the well-foundedness of $\leq_{\mathrm{W}}$ (which holds under AD), but the proof under $\mathsf{SLO}^{\mathbf{B}}$ is not just a matter of re-writing the usual proof. Moreover, knowing that $\mathsf{SLO}^{\mathbf{B}}$ is strong enough to civilize the Borel–Wadge hierarchy may prove to be useful in trying to show that AD follows from $\mathsf{SLO}^{\mathbf{B}}$. The second—and more important—reason to use $\mathsf{SLO}^{\mathbf{B}}$ or $\mathsf{SLO}^{\mathrm{W}}$ instead of AD is that there is no analogue of the Wadge/Lipschitz games for Borel functions, hence many of the standard proofs for the Wadge hierarchy do not generalize in a straightforward way to the Borel set-up. On the other hand, the ideas needed to prove standard consequences of AD under more parsimonious assumptions turn out to be useful for the study of the Borel–Wadge hierarchy.

The results in this paper are closely related to the ones in [And03], where it is shown that $\mathsf{SLO}^{\mathrm{W}}$ is strong enough to prove many of the standard results on the Wadge hierarchy. The second author realized that the techniques used in that paper could be extended to the Borel context and proved the results in §3. Then the first author found a simpler proof of them. The results in §4 are joint work.

2. Preliminaries. Our base theory is $\mathsf{ZF}+\mathsf{DC}(\mathbb{R})$, the Zermelo–Frænkel set theory augmented with the axiom of dependent choices over the reals. For certain proofs we will need to assume BP, the assertion that all sets of reals have the property of Baire.

2.1. *Notation.* Our set-theoretic notation is standard—for all unexplained facts on Descriptive Set Theory the reader should consult [Mos80] and [Kec95].

A *tree* on a non-empty set X is a non-empty $T \subseteq {}^{<\omega}X$ closed under subsequences. A tree T is *pruned* if $\forall t \in T\, \exists s \in T\, (t \subset s)$. A *branch* of T is a $b \in {}^{\omega}X$ such that $\forall n\, (b{\restriction}n \in T)$, and $[T]$ is the set of all branches of T. If

$s \in T$, then $T_{\lfloor s \rfloor} = \{t \mid s^\frown t \in T\}$ is a tree. We will mostly be interested in trees on ω.

As customary in the subject, $\mathbb{R}$ is identified with the Baire space ${}^\omega\omega$ endowed with the topology given by the usual metric $d(x, y) = 2^{-n}$ if $x \restriction n = y \restriction n$ and $x(n) \neq y(n)$, and $d(x, y) = 0$ if $x = y$. The basic open neighborhood determined by $s \in {}^{<\omega}\omega$ is $\boldsymbol{N}_s = \{x \in \mathbb{R} \mid s \subset x\}$, and a non-empty closed set is of the form $[T]$ where T is a pruned tree. Given $A \subseteq \mathbb{R}$ and $s \in {}^{<\omega}\omega$, let $s^\frown A = \{s^\frown x \mid x \in A\}$, and let $A_{\lfloor s \rfloor} = \{x \in \mathbb{R} \mid s^\frown x \in A\}$. When $s = \langle n \rangle$ we will write $A_{\lfloor n \rfloor}$ rather than $A_{\lfloor \langle n \rangle \rfloor}$. For $A_n \subseteq \mathbb{R}$ let

$$\bigoplus\nolimits_n A_n = \bigcup\nolimits_n \langle n \rangle^\frown A_n,$$

and let $B \oplus C = \bigoplus_n A_n$ with $A_{2n} = B$ and $A_{2n+1} = C$.

A set $F \subseteq {}^\omega 2$ is a *flip-set* iff

$$\forall x, y \in {}^\omega 2\, (\exists! k \in \omega\, (x(k) \neq y(k)) \Rightarrow (x \in F \Leftrightarrow y \notin F)).$$

It is easy to see that a flip-set neither has the property of Baire nor is Lebesgue measurable.

A map $\varphi : {}^{<\omega}\omega \to {}^{<\omega}\omega$ is:

- *monotone* if $s \subseteq t \Rightarrow \varphi(s) \subseteq \varphi(t)$;
- *Lipschitz* if it is monotone and $\mathrm{lh}(\varphi(s)) = \mathrm{lh}(s)$;
- *continuous* if it is monotone and $\lim_{n\to\infty} \mathrm{lh}(\varphi(x \restriction n)) = \infty$ for any $x \in \mathbb{R}$.

Clearly, if φ is Lipschitz then it is also continuous, and in both cases we can define the induced function

$$f_\varphi : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \bigcup\nolimits_n \varphi(x \restriction n).$$

If φ is continuous then so is f_φ, and if φ is Lipschitz then f_φ is Lipschitz with constant ≤ 1. Recall that a function $f : (X, d_X) \to (Y, d_Y)$ between metric spaces is *Lipschitz with constant* C if $d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2)$ for all $x_1, x_2 \in X$. If the constant C is < 1 then we will say that f is a *contraction*. Any $f : \mathbb{R} \to \mathbb{R}$ which is continuous, respectively Lipschitz with constant ≤ 1, is of the form f_φ with φ continuous, respectively Lipschitz. If $f : \mathbb{R} \to \mathbb{R}$ is a contraction, then the Lipschitz constant is $\leq 1/2$ and $f = f_\varphi$ for some φ monotone and such that $\forall s\, (\mathrm{lh}(\varphi(s)) = \mathrm{lh}(s) + 1)$. Let $\mathrm{Lip} = \{f \in {}^{\mathbb{R}}\mathbb{R} \mid f \text{ is Lipschitz with constant} \leq 1\}$.

2.2. *Reducibilities.* Let $\mathcal{F} \subseteq {}^{\mathbb{R}}\mathbb{R}$ be a family of functions closed under composition and containing the identity function. For $A, B \subseteq \mathbb{R}$, say that A is $\mathcal{F}$-*reducible* to B, in symbols $A \leq_{\mathcal{F}} B$, if $\forall x \in \mathbb{R}\, (x \in A \Leftrightarrow f(x) \in B)$ for some $f \in \mathcal{F}$. By our hypothesis $\leq_{\mathcal{F}}$ is reflexive and transitive. Set $A \equiv_{\mathcal{F}} B$ iff $A \leq_{\mathcal{F}} B$ & $B \leq_{\mathcal{F}} A$, and $A <_{\mathcal{F}} B$ iff $A \leq_{\mathcal{F}} B$ & $B \not\leq_{\mathcal{F}} A$. An $\mathcal{F}$-*degree* is an equivalence class of $\equiv_{\mathcal{F}}$, and $[A]_{\mathcal{F}} = \{B \mid B \equiv_{\mathcal{F}} A\}$ is the $\mathcal{F}$-degree of A. A set A is $\mathcal{F}$-*self-dual* iff $A \leq_{\mathcal{F}} \neg A$ iff $A \equiv_{\mathcal{F}} \neg A$, otherwise it is

$\mathcal{F}$*-non-self-dual*. Since self-duality is invariant under $\equiv_{\mathcal{F}}$, it can be applied to $\mathcal{F}$-degrees as well. The *dual* of $[A]_{\mathcal{F}}$ is $[\neg A]_{\mathcal{F}}$, and a pair of distinct degrees of the form $([A]_{\mathcal{F}}, [\neg A]_{\mathcal{F}})$ is a *non-self-dual pair*. The pre-order $\leq_{\mathcal{F}}$ induces a partial order $\leq$ on the $\mathcal{F}$-degrees: $[A]_{\mathcal{F}} \leq [B]_{\mathcal{F}}$ whenever $A \leq_{\mathcal{F}} B$. Similarly define $[A]_{\mathcal{F}} < [B]_{\mathcal{F}} \Leftrightarrow ([A]_{\mathcal{F}} \leq [B]_{\mathcal{F}} \ \& \ [A]_{\mathcal{F}} \neq [B]_{\mathcal{F}}) \Leftrightarrow A <_{\mathcal{F}} B$. Notice that $[\mathbb{R}]_{\mathcal{F}} = \{\mathbb{R}\}$ and $[\emptyset]_{\mathcal{F}} = \{\emptyset\}$ are the $<$-least $\mathcal{F}$-degrees and form a non-self-dual pair. We say that $[A]_{\mathcal{F}}$ is a *successor degree* if there is a $[B]_{\mathcal{F}} < [A]_{\mathcal{F}}$ such that $[B]_{\mathcal{F}} < [C]_{\mathcal{F}} < [A]_{\mathcal{F}}$ for no $C \subseteq \mathbb{R}$. If an $\mathcal{F}$-degree is not a successor and it is neither $[\mathbb{R}]_{\mathcal{F}}$ nor $[\emptyset]_{\mathcal{F}}$, then we say it is a *limit degree*. A limit degree is of *countable cofinality* if it is the least upper bound of an increasing sequence $[A_0]_{\mathcal{F}} < [A_1]_{\mathcal{F}} < \dots$ of $\mathcal{F}$-degrees.

$\mathsf{SLO}^{\mathcal{F}}$ is the statement:

$$\forall A, B \subseteq \mathbb{R}\,(A \leq_{\mathcal{F}} B \ \vee \ \neg B \leq_{\mathcal{F}} A).$$

Thus if each degree is identified with its dual, then $\leq$ is a linear order on the $\mathcal{F}$-degrees. If $\mathcal{F}$ is the set of all functions from $\mathbb{R}$ to $\mathbb{R}$, then the structure of the $\mathcal{F}$-degrees is trivial, since there are only three degrees: $[\mathbb{R}]_{\mathcal{F}}$, $[\emptyset]_{\mathcal{F}}$, and $\mathscr{P}(\mathbb{R}) \setminus \{\mathbb{R}, \emptyset\}$. Thus it is natural to put some restriction on the size of $\mathcal{F}$. For example, if $\mathcal{F}$ is the surjective image of $\mathbb{R}$, then a straightforward generalization of [Sol78, Lemma 0.2] (see also [And03, Lemma 18]) shows that there is no largest $\mathcal{F}$-degree, assuming $\mathsf{SLO}^{\mathcal{F}}$. More precisely

Lemma 1 (Solovay). *Suppose there is a surjection $\mathbb{R} \twoheadrightarrow \mathcal{F}$ and that $\mathsf{SLO}^{\mathcal{F}}$ holds. Then there is a map $J : \mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R})$ such that*

$$\forall A \subseteq \mathbb{R}\,(A <_{\mathcal{F}} J(A) \ \& \ \neg A <_{\mathcal{F}} J(A)).$$

In order to prove more results on the $\mathcal{F}$-degrees we must impose further restrictions on the set $\mathcal{F}$.

Definition 2. $\mathcal{F} \subseteq {}^{\mathbb{R}}\mathbb{R}$ is *amenable* if either $\mathcal{F} = \mathrm{Lip}$, or else:

(1) there is a surjection $\mathbb{R} \twoheadrightarrow \mathcal{F}$,
(2) $\mathcal{F} \supseteq \mathrm{Lip}$,
(3) $\mathcal{F}$ is closed under composition,
(4) if each $f_n \in \mathcal{F}$ then $\bigoplus_n f_n \in \mathcal{F}$, where

$$\textstyle\bigoplus_n f_n\,(x) = f_{x(0)}(x^-),$$

and $x^- = \langle x(n+1) \mid n \in \omega \rangle$.

Typical examples of amenable $\mathcal{F}$ are the collections of all Lipschitz functions, all continuous functions, and all Borel functions. The "$\mathcal{F}$" in $\leq_{\mathcal{F}}$, $[A]_{\mathcal{F}}$, $\mathsf{SLO}^{\mathcal{F}}$ etc. will be replaced by "L" in the Lipschitz case, by "W" in the continuous case (after Wadge), and by "**B**" in the Borel case. Notice that Lip satisfies (1), (2), and (3), but not (4).

Lemma 3. *Let $\mathcal{F} \neq \mathrm{Lip}$ be amenable and let $A \subseteq \mathbb{R}$.*

(a) *$A \oplus \neg A$ is $\mathcal{F}$-self-dual and $A, \neg A \leq_{\mathcal{F}} A \oplus \neg A$. Moreover if $A, \neg A \leq_{\mathcal{F}} C$, then $A \oplus \neg A \leq_{\mathcal{F}} C$. In particular $[A \oplus \neg A]_{\mathcal{F}}$ is the $\leq_{\mathcal{F}}$-least degree above $[A]_{\mathcal{F}}$ and $[\neg A]_{\mathcal{F}}$.*

(b) *Assume $\mathsf{SLO}^{\mathcal{F}}$ and suppose $[A]_{\mathcal{F}}$ is limit of countable cofinality. Then $[A]_{\mathcal{F}}$ is self-dual.*

Proof. (a) The first part is trivial by (2) of Definition 2. For the second part, notice that if $f^{-1}``C = A$ and $g^{-1}``C = \neg A$, then $\bigoplus_n f_n$ witnesses $A \oplus \neg A \leq_{\mathcal{F}} C$, where $f_{2n} = f$ and $f_{2n+1} = g$.

(b) Let $A_0 <_{\mathcal{F}} A_1 <_{\mathcal{F}} \ldots$ witness that $[A]_{\mathcal{F}}$ is limit of countable cofinality. Since $A_{n+1} \not\leq_{\mathcal{F}} A_n$, there is $f'_n \in \mathcal{F}$ witnessing $\neg A_n \leq_{\mathcal{F}} A_{n+1}$ by $\mathsf{SLO}^{\mathcal{F}}$. The functions $f_n(x) = \langle n+1 \rangle ^\frown f'_n(x)$ belong to $\mathcal{F}$ by (2) and (3) of Definition 2. Therefore $\bigoplus_n f_n$ witnesses that $\bigoplus_n \neg A_n \leq_{\mathcal{F}} \bigoplus_n A_n$, that is, $\bigoplus_n A_n$ is $\mathcal{F}$-self-dual. Clearly (2) implies that $A_i \leq_{\mathcal{F}} \bigoplus_n A_n$ for each i, and if g_n witnesses $A_n \leq_{\mathcal{F}} C$ then $\bigoplus_n g_n$ witnesses $\bigoplus_n A_n \leq_{\mathcal{F}} C$. In other words $\bigoplus_n A_n$ is a least upper bound of the A_n's. Therefore $\bigoplus_n A_n \equiv_{\mathcal{F}} A$. ∎

The lemma is still true if $\mathcal{F} = \mathrm{Lip}$ (and hence it is true for all amenable $\mathcal{F}$) but the argument is more involved and $\mathsf{SLO}^{\mathrm{L}}$ must be assumed also for case (a)—see [And03].

The *Lipschitz game* on $A, B \subseteq \mathbb{R}$, $G_{\mathrm{L}}(A, B)$, introduced by Wadge in [Wad83] is the game on ω where **I** plays a real a, **II** plays a real b, and **II** wins iff $a \in A \Leftrightarrow b \in B$. Wadge's Lemma is the simple, but fundamental observation that a winning strategy for **II** yields a Lipschitz map witnessing $A \leq_{\mathrm{L}} B$, while a winning strategy for **I** yields a Lipschitz map (in fact: a contraction) witnessing $\neg B \leq_{\mathrm{L}} A$. Therefore AD implies $\mathsf{SLO}^{\mathrm{L}}$, and since the smaller the $\mathcal{F}$ the stronger the $\mathsf{SLO}^{\mathcal{F}}$,

$$\mathsf{AD} \Rightarrow \mathsf{SLO}^{\mathrm{L}} \Rightarrow \mathsf{SLO}^{\mathrm{W}} \Rightarrow \mathsf{SLO}^{\mathbf{B}}.$$

We do not know whether any of these implications can be reversed—see [And03] for more on this. In fact, a well known open problem (probably first formulated by R. Solovay) asks whether $\mathsf{SLO}^{\mathrm{L}}$ or even $\mathsf{SLO}^{\mathrm{W}}$ implies AD, assuming $\mathrm{V} = \mathrm{L}(\mathbb{R})$. A similar question can be asked for $\mathsf{SLO}^{\mathbf{B}}$ or, more boldly, for $\mathsf{SLO}^{\mathcal{F}}$:

Open Problem 4. *Assume $\mathrm{V} = \mathrm{L}(\mathbb{R})$. Does $\mathsf{SLO}^{\mathbf{B}} \Rightarrow \mathsf{AD}$? Does $\mathsf{SLO}^{\mathcal{F}} \Rightarrow \mathsf{AD}$ if $\mathcal{F}$ is amenable?*

A less ambitious goal would be to prove some of the standard consequences of AD (like BP and LM, the assertion that all sets of reals are Lebesgue measurable) from some form of semi-linear ordering principle. This would yield some evidence for positive solutions to these open problems. For example it is known that the perfect set property [Wad83] and the axiom of countable choices for sets of reals [And03] follow from $\mathsf{SLO}^{\mathrm{W}}$, but the following seems to be open:

Open Problem 5. *Assume* V = L(ℝ) *and let* $\mathcal{F}$ *be amenable. Does* $\mathsf{SLO}^{\mathcal{F}} \Rightarrow \mathsf{BP}$*? Does* $\mathsf{SLO}^{\mathcal{F}} \Rightarrow \mathsf{LM}$*?*

This is open even when $\mathcal{F}$ is the smallest amenable set of functions (and hence $\mathsf{SLO}^{\mathcal{F}}$ is the strongest semi-linear ordering principle), that is, when $\mathcal{F} = \mathrm{Lip}$, the collection of all Lipschitz functions.

More partial evidence for a positive answer to Open Problem 4 would be provided by a proof of the equivalence between the various semi-linear ordering principles, say between $\mathsf{SLO}^{\mathrm{L}}$, $\mathsf{SLO}^{\mathrm{W}}$, and $\mathsf{SLO}^{\mathbf{B}}$; again see [And03].

2.3. *The Wadge and Lipschitz hierarchies.* Assuming $\mathsf{AD} + \mathsf{DC}(\mathbb{R})$, the following properties of the Wadge degrees hold:

(1) $<_{\mathrm{W}}$ is well-founded,

(2) immediately above a self-dual degree there is a non-self-dual pair of degrees, and immediately above a non-self-dual pair of degrees there is a self-dual degree,

(3) at limit levels of countable cofinality there is a single self-dual degree, and at uncountable cofinality there is a non-self-dual pair.

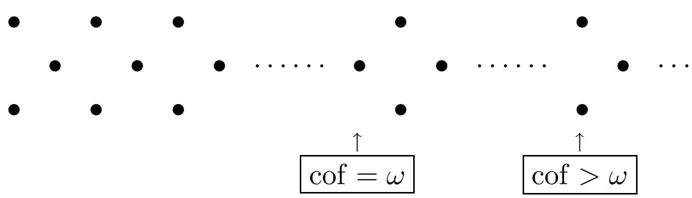
For the Lipschitz degrees we have the following:

(4) $<_{\mathrm{L}}$ is well-founded,

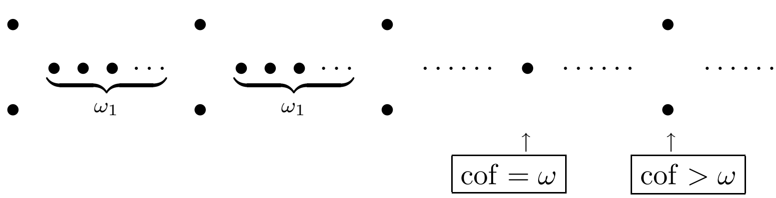
(5) every self-dual Wadge degree is the union of ω_1 consecutive Lipschitz self-dual degrees, while the non-self-dual pairs of Wadge degrees coincide exactly with the non-self-dual pairs of Lipschitz degrees,

(6) at limit levels of countable cofinality there is a single self-dual degree, and at uncountable cofinality there is a non-self-dual pair.

Therefore the Wadge hierarchy looks like this:



and the Lipschitz hierarchy looks like this:



with each ω_1-block of self-dual Lipschitz degrees collapsing to a single self-dual Wadge degree. In [And03] it is shown that (1)–(3) follow from $\mathsf{SLO}^{\mathrm{W}} + \mathsf{BP}$, and that (4)–(6) follow from $\mathsf{SLO}^{\mathrm{L}} + \mathsf{BP}$. Therefore, if $\mathcal{F}$ is amenable—so that the $\mathcal{F}$-degrees are coarser than (or equal to) the Lipschitz degrees—and if AD (or even just $\mathsf{SLO}^{\mathrm{L}} + \mathsf{BP}$) is assumed, then $<_{\mathcal{F}}$ is well-founded and every non-self-dual $\mathcal{F}$-degree is a non-self-dual Lipschitz degree, i.e., $A \not\equiv_{\mathcal{F}} \neg A \Rightarrow [A]_{\mathcal{F}} = [A]_{\mathrm{L}}$.

3. The Borel–Wadge hierarchy. We now focus on Borel–Wadge degrees. Our first goal is to prove that the well-foundedness of $<_{\mathbf{B}}$ follows from $\mathsf{SLO}^{\mathbf{B}} + \mathsf{BP}$. The standard proof of the non-existence of an infinite $<_{\mathrm{L}}$-descending sequence $\langle A_n \mid n \in \omega \rangle$ uses AD to pick winning strategies for $\mathbf{I}$ in $G_{\mathrm{L}}(A_n, A_{n+1})$, and in $G_{\mathrm{L}}(A_n, \neg A_{n+1})$. By pitting them against each other, a flip-set is constructed, contradicting BP. If we start from an infinite $<_{\mathbf{B}}$-descending sequence $\langle A_n \mid n \in \omega \rangle$ we would like to argue, assuming $\mathsf{SLO}^{\mathbf{B}}$, that $\mathbf{I}$ wins $G_{\mathrm{L}}(A_n, A_{n+1})$ and $G_{\mathrm{L}}(A_n, \neg A_{n+1})$, and proceed as before. In order to do this we need a few preliminary results.

A topological space is 0-*dimensional* if its topology is generated by the clopen sets. A metric space (X, d) is *Polish* if it is separable and d is complete. The collection of Borel subsets of (X, d) is denoted by $\mathbf{B}(X, d)$.

LEMMA 6. *Suppose (X, d) is a Polish space and $\langle A_n \mid n \in \omega \rangle$ is a sequence of Borel subsets of (X, d). Then there is a metric d' on X such that*

(1) *(X, d') is Polish and 0-dimensional*;

(2) *the new topology is finer than the old one*, *i.e.*, *every d-open set is also d'-open*;

(3) *each A_n is d'-clopen*;

(4) *the two topologies give rise to the same Borel sets*, *that is*, $\mathbf{B}(X, d) = \mathbf{B}(X, d')$.

See [Kec95, Theorem 13.1 and Exercise 13.5] for a proof. An easy consequence of this is the following result—see [Kec95, Theorem 13.11].

LEMMA 7. *Let (X, d) be a Polish space*, *let $B \in \mathbf{B}(X, d)$*, *and let $f : B \to B$ be a Borel function. There is a metric d' on B such that*

(1) *(B, d') is Polish and 0-dimensional*;

(2) *the topology τ' generated by d' on B refines the topology that B inherits from X*, *i.e.*, *$\tau' \supseteq \{U \cap B \mid U \in \tau\}$*, *where τ is the topology on X*;

(3) *(B, d') has the same Borel structure as B*, *that is*: *for every $C \subseteq B$*,

$$C \in \mathbf{B}(X, d) \;\Leftrightarrow\; C \in \mathbf{B}(B, d');$$

(4) *$f : (B, d') \to (B, d)$ is continuous.*

By [Kec95, Theorem 7.8] every 0-dimensional Polish space is homeomorphic to a closed subset of the Baire space, so using Lemmata 6 and 7 will not take us outside of $\mathscr{P}(\mathbb{R})$.

Lemma 8. (a) *If* $A \leq_{\mathbf{B}} B$ *then there is* $A^* \equiv_{\mathbf{B}} A$ *such that* $A^* \leq_{\mathrm{L}} A$ *and* $A^* \leq_{\mathrm{L}} B$.

(b) *Assume* $\mathsf{SLO}^{\mathbf{B}}$ *and* $A <_{\mathbf{B}} B$. *Then there is* $A^* \equiv_{\mathbf{B}} A$ *such that* **I** *has a winning strategy in* $G_{\mathrm{L}}(\neg B, A^*)$ *and in* $G_{\mathrm{L}}(B, A^*)$.

Proof. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be Borel such that $f^{-1}``B = A$. By Lemma 7 there is a 0-dimensional Polish topology τ on $\mathbb{R}$ that is finer than the standard one and makes f continuous. Let $G : C \to (\mathbb{R}, \tau)$ be a homeomorphism with $C \subseteq \mathbb{R}$ a closed set, and by [Kec95, Proposition 2.8] let $\pi : \mathbb{R} \twoheadrightarrow C$ be Lipschitz and such that $\pi \restriction C$ is the identity. Let

$$A' = (G \circ \pi)^{-1}``A.$$

Then $A' \leq_{\mathrm{W}} A$ via $G \circ \pi$, and $A \leq_{\mathbf{B}} A'$ via $G^{-1} : \mathbb{R} \to C \subseteq \mathbb{R}$. Since $f \circ G \circ \pi : \mathbb{R} \to \mathbb{R}$ is continuous and

$$x \in A' \iff G(\pi(x)) \in A \iff f(G(\pi(x))) \in B$$

we have $A' \leq_{\mathrm{W}} B$. We need the following result from [And03, Lemma 19, part (a)]:

Lemma 9. *If* $A' \leq_{\mathrm{W}} A$ *then there is* $A'' \equiv_{\mathrm{W}} A'$ *such that* $A'' \leq_{\mathrm{L}} A'$ *and* $A'' \leq_{\mathrm{L}} A$.

Let A'' be as in Lemma 9. Since $A'' \leq_{\mathrm{L}} A' \leq_{\mathrm{W}} B$, we have $A'' \leq_{\mathrm{W}} B$, so by Lemma 9 again there is A^* such that $A^* \leq_{\mathrm{L}} A''$, $A'' \leq_{\mathrm{W}} A^*$, and $A^* \leq_{\mathrm{L}} B$, which is what we had to prove.

(b) If $\forall n\, (A \not\leq_{\mathbf{B}} B_{\lfloor n \rfloor})$ or $\forall n\, (A \not\leq_{\mathbf{B}} \neg B_{\lfloor n \rfloor})$, then by $\mathsf{SLO}^{\mathbf{B}}$ we would have $\forall n\, (B_{\lfloor n \rfloor} \leq_{\mathbf{B}} \neg A)$ or $\forall n\, (B_{\lfloor n \rfloor} \leq_{\mathbf{B}} A)$, hence $B \leq_{\mathbf{B}} \neg A$ or $B \leq_{\mathbf{B}} A$, contradicting our assumption in either case. Therefore there are $n_0, m_0 \in \omega$ such that $A \leq_{\mathbf{B}} B_{\lfloor n_0 \rfloor}$ and $A \leq_{\mathbf{B}} \neg B_{\lfloor m_0 \rfloor}$ via Borel functions f and g. By successive applications of Lemma 7 there is a 0-dimensional Polish topology on $\mathbb{R}$ that is finer than the standard one and makes f and g continuous. Arguing as in part (a) shows that there is an $A' \leq_{\mathrm{L}} A$ such that $A \leq_{\mathbf{B}} A'$ and $A' \leq_{\mathrm{L}} B_{\lfloor n_0 \rfloor}$, and since $A' \leq_{\mathbf{B}} \neg B_{\lfloor m_0 \rfloor}$, there is $A^* \leq_{\mathrm{L}} A'$ such that $A' \leq_{\mathbf{B}} A^*$ and $A^* \leq_{\mathrm{L}} \neg B_{\lfloor m_0 \rfloor}$. By playing m_0 and then following the reduction witnessing $A^* \leq_{\mathrm{L}} \neg B_{\lfloor m_0 \rfloor}$, **I** wins $G_{\mathrm{L}}(B, A^*)$; similarly **I** has a winning strategy in $G_{\mathrm{L}}(\neg B, A^*)$. ∎

Corollary 10. *Assume* $\mathsf{SLO}^{\mathbf{B}} + \mathsf{BP}$. *Then* $<_{\mathbf{B}}$ *is a well-founded relation on* $\mathscr{P}(\mathbb{R})$.

Proof. Suppose $\langle A_n \mid n \in \omega \rangle$ is a $<_{\mathbf{B}}$-descending sequence of sets. Then $A_{n+1} \leq_{\mathbf{B}} A_n$ and $A_n \not\leq_{\mathbf{B}} A_{n+1}$, and hence, by $\mathsf{SLO}^{\mathbf{B}}$, $A_{n+1} <_{\mathbf{B}} A_n$ and $\neg A_{n+1} <_{\mathbf{B}} A_n$. By Lemma 8 we can construct inductively $A_0^* = A_0$ and

$A_n^* \equiv_{\mathbf{B}} A_n$ such that **I** has a winning strategy σ_n^1 in $G_{\mathrm{L}}(A_n^*, A_{n+1}^*)$ and σ_n^0 in $G_{\mathrm{L}}(\neg A_n^*, A_{n+1}^*)$. For any $z \in {}^\omega 2$ let $x_n = x_n^z$ be the real in the nth row of the following diagram where **I** uses $\sigma_n^{z(n)}$ on the nth row against his opponent on the $(n+1)$st row:

$$\begin{array}{ccccc} \sigma_0^{z(0)} & x_0(0) & x_0(1) & \cdots & = x_0 \\ \sigma_1^{z(1)} & x_1(0) & x_1(1) & \cdots & = x_1 \\ \vdots & \vdots & \vdots & & \vdots \end{array}$$

Thus x_n^z is the result of applying $\sigma_n^{z(n)}$ to x_{n+1}^z. Then $\{z \in {}^\omega 2 \mid x_0^z \in A_0^*\}$ is a flip-set, contradicting BP.

Lastly, to show that $<_{\mathbf{B}}$ is well-founded on $\mathscr{P}(\mathbb{R})$, it is enough to show that $<_{\mathbf{B}}$ is well-founded on $\{B \in \mathscr{P}(\mathbb{R}) \mid B \leq_{\mathbf{B}} A\}$ for any $A \subseteq \mathbb{R}$. So fix $A \subseteq \mathbb{R}$. Since there is a surjection $\mathbb{R} \twoheadrightarrow \{f \in {}^{\mathbb{R}}\mathbb{R} \mid f \text{ is Borel}\}$, $x \mapsto f_x$, consider the pre-order on $\mathbb{R}$ defined by

$$x \prec y \;\Leftrightarrow\; f_x^{-1}``A <_{\mathbf{B}} f_y^{-1}``A.$$

Then $<_{\mathbf{B}}$ is well-founded on $\{B \in \mathscr{P}(\mathbb{R}) \mid B \leq_{\mathbf{B}} A\}$ iff $\prec$ is well-founded on $\mathbb{R}$, which by $\mathsf{DC}(\mathbb{R})$ is equivalent to the non-existence of an infinite $\prec$-descending sequence. But any $\prec$-descending sequence in $\mathbb{R}$ yields a $<_{\mathbf{B}}$-descending sequence in $\{B \in \mathscr{P}(\mathbb{R}) \mid B \leq_{\mathbf{B}} A\}$, hence we are done by the first part of the proof. ∎

Thus, assuming $\mathsf{SLO}^{\mathbf{B}} + \mathsf{BP}$, the canonical rank function for the well-founded relation $<_{\mathbf{B}}$ on $\mathscr{P}(\mathbb{R})$ can be defined. It is called the *Borel–Wadge rank* and it is denoted by $A \mapsto \|A\|_{\mathbf{B}}$. It is immediate that $[A]_{\mathbf{B}}$ is a limit degree iff $\|A\|_{\mathbf{B}}$ is a limit ordinal, and that $[A]_{\mathbf{B}}$ is of countable cofinality iff $\mathrm{cof}(\|A\|_{\mathbf{B}}) = \omega$. For technical reasons (see [And03, Proposition 13]) it is convenient to assume that the Wadge rank $\|A\|_{\mathrm{W}}$ of a set is a non-zero ordinal, and hence, by analogy, we make the same assumption on the Borel–Wadge rank. Thus $\|\emptyset\|_{\mathbf{B}} = \|\mathbb{R}\|_{\mathbf{B}} = 1$.

The tree $\boldsymbol{T}(A) = \{s \in {}^{<\omega}\omega \mid A_{\lfloor s \rfloor} \equiv_{\mathrm{W}} A\}$ is a standard tool to investigate the structure of the Wadge degrees. For example $[A]_{\mathrm{W}}$, the Wadge degree of A, is self-dual iff $\boldsymbol{T}(A)$ is well-founded, i.e., if the converse of the extension relation on $\boldsymbol{T}(A)$ is well-founded. Notice that if $\boldsymbol{T}(A)$ is well-founded, then

$$\{\boldsymbol{N}_s \mid s \notin \boldsymbol{T}(A) \;\&\; s{\restriction}\mathrm{lh}(s) - 1 \in \boldsymbol{T}(A)\}$$

is a partition of $\mathbb{R}$ into countably many clopen sets D such that $D \cap A <_{\mathrm{W}} A$. This suggests the correct generalization of $\boldsymbol{T}(A)$ to the Borel context.

DEFINITION 11. Let $B \subseteq \mathbb{R}$. A *Borel partition* of B is a family $\{B_n \mid n < N\}$ of non-empty pairwise disjoint Borel sets such that $B = \bigcup_{n<N} B_n$ and $2 \leq N \leq \omega$.

First a trivial but useful fact:

Lemma 12. *Let $B \subseteq B'$ be Borel sets. If $A \cap B' \neq \mathbb{R}$, then $A \cap B \leq_{\mathbf{B}} A \cap B'$. In particular if B is Borel and $A \neq \mathbb{R}$, then $A \cap B \leq_{\mathbf{B}} A$.*

Then:

Lemma 13. *Let $\{B_n \mid n < N\}$ be a Borel partition of $\mathbb{R}$, and let $A \neq \mathbb{R}$.*

(a) *$\forall n < N\,(A \cap B_n \leq_{\mathbf{B}} A)$, and if C is such that $\forall n < N\,(A \cap B_n \leq_{\mathbf{B}} C)$, then $A \leq_{\mathbf{B}} C$. In other words: $[A]_{\mathbf{B}}$ is the $\leq_{\mathbf{B}}$-least upper bound of $\{[B_n \cap A]_{\mathbf{B}} \mid n < N\}$.*

(b) *Assume $\mathsf{SLO}^{\mathbf{B}}$. If $\forall n < N\,(A \cap B_n <_{\mathbf{B}} A)$ then $A \leq_{\mathbf{B}} \neg A$. Moreover, if $N < \omega$ then $[A]_{\mathbf{B}}$ is a successor degree.*

Proof. (a) The first part follows from Lemma 12. If g_n witnesses $B_n \cap A \leq_{\mathbf{B}} C$, then $g = \bigcup_n g_n \restriction B_n$ is Borel and witnesses $A \leq_{\mathbf{B}} C$.

(b) $A \cap B_n <_{\mathbf{B}} A$ implies, by $\mathsf{SLO}^{\mathbf{B}}$, that there are Borel functions $f_n : \mathbb{R} \to \mathbb{R}$ witnessing $A \cap B_n \leq_{\mathbf{B}} \neg A$. Then $f = \bigcup_n f_n \restriction B_n$ is Borel and $f^{-1}``\neg A = A$. Suppose now, towards a contradiction, that $[A]_{\mathbf{B}}$ is a limit degree and $N < \omega$. Let $C_0 = B_0 \cap A$ and, for $n+1 < N$, let $C_{n+1} = C_n \oplus (B_{n+1} \cap A)$. By induction, using Lemma 3(a) and the fact that $[A]_{\mathbf{B}}$ is limit, we see that $\forall n < N\,(C_n <_{\mathbf{B}} A)$. But $B_n \cap A \leq_{\mathbf{B}} C_{N-1}$ for $n < N$, hence $A \leq_{\mathbf{B}} C_{N-1}$ by part (a): a contradiction. ∎

Definition 14. For $A \subseteq \mathbb{R}$, let

$$\mathcal{I}(A) = \{B \mid B \text{ is Borel and } \exists \langle B_n \mid n \in \omega \rangle \text{ Borel sets such that } B = \textstyle\bigcup_n B_n \text{ and } B_n \cap A <_{\mathbf{B}} A\}.$$

By Lemma 12, the sets B_n in the definition can be taken to form a partition of B, since $\mathcal{I}(A)$ is empty when $A = \mathbb{R}$ or $A = \emptyset$. The following result can be easily verified.

Lemma 15. *Assume $\mathsf{SLO}^{\mathbf{B}}$.*

(a) *If $B \in \mathcal{I}(A)$ and $C \subseteq B$ is Borel, then $C \in \mathcal{I}(A)$.*

(b) *If $B_n \in \mathcal{I}(A)$, then $\bigcup_n B_n \in \mathcal{I}(A)$.*

Recall that a *σ-ideal of Borel sets* is a non-empty collection $\mathcal{J}$ of Borel subsets of $\mathbb{R}$, closed under Borel subsets and countable unions. A σ-ideal of Borel sets $\mathcal{J}$ is *proper* if $\mathbb{R} \notin \mathcal{J}$. Then Lemma 15 says that $\mathcal{I}(A)$ is a σ-ideal of Borel sets, and Lemma 13(b) says that if $\mathcal{I}(A)$ is not proper, then $[A]_{\mathbf{B}}$ is self-dual.

Theorem 16. *Assume BP and suppose $A \leq_{\mathbf{B}} \neg A$. Then there is a Borel partition $\{B_n \mid n \in \omega\}$ of $\mathbb{R}$ such that $\forall n < N\,(B_n \cap A <_{\mathbf{B}} A)$.*

Proof. Towards a contradiction, suppose that $\mathbb{R} \notin \mathcal{I} = \mathcal{I}(A)$.

CLAIM 16.1. *If B is Borel and $B \notin \mathcal{I}$, then there is a Borel function $f : B \to B$ witnessing*

$$\forall x \in B\,(x \in A \cap B \Leftrightarrow f(x) \in \neg A \cap B).$$

Proof of Claim. By case assumption $B \neq \emptyset$, and if $B = \mathbb{R}$ the result follows at once, so we may assume $B \neq \emptyset, \mathbb{R}$. By Lemma 12, $A \cap B \leq_{\mathbf{B}} A$ and $\neg A \cap B \leq_{\mathbf{B}} \neg A$. If $A \cap B <_{\mathbf{B}} A$, then taking $B_n = B$ in Definition 14, we would have $B \in \mathcal{I}$: a contradiction. Therefore $\neg A \cap B \leq_{\mathbf{B}} \neg A \leq_{\mathbf{B}} A \equiv_{\mathbf{B}} A \cap B$. Let $h : \mathbb{R} \to \mathbb{R}$ be a Borel function witnessing that $\neg A \cap B \leq_{\mathbf{B}} A \cap B$, and let $k : \mathbb{R} \twoheadrightarrow B$ be defined as

$$k(x) = \begin{cases} x & \text{if } x \in B, \\ b & \text{otherwise,} \end{cases}$$

where b is some fixed element of $\neg A \cap B$. Then $f = (k \circ h)\restriction B$ is the required function. ∎

We will construct a sequence of Borel sets $\mathbb{R} = B_0 \supseteq B_1 \supseteq \ldots$ such that $B_n \notin \mathcal{I}$. Using the Claim, let $f_n : B_n \to B_n$ be Borel and such that

$$\forall x \in B_n\,(x \in A \cap B_n \Leftrightarrow f_n(x) \in \neg A \cap B_n).$$

We will also choose a separable complete metric d_n on B_n such that d_0 is the usual metric on $\mathbb{R}$, and the topologies τ_n generated by the metrics d_n are all 0-dimensional and τ_{n+1} refines τ_n, that is, $\{U \cap B_{n+1} \mid U \in \tau_n\} \subseteq \tau_{n+1}$. We also require that $f_n\restriction B_{n+1} : (B_{n+1}, d_{n+1}) \to (B_n, d_n)$ be continuous, and that for any $m \leq n$ and every $a, b \in B_{n+1}$,

$$d_m(g_m \circ \ldots \circ g_n(a), g_m \circ \ldots \circ g_n(b)) < 2^{-n}, \tag{1}$$

where each g_i is either $f_i\restriction B_{i+1}$ or the identity on B_{i+1}. Then we can apply the Martin–Monk method as follows:

Fix $z \in {}^{\omega}2$ and let

$$g_n = \begin{cases} f_n\restriction B_{n+1} & \text{if } z(n) = 1, \\ \mathrm{id}\restriction B_{n+1} & \text{if } z(n) = 0. \end{cases}$$

For each n, pick $y_{n+1} \in B_{n+1}$ and let

$$x_m^n = g_m \circ \ldots \circ g_n(y_{n+1}) \in B_m$$

for all $m \leq n$. By construction, for any fixed m,

$$\forall n > m\,(g_m(x_{m+1}^n) = x_m^n) \tag{2}$$

and $\{x_m^n \mid n \geq m\} \subseteq B_m$ is a Cauchy sequence with respect to d_m, since $d_m(x_m^n, x_m^k) < 2^{-\min(n,k)}$ by (1). Therefore we get an

$$x_m = \lim_{n\to\infty} x_m^n \in B_m$$

and by continuity of $g_m : (B_{m+1}, d_{m+1}) \to (B_m, d_m)$ and by (2),

$$g_m(x_{m+1}) = x_m.$$

Naturally x_m really depends on $z \in {}^{\omega}2$, so we should write $x_m = x_m(z)$. By construction, if $\forall n > n_0\, (z(n) = w(n))$ then

$$\forall n > n_0\, (x_n(z) = x_n(w))$$

and if $z(n_0) \neq w(n_0)$ then

$$x_{n_0}(z) \in A \cap B_{n_0} \;\Leftrightarrow\; x_{n_0}(w) \notin A \cap B_{n_0}.$$

The usual argument shows that $\{z \in {}^{\omega}2 \mid x_0(z) \in A\}$ is a flip-set, contradicting the property of Baire.

Therefore it is enough to construct the B_n's and d_n's. As required, set $B_0 = \mathbb{R}$, d_0 the usual distance on $\mathbb{R}$, and let $f_0 : \mathbb{R} \to \mathbb{R}$ be a Borel function witnessing $A \leq_{\mathbf{B}} \neg A$.

Suppose B_m, f_m, and d_m have been defined for all $m \leq n$. Fix an $s \in {}^{n+1}2$, and for $i \leq n$, let $g_i^s = g_i$ be f_i or the identity, depending on whether $s(i) = 1$ or $s(i) = 0$. For each $m \leq n$ let $\{C_m^i \mid i \in \omega\}$ be a Borel partition of B_m such that d_m-diam$(C_m^i) < 2^{-n}$. We now inductively construct $B_n \supseteq B^0 \supseteq B^1 \supseteq \ldots \supseteq B^n$ as follows. By the σ-additivity of $\mathfrak{I}$ there is $i_0 \in \omega$ such that

$$B^0 = (g_0 \circ \ldots \circ g_n)^{-1} \text{``} C_0^{i_0} \notin \mathfrak{I},$$

and by σ-additivity again, inductively choose $i_m \in \omega$ such that

$$B^{m+1} = B^m \cap (g_m \circ \ldots \circ g_n)^{-1} \text{``} C_m^{i_m} \notin \mathfrak{I}$$

for $m < n$. Since the construction above depends on the chosen $s \in {}^{n+1}2$, let $B(s) = B^n$. Now we can repeat the construction above for each element of ${}^{n+1}2$: let $\langle s_i \mid 1 \leq i \leq 2^{n+1} \rangle$ be an enumeration of ${}^{n+1}2$, and construct $B(s_1)$ as above, then construct $B(s_2)$ using $B(s_1)$ instead of B_n, and so on. This gives a sequence of Borel sets not in $\mathfrak{I}$

$$B_n \supseteq B(s_1) \supseteq \ldots \supseteq B(s_{2^n}) = B_{n+1}$$

and by construction, for any $a, b \in B_{n+1}$, any $m \leq n$, and any $s \in {}^{n+1}2$,

$$d_m(g_m^s \circ \ldots \circ g_n^s(a), g_m^s \circ \ldots \circ g_n^s(b)) < 2^{-n}.$$

Since $B_{n+1} \notin \mathfrak{I}$, we have $A \cap B_{n+1} \leq_{\mathbf{B}} \neg A \cap B_{n+1}$; let $f_{n+1} : B_{n+1} \to B_{n+1}$ witness this. In order to complete the construction we need to prove the existence of d_{n+1} on B_{n+1}. This follows at once from Lemma 7 if we take $(X, d) = (B_n, d_n)$, $f = f_{n+1}$, and $B = B_{n+1}$. ■

Notice that the property of Baire is used in a "local way" in the proof of Theorem 16: if $\prod_n B_n \subset {}^{\omega}\mathbb{R}$ is endowed with the product topology of the

τ_n's, rather than with the topology induced by ${}^\omega\mathbb{R}$, then the map

$$ {}^\omega 2 \to \prod_n B_n, \qquad z \mapsto \langle x_m(z) \mid m \in \omega \rangle, $$

is continuous by (1), hence the map ${}^\omega 2 \to B_0 = \mathbb{R}$, $z \mapsto x_0(z)$, is continuous, and hence the flip-set is the continuous pre-image of A. Therefore the proof only requires the property of Baire for sets which are Wadge reducible to A.

Suppose $[A]_{\mathbf{B}}$ is limit and self-dual, and let $\{B_n \mid n < \omega\}$ be a Borel partition of $\mathbb{R}$ as in Theorem 16. If $C <_{\mathbf{B}} A$ were an upper bound for the $A \cap B_n$'s, i.e., $\forall n\, (A \cap B_n \leq_{\mathbf{B}} C)$, then Lemma 13 implies that $A \leq_{\mathbf{B}} C$, a contradiction. Since by Lemma 12,

$$ A \cap B_0 \leq_{\mathbf{B}} A \cap (B_0 \cup B_1) \leq_{\mathbf{B}} A \cap (B_0 \cup B_1 \cup B_2) \leq_{\mathbf{B}} \ldots \leq_{\mathbf{B}} A, $$

and $A \cap B_n \leq_{\mathbf{B}} A \cap \bigcup_{i \leq n} B_i$, it follows that $\|A\|_{\mathbf{B}} = \sup_n \|A \cap \bigcup_{i \leq n} B_i\|_{\mathbf{B}}$. Therefore if $[A]_{\mathbf{B}}$ is limit and $\mathrm{cof}(\|A\|_{\mathbf{B}}) > \omega$, then $[A]_{\mathbf{B}}$ is non-self-dual.

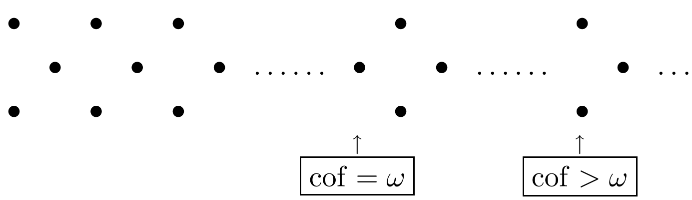
We have already seen that immediately above a non-self-dual pair $([A]_{\mathbf{B}}, [\neg A]_{\mathbf{B}})$ there is a self-dual degree $[A \oplus \neg A]_{\mathbf{B}}$. We will now argue that immediately above a self-dual degree there is a non-self-dual pair. This amounts to proving that if $[A]_{\mathbf{B}} < [B]_{\mathbf{B}}$ are both self-dual then $A <_{\mathbf{B}} C <_{\mathbf{B}} B$ for some C. Let $\{D_n \mid n \in \omega\}$ be a Borel partition of $\mathbb{R}$ such that $\forall n\, (B \cap D_n <_{\mathbf{B}} B)$. If $B \cap D_n \leq_{\mathbf{B}} A$ for all n then $B \leq_{\mathbf{B}} A$, which is absurd, so let $n_0 \in \omega$ be such that $B \cap D_{n_0} \not\leq_{\mathbf{B}} A$. By $\mathsf{SLO}^{\mathbf{B}}$, $\neg A \leq_{\mathbf{B}} B \cap D_{n_0}$ and since $A \equiv_{\mathbf{B}} \neg A$, then $A <_{\mathbf{B}} B \cap D_{n_0} <_{\mathbf{B}} B$. Thus we have proved:

COROLLARY 17. *Assume* $\mathsf{SLO}^{\mathbf{B}} + \mathsf{BP}$.

(a) *A limit Borel–Wadge degree of uncountable cofinality is non-self-dual.*

(b) *Immediately above a self-dual Borel–Wadge degree there is a non-self-dual pair.*

Therefore the structure of the Borel degrees is isomorphic to the structure of the Wadge degrees:



At the bottom of the hierarchy there is the non-self-dual pair $([\mathbb{R}]_{\mathbf{B}}, [\emptyset]_{\mathbf{B}})$ which—as already pointed out in Section 2.2—is $(\{\mathbb{R}\}, \{\emptyset\})$. Immediately above it there is the least self-dual degree, $\mathbf{\Delta}^1_1 \setminus \{\emptyset, \mathbb{R}\}$. We call these three degrees $[\mathbb{R}]_{\mathbf{B}}$, $[\emptyset]_{\mathbf{B}}$, and $\mathbf{\Delta}^1_1 \setminus \{\emptyset, \mathbb{R}\}$ *trivial*; in other words, $[A]_{\mathbf{B}}$ is non-trivial just in case $\|A\|_{\mathbf{B}} \geq 3$. Lastly we prove the converse to the second half of Lemma 13(b).

Proposition 18. *Assume* $\mathsf{SLO}^{\mathbf{B}} + \mathsf{BP}$. *If* $[A]_{\mathbf{B}}$ *is a non-trivial self-dual successor degree, then there is a Borel partition of* $\mathbb{R}$, $\{B_0, B_1\}$, *such that* $B_i \cap A <_{\mathbf{B}} A$ *for* $i = 0, 1$.

Proof. Let $[C]_{\mathbf{B}}$ be the immediate predecessor of $[A]_{\mathbf{B}}$. Then $[C]_{\mathbf{B}}$ is non-self-dual and $[C \oplus \neg C]_{\mathbf{B}}$ is its immediate successor, that is, $A \equiv_{\mathbf{B}} C \oplus \neg C$. Let $f : \mathbb{R} \to \mathbb{R}$ be Borel witnessing $A \leq_{\mathbf{B}} C \oplus \neg C$, let $D_0 = \{x \in \mathbb{R} \mid x(0)$ is even$\}$ and $D_1 = \{x \in \mathbb{R} \mid x(0)$ is odd$\}$, and let $B_i = f^{-1}``D_i$. Then $\{B_0, B_1\}$ is a Borel partition of $\mathbb{R}$, and f witnesses $B_i \cap A \leq_{\mathbf{B}} D_i \cap (C \oplus \neg C)$. Since $C \neq \mathbb{R}, \emptyset$, it follows that $D_0 \cap (C \oplus \neg C) \equiv_{\mathbf{B}} C$, and since $C <_{\mathbf{B}} A$ we have $B_0 \cap A <_{\mathbf{B}} A$. Similarly, $B_1 \cap A \leq_{\mathbf{B}} D_1 \cap (C \oplus \neg C) \equiv_{\mathbf{B}} \neg C <_{\mathbf{B}} A$. ∎

Corollary 19. *Assume* $\mathsf{SLO}^{\mathbf{B}} + \mathsf{BP}$. $[A]_{\mathbf{B}}$ *is a non-trivial self-dual successor degree iff there is a Borel partition* $\{B_n \mid n < N\}$ *of* $\mathbb{R}$ *such that* $\forall n < N\, (B_n \cap A <_{\mathbf{B}} A)$. *Moreover* $[A]_{\mathbf{B}}$ *is a successor degree iff* N *can be taken to be finite, and in fact* N *can be taken to be* 2.

4. Non-self-dual pointclasses. Assuming AD, Steel and Van Wesep independently showed that $A \leq_{\mathrm{W}} \neg A \Rightarrow A \leq_{\mathrm{L}} \neg A$, hence $[A]_{\mathrm{W}} = [A]_{\mathrm{L}}$ if $A \not\equiv_{\mathrm{W}} \neg A$. In fact both statements are provable assuming $\mathsf{SLO}^{\mathrm{L}} + \mathsf{BP}$—see [And03]. The analogue of the first statement for $\leq_{\mathbf{B}}$ and $\leq_{\mathrm{W}}$ is clearly false: if U is open but not closed, then $U \leq_{\mathbf{B}} \neg U$ but $U \not\leq_{\mathrm{W}} \neg U$. Nevertheless the second statement can be generalized to the Borel case.

Proposition 20. *Assume* $\mathsf{SLO}^{\mathrm{W}} + \mathsf{BP}$. *If* $A \not\equiv_{\mathbf{B}} \neg A$, *then* $[A]_{\mathbf{B}} = [A]_{\mathrm{W}}$.

Proof. If $B \in [A]_{\mathbf{B}}$ and $B \not\equiv_{\mathrm{W}} A$, then, since $A \equiv_{\mathrm{W}} \neg B$ cannot hold, either $B <_{\mathrm{W}} A$ or $A <_{\mathrm{W}} B$. For the sake of definiteness, assume the former. Then $C \in [A]_{\mathbf{B}}$ for any $B \leq_{\mathrm{W}} C <_{\mathrm{W}} A$. Since by $\mathsf{SLO}^{\mathrm{W}} + \mathsf{BP}$ we can certainly find such a C which is Wadge self-dual, we conclude that $A \equiv_{\mathbf{B}} C \equiv_{\mathrm{W}} \neg C \equiv_{\mathbf{B}} \neg A$, a contradiction. ∎

Each non-self-dual pair of Wadge–Borel degrees is a non-self-dual pair of Wadge degrees, but not vice versa. Notice that assuming $\mathsf{SLO}^{\mathrm{L}} + \mathsf{BP}$ the conclusion for non-self-dual $[A]_{\mathbf{B}}$ can be strengthened to $[A]_{\mathbf{B}} = [A]_{\mathrm{L}}$. On the other hand, the self-dual Borel degrees are obtained by glueing together many Wadge degrees. For example, the first self-dual degree is the collection of all Borel sets except for $\mathbb{R}$ and $\emptyset$. For $A \equiv_{\mathbf{B}} \neg A$ let $h([A]_{\mathbf{B}})$ be the length of the interval of Wadge degrees used to construct $[A]_{\mathbf{B}}$. Thus, if $A \in \mathbf{\Delta}^1_1 \setminus \{\emptyset, \mathbb{R}\}$, then $[A]_{\mathbf{B}} = \mathbf{\Delta}^1_1 \setminus \{\emptyset, \mathbb{R}\}$ and

$$\begin{aligned} h([A]_{\mathbf{B}}) &= \delta \\ &= \text{the length of the Wadge hierarchy restricted to } \mathbf{\Delta}^1_1 \\ &= \|B\|_{\mathrm{W}}, \text{ where } B \in \mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1 \setminus \mathbf{\Delta}^1_1. \end{aligned}$$

In analogy with the case of the Lipschitz-vs-Wadge hierarchies, where each self-dual Wadge degree is the union of ω_1 consecutive Lipschitz degrees, it is tempting to conjecture that $h([A]_{\mathbf{B}}) = \delta$ for any self-dual $[A]_{\mathbf{B}}$. However this is not true. In fact, $h([A]_{\mathbf{B}}) > \|A\|_{\mathrm{W}}$ for all self-dual $[A]_{\mathbf{B}}$, and therefore the Borel–Wadge hierarchy is obtained by collapsing to a single self-dual Borel–Wadge degree larger and larger blocks of the Wadge hierarchy. To see this we need to recall the definition—due to Wadge—of addition of sets of reals. For $A, B \subseteq \mathbb{R}$ let

$$A + B = \{(s+1)^\frown\langle 0\rangle^\frown(x+1) \mid s \in {}^{<\omega}\omega \ \&\ x \in A\} \cup \{x+1 \mid x \in B\},$$

where $y+1 = \langle y(n)+1 \mid n < \mathrm{lh}(y)\rangle$ for any (finite or infinite) sequence y. If $A \equiv_{\mathrm{W}} \neg A$ and assuming $\mathsf{SLO}^{\mathrm{W}} + \mathsf{BP}$, we have $\|A+B\|_{\mathrm{W}} = \|A\|_{\mathrm{W}} + \|B\|_{\mathrm{W}}$ (see [Wad83] or [And03] for more on this). In particular, if $B \leq_{\mathbf{B}} A$ and f witnesses this, then

$$g(x) = \begin{cases} f(x-1) & \text{if } \forall n\,(x(n) \neq 0), \\ y & \text{if } x = (s+1)^\frown\langle 0\rangle^\frown y \text{ for some } s \in {}^{<\omega}\omega, \end{cases}$$

is Borel and witnesses $A+B \leq_{\mathbf{B}} A$. Therefore $h([A]_{\mathbf{B}}) \geq \|A\|_{\mathrm{W}} + \|A\|_{\mathrm{W}} > \|A\|_{\mathrm{W}}$.

Assuming $\mathsf{SLO}^{\mathrm{W}}$ we can now describe the first few Borel degrees. Immediately above the trivial degrees $[\mathbb{R}]_{\mathbf{B}}$, $[\emptyset]_{\mathbf{B}}$, and $\mathbf{\Delta}^1_1 \setminus \{\mathbb{R}, \emptyset\}$ there is, by Proposition 20, the non-self-dual pair $(\mathbf{\Sigma}^1_1 \setminus \mathbf{\Delta}^1_1, \mathbf{\Pi}^1_1 \setminus \mathbf{\Delta}^1_1)$. At the next level we have a self-dual degree: it is the collection of all Borel-separated-unions of a true $\mathbf{\Sigma}^1_1$ and a true $\mathbf{\Pi}^1_1$,

$$\{A \cup B \mid A \in \mathbf{\Sigma}^1_1 \setminus \mathbf{\Delta}^1_1 \ \&\ B \in \mathbf{\Pi}^1_1 \setminus \mathbf{\Delta}^1_1 \ \&\ \exists C \in \mathbf{\Delta}^1_1\,(A \subseteq C \ \&\ B \cap C = \emptyset)\}.$$

In order to compute the next non-self-dual pair of degrees it is more convenient to work with pointclasses rather than with degrees. Recall that a collection $\mathbf{\Gamma} \subseteq \mathscr{P}(\mathbb{R})$ of sets is a *boldface pointclass* if it is non-empty and closed under continuous pre-images. It is self-dual if it is closed under complements, otherwise it is non-self-dual. The *dual* of $\mathbf{\Gamma}$ is the pointclass $\breve{\mathbf{\Gamma}} = \{\neg A \mid A \in \mathbf{\Gamma}\}$, and let $\mathbf{\Delta} = \mathbf{\Delta}_{\mathbf{\Gamma}}$ be the pointclass $\mathbf{\Gamma} \cap \breve{\mathbf{\Gamma}}$. Under $\mathsf{SLO}^{\mathrm{W}}$, the non-self-dual boldface pointclasses are of the form $\{X \subseteq \mathbb{R} \mid X \leq_{\mathrm{W}} A\}$ with $[A]_{\mathrm{W}}$ non-self-dual, while the self-dual ones are of the form $\{X \subseteq \mathbb{R} \mid X <_{\mathrm{W}} A\}$ with $A \neq \mathbb{R}, \emptyset$. Conversely, if $\mathbf{\Gamma}$ is non-self-dual, then $\mathbf{\Gamma} \setminus \breve{\mathbf{\Gamma}}$ is a non-self-dual Wadge degree by $\mathsf{SLO}^{\mathrm{W}}$. Therefore $\mathsf{SLO}^{\mathrm{W}}$ implies that boldface pointclasses are (essentially) well-ordered under inclusion: either $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$ or $\mathbf{\Lambda} \subseteq \breve{\mathbf{\Gamma}}$. By Proposition 20 and the discussion following its proof, $[A]_{\mathbf{B}}$ is a non-self-dual degree iff $\{X \mid X \leq_{\mathrm{W}} A\}$ is closed under Borel pre-images. A set U is $\mathbf{\Gamma}$-*universal* if it $\mathbb{R}$-parametrizes $\mathbf{\Gamma}$ and belongs to $\mathbf{\Gamma}$, i.e., $U \subseteq \mathbb{R}^2$, $\mathbf{\Gamma} = \{U_x \mid x \in \mathbb{R}\}$ where $U_x = \{y \mid (x,y) \in U\}$, and U (or better: its image under the standard homeomorphism $\mathbb{R}^2 \approx \mathbb{R}$) is in $\mathbf{\Gamma}$. If $\mathbf{\Gamma}$ has a universal set then it is non-self-dual. Conversely, $\mathsf{SLO}^{\mathrm{L}} + \mathsf{BP}$ implies every non-self-dual $\mathbf{\Gamma}$ has a universal set: choose an $\mathbb{R}$-parametrization $\langle \boldsymbol{g}_x \mid x \in \mathbb{R}\rangle$ of all Lipschitz functions such that $(x, y) \mapsto \boldsymbol{g}_x(y)$ is continuous; by the theorem of Steel and Van Wesep mentioned at the beginning of this section, $\boldsymbol{\Gamma} = \{X \mid X \leq_{\mathrm{L}} A\}$ for some $A \not\equiv_{\mathrm{L}} \neg A$, and let $U = \{(x, y) \mid \boldsymbol{g}_x(y) \in A\}$. Similarly, by choosing a parametrization $\langle \boldsymbol{f}_x \mid x \in \mathbb{R}\rangle$ of all continuous functions such that $(x, y) \mapsto \boldsymbol{f}_x(y)$ is Borel, $\mathsf{SLO}^{\mathrm{W}}$ implies that if $[A]_{\mathbf{B}}$ is non-self-dual then $\boldsymbol{\Gamma} = \{X \mid X \leq_{\mathrm{W}} A\}$ has a universal set.

Wadge gave a concrete description of the next non-self-dual pair of boldface pointclasses above a non-self-dual $\boldsymbol{\Gamma}$: Suppose $\boldsymbol{\Gamma} = \{X \mid X \leq_{\mathrm{W}} A\}$ with A non-self-dual and let

$$\boldsymbol{\Gamma}^{\triangledown} = \{(U \cap X) \cup (U' \setminus X') \mid U, U' \in \boldsymbol{\Sigma}^0_1 \;\&\; U \cap U' = \emptyset \;\&\; X, X' \in \boldsymbol{\Gamma}\}.$$

Then $\boldsymbol{\Gamma} \cup \breve{\boldsymbol{\Gamma}} \subseteq \boldsymbol{\Gamma}^{\triangledown}$, and $\boldsymbol{\Gamma}^{\triangledown}$ and its dual $(\boldsymbol{\Gamma}^{\triangledown})\breve{}$ are the least non-self-dual pair of pointclasses above $\boldsymbol{\Gamma} \cup \breve{\boldsymbol{\Gamma}}$. Notice that the self-dual pointclass $\boldsymbol{\Delta}_{\boldsymbol{\Gamma}^{\triangledown}} = \{X \mid X \leq_{\mathrm{W}} B \oplus \neg B\}$ is made up of those $(U \cap X) \cup (U' \setminus X') \in \boldsymbol{\Gamma}^{\triangledown}$ such that $U \subseteq C \subseteq \neg U'$ for some clopen set C. This suggests the following definition.

For $\boldsymbol{\Gamma}$ a boldface pointclass closed under Borel pre-images let

$$\boldsymbol{\Gamma}^{*} = \{(P \cap X) \cup (P' \setminus X') \mid P, P' \in \boldsymbol{\Pi}^1_1 \;\&\; P \cap P' = \emptyset \;\&\; X, X' \in \boldsymbol{\Gamma}\},$$

and let $\boldsymbol{\Delta}^* = \boldsymbol{\Delta}_{\boldsymbol{\Gamma}^*}$. Taking $P = \mathbb{R}$ or $P' = \mathbb{R}$ we find that $\boldsymbol{\Gamma} \subseteq \boldsymbol{\Gamma}^*$ and $\breve{\boldsymbol{\Gamma}} \subseteq \boldsymbol{\Gamma}^*$.

Lemma 21. *Assume* $\mathsf{SLO}^{\mathrm{W}}$ *and let* $\boldsymbol{\Gamma}$ *be a non-self-dual pointclass closed under Borel pre-images. Then* $\boldsymbol{\Gamma}^*$ *is non-self-dual and is closed under Borel pre-images.*

Proof. As both $\boldsymbol{\Gamma}$ and $\boldsymbol{\Pi}^1_1$ have universal sets, $\boldsymbol{\Gamma}^*$ also has a universal set, hence it is non-self-dual. Since both $\boldsymbol{\Gamma}$ and $\boldsymbol{\Pi}^1_1$ are closed under Borel pre-images, it follows that $\boldsymbol{\Gamma}^*$ is closed under Borel pre-images. ∎

Lemma 22. *Let* $[A]_{\mathbf{B}}$ *be non-self-dual and let* $\boldsymbol{\Gamma} = \{X \mid X \leq_{\mathrm{W}} A\}$. *Then any set in* $\boldsymbol{\Delta}^*$ *is Borel reducible to* $A \oplus \neg A$.

Proof. Let $Y \in \boldsymbol{\Delta}^*$ and let $P_1, P_1' \in \boldsymbol{\Pi}^1_1$ and $X_1, X_1' \in \boldsymbol{\Gamma}$ witness that

$$Y = (P_1 \cap X_1) \cup (P_1' \setminus X_1') \in \boldsymbol{\Gamma}^*,$$

and let $P_2, P_2' \in \boldsymbol{\Pi}^1_1$, $X_2, X_2' \in \boldsymbol{\Gamma}$ witness that

$$\neg Y = (P_2 \cap X_2) \cup (P_2' \setminus X_2') \in \boldsymbol{\Gamma}^*.$$

Then $P_1 \cup P_1' \cup P_2 \cup P_2' = \mathbb{R}$. By Reduction for $\boldsymbol{\Pi}^1_1$, let $\{B_1, B_1', B_2, B_2'\}$ be a Borel partition of $\mathbb{R}$ such that $B_1 \subseteq P_1$, $B_1' \subseteq P_1'$, $B_2 \subseteq P_2$, and $B_2' \subseteq P_2'$. Then

$$x \in B_1 \;\Rightarrow\; (x \in Y \Leftrightarrow x \in X_1),$$
$$x \in B_1' \;\Rightarrow\; (x \in Y \Leftrightarrow x \notin X_1'),$$

$$x \in B_2 \;\Rightarrow\; (x \in Y \Leftrightarrow x \notin X_2),$$
$$x \in B_2' \;\Rightarrow\; (x \in Y \Leftrightarrow x \in X_2').$$

This implies the desired conclusion. ■

THEOREM 23. *Assume* $\mathsf{SLO}^{\mathrm{W}}$. *Let* $[A]_{\mathbf{B}}$ *be non-self-dual and let* $\mathbf{\Gamma} = \{X \mid X \leq_{\mathrm{W}} A\}$. *Then*

$$\begin{aligned}\mathbf{\Delta}^* &= \{X \mid X \leq_{\mathbf{B}} A \oplus \neg A\}\\ &= \{X \cup \neg X' \mid \exists B \in \mathbf{\Delta}^1_1\,(X \subseteq B \subseteq \neg X')\ \&\ X, X' \in \mathbf{\Gamma}\}\end{aligned}$$

and $\mathbf{\Delta}^* \setminus (\mathbf{\Gamma} \cup \breve{\mathbf{\Gamma}}) = [A \oplus \neg A]_{\mathbf{B}}$, *i.e., it is the self-dual degree immediately above* $([A]_{\mathbf{B}}, [\neg A]_{\mathbf{B}})$ *and* $(\mathbf{\Gamma}^* \setminus \mathbf{\Delta}^*, (\mathbf{\Gamma}^*)\breve{} \setminus \mathbf{\Delta}^*)$ *is the next non-self-dual pair above it.*

Proof. It is easy to check that $A \oplus \neg A \in \mathbf{\Delta}^*$ and that the sets which are Borel-reducible to $A \oplus \neg A$ are of the form $X \cup \neg X'$ with $X, X' \in \mathbf{\Gamma}$ Borel separated. Therefore we are done by Lemmata 21 and 22. ■

Wadge's analysis shows that if $\mathbf{\Gamma}_n$ is an increasing sequence of boldface pointclasses, then

$$\mathbf{\Gamma} = \{\textstyle\bigcup_n (U_n \cap X_n) \mid U_n \in \mathbf{\Sigma}^0_1 \text{ are pairwise disjoint and } X_n \in \mathbf{\Gamma}_n\}$$

is non-self-dual, and $\mathbf{\Gamma}$ and its dual are the least non-self-dual pointclasses above the $\mathbf{\Gamma}_n$'s.

Similarly, if $\langle \mathbf{\Gamma}_n \mid n \in \omega \rangle$ is a strictly increasing sequence of pointclasses closed under Borel pre-images, then let

$$\mathbf{\Delta} = \{\textstyle\bigcup_n (B_n \cap X_n) \mid B_n \in \mathbf{\Delta}^1_1 \text{ are pairwise disjoint and } X_n \in \mathbf{\Gamma}_n\}$$

and let

$$\mathbf{\Lambda} = \{\textstyle\bigcup_n (P_n \cap X_n) \mid P_n \in \mathbf{\Pi}^1_1 \text{ are pairwise disjoint and } X_n \in \mathbf{\Gamma}_n\}.$$

If $A_{n+1} \in \mathbf{\Gamma}_{n+1} \setminus \mathbf{\Gamma}_n$ and there are pairwise disjoint Borel sets B_n such that $A_n \subseteq B_n$, then it is not hard to see that $\bigcup_n A_n$ is Borel self-dual, that

$$\mathbf{\Delta} = \{X \mid X \leq_{\mathbf{B}} \textstyle\bigcup_n A_n\}$$

is self-dual, that $\bigcup_n \mathbf{\Gamma}_n \subset \mathbf{\Delta}$, and that there is no pointclass closed under Borel pre-images in between. Arguing as above we get:

THEOREM 24. *Assume* $\mathsf{SLO}^{\mathrm{W}} + \mathsf{BP}$ *and suppose* $\mathbf{\Gamma}_n$, $\mathbf{\Delta}$ *and* $\mathbf{\Lambda}$ *are as above. Then* $\mathbf{\Lambda}$ *and* $\breve{\mathbf{\Lambda}}$ *are closed under Borel pre-images and are the least non-self-dual pair of boldface pointclasses above the* $\mathbf{\Gamma}_n$*'s, and* $\mathbf{\Delta} = \mathbf{\Delta}_{\mathbf{\Lambda}}$.

We can now give a complete description of the first ω_1 levels of the $\leq_{\mathbf{B}}$ hierarchy. By Theorem 23 the least non-self-dual pair of pointclasses closed under $\leq_{\mathbf{B}}$ and above $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ is $(\mathbf{\Gamma}, \breve{\mathbf{\Gamma}})$, where $Y \in \mathbf{\Gamma}$ iff $Y = P_1 \cup (P_2 \setminus P_3)$ with $P_1, P_2, P_3 \in \mathbf{\Pi}^1_1$ and $P_1 \cap P_2 = \emptyset$. Without loss of generality we may assume $P_3 \subseteq P_2$, so $Y = (P_1 \cup P_2) \setminus P_3$, hence $\mathbf{\Gamma}$ is the collection $\mathrm{Diff}(2; \mathbf{\Pi}^1_1)$ of all differences of $\mathbf{\Pi}^1_1$ sets. Inductively, using Theorem 24, one can show that the αth pair of non-self-dual pointclasses closed under Borel reducibility is $(\mathrm{Diff}(\alpha; \mathbf{\Pi}^1_1), \breve{\mathrm{Diff}}(\alpha; \mathbf{\Pi}^1_1))$.

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