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Typical multifractal box dimensions of measures

by

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Abstract. We study the typical behaviour (in the sense of Baire's category) of the multifractal box dimensions of measures on $\mathbb{R}^d$. We prove that in many cases a typical measure μ is as irregular as possible, i.e. the lower multifractal box dimensions of μ attain the smallest possible value and the upper multifractal box dimensions of μ attain the largest possible value.

1. Statement of results. In this paper we study the typical (in the sense of Baire) multifractal box dimensions of measures. In Section 1.1 we define multifractal box dimensions of sets and in Section 1.2 we define multifractal box dimensions of measures. Finally, in Section 1.3 we state our main results. Section 2 contains applications to typical multifractal box dimensions of measures on self-similar sets, and the proofs of the main results are given in Sections 3 and 4.

1.1. Multifractal box dimensions of sets. Fix a Borel probability measure π on $\mathbb{R}^d$ with support K. For a bounded subset E of K, the *multifractal box dimensions* of E with respect to π are defined as follows. For $r > 0$ and a real number q write

$$M_\pi^q(E;r) = \inf_{\substack{(B(x_i,r))_i \text{ is a cover of } E \\ x_i \in K}} \sum_i \pi(B(x_i,r))^q. \tag{1.1}$$

The *lower* and *upper covering multifractal box dimensions* of E of order q with respect to π are defined by

$$\begin{aligned} \underline{\dim}_{\pi,\mathsf{B}}^q(E) &= \liminf_{r \searrow 0} \frac{\log M_\pi^q(E;r)}{-\log r}, \\ \overline{\dim}_{\pi,\mathsf{B}}^q(E) &= \limsup_{r \searrow 0} \frac{\log M_\pi^q(E;r)}{-\log r}. \end{aligned} \tag{1.2}$$

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The main significance of multifractal box dimensions is their relationship with the so-called multifractal spectrum of π. During the 1990's there has been an enormous interest in computing multifractal box dimensions and multifractal spectra of measures, and within the last 15 years the multifractal box dimensions and the multifractal spectra of various classes of measures in $\mathbb{R}^d$ exhibiting some degree of self-similarity have been computed rigorously (cf. [Fa, Pe] and the references therein).

1.2. Multifractal box dimensions of measures. While multifractal box dimensions of measures have played a central role in multifractal analysis for the past 15 years, recently their importance in the study of fractal geometry and dynamical systems has been recognized (see, for example, [Pe, Yo]). The multifractal box dimensions of a Borel probability measure μ on $\mathbb{R}^d$ with respect to π are defined as follows. For a real number q, we define the *small* and *big lower multifractal box dimensions* of μ of order q with respect to the measure π by

$$\begin{aligned} \underline{\dim}^q_{*\pi,\mathsf{B}}(\mu) &= \inf_{\mu(E)>0} \underline{\dim}^q_{\pi,\mathsf{B}}(E), \\ \underline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) &= \lim_{\varepsilon\searrow 0} \inf_{\mu(E)>1-\varepsilon} \underline{\dim}^q_{\pi,\mathsf{B}}(E). \end{aligned} \tag{1.3}$$

Similarly, the *small* and *big upper multifractal box dimensions* of μ of order q with respect to π are

$$\begin{aligned} \overline{\dim}^q_{*\pi,\mathsf{B}}(\mu) &= \inf_{\mu(E)>0} \overline{\dim}^q_{\pi,\mathsf{B}}(E), \\ \overline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) &= \lim_{\varepsilon\searrow 0} \inf_{\mu(E)>1-\varepsilon} \overline{\dim}^q_{\pi,\mathsf{B}}(E). \end{aligned} \tag{1.4}$$

1.3. Typical multifractal box dimensions of measures. In this paper we study the multifractal box dimensions of a typical measure in the sense of Baire. For a compact subset K of $\mathbb{R}^d$, we denote the family of Borel probability measures on K by $\mathcal{P}(K)$ and we equip $\mathcal{P}(K)$ with the weak topology. We will say that *a typical probability measure on K has property* P if the set of probability measures that do not have property P is meagre with respect to the weak topology on $\mathcal{P}(K)$. The typical behaviour of various other quantities related to multifractal analysis has also been studied. In particular, the local dimension of a typical measure has been studied by Haase [Ha] and investigated further by Genyuk [Ge]. We also note that the multifractal spectrum of a typical continuous function has been studied by several authors (cf. [BuNa, Ja1, Ja2]).

To state the main results in the paper, we need a few definitions. Firstly, the *upper moment scaling function* of π is defined by

$$\overline{\tau}_\pi(q) = \overline{\dim}^q_{\pi,\mathsf{B}}(K). \tag{1.5}$$

We also define its local versions: first the *local upper box dimension* of K at x is defined by

$$\overline{\dim}^q_{\pi,\mathsf{B},\mathsf{loc}}(x;K)=\lim_{r\searrow 0}\overline{\dim}^q_{\pi,\mathsf{B}}(B(x,r)\cap K);\tag{1.6}$$

then we define the *local upper moment scaling functions* of π by

$$\begin{aligned}\overline{\tau}_{\pi,\mathsf{loc}}(q)&=\inf_{x\in K}\overline{\dim}^q_{\pi,\mathsf{B},\mathsf{loc}}(x;K),\\ \overline{T}_{\pi,\mathsf{loc}}(q)&=\sup_{x\in K}\overline{\dim}^q_{\pi,\mathsf{B},\mathsf{loc}}(x;K).\end{aligned}\tag{1.7}$$

Finally, let

$$\begin{aligned}\underline{D}_\pi&=\liminf_{r\searrow 0}\frac{\log\sup_{x\in K}\pi(B(x,r))}{\log r},\\ \overline{D}_\pi&=\limsup_{r\searrow 0}\frac{\log\inf_{x\in K}\pi(B(x,r))}{\log r}.\end{aligned}\tag{1.8}$$

Proposition 1.1 below gives the relationships between the dimensions introduced in (1.5)–(1.8).

PROPOSITION 1.1. *Let π be a Borel probability measure on $\mathbb{R}^d$ with compact support K. We have*

$$\begin{array}{ccccll}\underline{D}_\pi\, q &\le& \overline{\tau}_{\pi,\mathsf{loc}}(q) &\le& \overline{\tau}_\pi(q) & \text{for all } q\le 0,\\ \wedge && \wedge &&&\\ -\overline{D}_\pi\, q &\le& \overline{T}_{\pi,\mathsf{loc}}(q) &\le& \overline{\tau}_\pi(q) & \text{for all } q\le 0,\end{array}$$

and

$$\begin{array}{ccccll}-\overline{D}_\pi\, q &\le& \overline{\tau}_{\pi,\mathsf{loc}}(q) &\le& \overline{\tau}_\pi(q) & \text{for all } q\ge 0,\\ \wedge && \wedge &&&\\ -\underline{D}_\pi\, q &\le& \overline{T}_{\pi,\mathsf{loc}}(q) &\le& \overline{\tau}_\pi(q) & \text{for all } q\ge 0.\end{array}$$

Proof. This follows easily from the definitions and the proof is therefore omitted. ∎

A measure π on $\mathbb{R}^d$ is called a *doubling measure* if there is a constant c such that

$$\sup_{\substack{x\in\operatorname{supp}\pi\\ r>0}}\frac{\pi(B(x,2r))}{\pi(B(x,r))}\le c.$$

We can now state the main results, Theorems 1.2 and 1.3, giving bounds for the multifractal box dimensions of measures $\mu\in\mathcal{P}(K)$. The first result, which is easily proved and only included for completeness, provides bounds that are valid for all measures, whereas the second result provides bounds that are valid for typical measures.

THEOREM 1.2 (Results for all measures in $\mathcal{P}(K)$). *Let π be a Borel probability measure on $\mathbb{R}^d$ with compact support K.*

(1) *All measures $\mu \in \mathcal{P}(K)$ satisfy*

$$-\underline{D}_\pi q \le \underline{\dim}^q_{*\pi,\mathsf{B}}(\mu) \le \underline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) \quad \text{for all } q \le 0,$$
$$-\overline{D}_\pi q \le \underline{\dim}^q_{*\pi,\mathsf{B}}(\mu) \le \underline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) \quad \text{for all } q \ge 0.$$

(2) *All measures $\mu \in \mathcal{P}(K)$ satisfy*

$$\overline{\dim}^q_{*\pi,\mathsf{B}}(\mu) \le \overline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) \le \overline{\tau}_\pi(q) \quad \text{for all } q.$$

Proof. This follows easily from the definitions. ∎

THEOREM 1.3 (Results for typical measures in $\mathcal{P}(K)$). *Let π be a Borel probability measure on $\mathbb{R}^d$ with compact support K.*

(1) *A typical measure $\mu \in \mathcal{P}(K)$ satisfies*

$$-\underline{D}_\pi\, q \le \underline{\dim}^q_{*\pi,\mathsf{B}}(\mu) \le \underline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) \le -\overline{D}_\pi\, q \quad \text{for all } q \le 0,$$
$$-\overline{D}_\pi\, q \le \underline{\dim}^q_{*\pi,\mathsf{B}}(\mu) \le \underline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) \le -\underline{D}_\pi\, q \quad \text{for all } q \ge 0.$$

(2) *If π is a doubling measure, then a typical measure $\mu \in \mathcal{P}(K)$ satisfies*

$$\overline{\tau}_{\pi,\mathsf{loc}}(q) \le \overline{\dim}^q_{*\pi,\mathsf{B}}(\mu) \le \overline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) \le \overline{\tau}_\pi(q) \quad \text{for all } q \le 0.$$

If π is a doubling measure and K does not contain isolated points, then a typical measure $\mu \in \mathcal{P}(K)$ satisfies

$$\overline{\tau}_{\pi,\mathsf{loc}}(q) \le \overline{\dim}^q_{*\pi,\mathsf{B}}(\mu) \le \overline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) \le \overline{\tau}_\pi(q) \quad \text{for all } q.$$

Part (1) of Theorem 1.3 is proved in Section 3, and part (2) in Section 4. Comparing the statements in Theorems 1.2 and 1.3, we see that a typical measure μ is very close to being as irregular as possible. Namely, for all q, the lower multifractal box dimensions $\underline{\dim}^q_{*\pi,\mathsf{B}}(\mu)$ and $\underline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu)$ are close to the smallest possible value, and the upper multifractal box dimensions $\overline{\dim}^q_{*\pi,\mathsf{B}}(\mu)$ and $\overline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu)$ are close to the largest possible value. Figure 1 below illustrates Theorem 1.3.

For $q = 0$, Theorem 1.3 gives the following interesting result due to Myjak & Rudnicki [MyRu]. To state it we need a few definitions. For a set $E \subseteq \mathbb{R}^d$, we let $\overline{\dim}_{\mathsf{B}}(E)$ denote the upper box dimension. Also, for a probability measure μ we define the *small* and *big lower multifractal box dimensions* of μ by

$$\begin{aligned} \underline{\dim}_{*\mathsf{B}}(\mu) &= \inf_{\mu(E)>0} \underline{\dim}_{\mathsf{B}}(E), \\ \underline{\dim}^*_{\mathsf{B}}(\mu) &= \lim_{\varepsilon\searrow 0} \inf_{\mu(E)>1-\varepsilon} \underline{\dim}_{\mathsf{B}}(E). \end{aligned} \tag{1.9}$$

Similarly, we define the *small* and *big upper multifractal box dimensions* of μ by

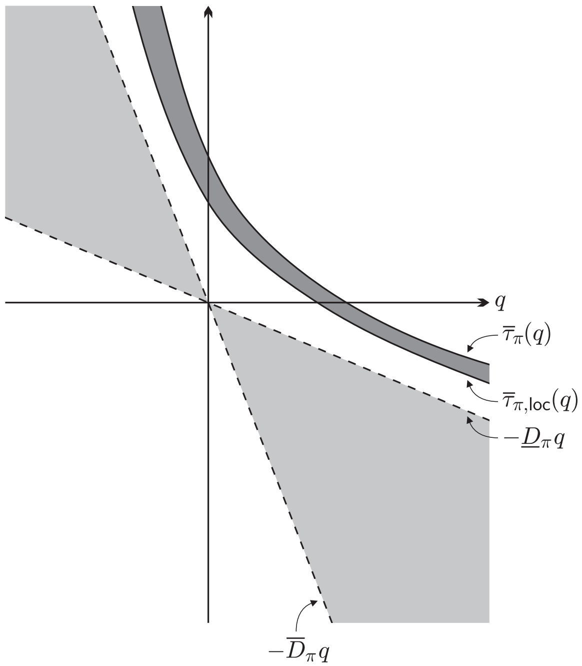

Fig. 1. This figure illustrates the statement in Theorem 1.3. (In the figure the graphs are drawn such that $-\overline{D}_\pi\, q \le \overline{\tau}_{\pi,\mathsf{loc}}(q)$ for all $q \le 0$ and $-\underline{D}_\pi\, q \le \overline{\tau}_{\pi,\mathsf{loc}}(q)$ for all $q \ge 0$; it follows from Proposition 1.1 that if $\overline{\tau}_{\pi,\mathsf{loc}}(q) = \overline{T}_{\pi,\mathsf{loc}}(q)$, then this is the case. The condition $\overline{\tau}_{\pi,\mathsf{loc}}(q) = \overline{T}_{\pi,\mathsf{loc}}(q)$ is, for example, satisfied if π is a self-similar measure satisfying the Strong Separation Condition; see Section 2.) Theorem 1.3 shows that the lower q box dimensions of a typical measure μ lie in the lightly shaded region bounded by the dashed lines, and if the measure π is doubling, then the upper q box dimensions of a typical measure μ lie in the darkly shaded region bounded by the solid curves.

$$\begin{aligned}\overline{\dim}_{*\mathsf{B}}(\mu) &= \inf_{\mu(E)>0} \overline{\dim}_{\mathsf{B}}(E),\\ \overline{\dim}^{*}_{\mathsf{B}}(\mu) &= \lim_{\varepsilon \searrow 0} \inf_{\mu(E)>1-\varepsilon} \overline{\dim}_{\mathsf{B}}(E).\end{aligned} \tag{1.10}$$

Finally, we define the *local upper box dimension* of E at x by

$$\overline{\dim}_{\mathsf{B},\mathsf{loc}}(x;E) = \lim_{r\searrow 0} \overline{\dim}_{\mathsf{B}}(B(x,r)\cap E). \tag{1.11}$$

It is clear that if we put $q = 0$ in (1.3), (1.4) and (1.6), then we obtain (1.9), (1.10) and (1.11), respectively. The following result due to Myjak & Rudnicki [MyRu] therefore follows from Theorem 1.3 by putting $q = 0$. This result gives bounds for the box dimensions of typical measures.

Corollary 1.4 (Results for typical measures in $\mathcal{P}(K)$ [MyRu]). *Let K be a compact set in $\mathbb{R}^d$. Write*

$$\overline{s}_{\mathsf{loc}} = \inf_{x\in K} \overline{\dim}_{\mathsf{B},\mathsf{loc}}(x;K), \qquad \overline{s} = \overline{\dim}_{\mathsf{B}}(K).$$

(1) *A typical measure $\mu \in \mathcal{P}(K)$ satisfies*

$$\underline{\dim}_{*\mathsf{B}}(\mu) = \underline{\dim}^*_{\mathsf{B}}(\mu) = 0.$$

(2) *A typical measure $\mu \in \mathcal{P}(K)$ satisfies*

$$\overline{s}_{\mathsf{loc}} \le \overline{\dim}_{*\mathsf{B}}(\mu) \le \overline{\dim}^*_{\mathsf{B}}(\mu) \le \overline{s}.$$

2. An application. Typical multifractal box dimensions of measures on self-similar sets. As an application of Theorem 1.3, we will now compute the multifractal box dimensions of typical measures on self-similar sets. Fix an integer N with $N \ge 2$. Next, let $S_i : \mathbb{R}^d \to \mathbb{R}^d$ for $i = 1, \dots, N$ be contracting similarities and let $(p_1, \dots, p_N)$ be a probability vector. For each i, we denote the Lipschitz constant of S_i by $r_i \in (0,1)$. Let K and π be the self-similar set and the self-similar measure associated with the list $(S_1, \dots, S_N, p_1, \dots, p_N)$, i.e. K is the unique non-empty compact subset of $\mathbb{R}^d$ such that

$$K = \bigcup_i S_i(K), \tag{2.1}$$

and π is the unique Borel probability measure on $\mathbb{R}^d$ such that

$$\pi = \sum_i p_i \pi \circ S_i^{-1} \tag{2.2}$$

(cf. [Fa, Hu]). It is well-known that $\operatorname{supp} \pi = K$ (cf. [Fa, Hu]). We say that the list $(S_1, \dots, S_n)$ satisfies the *Open Set Condition* (OSC) if there exists an open non-empty and bounded subset U of $\mathbb{R}^d$ with $S_i U \subseteq U$ for all i and $S_i U \cap S_j U = \emptyset$ for all i, j with $i \ne j$. Also, we say that the list $(S_1, \dots, S_N)$ satisfies the *Strong Separation Condition* (SSC) if $S_i K \cap S_j K = \emptyset$ for all i, j with $i \ne j$. Define $\beta : \mathbb{R} \to \mathbb{R}$ by

$$\sum_i p_i^q r_i^{\beta(q)} = 1. \tag{2.3}$$

The next result computes the multifractal box dimensions for typical measures $\mu \in \mathcal{P}(K)$ supported on a self-similar set K satisfying the OSC.

THEOREM 2.1. *Let K and π be as in* (2.1) *and* (2.2), *and assume that the OSC is satisfied. Next, let*

$$s_{\min} = \min_i \frac{\log p_i}{\log r_i} \quad \text{and} \quad s_{\max} = \max_i \frac{\log p_i}{\log r_i}.$$

(1) *Results for all measures $\mu \in \mathcal{P}(K)$.*

- *All measures $\mu \in \mathcal{P}(K)$ satisfy*

$$-s_{\min}\, q \le \underline{\dim}^q_{*\pi,\mathsf{B}}(\mu) \le \underline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) \quad \text{for all } q \le 0,$$
$$-s_{\max}\, q \le \underline{\dim}^q_{*\pi,\mathsf{B}}(\mu) \le \underline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) \quad \text{for all } q \ge 0.$$

- *All measures* $\mu \in \mathcal{P}(K)$ *satisfy*

$$\overline{\dim}^{q}_{*\pi,\mathsf{B}}(\mu) \le \overline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) \le \beta(q) \quad \textit{for all } q.$$

(2) *Results for typical measures* $\mu \in \mathcal{P}(K)$.

- *A typical measure* $\mu \in \mathcal{P}(K)$ *satisfies*

$$\begin{aligned} -s_{\min} q \le \underline{\dim}^{q}_{*\pi,\mathsf{B}}(\mu) \le \underline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) \le -s_{\max} q \quad &\textit{for all } q \le 0, \\ -s_{\max} q \le \underline{\dim}^{q}_{*\pi,\mathsf{B}}(\mu) \le \underline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) \le -s_{\min} q \quad &\textit{for all } q \ge 0. \end{aligned}$$

- *If* π *is a doubling measure* (this is, for example, easily seen to be the case if the SSC is satisfied; see [Yu] for a proof of this and for other conditions guaranteeing that π is a doubling measure), *then a typical measure* $\mu \in \mathcal{P}(K)$ *satisfies*

$$\overline{\dim}^{q}_{*\pi,\mathsf{B}}(\mu) = \overline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) = \beta(q) \quad \textit{for all } q.$$

Before proving Theorem 2.1 we make various comments and list two corollaries.

If we define $e_{\min}, e_{\max} \ge 0$ by

$$\sum_{i,\, \frac{\log p_i}{\log r_i} = s_{\min}} r_i^{e_{\min}} = 1, \qquad \sum_{i,\, \frac{\log p_i}{\log r_i} = s_{\max}} r_i^{e_{\max}} = 1,$$

then it is well known (see, for example, [CaMa]) that

$$\begin{aligned} \beta(q) - (e_{\min} - s_{\min} q) \searrow 0 \quad &\text{as } q \to \infty, \\ \beta(q) - (e_{\max} - s_{\max} q) \searrow 0 \quad &\text{as } q \to -\infty; \end{aligned}$$

see Figure 2 below. In particular, together with Theorem 2.1 this shows that if the OSC is satisfied and π is a doubling measure, then

$$\begin{aligned} \overline{\dim}^{q}_{*\pi,\mathsf{B}}(\mu) - \underline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) \ge e_{\max} \quad &\text{for all } q \le 0, \\ \overline{\dim}^{q}_{*\pi,\mathsf{B}}(\mu) - \underline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) \ge e_{\min} \quad &\text{for all } q \ge 0, \end{aligned}$$

for a typical measure $\mu \in \mathcal{P}(K)$. The reader is referred to Figure 2 for an illustration of this.

We now list two corollaries of Theorem 2.1.

COROLLARY 2.2. *Let* K *be as in* (2.1), *and assume that the OSC is satisfied. Next, let* $s \in \mathbb{R}$ *be defined by* $\sum_i r_i^s = 1$, *i.e.* $s = \dim_{\mathsf{H}}(K) = \dim_{\mathsf{B}}(K)$, *and let* $\mathcal{H}^s$ *denote the* s*-dimensional Hausdorff measure and write* $\mathcal{H}^s \llcorner K$ *for its restriction to* K.

(1) *Results for all measures* $\mu \in \mathcal{P}(K)$.

- *All measures* $\mu \in \mathcal{P}(K)$ *satisfy*

$$-sq \le \underline{\dim}^{q}_{*\mathcal{H}^s \llcorner K,\mathsf{B}}(\mu) \le \underline{\dim}^{*q}_{\mathcal{H}^s \llcorner K,\mathsf{B}}(\mu) \quad \textit{for all } q.$$

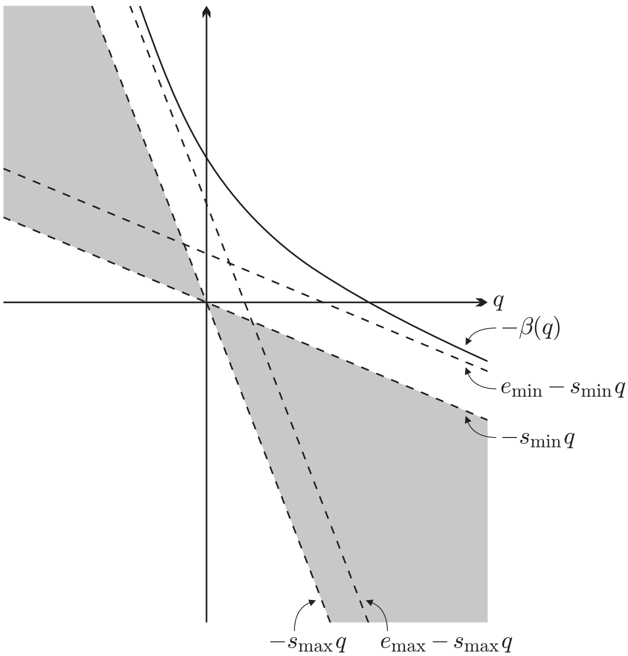

Fig. 2. This figure illustrates the statement in Theorem 2.1. Theorem 2.1 shows that if the OSC is satisfied, then the lower q box dimensions of a typical measure μ lie in the shaded region bounded by the dashed lines, and if the OSC is satisfied and the measure π is a doubling measure, then the upper q box dimensions of a typical measure μ equal $\beta(q)$.

- *All measures $\mu \in \mathcal{P}(K)$ satisfy*
$$\overline{\dim}^{q}_{*\mathcal{H}^s\llcorner K,\mathsf{B}}(\mu) \le \overline{\dim}^{*q}_{\mathcal{H}^s\llcorner K,\mathsf{B}}(\mu) \le s(1-q) \quad \textit{for all } q.$$

(2) *Results for typical measures $\mu \in \mathcal{P}(K)$.*

- *A typical measure $\mu \in \mathcal{P}(K)$ satisfies*
$$\underline{\dim}^{q}_{*\mathcal{H}^s\llcorner K,\mathsf{B}}(\mu) = \underline{\dim}^{*q}_{\mathcal{H}^s\llcorner K,\mathsf{B}}(\mu) = -sq \quad \textit{for all } q.$$
- *A typical measure $\mu \in \mathcal{P}(K)$ satisfies*
$$\overline{\dim}^{q}_{*\mathcal{H}^s\llcorner K,\mathsf{B}}(\mu) = \overline{\dim}^{*q}_{\mathcal{H}^s\llcorner K,\mathsf{B}}(\mu) = s(1-q) \quad \textit{for all } q.$$

Proof. Define the probability vector $(p_1,\dots,p_N)$ by $(p_1,\dots,p_N) = (r_1^s,\dots,r_N^s)$, and let π be the self-similar measure satisfying (2.2). For this particular choice of $(p_1,\dots,p_N)$ it follows from [Hu] that $\pi = \mathcal{H}^s\llcorner K/\mathcal{H}^s(K)$, and it follows from [Yu, Corollary 1.2] that π is a doubling measure. Corollary 2.2 therefore follows immediately from Theorem 2.1. ∎

It follows from Corollary 2.2 that a typical measure μ on a self-similar set satisfying the OSC is as irregular as possible. Namely, for all q, the lower multifractal box dimensions $\underline{\dim}^{q}_{*\mathcal{H}^s\llcorner K,\mathsf{B}}(\mu)$ and $\underline{\dim}^{*q}_{\mathcal{H}^s\llcorner K,\mathsf{B}}(\mu)$ attain the smallest possible value, and the upper multifractal box dimensions $\overline{\dim}^{q}_{*\mathcal{H}^s \llcorner K,\mathsf{B}}(\mu)$ and $\overline{\dim}^{*q}_{\mathcal{H}^s \llcorner K,\mathsf{B}}(\mu)$ attain the largest possible value. Specializing even further we obtain the following result about the multifractal box dimensions of measures on the unit cube in $\mathbb{R}^d$.

COROLLARY 2.3. *Let $I^d = [0,1]^d$ be the closed unit cube in $\mathbb{R}^d$. Next, let $\mathcal{L}^d$ denote the d-dimensional Lebesgue measure and write $\mathcal{L}^d \llcorner I^d$ for its restriction to I^d.*

(1) *A typical measure $\mu \in \mathcal{P}(I^d)$ satisfies*

$$\underline{\dim}^{q}_{*\mathcal{L}^d \llcorner I^d,\mathsf{B}}(\mu) = \underline{\dim}^{*q}_{\mathcal{L}^d \llcorner I^d,\mathsf{B}}(\mu) = -dq \quad \textit{for all } q.$$

(2) *A typical measure $\mu \in \mathcal{P}(I^d)$ satisfies*

$$\overline{\dim}^{q}_{*\mathcal{L}^d \llcorner I^d,\mathsf{B}}(\mu) = \overline{\dim}^{*q}_{\mathcal{L}^d \llcorner I^d,\mathsf{B}}(\mu) = d(1-q) \quad \textit{for all } q.$$

Proof. This follows immediately from Corollary 2.2. ∎

We now turn towards the proof of Theorem 2.1. We start by introducing the following notation. If $(S_1,\dots,S_N)$ is a list of similarities and r_i denotes the Lipschitz constant of S_i, then we will write $S_{\mathbf{i}} = S_{i_1} \circ \dots \circ S_{i_n}$ and $r_{\mathbf{i}} = r_{i_1} \cdots r_{i_n}$ for all lists $\mathbf{i} = i_1 \dots i_n$ with entries $i_k \in \{1,\dots,N\}$. Also, if $\mathbf{i} = i_1 \dots i_n$ is a list with entries $i_k \in \{1,\dots,N\}$ we will write $|\mathbf{i}| = n$ for the "length" of $\mathbf{i}$. Finally, if $(p_1,\dots,p_N)$ is a probability vector, then we will write $p_{\mathbf{i}} = p_{i_1} \cdots p_{i_n}$ for all lists $\mathbf{i} = i_1 \dots i_n$ with entries $i_k \in \{1,\dots,N\}$.

In order to prove Theorem 2.1 we need the following result.

PROPOSITION 2.4. *Let π and K be as in* (2.1) *and* (2.2), *and assume that the OSC is satisfied. Then the following hold.*

(1) *The set K does not have isolated points.*
(2) $\underline{D}_\pi = \min_i \frac{\log p_i}{\log r_i}$ *and* $\overline{D}_\pi = \max_i \frac{\log p_i}{\log r_i}$.
(3) $\overline{\tau}_{\pi,\mathsf{loc}}(q) = \overline{\tau}_\pi(q) = \beta(q)$ *for all* $q \in \mathbb{R}$.

Proof. (1) This is well-known: see, for example, [Fa].

(2) This is well-known: see, for example, [Pat].

(3) It is clear that $\overline{\tau}_{\pi,\mathsf{loc}}(q) \le \overline{\tau}_\pi(q)$. Hence, it suffices to show that $\beta(q) \le \overline{\tau}_{\pi,\mathsf{loc}}(q)$ and $\overline{\tau}_\pi(q) \le \beta(q)$. The latter follows immediately from the fact that

$$\lim_{r \searrow 0} \frac{\log M^q_\pi(K;r)}{-\log r} = \beta(q) \tag{2.4}$$

for all $q \in \mathbb{R}$ (see [Pat]).

Next, we prove that $\beta(q) \le \overline{\tau}_{\pi,\mathsf{loc}}(q)$. Therefore fix $x \in K$ and $r > 0$. We may clearly choose a list $\mathbf{i} = i_1 \dots i_n$ with entries $i_k \in \{1,\dots,N\}$ such that $S_{\mathbf{i}}(K) \subseteq B(x,r) \cap K$. Hence,

$$\overline{\dim}^{q}_{\pi,\mathsf{B}}(S_{\mathbf{i}}(K)) \le \overline{\dim}^{q}_{\pi,\mathsf{B}}(B(x,r) \cap K). \tag{2.5}$$

Next, we show that

$$\overline{\dim}^q_{\pi,\mathsf{B}}(K) = \overline{\dim}^q_{\pi,\mathsf{B}}(S_{\mathbf{i}}(K)). \tag{2.6}$$

Indeed, since $S_{\mathbf{i}}(K) \subseteq K$, we conclude that $\overline{\dim}^q_{\pi,\mathsf{B}}(S_{\mathbf{i}}(K)) \le \overline{\dim}^q_{\pi,\mathsf{B}}(K)$. We will now show that $\underline{\dim}^q_{\pi,\mathsf{B}}(K) \le \underline{\dim}^q_{\pi,\mathsf{B}}(S_{\mathbf{i}}(K))$. We first note that it follows from [MoRe, Theorem 2.1] (see also [Pat]) that $\pi(S_{\mathbf{i}}(K)) \cap S_{\mathbf{j}}(K)) = 0$ for all $\mathbf{j}$ with $\mathbf{i} \ne \mathbf{j}$ and $|\mathbf{i}| = |\mathbf{j}|$. This implies that if $x \in K$ and $r > 0$, then

$$\begin{aligned}\pi(S_{\mathbf{i}}B(x,r)) &= \sum_{|\mathbf{j}|=|\mathbf{i}|} p_{\mathbf{j}}\pi(S_{\mathbf{j}}^{-1}S_{\mathbf{i}}B(x,r)) = p_{\mathbf{i}}\pi(S_{\mathbf{i}}^{-1}S_{\mathbf{i}}B(x,r)) \\ &= p_{\mathbf{i}}\pi(B(x,r)).\end{aligned} \tag{2.7}$$

Next, fix $\rho > 0$ and let $(B(x_i,\rho))_i$ be a centred cover of $S_{\mathbf{i}}(K)$. Then $(S_{\mathbf{i}}^{-1}B(x_i,\rho))_i$ is a cover of K. Also, for each i, we can thus choose $y_i \in K$ such that $x_i = S_{\mathbf{i}}y_i$, whence $S_{\mathbf{i}}^{-1}B(x_i,\rho) = S_{\mathbf{i}}^{-1}B(S_{\mathbf{i}}y_i,\rho) = B(y_i, r_{\mathbf{i}}^{-1}\rho)$. Therefore $(B(y_i, r_{\mathbf{i}}^{-1}\rho))_i$ is a cover of K, and so (using (2.7)) we obtain

$$\begin{aligned}M^q_\pi(K; r_{\mathbf{i}}^{-1}\rho) &\le \sum_i \pi(B(y_i, r_{\mathbf{i}}^{-1}\rho))^q = \sum_i p_{\mathbf{i}}^{-q}\pi(S_{\mathbf{i}}B(y_i, r_{\mathbf{i}}^{-1}\rho))^q \\ &= p_{\mathbf{i}}^{-q}\sum_i \pi(B(S_{\mathbf{i}}y_i,\rho))^q = p_{\mathbf{i}}^{-q}\sum_i \pi(B(x_i,\rho))^q.\end{aligned}$$

Taking infimum over all centred covers $(B(x_i,\rho))_i$ of $S_{\mathbf{i}}(K)$ now gives

$$M^q_\pi(K; r_{\mathbf{i}}^{-1}\rho) \le p_{\mathbf{i}}^{-q} M^q_\pi(S_{\mathbf{i}}(K);\rho)$$

for all $\rho > 0$. This clearly implies that $\overline{\dim}^q_{\pi,\mathsf{B}}(K) \le \overline{\dim}^q_{\pi,\mathsf{B}}(S_{\mathbf{i}}(K))$, and completes the proof of (2.6).

Combining (2.4)–(2.6) now shows that

$$\beta(q) = \overline{\dim}^q_{\pi,\mathsf{B}}(K) = \overline{\dim}^q_{\pi,\mathsf{B}}(S_{\mathbf{i}}(K)) \le \overline{\dim}^q_{\pi,\mathsf{B}}(B(x,r) \cap K).$$

Finally, taking infimum over all $x \in K$ and all $r > 0$ gives the desired result. ∎

We can now prove Theorem 2.1.

Proof of Theorem 2.1. Theorem 2.1 follows immediately from Theorem 1.2, Theorem 1.3 and Proposition 2.4. ∎

3. Proof of part (1) in Theorem 1.3. The purpose of this section is to prove

THEOREM 1.3(1). *Let π be a Borel probability measure on $\mathbb{R}^d$ with compact support K. A typical measure $\mu \in \mathcal{P}(K)$ satisfies*

$$\begin{aligned}-\underline{D}_\pi q \le \underline{\dim}^q_{*\pi,\mathsf{B}}(\mu) \le \underline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) \le -\overline{D}_\pi q \quad &\textit{for all } q \le 0,\\ -\overline{D}_\pi q \le \underline{\dim}^q_{*\pi,\mathsf{B}}(\mu) \le \underline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) \le -\underline{D}_\pi q \quad &\textit{for all } q \ge 0.\end{aligned}$$

It is well-known (cf., for example, [Ed, pp. 105–107] or [Par, p. 51, Theorem 6.8]) that the weak topology on $\mathcal{P}(K)$ is induced by the metric L on $\mathcal{P}(K)$ defined as follows. Let $\mathrm{Lip}(K)$ denote the family of Lipschitz functions $f : K \to \mathbb{R}$ with $|f| \le 1$ and $\mathrm{Lip}(f) \le 1$ where $\mathrm{Lip}(f)$ denotes the Lipschitz constant of f. The metric L is now defined by

$$L(\mu,\nu) = \sup_{f\in \mathrm{Lip}(K)} \left| \int f\, d\mu - \int f\, d\nu \right|$$

for $\mu,\nu \in \mathcal{P}(K)$. We will always equip $\mathcal{P}(K)$ with the metric L and all balls in $\mathcal{P}(K)$ will be with respect to the metric L, i.e. if $\mu \in \mathcal{P}(K)$ and $r > 0$, we will write $B(\mu, r) = \{\nu \in \mathcal{P}(K) \mid L(\mu,\nu) < r\}$ for the ball with centre at μ and radius r.

We now turn to the proof of part (1) of Theorem 1.3. Let

$$\begin{aligned}
\Gamma = \{\mu \in \mathcal{P}(K) \mid\ & -\underline{D}_\pi q \le \underline{\dim}^{q}_{*\pi,\mathsf{B}}(\mu) \le \underline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) \le -\overline{D}_\pi q \text{ for all } q \le 0,\\
& -\overline{D}_\pi q \le \underline{\dim}^{q}_{*\pi,\mathsf{B}}(\mu) \le \underline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) \le -\underline{D}_\pi q \text{ for all } q \ge 0\}\\
= \{\mu \in \mathcal{P}(K) \mid\ & \underline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) \le -\overline{D}_\pi q \text{ for all } q \le 0,\\
& \underline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) \le -\underline{D}_\pi q \text{ for all } q \ge 0\}
\end{aligned}$$

We must now prove that Γ is a co-meagre subset of $\mathcal{P}(K)$. It clearly suffices to construct a set $M \subseteq \mathcal{P}(K)$ satisfying the following conditions:

(1) $M \subseteq \Gamma$;
(2) M is dense in $\mathcal{P}(K)$;
(3) M is $\mathcal{G}_\delta$ in $\mathcal{P}(K)$.

For a positive integer n, write

$$G_n = \bigcup_{\substack{\mu\in\mathcal{P}(K)\\ |\operatorname{supp}\mu|<\infty}} B\left(\mu, \frac{1}{3^{|\operatorname{supp}\mu|+n}}\right),$$

and define

$$M = \bigcap_n G_n.$$

Below we show that the set M has the above three properties (1)–(3) (Propositions 3.1–3.3).

Proposition 3.1. *$M \subseteq \Gamma$.*

Proof. Let $\mu \in M = \bigcap_n G_n$. To prove that $\mu \in \Gamma$, we first make a few observations. Since $\mu \in \bigcap_n G_n$, for each positive integer n we can find a measure $\mu_n \in \mathcal{P}(K)$ with $|\operatorname{supp}\mu_n| < \infty$ such that

$$L(\mu,\mu_n) \le \frac{1}{3^{|\operatorname{supp}\mu_n|+n}}.$$

Now put

$$E_n = \bigcup_{x\in\operatorname{supp}\mu_n}\left(B\left(x, \frac{1}{2^{|\operatorname{supp}\mu_n|+n}}\right)\cap K\right), \qquad F_m = \bigcap_{n\ge m} E_n.$$

Next, we prove the following two claims.

CLAIM 1. *We have* $\mu(E_n) \ge 1 - (2/3)^n$.

Proof of Claim 1. Define $f_n : K \to \mathbb{R}$ by

$$f_n(x) = \max\left(\frac{1}{2^{|\operatorname{supp}\mu_n|+n}} - \operatorname{dist}(x, \operatorname{supp}\mu_n), 0\right).$$

It is clear that $f_n \in \operatorname{Lip}(K)$, and we therefore conclude that

$$\begin{aligned} L(\mu,\mu_n) &\ge \int f_n\,d\mu_n - \int f_n\,d\mu \ge \int_{\operatorname{supp}\mu_n} \frac{1}{2^{|\operatorname{supp}\mu_n|+n}}\,d\mu_n - \int_{E_n} \frac{1}{2^{|\operatorname{supp}\mu_n|+n}}\,d\mu \\ &= \frac{1}{2^{|\operatorname{supp}\mu_n|+n}} - \frac{1}{2^{|\operatorname{supp}\mu_n|+n}}\mu(E_n), \end{aligned}$$

and so

$$\begin{aligned} \mu(E_n) &\ge 1 - 2^{|\operatorname{supp}\mu_n|+n} L(\mu,\mu_n) \\ &> 1 - 2^{|\operatorname{supp}\mu_n|+n}\,\frac{1}{3^{|\operatorname{supp}\mu_n|+n}} = 1 - \left(\frac{2}{3}\right)^n. \end{aligned}$$

CLAIM 2. *We have* $\mu(F_m) \to 1$.

Proof of Claim 2. This follows from Claim 1: indeed, $\mu(K\setminus F_m) = \mu(\bigcup_{n\ge m}(K\setminus E_n)) \le \sum_{n\ge m}\mu(K\setminus E_n) \le \sum_{n\ge m}\left(\frac{2}{3}\right)^n \to 0$.

We can now prove that $\mu \in \Gamma$, i.e. we must show that

$$\lim_{\varepsilon\searrow 0}\ \inf_{\mu(E)>1-\varepsilon} \underline{\dim}^q_{\pi,\mathsf{B}}(E) = \underline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) \le \begin{cases} -\overline{D}_\pi q & \text{for } q \le 0, \\ -\underline{D}_\pi q & \text{for } q \ge 0. \end{cases}$$

We therefore fix $\varepsilon > 0$. We must show that there is a subset $E \subseteq K$ with

$$\mu(E) > 1 - \varepsilon \tag{3.1}$$

and

$$\underline{\dim}^q_{\pi,\mathsf{B}}(E) \le \begin{cases} -\overline{D}_\pi\, q & \text{for } q \le 0, \\ -\underline{D}_\pi\, q & \text{for } q \ge 0. \end{cases}$$

It follows from Claim 2 that we can choose a positive integer m_0 such that $\mu(F_{m_0}) > 1-\varepsilon$. Now put $E = F_{m_0}$, so E satisfies (3.1). To show that it also satisfies (3.2), fix $n \ge m_0$ and write $r_n = 1/2^{|\operatorname{supp}\mu_n|+n}$. It is clear that

$F_{m_0} \subseteq E_n = \bigcup_{x \in \operatorname{supp} \mu_n} (B(x, r_n) \cap K)$. This implies that

$$M_\pi^q(F_m; r_n) \le \sum_{x \in \operatorname{supp} \mu_n} \pi(B(x, r_n))^q$$
$$\le \begin{cases} |\operatorname{supp} \mu_n| \big(\inf_{x \in K} \pi(B(x, r_n))\big)^q & \text{for } q \le 0, \\ |\operatorname{supp} \mu_n| \big(\sup_{x \in K} \pi(B(x, r_n))\big)^q & \text{for } q \ge 0, \end{cases}$$

and so

$$\underline{\dim}_{\pi,\mathsf{B}}^q(E) = \liminf_{r \searrow 0} \frac{\log M_\pi^q(F_{m_0}; r)}{-\log r} \le \liminf_n \frac{\log M_\pi^q(F_{m_0}; r_n)}{-\log r_n}$$
$$\le \begin{cases} \liminf_n \dfrac{\log\big(|\operatorname{supp} \mu_n| (\inf_{x \in K} \pi(B(x, r_n)))^q\big)}{-\log r_n} & \text{for } q \le 0, \\ \liminf_n \dfrac{\log\big(|\operatorname{supp} \mu_n| (\sup_{x \in K} \pi(B(x, r_n)))^q\big)}{-\log r_n} & \text{for } q \ge 0, \end{cases}$$
$$= \begin{cases} \liminf_n \left(\dfrac{\log |\operatorname{supp} \mu_n|}{(|\operatorname{supp} \mu_n| + n) \log 2} + q \dfrac{\log \inf_{x \in K} \pi(B(x, r_n))}{-\log r_n} \right) & \text{for } q \le 0, \\ \liminf_n \left(\dfrac{\log |\operatorname{supp} \mu_n|}{(|\operatorname{supp} \mu_n| + n) \log 2} + q \dfrac{\log \sup_{x \in K} \pi(B(x, r_n))}{-\log r_n} \right) & \text{for } q \ge 0, \end{cases}$$
$$\le \begin{cases} -\overline{D}_\pi q & \text{for } q \le 0, \\ -\underline{D}_\pi q & \text{for } q \ge 0. \end{cases}$$

This completes the proof of Proposition 3.1. ∎

Proposition 3.2. *M is dense in $\mathcal{P}(K)$.*

Proof. Since $\mathcal{P}(K)$ is a complete metric space and $M = \bigcap_n G_n$ where each G_n is open, it suffices to show that G_n is dense for all n. We therefore fix a positive integer n. Next, let $\mu \in \mathcal{P}(K)$ and $r > 0$. We must now find a measure $\lambda \in G_n$ such that $L(\mu, \lambda) < r$. Indeed, it is clear that we can find $\lambda \in \mathcal{P}(K)$ such that $|\operatorname{supp} \lambda| < \infty$ and $L(\mu, \lambda) < r$. Also, since $|\operatorname{supp} \lambda| < \infty$, we conclude that $\lambda \in B(\lambda, 1/3^{|\operatorname{supp} \lambda| + n}) \subseteq G_n$. ∎

Proposition 3.3. *M is $\mathcal{G}_\delta$ in $\mathcal{P}(K)$.*

Proof. This is clear. ∎

4. Proof of part (2) of Theorem 1.3. The purpose of this section is to prove

Theorem 1.3(2). *Let π be a Borel probability measure on $\mathbb{R}^d$ with compact support K. If π is doubling, then a typical measure $\mu \in \mathcal{P}(K)$ satisfies*

$$\overline{\tau}_{\pi,\mathsf{loc}}(q) \le \overline{\dim}_{*\pi,\mathsf{B}}^q(\mu) \le \overline{\dim}_{\pi,\mathsf{B}}^{*q}(\mu) \le \overline{\tau}_\pi(q) \quad \textit{for all } q \le 0.$$

If π is doubling and K does not contain isolated points, then a typical measure $\mu \in \mathcal{P}(K)$ satisfies

$$\overline{\tau}_{\pi,\mathsf{loc}}(q) \le \overline{\dim}^{q}_{*\pi,\mathsf{B}}(\mu) \le \overline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu) \le \overline{\tau}_{\pi}(q) \quad \text{for all } q.$$

Before proving this result we state and prove a few auxiliary results. We begin with a definition. Let X be a metric space and let π be a Borel probability measure on X. For a bounded subset E of X, $r > 0$ and a real number q write

$$N^q_\pi(E;r) = \sup_{(B(x_i,r))_i \text{ is a centred packing of } E} \sum_i \pi(B(x_i,r))^q;$$

recall that a family of balls $(B(x_i,r))_i$ is called a *centred packing* of E if $x_i \in E$ for all i and $|x_i - x_j| > 2r$ for all $i \ne j$. We now have the following alternative expressions for $\underline{\dim}^q_{\pi,\mathsf{B}}(E)$ and $\overline{\dim}^q_{\pi,\mathsf{B}}(E)$.

LEMMA 4.1. *Let π be a Borel probability measure on $\mathbb{R}^d$ with compact support K. If π is doubling, then*

$$\underline{\dim}^q_{\pi,\mathsf{B}}(E) = \liminf_{r \searrow 0} \frac{\log N^q_\pi(E;r)}{-\log r},$$
$$\overline{\dim}^q_{\pi,\mathsf{B}}(E) = \limsup_{r \searrow 0} \frac{\log N^q_\pi(E;r)}{-\log r},$$

for all $E \subseteq K$ and all $q \in \mathbb{R}$.

Proof. The proof uses standard arguments and is therefore omitted. ∎

Next, we list a few more auxiliary results.

LEMMA 4.2. *Let π be a Borel probability measure on $\mathbb{R}^d$ with compact support K, and let $E \subseteq K$.*

(1) *If π is doubling, then the map $q \mapsto \overline{\dim}^q_{\pi,\mathsf{B}}(E)$ is convex, and therefore in particular continuous.*
(2) *The map $q \mapsto \overline{\tau}_{\pi,\mathsf{loc}}(q)$ is decreasing.*

Proof. (1) Let $q_1, q_2 \in \mathbb{R}$ and $t_1, t_2 \ge 0$ with $t_1 + t_2 = 1$. Fix $r > 0$. For each centred packing $(B(x_i,r))_i$ of E, it follows from Hölder's inequality that

$$\begin{aligned}\sum_i \pi(B(x_i,r))^{t_1q_1+t_2q_2} &\le \Big(\sum_i \pi(B(x_i,r))^{q_1}\Big)^{t_1}\Big(\sum_i \pi(B(x_i,r))^{q_2}\Big)^{t_2}\\ &\le N^{q_1}_\pi(E;r)^{t_1} N^{q_2}_\pi(E;r)^{t_2}.\end{aligned}$$

Taking supremum over all centred packings $(B(x_i,r))_i$ of E now gives

$$N^{t_1q_1+t_2q_2}_\pi(E;r) \le N^{q_1}_\pi(E;r)^{t_1}\, N^{q_2}_\pi(E;r)^{t_2}.$$

Using this inequality and Lemma 4.1, we conclude that

$$\begin{aligned}
\overline{\dim}_{\pi,\mathsf{B}}^{t_1q_1+t_2q_2}(E) &= \limsup_{r\searrow 0} \frac{\log N_\pi^{t_1q_1+t_2q_2}(E;r)}{-\log r} \\
&\le \limsup_{r\searrow 0} \frac{\log(N_\pi^{q_1}(E;r)^{t_1} N_\pi^{q_2}(E;r)^{t_2})}{-\log r} \\
&= \limsup_{r\searrow 0} \left(t_1 \frac{\log N_\pi^{q_1}(E;r)}{-\log r} + t_2 \frac{\log N_\pi^{q_2}(E;r)}{-\log r} \right) \\
&\le t_1 \limsup_{r\searrow 0} \frac{\log N_\pi^{q_1}(E;r)}{-\log r} + t_2 \limsup_{r\searrow 0} \frac{\log N_\pi^{q_2}(E;r)}{-\log r} \\
&= t_1 \overline{\dim}_{\pi,\mathsf{B}}^{q_1}(E) + t_2 \overline{\dim}_{\pi,\mathsf{B}}^{q_2}(E).
\end{aligned}$$

(2) This follows immediately from the definitions. ∎

PROPOSITION 4.3. *Let π be a Borel probability measure on $\mathbb{R}^d$ with compact support K. If π is doubling, then*

$$\begin{aligned}
&\{\mu \in \mathcal{P}(K) \mid \overline{\tau}_{\pi,\mathsf{loc}}(q) \le \overline{\dim}_{*\pi,\mathsf{B}}^{q}(\mu) \text{ for all } q\} \\
&\qquad\qquad = \bigcap_{q\in\mathbb{Q}} \{\mu \in \mathcal{P}(K) \mid \overline{\tau}_{\pi,\mathsf{loc}}(q) \le \overline{\dim}_{*\pi,\mathsf{B}}^{q}(\mu)\}.
\end{aligned}$$

Proof. It is clear that the left-hand side is contained in the right-hand side, so it suffices to prove the other inclusion. We therefore fix $\mu \in \mathcal{P}(K)$ such that

$$\overline{\tau}_{\pi,\mathsf{loc}}(q) \le \overline{\dim}_{*\pi,\mathsf{B}}^{q}(\mu) \tag{4.1}$$

for all $q \in \mathbb{Q}$. We must now prove (4.1) holds for all $q \in \mathbb{R}$. Fix $q \in \mathbb{R}$ and a set $E \subseteq K$ with $\mu(E) > 0$. Next, choose a sequence $(q_n)_n \subseteq \mathbb{Q}$ such that $q_n \searrow q$. Since $q_n \ge q$ for all n and the function $p \mapsto \overline{\tau}_{\pi,\mathsf{loc}}(p)$ is decreasing (by Lemma 4.2), we conclude that

$$\overline{\tau}_{\pi,\mathsf{loc}}(q) \le \liminf_n \overline{\tau}_{\pi,\mathsf{loc}}(q_n). \tag{4.2}$$

Next, since $q_n \in \mathbb{Q}$, it follows from (4.1) that

$$\overline{\tau}_{\pi,\mathsf{loc}}(q_n) \le \overline{\dim}_{*\pi,\mathsf{B}}^{q_n}(\mu) = \inf_{\mu(F)>0} \overline{\dim}_{\pi,\mathsf{B}}^{q_n}(F) \le \overline{\dim}_{\pi,\mathsf{B}}^{q_n}(E). \tag{4.3}$$

Finally, since $q_n \to q$ and the function $p \mapsto \overline{\dim}_{\pi,\mathsf{B}}^{p}(E)$ is continuous (by Lemma 4.2), we conclude that

$$\liminf_n \overline{\dim}_{\pi,\mathsf{B}}^{q_n}(E) = \overline{\dim}_{\pi,\mathsf{B}}^{q}(E). \tag{4.4}$$

Combining (4.2)–(4.4) gives

$$\overline{\tau}_{\pi,\mathsf{loc}}(q) \le \liminf_n \overline{\tau}_{\pi,\mathsf{loc}}(q_n) \le \liminf_n \overline{\dim}_{\pi,\mathsf{B}}^{q_n}(E) = \overline{\dim}_{\pi,\mathsf{B}}^{q}(E).$$

Taking infimum over all $E\subseteq K$ with $\mu(E)>0$ yields

$$\overline{\tau}_{\pi,\mathsf{loc}}(q)\le\inf_{\mu(E)>0}\overline{\dim}^q_{\pi,\mathsf{B}}(E)=\overline{\dim}^q_{*\pi,\mathsf{B}}(\mu).\ \blacksquare$$

We can now prove part (2) in Theorem 1.3. Let

$$\Gamma=\{\mu\in\mathcal{P}(K)\mid\overline{\tau}_{\pi,\mathsf{loc}}(q)\le\overline{\dim}^q_{*\pi,\mathsf{B}}(\mu)\le\overline{\dim}^{*q}_{\pi,\mathsf{B}}(\mu)\le\overline{\tau}_{\pi}(q)\text{ for all }q\}.$$

To prove that Γ is co-meagre in $\mathcal{P}(K)$, for $q,t,u\in\mathbb{R}$ write

$$\Gamma^q_{t,u}=\{\mu\in\mathcal{P}(K)\mid t\le\inf_{\mu(E)>u}\overline{\dim}^q_{\pi,\mathsf{B}}(E)\},$$

and observe that it follows from Theorem 1.2 and Proposition 4.3 that

$$\begin{aligned}\Gamma&=\{\mu\in\mathcal{P}(K)\mid\overline{\tau}_{\pi,\mathsf{loc}}(q)\le\overline{\dim}^q_{*\pi,\mathsf{B}}(\mu)\text{ for all }q\}\\&=\bigcap_{q\in\mathbb{Q}}\{\mu\in\mathcal{P}(K)\mid\overline{\tau}_{\pi,\mathsf{loc}}(q)\le\inf_{\mu(E)>0}\overline{\dim}^q_{\pi,\mathsf{B}}(E)\}\\&=\bigcap_{q\in\mathbb{Q}}\ \bigcap_{\substack{t,u\in\mathbb{Q}\\ t<\overline{\tau}_{\pi,\mathsf{loc}}(q)\\ u>0}}\{\mu\in\mathcal{P}(K)\mid t\le\inf_{\mu(E)>u}\overline{\dim}^q_{\pi,\mathsf{B}}(E)\}=\bigcap_{q\in\mathbb{Q}}\ \bigcap_{\substack{t,u\in\mathbb{Q}\\ t<\overline{\tau}_{\pi,\mathsf{loc}}(q)\\ u>0}}\Gamma^q_{t,u}.\end{aligned}$$

Hence it suffices to prove that $\Gamma^q_{t,u}$ is co-meagre in $\mathcal{P}(K)$ for all $q,t,u\in\mathbb{Q}$ with $t<\overline{\tau}_{\pi,\mathsf{loc}}(q)$ and $u>0$. We fix such q,t,u and divide the proof into two cases.

Case 1: $\overline{\tau}_{\pi,\mathsf{loc}}(q)=\inf_E\overline{\dim}^q_{\pi,\mathsf{B}}(E)$. Since $t<\overline{\tau}_{\pi,\mathsf{loc}}(q)=\inf_E\overline{\dim}^q_{\pi,\mathsf{B}}(E)$, we conclude that

$$\Gamma^q_{t,u}=\big\{\mu\in\mathcal{P}(K)\mid t\le\inf_{\mu(E)>u}\overline{\dim}^q_{\pi,\mathsf{B}}(E)\big\}=\mathcal{P}(K),$$

and the set $\Gamma^q_{t,u}$ is therefore (trivially) co-meagre in $\mathcal{P}(K)$.

Case 2: $\overline{\tau}_{\pi,\mathsf{loc}}(q)>\inf_E\overline{\dim}^q_{\pi,\mathsf{B}}(E)$. In order to show that $\Gamma^q_{t,u}$ is co-meagre in $\mathcal{P}(K)$, it clearly suffices to construct a set $M^q_{t,u}\subseteq\mathcal{P}(K)$ satisfying the following conditions:

(1) $M^q_{t,u}\subseteq\Gamma^q_{t,u}$;
(2) $M^q_{t,u}$ is dense in $\mathcal{P}(K)$;
(3) $M^q_{t,u}$ is $\mathcal{G}_\delta$ in $\mathcal{P}(K)$.

For all $x\in K$ and all $s>0$, it follows from Lemma 4.1 that

$$\begin{aligned}t<\overline{\tau}_{\pi,\mathsf{loc}}(q)&\le\overline{\dim}^q_{\pi,\mathsf{B},\mathsf{loc}}(x;K)\le\overline{\dim}^q_{\pi,\mathsf{B}}(B(x,s)\cap K)\\&=\limsup_{r\searrow0}\frac{\log N^q_\pi(B(x,s)\cap K;r)}{-\log r},\end{aligned}$$

and we can therefore choose $r_{x,s}<s$ such that

$$t<\frac{\log N^q_\pi(B(x,s)\cap K;r_{x,s})}{-\log r_{x,s}}.$$

This implies that

$$N_\pi^q(B(x,s)\cap K; r_{x,s}) \geq r_{x,s}^{-t}. \tag{4.5}$$

We may also choose a centred packing $(B(y, r_{x,s}))_{y\in\Lambda_{x,s}}$ of $B(x,s)\cap K$ such that

$$\sum_{y\in\Lambda_{x,s}} \pi(B(y,r_{x,s}))^q \geq \frac{1}{2} N_\pi^q(B(x,s)\cap K; r_{x,s}). \tag{4.6}$$

Now define the measure $\mu_{x,s}\in\mathcal{P}(K)$ by

$$\mu_{x,s} = \frac{1}{\sum_{y\in\Lambda_{x,s}} \pi(B(y,r_{x,s}))^q} \sum_{y\in\Lambda_{x,s}} \pi(B(y,r_{x,s}))^q \delta_y.$$

Next, for each $F\subseteq K$ with $|F|<\infty$, we define $\mu_{F,s}\in\mathcal{P}(K)$ and $r_{F,s}>0$ by

$$\mu_{F,s} = \frac{1}{|F|}\sum_{x\in F}\mu_{x,s}, \qquad r_{F,s} = \min_{x\in F} r_{x,s}.$$

Let $(s_n)_n$ be a sequence of real numbers with $s_n>0$ for all n and $s_n\to 0$. Finally, for a positive integer n, we put

$$G_n = \bigcup_{\substack{m\geq n\\ F\subseteq K\\ |F|<\infty}} B(\mu_{F,s_m}, (u/6)r_{F,s_m}),$$

and define the set $M_{t,u}^q\subseteq\mathcal{P}(K)$ by

$$M_{t,u}^q = \bigcap_n G_n.$$

Below we show that $M_{t,u}^q$ has the above three properties (1)–(3) (Propositions 4.4–4.7).

PROPOSITION 4.4. $M_{t,u}^q\subseteq\Gamma_{t,u}^q$.

Proof. Let $\mu\in M_{t,u}^q$. To show that $\mu\in\Gamma_{t,u}^q$, i.e. we must show that

$$t\leq \inf_{\mu(E)>u}\overline{\dim}_{\pi,\mathsf{B}}(E).$$

We therefore fix $E\subseteq K$ with $\mu(E)>u$. We must prove that

$$t\leq\overline{\dim}_{\pi,\mathsf{B}}(E).$$

Since $\mu\in\bigcap_n G_n$, for each positive integer n there is a positive integer m_n and a set $F_n\subseteq K$ with $|F_n|<\infty$ such that

$$L(\mu,\mu_{F_n,s_{m_n}}) \leq \frac{u}{6} r_{F_n,s_{m_n}}.$$

Write $\rho_n = \frac{1}{3} r_{F_n,s_{m_n}}$ and let E_n denote the ρ_n-neighbourhood of E, i.e. $E_n=\{y\in K \mid \mathrm{dist}(x,E)<\rho_n\}$.

CLAIM 1. *We have*

$$\sum_{y\in\Lambda_{x_n,s_{m_n}}\cap E_n}\pi(B(y,r_{x_n,s_{m_n}}))^q\ge\frac{u}{4}r_{x_n,s_{m_n}}^{-t}.$$

Proof of Claim 1. Define $f_n:K\to\mathbb{R}$ by

$$f_n(y)=\max(\rho_n-\operatorname{dist}(y,E_n),0).$$

It is clear that $f_n\in\operatorname{Lip}(K)$, and we therefore conclude that

$$\begin{aligned}L(\mu,\mu_{F_n,s_{m_n}})&\ge\int f_n\,d\mu-\int f_n\,d\mu_{F_n,s_{m_n}}\ge\int_E\rho_n\,d\mu-\int_{E_n}\rho_n\,d\mu_{F_n,s_{m_n}}\\&=\rho_n\mu(E)-\rho_n\mu_{F_n,s_{m_n}}(E_n),\end{aligned}$$

and so

$$\mu_{F_n,s_{m_n}}(E_n)\ge\mu(E)-\frac{1}{\rho_n}L(\mu,\mu_{F_n,s_{m_n}})>u-\frac{1}{\rho_n}\frac{u}{6}r_{F_n,s_{m_n}}=\frac{u}{2}.$$

By the definition of $\mu_{F_n,s_{m_n}}$, this implies that

$$\frac{1}{|F_n|}\sum_{y\in F_n}\mu_{y,s_{m_n}}(E_n)=\mu_{F_n,s_{m_n}}(E_n)>\frac{u}{2},$$

and so there is an element $x_n\in F_n$ such that

$$\mu_{x_n,s_{m_n}}(E_n)>\frac{u}{2}.$$

By the definition of $\mu_{x_n,s_{m_n}}$, this implies that

$$\frac{1}{\sum_{y\in\Lambda_{x_n,s_{m_n}}}\pi(B(y,r_{x_n,s_{m_n}}))^q}\sum_{y\in\Lambda_{x_n,s_{m_n}}\cap E_n}\pi(B(y,r_{x_n,s_{m_n}}))^q\\=\mu_{x_n,s_{m_n}}(E_n)>\frac{u}{2}.$$

We see from this inequality and (4.5) and (4.6) that

$$\begin{aligned}\sum_{y\in\Lambda_{x_n,s_{m_n}}\cap E_n}\pi(B(y,r_{x_n,s_{m_n}}))^q&\ge\frac{u}{2}\sum_{y\in\Lambda_{x_n,s_{m_n}}}\pi(B(y,r_{x_n,s_{m_n}}))^q\\&\ge\frac{u}{2}\frac{1}{2}N_\pi^q(B(x_n,s_{m_n})\cap K;r_{x_n,s_{m_n}})\\&\ge\frac{u}{4}r_{x_n,s_{m_n}}^{-t}.\end{aligned}$$

This completes the proof of Claim 1.

Next, for each $y\in E_n$, we may choose $x_y\in E$ such that

$$|y-x_y|<\rho_n=\frac{1}{3}r_{F_n,s_{m_n}}\le\frac{1}{3}r_{x_n,s_{m_n}}.$$

CLAIM 2. *The family* $(B(x_y,\frac{1}{3}r_{x_n,s_{m_n}}))_{y\in\Lambda_{x_n,s_{m_n}}\cap E_n}$ *is a centred packing of* E.

Proof of Claim 2. It is clear that $y_x \in E$ for all $y \in \Lambda_{x_n,s_{m_n}} \cap E_n$. Next, we show that $|x_{y_1} - x_{y_2}| > \frac{2}{3}r_{x_n,s_{m_n}}$ for all $y_1, y_2 \in \Lambda_{x_n,s_{m_n}} \cap E_n$ with $y_1 \neq y_2$. Indeed, if there are $y_1, y_2 \in \Lambda_{x_n,s_{m_n}} \cap E_n$ with $y_1 \neq y_2$ such that $|x_{y_1} - x_{y_2}| \leq \frac{2}{3}r_{x_n,s_{m_n}}$ then

$$\begin{aligned}|y_1 - y_2| &\leq |y_1 - x_{y_1}| + |x_{y_1} - x_{y_2}| + |x_{y_2} - y_2| \\ &\leq \tfrac{1}{3}r_{x_n,s_{m_n}} + \tfrac{2}{3}r_{x_n,s_{m_n}} + \tfrac{1}{3}r_{x_n,s_{m_n}} = \tfrac{4}{3}r_{x_n,s_{m_n}} \leq 2r_{x_n,s_{m_n}},\end{aligned}$$

contradicting the fact that $(B(y, r_{x_n,s_{m_n}}))_{y\in\Lambda_{x_n,s_{m_n}}}$ is a packing. This completes the proof of Claim 2.

Since π is doubling it is easily seen that there is a constant $c_0 > 0$ such that

$$\frac{\pi(B(y,4r))}{\pi(B(y,r))} \leq c_0 \tag{4.7}$$

for all $y \in \operatorname{supp}\pi = K$ and all $r > 0$. We now deduce from Claim 2 and (4.7) that

$$\begin{aligned}&N^q_{\pi,\mathsf{B}}(E; \tfrac{1}{3}r_{x_n,s_{m_n}}) \\ &\geq \sum_{y\in\Lambda_{x_n,s_{m_n}}\cap E_n} \pi(B(x_y, \tfrac{1}{3}r_{x_n,s_{m_n}}))^q \\ &= \begin{cases} \displaystyle\sum_{y\in\Lambda_{x_n,s_{m_n}}\cap E_n} \pi(B(x_y, \tfrac{1}{3}r_{x_n,s_{m_n}}))^q & \text{for } q \leq 0, \\ \displaystyle\sum_{y\in\Lambda_{x_n,s_{m_n}}\cap E_n} \left(\frac{\pi(B(x_y, \frac{1}{3}r_{x_n,s_{m_n}}))}{\pi(B(x_y, \frac{4}{3}r_{x_n,s_{m_n}}))}\right)^q \pi(B(x_y, \tfrac{4}{3}r_{x_n,s_{m_n}}))^q & \text{for } q > 0; \end{cases} \\ &\geq \begin{cases} \displaystyle\sum_{y\in\Lambda_{x_n,s_{m_n}}\cap E_n} \pi(B(x_y, \tfrac{1}{3}r_{x_n,s_{m_n}}))^q & \text{for } q \leq 0, \\ \displaystyle\frac{1}{c_0^q}\sum_{y\in\Lambda_{x_n,s_{m_n}}\cap E_n} \pi(B(x_y, \tfrac{4}{3}r_{x_n,s_{m_n}}))^q & \text{for } q > 0. \end{cases}\end{aligned} \tag{4.8}$$

However, if $y \in \Lambda_{x_n,s_{m_n}} \cap E_n$, then $|y - x_y| < \frac{1}{3}r_{x_n,s_{m_n}}$, whence $B(y, r_{x_n,s_{m_n}}) \subseteq B(x_y, \frac{4}{3}r_{x_n,s_{m_n}})$ and $B(x_y, \frac{1}{3}r_{x_n,s_{m_n}}) \subseteq B(y, r_{x_n,s_{m_n}})$. It follows from this and (4.8) and Claim 1 that

$$N^q_{\pi,\mathsf{B}}(E; \tfrac{1}{3}r_{x_n,s_{m_n}}) \geq \begin{cases} \displaystyle\sum_{y\in\Lambda_{x_n,s_{m_n}}\cap E_n} \pi(B(y, r_{x_n,s_{m_n}}))^q & \text{for } q \leq 0, \\ \displaystyle\frac{1}{c_0^q}\sum_{y\in\Lambda_{x_n,s_{m_n}}\cap E_n} \pi(B(y, r_{x_n,s_{m_n}}))^q & \text{for } q > 0; \end{cases}$$

$$\geq c_1 \sum_{y\in \varLambda_{x_n,s_{m_n}}\cap E_n} \pi(B(y,r_{x_n,s_{m_n}}))^q$$
$$\geq c_1 \frac{u}{4} r_{x_n,s_{m_n}}^{-t} = c_2(\tfrac{1}{3} r_{x_n,s_{m_n}})^{-t},$$

where $c_1 = \min(1, 1/c_0^q)$ and $c_2 = c_1 u 3^{-t}/4$. We immediately conclude from this and Lemma 4.1 that

$$\begin{aligned}\overline{\dim}_{\pi,\mathsf{B}}(E) &= \limsup_{r\searrow 0} \frac{\log N_\pi^q(E;r)}{-\log r}\\ &\geq \limsup_n \frac{\log N_{\pi,\mathsf{B}}^q(E;\frac{1}{3}r_{x_n,s_{m_n}})}{-\log \frac{1}{3} r_{x_n,s_{m_n}}}\\ &\geq \limsup_n \frac{\log c_2(\frac{1}{3}r_{x_n,s_{m_n}})^{-t}}{-\log \frac{1}{3} r_{x_n,s_{m_n}}} = t.\end{aligned}$$

This completes the proof of Proposition 4.4. ∎

PROPOSITION 4.5. *If K does not have any isolated points and $q \in \mathbb{R}$, then $M_{t,u}^q$ is dense in $\mathcal{P}(K)$.*

Proof. For $F \subseteq K$ with $|F| < \infty$ write

$$\mu_F = \frac{1}{|F|}\sum_{x\in F}\delta_x.$$

We first prove that if $s > 0$, then

$$L(\mu_F, \mu_{F,s}) \leq s. \tag{4.9}$$

Indeed, writing $w_{x,y} = \pi(B(y, r_{x,s}))^q$ for $x \in F$ and $y \in \varLambda_{x,s}$, we have

$$L(\mu_F,\mu_{F,s}) = \sup_{f\in \mathrm{Lip}(K)} \left|\int f\,d\mu_F - \int f\,d\mu_{F,s}\right| \tag{4.10}$$
$$= \sup_{f\in \mathrm{Lip}(K)} \left|\frac{1}{|F|}\sum_{x\in F} f(x) - \frac{1}{|F|}\sum_{x\in F}\frac{1}{\sum_{y\in \varLambda_{x,s}} w_{x,y}}\sum_{y\in\varLambda_{x,s}} w_{x,y} f(y)\right|$$
$$\leq \sup_{f\in \mathrm{Lip}(K)} \frac{1}{|F|}\sum_{x\in F}\left| f(x) - \frac{1}{\sum_{y\in \varLambda_{x,s}} w_{x,y}}\sum_{y\in\varLambda_{x,s}} w_{x,y} f(y)\right|$$
$$= \sup_{f\in \mathrm{Lip}(K)} \frac{1}{|F|}\sum_{x\in F}\left| \frac{1}{\sum_{y\in \varLambda_{x,s}} w_{x,y}}\sum_{y\in\varLambda_{x,s}} w_{x,y} f(x) - \frac{1}{\sum_{y\in \varLambda_{x,s}} w_{x,y}}\sum_{y\in\varLambda_{x,s}} w_{x,y} f(y)\right|$$
$$\leq \sup_{f\in \mathrm{Lip}(K)} \frac{1}{|F|}\sum_{x\in F} \frac{1}{\sum_{y\in \varLambda_{x,s}} w_{x,y}}\sum_{y\in\varLambda_{x,s}} w_{x,y} |f(x) - f(y)|.$$

However, if $f \in \mathrm{Lip}(K)$ and $x \in F$ and $y \in \Lambda_{x,s}$, then $|f(x) - f(y)| \leq |x - y| \leq s$. It follows from this and (4.10) that

$$L(\mu_F, \mu_{F,s}) \leq \sup_{f \in \mathrm{Lip}(K)} \frac{1}{|F|} \sum_{x \in F} \frac{1}{\sum_{y \in \Lambda_{x,s}} w_{x,y}} \sum_{y \in \Lambda_{x,s}} w_{x,y} s = s.$$

This completes the proof of (4.9).

We now turn to the proof of Proposition 4.5. Since $\mathcal{P}(K)$ is a complete metric space and $M^q_{t,u} = \bigcap_n G_n$ where each G_n is open, it suffices to show that G_n is dense for all n. We therefore fix a positive integer n. Next, let $\mu \in \mathcal{P}(K)$ and $r > 0$. We must find a measure $\lambda \in G_n$ such that $L(\mu, \lambda) < r$. Since K does not contain isolated points we may choose a set $F \subseteq K$ with $|F| < \infty$ such that $L(\mu, \mu_F) < r/2$. Next, since $s_m \to 0$, we may choose a positive integer m_0 with $m_0 \geq n$ such that $s_{m_0} < r/2$. Now put $\lambda = \mu_{F,s_{m_0}}$. Then clearly, by (4.9),

$$L(\mu, \lambda) \leq L(\mu, \mu_F) + L(\mu_F, \mu_{F,s_{m_0}}) < \frac{1}{2} + s_{m_0} \leq \frac{r}{2} + \frac{r}{2} = r$$

and $\lambda = \mu_{F,s_{m_0}} \in B(\mu_{F,s_{m_0}}, (u/6) r_{F,s_{m_0}}) \subseteq G_n$. ∎

Proposition 4.6. *If $q \leq 0$, then $M^q_{t,u}$ is dense in $\mathcal{P}(K)$.*

Proof. By Proposition 4.5, it suffices to prove that K does not have isolated points. Indeed, if x_0 is an isolated point of K, it is not difficult to see that $\overline{\dim}^q_{\pi,\mathsf{B},\mathsf{loc}}(x_0; K) = 0$, and so

$$\overline{\tau}_{\pi,\mathsf{loc}}(q) \leq 0. \tag{4.11}$$

Furthermore, since $q \leq 0$, it is also not difficult to see that $\overline{\dim}^q_{\pi,\mathsf{B}}(E) \geq 0$ for all $E \subseteq K$, whence

$$0 \leq \inf_E \overline{\dim}^q_{\pi,\mathsf{B}}(E). \tag{4.12}$$

Combining (4.11) and (4.12) shows that

$$\overline{\tau}_{\pi,\mathsf{loc}}(q) \leq \inf_E \overline{\dim}^q_{\pi,\mathsf{B}}(E), \tag{4.13}$$

contrary to the assumption of Case 2. ∎

Proposition 4.7. *$M^q_{t,u}$ is $\mathcal{G}_\delta$ in $\mathcal{P}(K)$.*

Proof. This is clear. ∎

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