FUNDAMENTA
MATHEMATICAE
211 (2011)

On the uniqueness of periodic decomposition

by

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Abstract. Let $a_1, \dots, a_k$ be arbitrary nonzero real numbers. An $(a_1, \dots, a_k)$-*decomposition* of a function $f : \mathbb{R} \to \mathbb{R}$ is a sum $f_1 + \dots + f_k = f$ where $f_i : \mathbb{R} \to \mathbb{R}$ is an a_i-periodic function. Such a decomposition is not unique because there are several solutions of the equation $h_1 + \dots + h_k = 0$ with $h_i : \mathbb{R} \to \mathbb{R}$ a_i-periodic. We will give solutions of this equation with a certain simple structure (*trivial solutions*) and study whether there exist other solutions or not. If not, we say that the $(a_1, \dots, a_k)$-decomposition is *essentially unique*. We characterize those periods for which essential uniqueness holds.

1. Introduction. We study finite sums of periodic functions. We fix some periods $a_1, \dots, a_k \in \mathbb{R} \setminus \{0\}$, and consider sums $f_1 + \dots + f_k = f$ where f_i is an a_i-periodic $\mathbb{R} \to \mathbb{R}$ function. Such a sum is called an $(a_1, \dots, a_k)$-*decomposition* of f. We say that a decomposition has a certain property (e.g. is bounded, integer-valued, measurable) if each function in the decomposition has that property.

A natural question is which functions have an $(a_1, \dots, a_k)$-decomposition. In [1] a necessary and sufficient condition was given. Another natural question is: to what extent is such a decomposition unique? One of our goals in this paper is to answer that question. If a function $f : \mathbb{R} \to \mathbb{R}$ has two periodic decompositions

$$f = f_1 + \dots + f_k = \tilde{f}_1 + \dots + \tilde{f}_k \quad (f_i \text{ and } \tilde{f}_i \text{ are } a_i\text{-periodic}),$$

then the difference of these decompositions ($h_i := f_i - \tilde{f}_i$) is a solution of the following *homogeneous equation*:

$$h_1 + \dots + h_k = 0 \quad (h_i \text{ is } a_i\text{-periodic}). \tag{1.1}$$

On the other hand, the sum of an $(a_1, \dots, a_k)$-decomposition and a solution of the homogeneous equation (briefly, a *homogeneous solution*) is another $(a_1, \dots, a_k)$-decomposition. Consequently, the homogeneous solutions tell us

2010 *Mathematics Subject Classification*: Primary 39B22; Secondary 26A99, 39A70.
Key words and phrases: periodic functions, periodic decomposition, uniqueness, difference operator, decomposition property.

DOI: 10.4064/fm211-3-2

to what extent the $(a_1, \dots, a_k)$-decomposition is unique. To determine the homogeneous solutions we do not really need to consider functions over $\mathbb{R}$. Instead, it suffices to solve (1.1) for functions over the additive subgroup generated by the periods: $\mathcal{A} = a_1\mathbb{Z} + \dots + a_k\mathbb{Z}$. Translating these solutions by some real number t gives us solutions on the coset $\mathcal{A}+t$. Now by choosing a solution separately on each coset of $\mathcal{A}$ we can get any solution over $\mathbb{R}$. (Note that $\mathcal{A}$ is always isomorphic to $\mathbb{Z}^d$ for some positive integer d.)

In the case of two periods ($k = 2$), the homogeneous equation is simply $h_1 = -h_2$. So h_1 and h_2 are both a_1- and a_2-periodic and they are the negative of each other. In the case of three or more periods one can get a solution by setting all the functions but two equal to constant 0 and choosing the remaining two functions to be the negative of each other and periodic with respect to both periods. If a solution can be written as the sum of such solutions, we say that it is *trivial*. More precisely:

Definition 1.1. A solution of the homogeneous equation (1.1) is *trivial* if it can be written in the form

$$h_i = \sum_{j=1}^{k} h_{i,j} \qquad (h_{i,i} = 0,\ h_{i,j} = -h_{j,i},\ h_{i,j} \text{ is } a_i\text{- and } a_j\text{-periodic}). \tag{1.2}$$

Again, the notion of trivial solutions can be defined for functions over some coset of $\mathcal{A}$. Clearly, a solution over $\mathbb{R}$ is trivial if and only if it is trivial on each coset.

Definition 1.2. If every solution of the homogeneous equation is trivial for some periods $a_1, \dots, a_k$, then the $(a_1, \dots, a_k)$-decomposition is unique in the sense that any two decompositions differ by a trivial homogeneous solution. In that case, we say that the decomposition is *essentially unique*.

Our main goal is to characterize those periods for which essential uniqueness holds. First we show that this is not always the case, in other words, there exist nontrivial homogeneous solutions. Let us consider periods a, b, c where a and b are independent over $\mathbb{Q}$ and $c = a + b$. It suffices to exhibit a nontrivial solution over the additive subgroup $\mathcal{A}$ spanned by these periods, $a\mathbb{Z} + b\mathbb{Z} = \{ax + by : x, y \in \mathbb{Z}\} \simeq \mathbb{Z} \times \mathbb{Z}$. The corresponding periods in $\mathbb{Z} \times \mathbb{Z}$ are $(1, 0)$, $(0, 1)$ and $(1, 1)$. A $(1, 0)$-periodic function $f(x, y)$ over $\mathbb{Z} \times \mathbb{Z}$ does not depend on x so we simply write $f(y)$. Similarly, we write $g(x)$ for a $(0, 1)$-periodic function and $h(x - y)$ for a $(1, 1)$-periodic function. Now the homogeneous equation is $f(y) + g(x) + h(x - y) = 0$. The functions in a trivial solution are constant because a function that is periodic with at least two of these periods must be constant on the whole $\mathbb{Z} \times \mathbb{Z}$. Thus setting $f(y) = y$, $g(x) = -x$ and $h(x - y) = x - y$ gives us a nontrivial solution of the homogeneous equation.

Now suppose that the periods a, b, c are as in the definition below.

DEFINITION 1.3. A triple a, b, c of real numbers is a *planar triple* if they are not linearly independent over $\mathbb{Q}$ but any two of them are linearly independent.

Then $c = r_1 a + r_2 b$ for some nonzero rational numbers r_1, r_2. It is not hard to modify the above example to obtain a homogeneous solution on $\mathcal{A} = a\mathbb{Z} + b\mathbb{Z} + c\mathbb{Z}$. This time a function in a trivial solution is not necessarily constant on $\mathcal{A}$ but rather has a finite image on it, which still implies that our solution cannot be trivial.

We will show that essential uniqueness holds if and only if there is no planar triple among the periods. The "if" part will be proved in Section 3. We sketch the proof for the simplest case here. Suppose that we have three linearly independent periods a, b, c. Due to the above observations, we can regard the problem over $\mathbb{Z}^3$ with periods $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. Then the homogeneous equation is basically $f(y, z) + g(x, z) + h(x, y) = 0$, and a solution being trivial means that there exist functions $p(x), q(y), r(z)$ such that $f(y, z) = q(y) - r(z)$, $g(x, z) = r(z) - p(x)$ and $h(x, y) = p(x) - q(y)$. (It clearly suffices to prove two of these equalities.) Now taking an arbitrary solution of the homogeneous equation above, we evaluate the equation at z and $z+1$ and compare: we obtain $f(y, z+1) - f(y, z) = g(x, z) - g(x, z+1)$. The left-hand side does not depend on x while the right-hand side does not depend on y, thus both equal some function $s(z)$ depending only on z. It follows that $f(y, z) = f(y, 0) + s(0) + \cdots + s(z-1)$ and $g(x, z) = g(x, 0) - s(0) - \cdots - s(z-1)$, so the solution is indeed trivial. When we have more periods (but any three of them are still linearly independent), then we can modify the above argument to get an inductive proof (see Lemma 3.2). The general case (when we can have periods with rational ratio) is more complicated and will be handled in Theorem 3.3.

For the "only if" part, we have already seen nontrivial solutions in the case when the periods form a planar triple. If we add more periods, we can extend the solution by adding zero functions. However, it might happen that because of these extra periods a nontrivial solution becomes trivial. In certain cases, it is not hard to see that the solution is still nontrivial, but in general things get more complicated. A relatively simple way of proving the existence of a nontrivial solution in the general case goes via another problem.

The starting point is the following question posed in [5]: does the existence of a real-valued periodic decomposition of an integer-valued function f imply the existence of an integer-valued periodic decomposition of f with the same periods? In [1] the question was answered in the affirmative.

However, the integer-valued decomposition is not necessarily as *nice* as the real-valued one. There exists a function $f : \mathbb{R} \to \{0, 1\}$ that can be written as the sum of three periodic bounded functions but it does not have a bounded integer-valued decomposition with the same periods ([5]). The goal of Section 4 is to determine those periods for which this cannot happen, i.e. for which the existence of a bounded real-valued periodic decomposition of an integer-valued function f implies the existence of a bounded integer-valued periodic decomposition of f with the same periods. (This problem was posed by T. Keleti [6, Problem 3.6].) It turns out that the above implication holds for any integer-valued f if and only if essential uniqueness holds for the periods.

THEOREM 1.4 (Main theorem). *For nonzero periods $a_1, \ldots, a_k$ the following assertions are equivalent:*

(i) *There is no planar triple among $a_1, \ldots, a_k$. (That is, any three pairwise linearly independent periods are linearly independent over $\mathbb{Q}$).*
(ii) *The $(a_1, \ldots, a_k)$-decomposition is essentially unique. (That is, every solution of the homogeneous equation* (1.1) *is of the form* (1.2).)
(iii) *For any function $f : \mathbb{R} \to \mathbb{Z}$ the following implication holds: if f decomposes into the sum of bounded real-valued a_i-periodic functions, then it also decomposes into the sum of bounded integer-valued a_i-periodic functions.*

We will prove (i)⇒(ii) in Theorem 3.3, (ii)⇒(iii) in Theorem 4.1 and (iii)⇒(i) in Theorem 4.4. In Section 5 we give a fourth equivalent assertion (Proposition 5.2).

As a corollary, we answer another problem of T. Keleti [6, Problem 3.5]. He studied the measurable version of (iii) and asked for which periods the existence of a bounded measurable real-valued $(a_1, \ldots, a_k)$-decomposition of a function $f : \mathbb{R} \to \mathbb{Z}$ implies the existence of a bounded measurable integer-valued $(a_1, \ldots, a_k)$-decomposition of f. In Theorem 5.4 we give a characterization.

Our motivation to investigate the solutions of the homogeneous equation (apart from the fact that we think it is a natural and interesting question) was that it can be very helpful in this kind of problems when one has a periodic decomposition and wants another decomposition with a certain given property.

2. Preliminary lemmas. Recall that two nonzero real numbers are said to be *commensurable* if their ratio is rational. Otherwise they are *incommensurable*, that is, linearly independent over $\mathbb{Q}$. (Linear independence will always be meant over the field of rational numbers throughout this paper.) Real numbers $a_1, \ldots, a_k$ are commensurable if any two of them are commensurable. Equivalently, $a_1, \ldots, a_k$ are commensurable if they have a common multiple, a nonzero real number m for which $m/a_i \in \mathbb{Z}$ ($i = 1, \ldots, k$). The common multiple with the smallest absolute value is the *least common multiple*. The sign of the least common multiple is irrelevant. One can define the greatest common divisor in a similar manner.

We will use two classes of linear operators that act on the set of $\mathbb{R} \to \mathbb{R}$ functions.

DEFINITION 2.1. For a real number a the *difference operator* Δ_a is defined by

$$(\Delta_a f)(x) = f(x+a) - f(x) \quad (x \in \mathbb{R})$$

where f is an arbitrary $\mathbb{R} \to \mathbb{R}$ function.

DEFINITION 2.2. Let a, m be real numbers with $m/a \in \mathbb{Z}^+$. The operator M_a^m gives the average of certain translates of the input function. For a function $f : \mathbb{R} \to \mathbb{R}$ let

$$(M_a^m f)(x) = \left(\frac{m}{a}\right)^{-1} \sum_{j=0}^{m/a-1} f(x+ja) \quad (x \in \mathbb{R}).$$

PROPOSITION 2.3. *The operators Δ_a and M_a^m have the following properties:*

- *A function f is a-periodic if and only if $\Delta_a f = 0$.*
- *Both Δ_a and M_a^m are linear operators.*
- *Both Δ_a and M_a^m commute with Δ_b for any $b \in \mathbb{R}$. Consequently, Δ_a and M_a^m map each b-periodic function to a b-periodic function.*
- *M_a^m maps each m-periodic function to an a-periodic function.*
- *M_a^m maps each a-periodic function to itself.*

Suppose that $\hat{f} = \Delta_a f$ for some functions $f, \hat{f}$ and a period a. We call f the *lift-up* of $\hat{f}$ with respect to a. It is obvious that two lift-ups of the same function differ by an a-periodic function. It is also clear that adding an a-periodic function to a lift-up gives another lift-up.

Given periods a, b and a b-periodic function, we would like to know whether we can lift up this function with respect to a in such a way that the lift-up is also b-periodic. As we will see, this can always be done provided that a and b are incommensurable (Lemma 2.6). For commensurable periods, we give a necessary and sufficient condition (Lemma 2.4), and we also study the case when this condition fails (Lemma 2.8).

The next lemma is a special case of [2, Lemma 10] (see also [1, Lemma 3.3]).

LEMMA 2.4. *Let $a, b \in \mathbb{R} \setminus \{0\}$ be commensurable periods and $\hat{f} : \mathbb{R} \to \mathbb{R}$ be a b-periodic function. There exists a function f such that $\Delta_a f = \hat{f}$ and $\Delta_b f = 0$ if and only if $\hat{f}(x) + \hat{f}(x+a) + \hat{f}(x+2a) + \dots + \hat{f}(x+m-a) = 0$ ($\forall x \in \mathbb{R}$) for any real number m with $m/a \in \mathbb{Z}^+$ and $m/b \in \mathbb{Z}$.*

In other words, $\hat{f}$ has a b-periodic lift-up with respect to a if and only if $M_a^m \hat{f} = 0$ for any m with $m/a \in \mathbb{Z}^+$ and $m/b \in \mathbb{Z}$.

Remark 2.5. Since $\hat{f}$ is b-periodic, the functions $M_a^m \hat{f}$ are clearly the same for any common multiple m of a and b, so it suffices to check the above condition for one single m.

Lemma 2.6. *Let $a, b_1, \ldots, b_r$ be linearly independent periods (over $\mathbb{Q}$). Suppose that $\hat{f}$ is a function for which $\Delta_{b_i} \hat{f} = 0$ $(i = 1, \ldots, r)$. Then there exists a function f such that $\Delta_a f = \hat{f}$ and $\Delta_{b_i} f = 0$ $(i = 1, \ldots, r)$.*

Proof. Take a point $x_0 \in \mathbb{R}$ and define $f(x_0)$ arbitrarily. From x_0, f can be extended uniquely over the set $a\mathbb{Z} + x_0$ with $\Delta_a f = \hat{f}$. Then we extend f periodically with all periods $b_1, \ldots, b_r$. In this manner we get a uniquely defined function over the set $\mathcal{A} + x_0$ where $\mathcal{A} = a\mathbb{Z} + b_1\mathbb{Z} + \cdots + b_r\mathbb{Z}$ is an additive subgroup of $\mathbb{R}$. The uniqueness follows from the fact that the points of $\mathcal{A}$ can be written uniquely as linear combinations of $a, b_1, \ldots, b_r$ with integer coefficients because of the linear independence of $a, b_1, \ldots, b_r$. By the b_i-periodicity of $\hat{f}$ $(i = 1, \ldots, r)$, the equation $\Delta_a f = \hat{f}$ holds not only on $a\mathbb{Z} + x_0$ but also on $\mathcal{A} + x_0$.

We have now defined the lift-up with the desired properties on a coset of $\mathcal{A}$. However, we can do this independently on each coset. ∎

Note that for commensurable periods a and b, a function f is a- and b-periodic if and only if it is (a, b)-periodic, where (a, b) stands for the greatest common divisor of a and b.

Definition 2.7. Let $a \in \mathbb{R}$. A function $L : \mathbb{R} \to \mathbb{R}$ is *a-linear* if $\Delta_a^2 L = \Delta_a \Delta_a L = 0$. (The name comes from the fact that L is a-linear if and only if $L|_{a\mathbb{Z}+x_0}$ is a linear function for any $x_0 \in \mathbb{R}$.)

Lemma 2.8. *Let a, b be commensurable periods and $\hat{f} : \mathbb{R} \to \mathbb{R}$ be a b-periodic function. There exists a lift-up of $\hat{f}$ with respect to a of the form $f + L$, where f is b-periodic and L is a-linear.*

Suppose that $\hat{f}$ is also c-periodic for a real number c that is incommensurable with a and b. Then f can be chosen to be b- and c-periodic.

Proof. Let m be the least common multiple of a and b (the one which has the same sign as a). We decompose $\hat{f}$ as

$$\hat{f} = (\hat{f} - M_a^m \hat{f}) + M_a^m \hat{f}.$$

Now we use Proposition 2.3. Since $\hat{f}$ is b-periodic (thus m-periodic), we infer that $M_a^m \hat{f}$ is a-periodic. Hence M_a^m maps it to itself: $M_a^m(M_a^m \hat{f}) = M_a^m \hat{f}$. Consequently, M_a^m maps the first summand above to 0. So that summand has a b-periodic lift-up f with respect to a by Lemma 2.4. The second summand is a-periodic, so every lift-up L of it is a-linear.

If $\hat{f}$ is b- and c-periodic, then so is $\hat{f} - M_a^m \hat{f}$. We have shown that the latter has a b-periodic lift-up f with respect to a. We need to prove that it has a lift-up g which is both b- and c-periodic. First we define g on the subgroup $\mathcal{A} = (a,b)\mathbb{Z}+c\mathbb{Z}$ where (a,b) is the greatest common divisor of a,b. By the incommensurability of (a,b) and c,

$$\mathcal{A} = \bigcup_{j\in\mathbb{Z}} (a,b)\mathbb{Z} + jc$$

is a disjoint union. Consider $f|_{(a,b)\mathbb{Z}}$ and let $g|_{(a,b)\mathbb{Z}+jc}$ be the translate of this function by jc for each $j \in \mathbb{Z}$. Obviously, $g|_{\mathcal{A}}$ is c-periodic. It is also b-periodic because f is b-periodic. Finally,

$$(\Delta_a g)(i(a,b)+jc) = (\Delta_a f)(i(a,b)) = (\Delta_a f)(i(a,b)+jc) \qquad (i,j \in \mathbb{Z}),$$

since $\Delta_a f = (\hat{f} - M_a^m \hat{f})$ is c-periodic. This means that g is also a lift-up.

So we have defined g on $\mathcal{A}$ with the desired properties. Of course, we can do the same on every coset of $\mathcal{A}$. ∎

We mention a stronger version of the previous lemma without proof.

Proposition 2.9 ([4]). *Suppose that a,b are commensurable and f satisfies $\Delta_a\Delta_b f = 0$. Then f can be written as*

$$f = L + f_a + f_b \qquad (L \textit{ is } (a,b)\textit{-linear, } \Delta_a f_a = \Delta_b f_b = 0).$$

The theorems of this paper concern functions on $\mathbb{R}$. One could study these problems for functions on an Abelian group. This is not our goal in this paper. Still, we will need a few simple lemmas about functions on Abelian groups. Let $\mathcal{A}$ be an Abelian group. For $a \in \mathcal{A}$ and a function $f : \mathcal{A} \to \mathbb{R}$, set $(\Delta_a f)(x) := f(x+a) - f(x)$ $(x \in \mathcal{A})$. We say that f is *a-periodic* if $\Delta_a f = 0$, and *a-linear* if $\Delta_a^2 f = 0$. Commensurability can also be defined: two elements a,b of $\mathcal{A}$ are *commensurable* if they have a common multiple (that is, there exist nonzero integers n_a, n_b such that $n_a a = n_b b$).

Lemma 2.10. *Suppose that a,b are commensurable elements of an Abelian group $\mathcal{A}$ and L is an a-linear function on $\mathcal{A}$. If L is b-periodic or bounded, then it is necessarily a-periodic too.*

Proof. Let $m = na$ for some positive integer n. The a-linearity of L means that $L(x+2a) - L(x+a) = L(x+a) - L(x)$ for all x. For a fixed x let $c = L(x+a) - L(x) = L(x+(i+1)a) - L(x+ia)$ for any integer i. This entails that

$$L(x+m) - L(x) = \sum_{i=0}^{n-1} (L(x+(i+1)a) - L(x+ia)) = nc.$$

If L is b-periodic, we choose m to be a common multiple of a and b. By b-periodicity $L(x+m) - L(x) = 0$, thus $c = 0$, so $L(x+a) = L(x)$ indeed.

If L is bounded with some bound $K \in \mathbb{R}_+$, then $|L(x+m)-L(x)| \leq 2K$, hence $|c| \leq 2K/n$ for all n, so c must be zero again. ∎

The following corollary is a simple special case of a theorem stating that the class of bounded $\mathcal{A} \to \mathbb{R}$ functions has the *decomposition property*, that is, for a bounded function $f : \mathcal{A} \to \mathbb{R}$ the equation $\Delta_{a_1} \ldots \Delta_{a_k} f = 0$ implies that f has a decomposition into the sum of bounded a_i-periodic functions. (This theorem was first proved by M. Laczkovich and Sz. Gy. Révész [7]; for an alternative proof see [3].) The case $a_1 = \cdots = a_k = a$ entails the following corollary for which we give a short proof for the sake of completeness.

Corollary 2.11. *Suppose that $\Delta_a^k f = 0$ for an element a of an Abelian group $\mathcal{A}$, a bounded function $f : \mathcal{A} \to \mathbb{R}$ and a positive integer k. Then f is a-periodic.*

Proof. For $k = 1$ this is obvious. For $k \geq 2$, consider the function $L = \Delta_a^{k-2} f$. It is bounded and a-linear, thus a-periodic by Lemma 2.10. Consequently,

$$0 = \Delta_a L = \Delta_a \Delta_a^{k-2} f = \Delta_a^{k-1} f.$$

We can repeat this argument until the exponent reaches 1, so $\Delta_a f = 0$. ∎

Lemma 2.12. *Let $\mathcal{A}$ be an Abelian group, and $a, b \in \mathcal{A}$. If f is an a-periodic function, then $\Delta_b f = \Delta_{b+ka} f$ for each integer k.*

Proof. Indeed,

$$(\Delta_b f)(x) = f(x+b) - f(x) = f(x+b+ka) - f(x) = (\Delta_{b+ka} f)(x). \ ∎$$

3. Homogeneous solutions. In this section we study the following problem: for which periods every solution of the homogeneous equation (1.1) is trivial? If that is the case, we say that the $(a_1, \ldots, a_k)$-decomposition is essentially unique. (A solution is trivial if it can be written in the form (1.2).) First we prove two special cases (Proposition 3.1 and Lemma 3.2) that we will use to prove the general case (Theorem 3.3).

Proposition 3.1. *If $a_1, \ldots, a_k$ are commensurable periods, then every solution of the homogeneous equation is trivial.*

Proof. The proof is by induction on k. The case $k = 1$ is obvious. Consider a homogeneous solution $h_1 + \cdots + h_k = 0$ ($\Delta_{a_i} h_i = 0$), and apply the operator Δ_{a_1} to this equation. We get

$$\hat{h}_2 + \hat{h}_3 + \cdots + \hat{h}_k = 0 \qquad (\hat{h}_i = \Delta_{a_1} h_i).$$

This is a homogeneous solution with periods $a_2, \ldots, a_k$. It must be trivial by the inductive assumption, which means that there exist functions $\hat{h}_{i,j}$ ($2 \leq i, j \leq k$) such that $\hat{h}_{i,j}$ is a_i- and a_j-periodic, $\hat{h}_{i,j} = -\hat{h}_{j,i}$ and $\hat{h}_i = \sum_{j=2}^{k} \hat{h}_{i,j}$ ($2 \leq i \leq k$). Since the periods are commensurable, being a_i- and a_j-periodic is the same as being (a_i, a_j)-periodic where (a_i, a_j) is the greatest common divisor of a_i and a_j.

Let us lift up the functions $\hat{h}_{i,j}$ with respect to a_1. Lemma 2.8 implies that these lift-ups can be written in the form $h_{i,j} + L_{i,j}$, where $h_{i,j}$ is (a_i, a_j)-periodic, and $L_{i,j}$ is a_1-linear.

The functions $\hat{h}_{i,j}$ and $\hat{h}_{j,i}$ are the negative of each other. It is clear that they can be lifted up in such a way that $h_{i,j} = -h_{j,i}$ too.

Moreover, we will show that the functions $L_{i,j}$ can be chosen such that

$$h_i = \sum_{j=2}^{k} h_{i,j} + L_{i,j} \qquad (i = 2, \ldots, k). \tag{3.1}$$

Since h_i and $\sum_{j=2}^{k} h_{i,j} + L_{i,j}$ are both lift-ups of $\hat{h}_i$ with respect to a_1, they differ by an a_1-periodic function. Add this function to $L_{i,j}$ for some index j. Then $L_{i,j}$ is still a_1-linear and $h_{i,j} + L_{i,j}$ is still a lift-up of $\hat{h}_{i,j}$, but now (3.1) holds.

Consider $\sum_{j=2}^{k} L_{i,j}$. It is clearly a_1-linear, and (3.1) implies that it is also a_i-periodic. Hence by Lemma 2.10 it is a_1-periodic. So let $h_{i,1} = \sum_{j=2}^{k} L_{i,j}$ and $h_{1,i} = -h_{i,1}$. We have defined all $h_{i,j}$s. They are periodic with the corresponding periods, $h_{i,j} = -h_{j,i}$ and $h_{i,1}$ was chosen such that $h_i = \sum_{j=1}^{k} h_{i,j}$ holds for $i = 2, \ldots, k$. Then this holds for $i = 1$ automatically. Indeed,

$$h_1 = -h_2 - \cdots - h_k = -\sum_{i=2}^{k} \sum_{j=1}^{k} h_{i,j} \overset{*}{=} -\sum_{i=2}^{k} h_{i,1} = \sum_{i=2}^{k} h_{1,i}.$$

The equality labelled $*$ holds because

$$\sum_{i=2}^{k} \sum_{j=2}^{k} h_{i,j} = 0,$$

which follows from $h_{i,j} = -h_{j,i}$ $(2 \le i, j \le k)$.

We have shown that the solution h_i $(1 \le i \le k)$ is of the form (1.2), so it is trivial. ∎

LEMMA 3.2. *Suppose that $a_1 \notin \langle a_i, a_j \rangle_{\mathbb{Q}}$ for every pair of indices $2 \le i, j \le k$ (in other words, either a_i, a_j are commensurable with each other but not with a_1, or a_1, a_i, a_j are linearly independent). If every homogeneous solution is trivial for the periods $a_2, \ldots, a_k$, then this also holds for the periods $a_1, \ldots, a_k$.*

Proof. We take an arbitrary homogeneous solution $\{h_i\}_{i=1}^{k}$. Then the functions $\hat{h}_i = \Delta_{a_1} h_i$ $(i = 2, \ldots, k)$ give a homogeneous solution with periods $a_2, \dots, a_k$. This must be a trivial solution, so there exist the corresponding functions $\hat{h}_{i,j} : \mathbb{R} \to \mathbb{R}$ $(2 \le i, j \le k)$.

We claim that $\hat{h}_{i,j}$ has a lift-up $h_{i,j}$ with respect to a_1 such that $\Delta_{a_i} h_{i,j} = \Delta_{a_j} h_{i,j} = 0$. If a_i and a_j are commensurable, then we need (a_i, a_j)-periodicity. Since a_1 is incommensurable with (a_i, a_j), we can apply Lemma 2.6 with $a = a_1$ and $b_1 = (a_i, a_j)$. If a_1, a_i and a_j are linearly independent, then Lemma 2.6 can be applied again, this time with $a = a_1$, $b_1 = a_i$, $b_2 = a_j$.

We can assume that $h_{j,i} = -h_{i,j}$. (Lift up $\hat{h}_{i,j}$ $(i < j)$ first. Clearly, $-h_{i,j}$ will be a lift-up of $\hat{h}_{j,i} = -\hat{h}_{i,j}$.) Let

$$h_{i,1} = h_i - h_{i,2} - h_{i,3} - \dots - h_{i,k}.$$

We claim that $h_{i,1}$ is a_1- and a_i-periodic. Indeed,

$$\Delta_{a_1} h_{i,1} = \hat{h}_i - \hat{h}_{i,2} - \dots - \hat{h}_{i,k} = 0,$$
$$\Delta_{a_i} h_{i,1} = \Delta_{a_i} h_i - \Delta_{a_i} h_{i,2} - \dots - \Delta_{a_i} h_{i,k} = 0 - 0 - \dots - 0 = 0.$$

Finally, let $h_{1,i} = -h_{i,1}$. One can easily check (as in Proposition 3.1) that the functions $h_{i,j}$ $(1 \le i, j \le k)$ satisfy (1.2). ∎

We are now in a position to prove the implication (i)⇒(ii) of Theorem 1.4.

THEOREM 3.3. *If there is no planar triple among the periods* $a_1, \dots, a_k$, *then the* $(a_1, \dots, a_k)$*-decomposition is essentially unique.* (*That is, if any three pairwise incommensurable periods of* $a_1, \dots, a_k$ *are linearly independent over* $\mathbb{Q}$, *then every solution of the homogeneous equation* (1.1) *is of the form* (1.2).)

Proof. The proof is by induction on k; the case $k = 1$ is obvious. Without loss of generality we can assume that the periods that are commensurable with a_1 are exactly $a_1, \dots, a_l$ for some integer $1 \le l \le k$.

CASE 1: $l = 1$. In this case there is no period that is commensurable with a_1. Consequently, if a_i and a_j are incommensurable for some indices $i, j \ge 2$, then a_1, a_i, a_j must be linearly independent. (Otherwise they would be a planar triple.) Thus Lemma 3.2 can be applied, and we are done.

CASE 2: $l \ge 2$. Take an arbitrary homogeneous solution h_i $(i = 1, \dots, k)$. The functions $\hat{h}_i = \Delta_{a_1} h_i$ form a (necessarily trivial) homogeneous solution with periods $a_2, \dots, a_k$. We consider the corresponding functions $\hat{h}_{i,j}$ $(2 \le i, j \le k)$ and we lift them up with respect to a_1.

If $l+1 \le i, j \le k$, then $a_1 \notin \langle a_i, a_j \rangle_{\mathbb{Q}}$. In this case there exists an a_i- and a_j-periodic lift-up $h_{i,j}$ as we have seen in the proof of Lemma 3.2.

If $l+1 \le i \le k$ and $2 \le j \le l$, then $\hat{h}_{i,j}$ is a_i- and a_j-periodic where a_1 is commensurable with a_j but not with a_i. By Lemma 2.8 there is a lift-up of the form $h_{i,j} + L_{i,j}$, where $h_{i,j}$ is a_i- and a_j-periodic, and $L_{i,j}$ is a_1-linear. As we have seen in the proof of Proposition 3.1, we can assume that

$$h_i = \sum_{j=2}^{k} h_{i,j} + \sum_{j=2}^{l} L_{i,j} = L_i + \sum_{j=2}^{k} h_{i,j} \qquad (l+1 \le i \le k)$$

where $L_i = \sum_{j=2}^{l} L_{i,j}$ is an a_1-linear and (by the above equation) a_i-periodic function.

Using $\sum_{i=l+1}^{k} \sum_{j=l+1}^{k} h_{i,j} = 0$ $(*)$, we get

$$-\sum_{i=1}^{l} h_i = \sum_{i=l+1}^{k} h_i = \sum_{i=l+1}^{k} \sum_{j=2}^{k} h_{i,j} + \sum_{i=l+1}^{k} L_i \overset{*}{=} \sum_{i=l+1}^{k} \sum_{j=2}^{l} h_{i,j} + \sum_{i=l+1}^{k} L_i.$$

Let $L_1 = \sum_{i=1}^{l} h_i + \sum_{i=l+1}^{k} \sum_{j=2}^{l} h_{i,j}$. Each summand is a_j-periodic for some $1 \le j \le l$, so L_1 is m-periodic where m denotes the least common multiple of $a_1, \dots, a_l$. On the other hand,

$$L_1 + L_{l+1} + L_{l+2} + \cdots + L_k = 0,$$

thus L_1 is a_1-linear (all the other summands are a_1-linear). Consequently, L_1 is a_1-periodic by Lemma 2.10. This means that the functions L_i $(i = 1, l+1, \dots, k)$ form a solution of the homogeneous equation with periods $a_1, a_{l+1}, a_{l+2}, \dots, a_k$. The number of these periods is at most $k-1$ because $l \ge 2$ by assumption. So it must be a trivial solution. Consequently, there exist functions $h'_{i,j}$ $(i, j \in \{1, l+1, \dots, k\})$ such that $h'_{i,j} = -h'_{j,i}$ is a_i- and a_j-periodic,

$$L_i = h'_{i,1} + \sum_{j=l+1}^{k} h'_{i,j} \qquad (i = 1, l+1, \dots, k).$$

For $i = l+1, \dots, k$ we set

$$h''_{i,j} = \begin{cases} h'_{i,1} & (j = 1), \\ h_{i,j} & (2 \le j \le l), \\ h_{i,j} + h'_{i,j} & (l+1 \le j \le k), \end{cases}$$

and $h''_{i,j} := -h''_{j,i}$ if $i \le l$ and $j \ge l+1$.

Now we define $h''_{i,j}$ in the case when both indices are at most l. Since

$$\sum_{i=1}^{l} h_i = -\sum_{i=l+1}^{k} h_i = -\sum_{i=l+1}^{k} \sum_{j=1}^{k} h''_{i,j} = \sum_{i=1}^{k} \sum_{j=l+1}^{k} h''_{i,j} = \sum_{i=1}^{l} \sum_{j=l+1}^{k} h''_{i,j},$$

the a_i-periodic functions $g_i := (h_i - \sum_{j=l+1}^{k} h''_{i,j})$ $(i = 1, \dots, l)$ form a homogeneous solution with periods $a_1, \dots, a_l$. These periods are commensurable, so this must be a trivial solution by Proposition 3.1. Let us take the corresponding functions $h''_{i,j}$ $(1 \le i, j \le l)$ and complement the already defined $h''_{i,j}$s with these functions. This shows that $h_1, \dots, h_k$ is a trivial solution. ■

4. Decomposition into bounded functions. In this section we determine those periods for which the implication below holds for any function $f : \mathbb{R} \to \mathbb{Z}$:

$$\begin{aligned} (4.1) \qquad & \exists f_1, \dots, f_k : \mathbb{R} \to \mathbb{R} \;\; f = f_1 + \dots + f_k,\, \Delta_{a_i} f_i = 0,\, f_i \text{ bounded } \stackrel{?}{\Rightarrow} \\ & \exists \tilde{f}_1, \dots, \tilde{f}_k : \mathbb{R} \to \mathbb{Z} \;\; f = \tilde{f}_1 + \dots + \tilde{f}_k,\, \Delta_{a_i} \tilde{f}_i = 0,\, \tilde{f}_i \text{ bounded.} \end{aligned}$$

4.1. Connection with the homogeneous equation. Using a theorem of B. Farkas et al. [1, Corollary 4.2], the implication (ii)⇒(iii) of Theorem 1.4 can be proved easily.

THEOREM 4.1. *If every solution of the homogeneous equation* (1.1) *is trivial for some periods $a_1, \dots, a_k$, then the implication* (4.1) *holds for any function $f : \mathbb{R} \to \mathbb{Z}$.*

Proof. Suppose that $f : \mathbb{R} \to \mathbb{Z}$ has a real-valued bounded periodic decomposition with periods $a_1, \dots, a_k$. We mentioned in the introduction that, for any periods, the existence of a real-valued periodic decomposition of an integer-valued function on $\mathbb{R}$ implies the existence of an integer-valued periodic decomposition with the same periods [1, Corollary 4.2]. So there exist decompositions

$$f = f_1 + \dots + f_k = g_1 + \dots + g_k,$$

where $f_i : \mathbb{R} \to \mathbb{R}$ is bounded, $g_i : \mathbb{R} \to \mathbb{Z}$ is not necessarily bounded and $\Delta_{a_i} f_i = \Delta_{a_i} g_i = 0$. The functions $h_i := f_i - g_i$ $(i = 1, \dots, k)$ form a homogeneous solution which must be trivial by assumption. Consider the corresponding functions $h_{i,j} : \mathbb{R} \to \mathbb{R}$ and define integer-valued functions $\tilde{h}_{i,j}$ close to $h_{i,j}$ by

$$\tilde{h}_{i,j}(x) = \begin{cases} [h_{i,j}(x)], & i < j, \\ \lceil h_{i,j}(x) \rceil = -[-h_{i,j}(x)] = -[h_{j,i}(x)], & i > j, \\ 0, & i = j. \end{cases}$$

Obviously, $|\tilde{h}_{i,j}(x) - h_{i,j}(x)| < 1$ for all i, j, x. It is also clear that the conditions $\Delta_{a_i} \tilde{h}_{i,j} = \Delta_{a_j} \tilde{h}_{i,j} = 0$ and $\tilde{h}_{i,j} = -\tilde{h}_{j,i}$ still hold. Set

$$\tilde{h}_i = \sum_{j=1}^{k} \tilde{h}_{i,j} \qquad (i = 1, \dots, k).$$

These are integer-valued functions that form a homogeneous solution ($\tilde{h}_1 + \cdots + \tilde{h}_k = 0$, $\Delta_{a_i}\tilde{h}_i = 0$). They are also close to h_i:

$$|\tilde{h}_i(x) - h_i(x)| \leq \sum_{1\leq j\leq k,\, j\neq i} |\tilde{h}_{i,j}(x) - h_{i,j}(x)| < k-1.$$

Clearly, the functions $\tilde{f}_i := g_i + \tilde{h}_i$ ($i = 1, \ldots, k$) form an integer-valued decomposition of f, and it is bounded because $\tilde{f}_i - f_i$ is bounded:

$$|\tilde{f}_i(x) - f_i(x)| = |(g_i(x) + \tilde{h}_i(x)) - (g_i(x) + h_i(x))| = |\tilde{h}_i(x) - h_i(x)| < k-1. \blacksquare$$

According to Theorem 3.3 every homogeneous solution is trivial for periods with no planar triple among them, so in this case the implication (4.1) holds. Now we show that it does not hold in other cases.

4.2. Negative results. A counter-example to (4.1) was given in [5] where the authors constructed a function that is the sum of three bounded periodic functions but it does not have a bounded integer-valued decomposition with the same periods. We generalize this example.

Given three periods a_1, a_2, a_3 forming a planar triple we take the two-dimensional $\mathbb{Q}$-linear subspace spanned by a_1, a_2, a_3. We will define functions on this subspace: f_i (a_i-periodic, bounded, real-valued) and g_i (a_i-periodic, integer-valued) for $i = 1, 2, 3$ in such a way that $f_1 + f_2 + f_3 = g_1 + g_2 + g_3 = f$. Then we extend all these functions over $\mathbb{R}$ by zero. What we will basically prove is that no matter how we add new periods $a_4, \ldots, a_k$, this extended function f will never have a bounded integer-valued $(a_1, \ldots, a_k)$-decomposition. We will do that in two steps: first we consider the case when all periods are contained in the subspace spanned by a_1, a_2, a_3 (Proposition 4.2), then we deal with periods outside that subspace (Lemma 4.3).

PROPOSITION 4.2. *Suppose that there are three pairwise incommensurable periods among $a_1, \ldots, a_k \in \mathbb{Q} \times \mathbb{Q} \setminus \{(0,0)\}$. Then there exists a $\mathbb{Q} \times \mathbb{Q} \to \mathbb{Z}$ function which has a bounded real-valued decomposition but no bounded integer-valued decomposition with periods $a_1, \ldots, a_k$.*

Proof. We can assume that a_1, a_2, a_3 are pairwise incommensurable. It follows that any two of them give a basis of $\mathbb{Q} \times \mathbb{Q}$, so we can also assume that $a_1 = (1, 0)$ and $a_2 = (0, 1)$. Denote the coordinates of a_i by $p_i, q_i \in \mathbb{Q}$. We also know that $p_3, q_3 \neq 0$ since a_3 is not commensurable with a_1, a_2.

Note that for any rational number r, functions of the form $(x, y) \mapsto f(y)$ are $(r, 0)$-periodic, functions of the form $(x, y) \mapsto f(x)$ are $(0, r)$-periodic, and functions of the form $(x, y) \mapsto f(-q_3x + p_3y)$ are (rp_3, rq_3)-periodic.

Fix an arbitrary irrational number t and consider the functions below.

(We write $[\cdot]$, $\{\cdot\}$ for the integer part and fractional part, respectively.)

$$\begin{aligned}
f_1(x,y) &= -\{tp_3y\}, & g_1(x,y) &= [tp_3y],\\
f_2(x,y) &= \{tq_3x\}, & g_2(x,y) &= -[tq_3x],\\
f_3(x,y) &= \{t(-q_3x+p_3y)\}, & g_3(x,y) &= -[t(-q_3x+p_3y)].
\end{aligned}$$

By our remarks $\Delta_{a_i} f_i = \Delta_{a_i} g_i = 0$ for $i = 1,2,3$. Using

$$(-tp_3y) + tq_3x + t(-q_3x+p_3y) = 0,$$

we get $f_1+f_2+f_3 = g_1+g_2+g_3$. Denote this sum by $f : \mathbb{Q}\times\mathbb{Q} \to \mathbb{Z}$. The functions f_1, f_2, f_3 form a bounded real-valued (a_1,a_2,a_3)-decomposition of f. Then f also has a bounded real-valued $(a_1,\dots,a_k)$-decomposition because the remaining functions in the decomposition (f_i, $i \geq 4$) can be chosen to be constant 0.

We claim that f has no bounded integer-valued $(a_1,\dots,a_k)$-decomposition. Assume towards a contradiction that there exist functions $\tilde{g}_i$ such that

$$f = \tilde{g}_1 + \dots + \tilde{g}_k \qquad (\tilde{g}_i : \mathbb{Q}\times\mathbb{Q} \to \mathbb{Z} \text{ is bounded and } a_i\text{-periodic}).$$

For the sake of simplicity, first we assume that every period is incommensurable with a_1, that is, $q_i \neq 0$ ($i \geq 2$).

Let M be a common multiple of $q_2 = 1, q_3, \dots, q_k$. Thus $M/q_i \in \mathbb{Z}$ for $i = 2,\dots,k$. We choose a positive integer N such that $Np_i \in \mathbb{Z}$ for $i \geq 2$. Setting $n_i = MN/q_i$, we have

$$n_ia_i = (n_ip_i, n_iq_i) = \left(\frac{M}{q_i}(Np_i), MN\right) \in \mathbb{Z}\times\{MN\} \qquad (i = 2,\dots,k).$$

Applying the operator $\mathcal{S} = \Delta_{n_2a_2}\dots\Delta_{n_ka_k}$ to f, we get

$$\mathcal{S}f = \mathcal{S}g_1 = \mathcal{S}\tilde{g}_1.$$

(Because $\mathcal{S}$ maps a_i-periodic functions to 0 for $i \geq 2$.) So $\mathcal{S}$ maps $h_1 := g_1 - \tilde{g}_1$ to 0. We claim that

$$\begin{aligned}
\mathcal{S}h_1 &= \Delta_{n_2a_2}\dots\Delta_{n_{k-1}a_{k-1}}\Delta_{n_ka_k}h_1 = \Delta_{n_2a_2}\dots\Delta_{n_{k-1}a_{k-1}}\Delta_{(MNp_k/q_k,MN)}h_1\\
&= \Delta_{n_2a_2}\dots\Delta_{n_{k-1}a_{k-1}}\Delta_{(0,MN)}h_1 = \dots = \Delta_{(0,MN)}\dots\Delta_{(0,MN)}\Delta_{(0,MN)}h_1.
\end{aligned}$$

As h_1 is $a_1 = (1,0)$-periodic and $(MNp_k/q_k) \in \mathbb{Z}$, Lemma 2.12 entails that

$$\Delta_{n_ka_k}h_1 = \Delta_{(MNp_k/q_k,MN)}h_1 = \Delta_{(0,MN)}h_1.$$

Thus $\Delta_{n_ka_k}$ can be replaced by $\Delta_{(0,MN)}$. Since $\Delta_{(0,MN)}h_1$ is also $(1,0)$-periodic, we can repeat the same argument to deduce that $\Delta_{n_{k-1}a_{k-1}}$ can also be replaced by $\Delta_{(0,MN)}$ and so on.

Finally we get

$$\Delta^{k-1}_{(0,MN)}h_1 = 0.$$

Now let us consider the function

$$L_1(x,y) = h_1(x,y) - tp_3y = (h_1(x,y) - [tp_3y]) - \{tp_3y\} = -\tilde{g}_1(x,y) - \{tp_3y\}.$$

It is bounded because both $\tilde{g}_1$ and $(x, y) \mapsto \{tp_3 y\}$ are bounded. On the other hand, $\Delta^{k-1}_{(0,MN)} L_1 = 0$ since this holds for both h_1 and $(x, y) \mapsto tp_3 y$. (The latter is $(0, r)$-linear for any $r \in \mathbb{Q}$, and $k - 1 \geq 2$.) Corollary 2.11 implies the $(0, MN)$-periodicity of L_1. Hence

$$h_1(0, MN) - h_1(0, 0) = L_1(0, MN) - L_1(0, 0) + tp_3 MN = tp_3 MN \notin \mathbb{Q},$$

though h_1 is an integer-valued function, a contradiction.

Now we turn to the case when at least one of $a_4, \dots, a_k$ is commensurable with a_1. First we change our notation a little. Let $a_2, \dots, a_k$ now denote those periods that are not commensurable with a_1; those commensurable with a_1 are $a_1 = (r_1, 0) = (1, 0), (r_2, 0), \dots, (r_l, 0)$. (Then the meaning of k changes as well.) Let m be the least common multiple of $r_1 = 1, r_2, \dots, r_l$. Our original argument needs to be changed at only one point. After the indirect assumption, we add up the functions corresponding to the periods $a_1 = (r_1, 0) = (1, 0), (r_2, 0), \dots, (r_l, 0)$. We get an $(m, 0)$-periodic function $\tilde{G}_1$. (The function corresponding to the period a_i is still denoted by $\tilde{g}_i$, $i \geq 2$.) This time we choose N such that $Np_i \in m\mathbb{Z}$. Since $H_1 := g_1 - \tilde{G}_1$ is m-periodic, we get a contradiction the same way. ∎

The next lemma deals with periods outside $\mathbb{Q} \times \mathbb{Q}$.

Lemma 4.3. *Let $d < D$ be positive integers, $a_1, \dots, a_l \in \mathbb{Q}^d \subset \mathbb{Q}^D$ and $a_{l+1}, \dots, a_k \in \mathbb{Q}^D \setminus \mathbb{Q}^d$. Suppose that there exists a function $f : \mathbb{Q}^d \to \mathbb{Z}$ which has a bounded real-valued $(a_1, \dots, a_l)$-decomposition, but no bounded integer-valued $(a_1, \dots, a_l)$-decomposition. Let $F : \mathbb{Q}^D \to \mathbb{Z}$ be the extension of f by zero. Then F has a bounded real-valued $(a_1, \dots, a_k)$-decomposition, but no bounded integer-valued $(a_1, \dots, a_k)$-decomposition.*

Proof. Let $f = f_1 + \dots + f_l$ be a bounded real-valued $(a_1, \dots, a_l)$-decomposition. By definition,

$$F(x) = \begin{cases} f(x), & x \in \mathbb{Q}^d, \\ 0, & x \in \mathbb{Q}^D \setminus \mathbb{Q}^d. \end{cases}$$

We can extend f_i to F_i the same way $(1 \leq i \leq l)$. Since $a_i \in \mathbb{Q}^d$, F_i is a_i-periodic $(1 \leq i \leq l)$. So $F_1 + \dots + F_l$ is a bounded real-valued $(a_1, \dots, a_l)$-decomposition of F. Setting $F_i = 0$ $(l < i \leq k)$, we also get a bounded real-valued $(a_1, \dots, a_k)$-decomposition of F.

To show that F has no bounded integer-valued $(a_1, \dots, a_k)$-decomposition, assume for contradiction that

$$F = G_1 + \dots + G_k \quad (G_i : \mathbb{Q}^D \to \mathbb{Z} \text{ is bounded and } a_i\text{-periodic}).$$

Consider the operator

$$\mathcal{S} := \Delta_{n_{l+1} a_{l+1}} \Delta_{n_{l+2} a_{l+2}} \dots \Delta_{n_k a_k}$$

where $n_{l+1}, n_{l+2}, \dots, n_k$ are positive integers. Clearly, $\mathcal{S}$ maps every a_i-periodic function to 0 for $l+1 \le i \le k$, thus

$$\mathcal{S}F = \mathcal{S}G_1 + \cdots + \mathcal{S}G_l,$$

where $\mathcal{S}G_i : \mathbb{Q}^D \to \mathbb{Z}$ is a bounded a_i-periodic function ($1 \le i \le l$). Consequently, $\mathcal{S}F$ has a bounded integer-valued $(a_1, \dots, a_l)$-decomposition. Our goal is to choose $n_{l+1}, n_{l+2}, \dots, n_k$ in such a way that the restriction of $\mathcal{S}F$ to $\mathbb{Q}^d$ is $(-1)^{k-l} f$. This will be a contradiction since, by assumption, f has no bounded integer-valued $(a_1, \dots, a_l)$-decomposition.

First let n_k be an arbitrary positive integer, and consider the function $(\Delta_{n_k a_k} F)(x) = F(x + n_k a_k) - F(x)$. It maps $x \in \mathbb{Q}^d$ to $-f(x)$, since $a_k \notin \mathbb{Q}^d$. On the other hand, it is supported by a band parallel to $\mathbb{Q}^d$, that is, it vanishes outside the set

$$\mathbb{Q}^d \times ([-K_{d+1}, K_{d+1}] \cap \mathbb{Q}) \times \cdots \times ([-K_D, K_D] \cap \mathbb{Q}) \subset \mathbb{Q}^D$$

for some rational numbers $K_{d+1}, \dots, K_D \ge 0$. (In this case K_j can be chosen to be $|(n_k a_k)_j|$, the absolute value of the jth coordinate of the point $n_k a_k$.) Now we choose n_{k-1} such that $n_{k-1} a_{k-1}$ lies outside this band. (This is possible because $a_{k-1} \notin \mathbb{Q}^d$, so there must be an index $j > d$ for which the jth coordinate of a_{k-1} is not 0, thus $(n_{k-1} a_{k-1})_j > K_j$ if n_{k-1} is large enough.) Then the restriction of $(\Delta_{n_{k-1} a_{k-1}} \Delta_{n_k a_k} F)(x)$ to $\mathbb{Q}^d$ equals f and there still exists a band that supports this function. (The new K_j can be chosen to be the sum of the old K_j and $|(n_{k-1} a_{k-1})_j|$.) Now we choose n_{k-2} such that $n_{k-2} a_{k-2}$ lies outside this new band and so on. Finally we get an operator $\mathcal{S}$ for which $\mathcal{S}F|_{\mathbb{Q}^d} = (-1)^{k-l} f$. ∎

Now we complete the proof of Theorem 1.4 by showing the remaining implication (iii)⇒(i).

THEOREM 4.4. *Let $a_1, \dots, a_k \in \mathbb{R} \setminus \{0\}$. Suppose that there is a planar triple (three pairwise incommensurable but linearly dependent real numbers) among them. Then there is an $\mathbb{R} \to \mathbb{Z}$ function that has a bounded real-valued $(a_1, \dots, a_k)$-decomposition, but no bounded integer-valued $(a_1, \dots, a_k)$-decomposition.*

Proof. We can assume that $\{a_1, a_2, a_3\}$ is a planar triple. These three periods span a two-dimensional $\mathbb{Q}$-linear subspace of $\mathbb{R}$. We can also assume that the periods lying in this subspace $\langle a_1, a_2, a_3 \rangle_{\mathbb{Q}} \cong \mathbb{Q} \times \mathbb{Q}$ are exactly $a_1, \dots, a_l$ for some integer $3 \le l \le k$. Let D denote the dimension of the $\mathbb{Q}$-linear subspace spanned by all the periods: $\langle a_1, \dots, a_k \rangle_{\mathbb{Q}} \cong \mathbb{Q}^D$. Obviously, $D \ge 2$.

By Proposition 4.2 there exists a function f over *this* $\mathbb{Q} \times \mathbb{Q}$ which has a bounded real-valued $(a_1, \dots, a_l)$-decomposition without having a bounded integer-valued $(a_1, \dots, a_l)$-decomposition. Then, by Lemma 4.3, there also exists $F : \mathbb{Q}^D \to \mathbb{Z}$ with a bounded real-valued $(a_1, \ldots, a_k)$-decomposition but without a bounded integer-valued $(a_1, \ldots, a_k)$-decomposition. Extending F by zero over $\mathbb{R}$, we get a function with the desired properties. ■

5. Corollaries and questions. We start this section with the following observation. Suppose that we have some periods $a_1, \ldots, a_l$ and a nontrivial homogeneous solution $h_1 + \cdots + h_l = 0$. Then it can be viewed as a solution with periods $a_1, \ldots, a_k$ for arbitrary extra periods $a_{l+1}, \ldots, a_k$ (we just complement the solution with zero functions, that is, we set $h_i \equiv 0$, $i = l+1, \ldots, k$). It can happen that the solution becomes trivial because of the extra periods. However, the following is true.

PROPOSITION 5.1. *For any planar triple a_1, a_2, a_3 there exists a nontrivial homogeneous solution $h_1 + h_2 + h_3 = 0$ that remains nontrivial even if we add arbitrary extra periods.*

Proof. The proof of Theorem 4.1 tells us how to obtain a nontrivial homogeneous solution: take a bounded decomposition and an integer-valued decomposition of the same function and take their difference; it will be nontrivial provided that the function has no bounded integer-valued decomposition.

We have also seen (see the second paragraph of Subsection 4.2) that for any planar triple a_1, a_2, a_3 there exists a bounded function $f : \mathbb{R} \to \mathbb{Z}$ with a bounded (a_1, a_2, a_3)-decomposition $f_1 + f_2 + f_3 = f$ and an integer-valued (a_1, a_2, a_3)-decomposition $g_1 + g_2 + g_3 = f$; furthermore, f has no bounded integer-valued $(a_1, \ldots, a_k)$-decomposition for any extra periods $a_4, \ldots, a_k$.

Consequently, for arbitrary extra periods $a_4, \ldots, a_k$, f has a bounded decomposition ($f_1 + f_2 + f_3 + 0 + \cdots + 0 = f$) and an integer-valued decomposition ($g_1 + g_2 + g_3 + 0 + \cdots + 0 = f$) but it has no $(a_1, \ldots, a_k)$-decomposition that is both bounded and integer-valued. It follows, in view of the first paragraph, that the homogeneous solution

$$(f_1 - g_1) + (f_2 - g_2) + (f_3 - g_3) + 0 + \cdots + 0 = 0$$

is nontrivial. Since this holds for an arbitrary choice of additional periods, we are done. ■

Next, we add one more statement to the list of equivalent assertions in Theorem 1.4. Recall the already mentioned theorem that the class of bounded $\mathbb{R} \to \mathbb{R}$ functions has the decomposition property: a function $f : \mathbb{R} \to \mathbb{R}$ has a bounded real-valued $(a_1, \ldots, a_k)$-decomposition if and only if f is bounded and satisfies $\Delta_{a_1} \ldots \Delta_{a_k} f = 0$. Using this, we can rephrase (iii) equivalently as follows.

PROPOSITION 5.2. *The following is also equivalent to* (i)–(iii) *of Theorem* 1.4*:*

(iii′) *If $f:\mathbb{R}\to\mathbb{Z}$ is bounded and $\Delta_{a_1}\dots\Delta_{a_k}f=0$, then f has a bounded integer-valued $(a_1,\dots,a_k)$-decomposition.*

As a corollary of Theorem 1.4 we answer another problem of T. Keleti who studied (Lebesgue) measurable periodic decompositions of integer-valued measurable functions in [6]. In [6, Theorem 2.5] he proved the equivalence of seven assertions. We will need the equivalence of two of them.

Theorem 5.3 ([6]). *Let $a_1,\dots,a_k\in\mathbb{R}\setminus\{0\}$. Let $B_1,\dots,B_n$ denote the equivalence classes of $\{a_1,\dots,a_k\}$ with respect to the relation defined by $a\sim b\Leftrightarrow a/b\in\mathbb{Q}$ and b_j be the least common multiple of the numbers in B_j. (In fact, b_j can be any element that is commensurable with the elements in B_j.) The following two statements are equivalent.*

(a) *If an (everywhere) integer-valued function f on $\mathbb{R}$ has a bounded measurable real-valued $(a_1,\dots,a_k)$-decomposition, then it also has an almost everywhere integer-valued bounded measurable $(a_1,\dots,a_k)$-decomposition.*
(b) *The real numbers $1/b_1,\dots,1/b_n$ are linearly independent over $\mathbb{Q}$.*

If we want an (everywhere) integer-valued bounded measurable decomposition, we have to fix the decomposition on an exceptional null set. To do this, as pointed out in [6], we need to use the original (nonmeasurable) version of this problem. Since we have solved it, we are able to answer the measurable version too.

Theorem 5.4. *Let $a_1,\dots,a_k$ nonzero real number, and let $b_1,\dots,b_n$ be as defined in the previous theorem. The implication*

f has a bounded measurable real-valued $(a_1,\dots,a_k)$-decomposition

$\Rightarrow$ f has a bounded measurable integer-valued $(a_1,\dots,a_k)$-decomposition

holds for any function $f:\mathbb{R}\to\mathbb{Z}$ if and only if the periods satisfy the following two conditions:

- *$1/b_1,\dots,1/b_n$ are linearly independent over $\mathbb{Q}$,*
- *any three of $b_1,\dots,b_n$ are linearly independent over $\mathbb{Q}$.*

(Note that the second condition holds if and only if there is no planar triple among $a_1,\dots,a_k$.)

Proof. If the first condition fails to hold then by Theorem 5.3 there exists an integer-valued function that has a bounded measurable real-valued $(a_1,\dots,a_k)$-decomposition, but it does not have a decomposition in which the functions are bounded, measurable and (almost everywhere) integer-valued.

If the second condition fails, then according to Theorem 4.4 there exists an integer-valued function f that has a bounded real-valued $(a_1,\dots,a_k)$-

decomposition, but no bounded integer-valued $(a_1, \ldots, a_k)$-decomposition. Moreover, f and the functions of its real-valued decomposition are all supported by a finite-dimensional $\mathbb{Q}$-linear subspace. Such a subspace is countable, so it has measure zero. However, every function supported by a null set is measurable. Consequently, f shows that the implication does not hold.

Now we suppose that both conditions are satisfied by $a_1, \ldots, a_k$. Let us take an integer-valued function f with a decomposition $f = f_1 + \cdots + f_k$ where f_i is bounded, measurable and a_i-periodic. Theorem 5.3 entails that there is a decomposition $f = g_1 + \cdots + g_k$ where g_i is bounded, measurable, almost everywhere integer-valued and a_i-periodic.

From this point on, the proof goes as in [6, Proposition 3.3]. Set

$$E_j := \{x \in \mathbb{R} : g_j(x) \notin \mathbb{Z}\}, \qquad E = \Big(\bigcup_{j=1}^{k} E_j\Big) + a_1\mathbb{Z} + \cdots + a_k\mathbb{Z}.$$

Clearly, E has measure zero. Consider the integer-valued function $F = f\chi_E$ which has a bounded real-valued decomposition $g_1\chi_E + \cdots + g_k\chi_E$. By Theorem 1.4 it also has a bounded integer-valued decomposition $F = G_1 + \cdots + G_k$. Then the functions

$$\tilde{g}_j(x) = g_j\chi_{\mathbb{R}\setminus E} + G_j\chi_E$$

give us a bounded, measurable, everywhere integer-valued periodic decomposition. ∎

Finally, we mention a few open problems. We have seen that if there is no planar triple among the periods, then we can get every solution of the homogeneous equation (1.1) by adding up solutions of a certain simple type (namely, solutions that contain only two nonzero functions). It would be nice to have a similar theorem in general (when we have no restriction on the periods). Let a solution be a *basic solution* if the periods that correspond to nonzero functions are in a plane (by which we mean that they span a one- or two-dimensional $\mathbb{Q}$-linear subspace). Our conjecture is that every homogeneous solution is the sum of basic solutions.

Problem 5.5. *Is it true that every solution of the homogeneous equation can be written as the sum of such solutions where the periods corresponding to the nonzero functions of the decomposition span a $\mathbb{Q}$-linear subspace with dimension at most 2?*

A positive answer to this question could be a first step towards describing all homogeneous solutions. In that case it would be enough to determine the basic solutions. It is easy to see that it suffices to do that on $\mathbb{Z} \times \mathbb{Z}$.

Problem 5.6. *Let $a_1, \ldots, a_k$ be nonzero elements of $\mathbb{Z} \times \mathbb{Z}$. Determine the solutions of the equation*

$$h_1 + \cdots + h_k = 0 \qquad (h_i : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R},\ \Delta_{a_i} h_i = 0).$$

The case of three periods is solved [4].

Acknowledgements. The author is grateful to the referee for the detailed and insightful review and for the numerous suggestions on how to make the paper more comprehensible to the reader.

The research was supported by the József Öveges Program of NKTH and KPI.

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Received 9 November 2007;
in revised form 9 May 2010