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Iterations of the Frobenius–Perron operator for parabolic random maps

by

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Abstract. We describe totally dissipative parabolic extensions of the one-sided Bernoulli shift. For the fractional linear case we obtain conservative and totally dissipative families of extensions. Here, the property of conservativity seems to be extremely unstable.

0. Introduction. Let σ be the one-sided (p,q)-Bernoulli shift on the space $\Omega=\{0,1\}^{\mathbb{N}}$, $\mathbb{N}=\{0,1,2,\dots\}$, with the (p,q)-measure μ_p on $(\Omega,\mathcal{B})$, where $\mathcal{B}$ is the Borel product σ-algebra and (p,q) is a probability vector. Let us consider two transformations T_0, T_1 of the interval $[0,1]$ onto itself such that $T_i\in C^2[0,1]$, $T_i'>0$, $T_i(0)=0$, $T_i(1)=1$ for $i=0,1$ and $T_0\ge I$, $T_1\le I$ where $I(x)=x$ for $x\in[0,1]$. Let S_i denote the inverse of T_i, $i=0,1$. We define the transformation

$$T(\omega,x)=(\sigma(\omega),S_{\omega(0)}(x)). \tag{1}$$

This transformation is a realization of the random map $T(x)=S_0(x)$ with probability p and $T(x)=S_1(x)$ with probability q, or a realization of the random walk on the unit interval. Let Λ denote the Lebesgue measure on $[0,1]$. It will cause no confusion to use the same letter for the Lebesgue measure on $\mathbb{R}$. Moreover, let us denote by P the restriction to $L^1(\Lambda)$ of the Frobenius–Perron operator with respect to $\mu_p\times\Lambda$. By using two different methods we investigate iterations of P. The first has been used for transformations T such that

$$T_i=(1-\varepsilon_i)x+\varepsilon_i g(x),\qquad i=0,1, \tag{2}$$

$g\in C^2[0,1]$, $g(0)=0$, $g(1)=1$, $(1-\sup g')^{-1}<\varepsilon_0,\varepsilon_1<(1-\inf g')^{-1}$. We additionally assume that there exists exactly one point x_0 for which

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$g'(x_0) = 1$ and $g'(x) < 1$ for $x < x_0$ or $g'(x) > 1$ for $x < x_0$. By the modification of P to P_h which relies on replacing Λ by the equivalent measure with density h we show that $P_h^n(1) \to 0$, which yields the total dissipativity of T for some parameters ε_i, $i = 0, 1$, and p for $g(x) = x^2$. We also observe that $(T, \mu_p \times \Lambda)$ for T given by (2) is either conservative and ergodic, or totally dissipative. The conservativity of different kinds of random maps is studied in [D-K-S]. In the second method we apply the isomorphism of fractional linear maps with translations of the real line $\mathbb{R}$. By using the results about conservativity of $\mathbb{R}$-extensions, we obtain either conservative and ergodic, or totally dissipative systems which have an equivalent σ-finite invariant measure. The conservative transformations appear to be isolated. Moreover, by repeating the approximation argument from [K3] we extend the area of dissipativity of T given by (2) for $g(x) = x^2$. The observation that fractional linear extensions are isomorphic to random walks on $\mathbb{R}$ allows us to improve the description of their ergodic properties included in [K3]. We finish our paper by completing the information about the example of [K2], i.e. the transformation T given by $T_0 = \frac{3}{2}x - \frac{1}{2}x^2$, $T_1 = x^2$ and $p \in (0, 1)$.

1. Total dissipativity. We start with a slightly more general situation, i.e. σ is the one-sided Markov shift on the space $\Omega = \{0, \ldots, s-1\}^N$, $s \geq 2$, with $(\Pi, \vec{p})$-measure $\mu_{\vec{p}}$. Here $\vec{p} = (p_0, \ldots, p_{s-1})$ is a probability vector, and $\Pi = (p_{ij})_{s\times s}$ is a stochastic matrix such that $\vec{p}\Pi = \vec{p}$. Let $\{S_i\}_{i=0}^{s-1}$ be a family of positively and negatively nonsingular transformations of a probability space $(Y, \mathcal{C}, m)$, i.e. $m(B) = 0 \Rightarrow m(S_i^{-1}(B)) = m(S_i(B)) = 0$ for $i = 0, \ldots, s-1$. This definition slightly differs from that of the two-sided nonsingularity [A]. We introduce the transformation

$$T(\omega, x) = (\sigma(\omega), S_{\omega(0)}(x)).$$

Let us denote by $\mathcal{C}(T)$ the conservative part of T and by $\mathcal{D}(T)$ the dissipative part. Moreover, let $\alpha = \{A_i : i = 0, \ldots, s-1\}$ where $A_i = \{\omega : \omega(0) = i\}$.

Theorem 1. *If $E \subset \mathcal{C}(T)$, $\mu_{\vec{p}} \times m(E) > 0$ and $T(E) \subset E$ then*

$$E = \bigcup_{i=0}^{s-1} A_i \times E_i \quad \text{for some } E_i \in \mathcal{C}.$$

Proof. Let $E_x = \{\omega : (\omega, x) \in E\}$. Suppose, on the contrary, that there exist $\epsilon > 0$ and i such that

$$B = \{x : 0 < \mu_{\vec{p}}(E_x \cap A_i) \leq (1-\epsilon)\mu_{\vec{p}}(A_i)\}$$

has positive measure m. Let $(\omega, x) \in E \cap A_i \times B$, where ω is a density point for E_x and $T^n(\omega, x)$ returns infinitely many times to $E \cap A_i \times B$ (because

$E \subset \mathcal{C}(T)$). By the choice of ω, there exists n_0 such that for $n \geq n_0$,

$$\mu_{\vec{p}}(A_n(\omega) \cap E_x) \geq (1 - \epsilon/2)\mu_{\vec{p}}(A_n(\omega))$$

where $\omega \in A_n(\omega) \in \bigvee_{i=0}^{n} \sigma^{-i}\alpha$. Let n_1 satisfy $n_1 > n_0$ and $T^{n_1}(\omega, x) \in E \cap A_i \times B$. Then

$$T^{n_1}(A_{n_1}(\omega) \cap E_x \times \{x\}) = \sigma^{n_1}(A_{n_1}(\omega) \cap E_x) \times \{S_\omega^{n_1}(x)\}$$

where $S_\omega^{n_1}(x) = S_{\omega(n_1-1)} \circ \dots \circ S_{\omega(0)}(x)$. Hence

$$E_{S_\omega^{n_1}(x)} \cap A_i \supset \sigma^{n_1}(A_{n_1}(\omega) \cap E_x).$$

Therefore

$$\begin{aligned}\mu_{\vec{p}}(E_{S_\omega^{n_1}(x)} \cap A_i) &\geq J_{\sigma^{n_1}}(\omega)\mu_{\vec{p}}(A_{n_1}(\omega) \cap E_x) \\ &\geq (1 - \epsilon/2)J_{\sigma^{n_1}}(\omega)\mu_{\vec{p}}(A_{n_1}(\omega)) = (1 - \epsilon/2)\mu_{\vec{p}}(A_i).\end{aligned}$$

Here J_σ denotes the Jacobian of σ. This contradicts our assumption. ■

COROLLARY 1. $\mathcal{C}(T) = \bigcup_{i=0}^{s-1} A_i \times B_i$ *and* $\mathcal{D}(T) = \bigcup_{i=0}^{s-1} A_i \times C_i$.

THEOREM 2. *If T is given by* (2) *then* $(T, \mu_p \times \Lambda)$ *is either conservative and ergodic*, *or totally dissipative.*

Proof. Let $\mu_p \times \Lambda(\mathcal{C}(T)) > 0$. Hence by Corollary 1, $\mathcal{C}(T) = \Omega \times B$. Moreover, $T(\Omega \times B) = \Omega \times B$, which implies $\Lambda(B \div (S_0B \cup S_1B)) = 0$. Therefore, $T(\Omega \times B) \subset \Omega \times B$ with respect to the measure $\mu_r \times \Lambda$ where $r\epsilon_0 + (1 - r)\epsilon_1 = 0$. The measure $\mu_r \times \Lambda$ is T-invariant and ergodic (for the proof see [K1]). Thus, $\Lambda(B) = 1$. We apply similar arguments to get the ergodicity of T. ■

Let $h : (0, 1) \to \mathbb{R}^+$ be a function from $C^1(0, 1)$ and ν_h be a measure on $[0, 1]$ such that $d\nu_h/d\Lambda = h$. For our applications we use $h(x) = x^{-\alpha}$ or $h(x) = (1 - x)^{-\alpha}$ for $\alpha \geq 1$. Let T be the transformation given by (1) and let $\mathcal{P}$ ($\mathcal{P}_h$ respectively) be the restriction to $L^1(\Lambda)$ ($L^1(\nu_h)$ respectively) of the Frobenius–Perron operator with respect to $\mu_p \times \Lambda$ ($\mu_p \times \nu_h$ respectively). The following relation holds between these operators:

$$\mathcal{P}_h f = h^{-1}\mathcal{P}(hf) \quad \text{for } f \in L^1(\nu_h).$$

We define

$$\begin{aligned}h^{-1}h(T_i)(0) &= \lim_{x \to 0^+} h^{-1}(x)h(T_i(x)), \\ h^{-1}h(T_i)(1) &= \lim_{x \to 1^-} h^{-1}(x)h(T_i(x)) \quad \text{for } i = 0, 1.\end{aligned}$$

Here we assume the existence of the above limits. The explicit form of $\mathcal{P}_h$ is

$$\begin{aligned}\mathcal{P}_h f(x) = {} & ph(T_0(x))h^{-1}(x)T_0'(x)f(T_0(x)) \\ & + (1 - p)h(T_1(x))h^{-1}(x)T_1'(x)f(T_1(x)),\end{aligned}$$

for $x \in [0, 1]$. Our aim is to obtain some conditions for dissipativity of T.

THEOREM 3. *Let T be given by* (1). *If for all $x \in [0,1]$,*

$$\gamma(x) = ph^{-1}(x)h(T_0(x))T_0'(x) + (1-p)h^{-1}(x)h(T_1(x))T_1'(x) \le \beta < 1$$

then T is totally dissipative.

Proof. By assumption, $P_h(1) \le \beta$. Therefore $P_h^n(1) \le \beta^n$. Let $f \in L^1(\Lambda)$ be such that $0 < f \le h$. Then

$$P^n(f) = P^n\left(h\,\frac{f}{h}\right) = hP_h^n\left(\frac{f}{h}\right) \le hP_h^n(1) \le \beta^n h.$$

Therefore, the measure of $\{x : \sum_{n=0}^{\infty} P^n f < \infty\}$ is equal to one. This proves the theorem. ∎

2. An application. Let us consider the transformations

$$T_0(x) = (1+\epsilon_0)x - \epsilon_0 x^2, \qquad T_1(x) = (1-\epsilon_1)x + \epsilon_1 x^2, \tag{3}$$

for $\epsilon_0, \epsilon_1 \in [0,1]$. For $h(x) = x^{-2}$ we determine p which satisfies

$$\gamma(x) = p\,\frac{1+\epsilon_0-2\epsilon_0 x}{(1+\epsilon_0-\epsilon_0 x)^2} + (1-p)\,\frac{1-\epsilon_1+2\epsilon_1 x}{(1-\epsilon_1+\epsilon_1 x)^2} < 1$$

for every $x \in [0,1]$. For this purpose we compute

$$\gamma'(x) = -2p\epsilon_0^2\,\frac{x}{(1+\epsilon_0-\epsilon_0 x)^3} - 2(1-p)\epsilon_1^2\,\frac{x}{(1-\epsilon_1+\epsilon_1 x)^3} \le 0.$$

Therefore, $\gamma(x) \le \gamma(0)$ for $x \in [0,1]$. Hence for $\epsilon_1 \ne 1$ we get

$$\beta = \gamma(0) < 1 \;\Leftrightarrow\; \frac{p}{1+\epsilon_0} + \frac{1-p}{1-\epsilon_1} < 1 \;\Leftrightarrow\; p > \frac{1+\epsilon_0}{\epsilon_1+\epsilon_0}\,\epsilon_1.$$

The same reasoning applies to the case $h(x) = (1-x)^{-2}$. For $\epsilon_0 \ne 1$ we get

$$\beta = \gamma(1) < 1 \;\Leftrightarrow\; \frac{p}{1-\epsilon_0} + \frac{1-p}{1+\epsilon_1} < 1 \;\Leftrightarrow\; p < \frac{1-\epsilon_0}{\epsilon_1+\epsilon_0}\,\epsilon_1.$$

Therefore as a consequence of Theorem 3 we get

COROLLARY 2. *If T is given by* (3) *then $(T, \mu_p \times \Lambda)$ is totally dissipative whenever*

$$p < \frac{1-\epsilon_0}{\epsilon_1+\epsilon_0}\,\epsilon_1 \quad \textit{or} \quad p > \frac{1+\epsilon_0}{\epsilon_1+\epsilon_0}\,\epsilon_1.$$

We can improve on the above by using $h(x) = x^{-(1+\alpha)}$ or $h(x) = (1-x)^{-(1+\alpha)}$ for $\alpha \in (0,1)$.

EXAMPLE. For

$$h(x) = x^{-1.4}, \qquad p \ge 0.77, \qquad \epsilon_0 = 0.9, \qquad \epsilon_1 = 0.7$$

we get $\gamma(x) < 1$ for every $x \in [0,1]$. Similarly for

$$h(x) = (1-x)^{-1.4}, \qquad p \le 0.4, \qquad \epsilon_0 = 0.5, \qquad \epsilon_1 = 1$$

we get $\gamma(x) < 1$ for each $x \in [0,1]$.

3. Fractional linear maps and $\mathbb{R}$-extensions. Let T be given by (1) where

$$T_0 = T_{\lambda_0} = \frac{x}{\lambda_0 x + 1 - \lambda_0}, \qquad \lambda_0 \in (0,1),$$
$$T_1 = T_{\lambda_1} = \frac{x}{\lambda_1 x + 1 - \lambda_1}, \qquad \lambda_1 < 0.$$

REMARK 1. T has an equivalent invariant σ-finite measure for every $p \in (0,1)$.

Proof. It is easy to see that the measure $\mu_p \times \nu$ where $d\nu/d\Lambda = 1/x(1-x)$ is T-invariant. ∎

Let us observe that the system $(\Omega \times [0,1], \mu_p \times \nu, T)$, where T and ν are considered above, is isomorphic to $(\Omega \times \mathbb{R}, \mu_p \times \Lambda, \hat{T})$ where

$$\hat{T}(\omega, u) = (\sigma(\omega), u + a_{\omega(0)}).$$

Here $a_0 = \ln(1-\lambda_0)$ and $a_1 = \ln(1-\lambda_1)$. The isomorphism is given by the map

$$\Omega \times \mathbb{R} \ni (\omega, u) \mapsto \left(\omega, \frac{e^u}{1+e^u}\right) \in \Omega \times [0,1].$$

Now we are in a position to use Corollary 8.15 of [A].

THEOREM 4. *T is conservative if and only if*

$$p = \frac{\ln(1-\lambda_1)}{\ln\left(\frac{1-\lambda_1}{1-\lambda_0}\right)}.$$

For other p, T is totally dissipative.

The second observation relies on the representation of $\hat{T}$ as a random walk on $\mathbb{R}$. Namely, $(\Omega \times \mathbb{R}, \mu_p \times \Lambda, \hat{T})$ is isomorphic to $(\mathbb{R}^N, \mu, \sigma)$ via the map

$$\Phi(\omega, u) = (u, u + a_{\omega(0)}, u + a_{\omega(0)} + a_{\omega(1)}, \ldots) \in \mathbb{R}^N.$$

Here σ is the one-sided shift and μ is determined by the "jump probability"

$$P = p\delta_{\{\ln(1-\lambda_0)\}} + (1-p)\delta_{\{\ln(1-\lambda_1)\}}$$

and Λ.

THEOREM 5. *$(\Omega \times [0,1], \mu_p \times \nu, T)$ is*

(i) *ergodic if and only if $\frac{\ln(1-\lambda_1)}{\ln(1-\lambda_0)}$ is irrational*,
(ii) *not exact.*

Proof. By results of [D-L] the random walk $(\mathbb{R}^N, \mu, \sigma)$ is ergodic if and only if the set

$$\{n \ln(1-\lambda_0) + m \ln(1-\lambda_1) : m, n \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$. But the above is equivalent to

$$\frac{\ln(1-\lambda_1)}{\ln(1-\lambda_0)} \notin \mathbb{Q}.$$

Moreover, $(\mathbb{R}^N, \mu, \sigma)$ is exact if and only if

$$\left\{ n \ln\left(\frac{1-\lambda_1}{1-\lambda_0}\right) : n \in \mathbb{Z} \right\}$$

is dense in $\mathbb{R}$. But this is impossible. ■

The isomorphism of T and $\hat{T}$ carries new information about iterations of T. Namely,

$$\hat{T}^n(\omega, u) = (\sigma^n(\omega), u + (n - S_n(\omega))a_0 + S_n(\omega)a_1)$$

where

$$S_n(\omega) = \sum_{k=0}^{n-1} \omega(k).$$

Therefore,

$$u + (n - S_n(\omega))a_0 + S_n(\omega)a_1 \to \infty$$

and simultaneously

$$1_{\Omega\times[0,b]}(T^n(\omega, x)) \to 0 \quad \text{for a.e. } \omega$$

when

$$p < \frac{\ln(1-\lambda_1)}{\ln\left(\frac{1-\lambda_1}{1-\lambda_0}\right)}.$$

Moreover,

$$u + (n - S_n(\omega))a_0 + S_n(\omega)a_1 \to -\infty$$

and at the same time

$$1_{\Omega\times[b,1]}(T^n(\omega, x)) \to 0 \quad \text{for a.e. } \omega$$

if

$$\frac{\ln(1-\lambda_1)}{\ln\left(\frac{1-\lambda_1}{1-\lambda_0}\right)} < p.$$

Here $b \in (0,1)$. We will apply the last observations to parabolic extensions T given by (3). It is easy to see that

$$\operatorname{sgn}(T_0(x) - T_{\lambda_0}(x)) = \operatorname{sgn}(\epsilon_0\lambda_0 x - \epsilon_0\lambda_0 + \epsilon_0 - \lambda_0)$$

and

$$\operatorname{sgn}(T_1(x) - T_{\lambda_1}(x)) = \operatorname{sgn}(-\epsilon_1\lambda_1 x + \epsilon_1\lambda_1 - \epsilon_1 - \lambda_1)$$

for $x \in (0,1)$. Therefore, we get

$$S_0(x) \ge T_{\lambda_0}^{-1}(x) \Leftrightarrow \epsilon_0 \le \lambda_0, \quad S_1(x) \ge T_{\lambda_1}^{-1}(x) \Leftrightarrow -\epsilon_1 \le \lambda_1$$

and

$$S_0(x) \le T_{\lambda_0}^{-1}(x) \;\Leftrightarrow\; \lambda_0 \le \frac{\epsilon_0}{1+\epsilon_0},$$
$$S_1(x) \le T_{\lambda_1}^{-1}(x) \;\Leftrightarrow\; \lambda_1 \le -\frac{\epsilon_1}{1-\epsilon_1}, \qquad \epsilon_1 < 1.$$

Therefore, for $\epsilon_1 < 1$ and

$$p > \frac{\ln(1-\epsilon_1)}{\ln\left(\frac{1-\epsilon_1}{1+\epsilon_0}\right)} = \min\left\{\frac{\ln(1-\lambda_1)}{\ln\left(\frac{1-\lambda_1}{1-\lambda_0}\right)} : (\lambda_0,\lambda_1) \in \left(0, \frac{\epsilon_0}{1+\epsilon_0}\right] \times \left(-\infty, -\frac{\epsilon_1}{1-\epsilon_1}\right]\right\}$$

we obtain

$$\lim_{n\to\infty} 1_{\Omega\times[b,1]}(T^n(\omega,x)) = 0 \qquad \text{for a.e. } \omega.$$

Similarly, for

$$p < \frac{\ln(1+\epsilon_1)}{\ln\left(\frac{1+\epsilon_1}{1-\epsilon_0}\right)} = \max\left\{\frac{\ln(1-\lambda_1)}{\ln\left(\frac{1-\lambda_1}{1-\lambda_0}\right)} : (\lambda_0,\lambda_1) \in [\epsilon_0, 1] \times [-\epsilon_1, 0]\right\}$$

we have

$$\lim_{n\to\infty} 1_{\Omega\times[0,b]}(T^n(\omega,x)) = 0 \qquad \text{for a.e. } \omega.$$

As a consequence we get

THEOREM 6. *If T is given by* (3) *then $(T, \mu_p \times \Lambda)$ is totally dissipative and the set of product measures in M_p is* $\mathrm{conv}\{\mu_p \times \delta_{\{0\}}, \mu_p \times \delta_{\{1\}}\}$ *whenever*

$$p < \frac{\ln(1+\epsilon_1)}{\ln\left(\frac{1+\epsilon_1}{1-\epsilon_0}\right)} \quad \textit{or} \quad p > \frac{\ln(1-\epsilon_1)}{\ln\left(\frac{1-\epsilon_1}{1+\epsilon_0}\right)}.$$

Here M_p denotes the set of T-invariant probability measures m such that $m|\mathcal{B} \times \{[0,1]\} = \mu_p$.

Proof. Let us assume the first inequality holds. Then

$$\left\{(\omega,x) : \sum_{n=1}^{\infty} 1_{\Omega\times[0,b]}(T^n(\omega,x)) < \infty\right\}$$

has measure one. Therefore, by the Halmos recurrence theorem [A], $[0,b] \subset \mathcal{D}_T$ for every $0 < b < 1$. Hence $\mathcal{D}_T = [0,1]$. Moreover,

$$A^n I(x) = \int_0^x P^n 1\, d\Lambda = \int_\Omega \int_0^1 P^n 1 \cdot 1_{\Omega\times[0,x]}\, d\Lambda\, d\mu_p$$
$$= \int_\Omega \int_0^1 1_{\Omega\times[0,x]}(T^n(\omega,t))\, d\Lambda\, d\mu_p \to 0.$$

Thus by Theorem 3 of [K3] we get the desired conclusion. The proof for the second inequality is similar. ■

4. The example. Let us consider the example from [K2]:

$$T_0(x)=\frac{3}{2}\,x-\frac{1}{2}\,x^2,\qquad T_1(x)=x^2,\qquad \text{i.e.}\quad \epsilon_0=\frac{1}{2},\qquad \epsilon_1=1,$$

and T given as in (1).

THEOREM 7. $M_p=\operatorname{conv}\{\mu_p\times\delta_{\{0\}},\mu_p\times\delta_{\{1\}},\mu_p\times\Lambda\}$ *for* $p=2/3$. *The set of product measures in* M_p *is* $\operatorname{conv}\{\mu_p\times\delta_{\{0\}},\mu_p\times\delta_{\{1\}}\}$ *for* $p\in(0,1/2]$. *Moreover*, $(T,\mu_p\times\Lambda)$ *is totally dissipative for* $p<1/2$.

REMARK 2. We only need to prove the case $p=1/2$. The other conclusions result from Theorem 6 and Theorem 3 of [K3] respectively.

We will need the considerations below. Define the operator $\mathcal{A}$ on $\mathcal{D}$ as follows:

$$\mathcal{A}F(x)=\frac{1}{2}\,F(T_0(x))+\frac{1}{2}\,F(T_1(x))\qquad\text{for } F\in\mathcal{D}.$$

Let ν_F denote the measure determined by F.

FACT ([K2]). *The measure* $\mu_{1/2}\times\nu_F$ *is* T*-invariant if and only if* $\mathcal{A}F$ $=F$.

LEMMA 1.

$$\lim_{n\to\infty}\mathcal{A}^nI(x)\le\frac{1}{2}\qquad\textit{for } x\in[0,1).$$

Proof. Since $T_1(T_0(x))\le I(x)$ for $x\in[0,1]$ we have

$$\begin{aligned}\mathcal{A}^nI(x)&\le\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}x^{2^{2k-n}}\\&\le\frac{1}{2^n}\sum_{k=0}^{E(n/2)}\binom{n}{k}x^{2^{2k-n}}+\frac{1}{2^n}\sum_{k=E(n/2)}^{E(n/2)+s}\binom{n}{k}\\&\quad+\Bigg[\frac{1}{2^n}\sum_{k=E(n/2)+s+1}^{n}\binom{n}{k}\Bigg]x^{2^s}\\&\le\frac{1}{2}+\frac{1}{2^n}\sum_{k=E(n/2)}^{E(n/2)+s}\binom{n}{k}+\frac{1}{2}\,x^{2^s},\end{aligned}$$

where $E(x)$ denotes the integer part of x. By the existence of $\lim_{n\to\infty}\mathcal{A}^nI$ (see Lemma 3 of [K2]) we get

$$\lim_{n\to\infty}\mathcal{A}^nI(x)\le\frac{1}{2}+\frac{1}{2}\,x^{2^s}\qquad\text{for any } s\ge1.$$

Therefore $\lim_{n\to\infty}\mathcal{A}^nI(x)\le1/2$ for $x\in[0,1)$. ■

LEMMA 2. *Let $\omega(x)$ be a polynomial such that $0 \le \omega(x) \le \delta$ for $x \in [0,1]$, $\omega(0) = 0$ and $\delta < 1$. Then* $\limsup_{n\to\infty} \mathcal{A}^n\omega(x) \le 1/2$ *for $x \in [0,1)$.*

Proof. We first observe that
$$\limsup_{n\to\infty} \mathcal{A}^n I^{1/k}(x) \le \frac{1}{2} \quad \text{for } x \in [0,1) \text{ and } k = 1,2,\ldots.$$
Let
$$d_n(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}x^{2^{2k-n}}.$$
By the proof of Lemma 1 we see that $\mathcal{A}^n I^{1/k}(x) \le d_n(x^{1/k})$ and naturally
$$\limsup_{n\to\infty} \mathcal{A}^n I^{1/k}(x) \le \frac{1}{2} + \frac{1}{2}x^{2^s/k} \quad \text{for } s = 1,2,\ldots.$$
Therefore $\limsup_{n\to\infty} \mathcal{A}^n I^{1/k} \le 1/2$ for $x \in [0,1)$. Let $\omega(x)$ be a polynomial satisfying our assumptions. Then $\omega(x) \le x^{1/2}$ for $x \in [0,\epsilon]$ and for some $\epsilon > 0$. If we take k such that $\epsilon^{1/k} > \delta$ then $\omega(x) \le x^{1/k}$ for $x \in [0,1]$. Hence $\mathcal{A}^n\omega \le \mathcal{A}^n I^{1/k}$ and as a result
$$\limsup_{n\to\infty} \mathcal{A}^n\omega(x) \le \limsup_{n\to\infty} \mathcal{A} I^{1/k}(x) \le \frac{1}{2} \quad \text{for } x \in [0,1). \ ∎$$

LEMMA 3. *Let T be given by* (2) *and let $\mu_p \times \nu_p$ be a T-invariant measure. If $\nu_p \notin \operatorname{conv}\{\delta_{\{0\}}, \delta_{\{1\}}\}$ then ν_F has the dense support property* (*equivalently F is* 1-1).

Proof. Let $T_0 = (1+\epsilon_0)x - \epsilon_0 g(x)$ and $T_1(x) = (1-\epsilon_1)x + \epsilon_1 g(x)$ for $\epsilon_0, \epsilon_1 > 0$. Then
$$T_0(x) = (1+\epsilon)x - \epsilon T_1(x) \quad \text{for } \epsilon = \epsilon_0/\epsilon_1.$$
Let (a,b) be a nonempty interval of maximal length such that $F|(a,b) =$ const. By assumptions we have $(a,b) \ne (0,1)$ and $T_0(b) - T_0(a) \le b - a$, $T_1(b) - T_1(a) \le b - a$. Here we use the fact that $F|(T_0(a), T_0(b)) =$ const and $F|(T_1(a), T_1(b)) =$ const by Lemma 1 of [K2]. In particular,
$$\begin{aligned} T_0(b) - T_0(a) \le b - a \ &\Leftrightarrow\ (1+\epsilon)b - \epsilon T_1(b) - (1+\epsilon)a + \epsilon T_1(a) \le b - a \\ &\Leftrightarrow\ T_1(b) - T_1(a) \ge b - a. \end{aligned}$$
Therefore, $T_1(b) - T_1(a) = b - a$ and by induction $T_1^n(b) - T_1^n(a) = b - a$ for $n = 1,2,\ldots$. Hence $a = b$. ∎

Proof of Theorem 7. Suppose, contrary to our claim, that there exists a product measure in $M_{1/2}$ outside $\operatorname{conv}\{\mu_p \times \delta_{\{0\}}, \mu_p \times \delta_{\{1\}}\}$. We may assume (by ergodic decomposition [Ki]) that there exists a distribution G such that $\mu_p \times \nu_G$ is ergodic and $\nu_G \notin \operatorname{conv}\{\delta_{\{0\}}, \delta_{\{1\}}\}$. Since ν_G has the dense support property (by Lemma 3) we see that G is continuous and increasing. Therefore for every $\epsilon > 0$ there exists a polynomial ω_ϵ such that

$$\omega_\epsilon(0) = 0, \quad 0 \le \omega_\epsilon(x) \le 1 - \epsilon \quad \text{for} \quad x \in [0,1] \quad \text{and} \quad \|G - \omega_\epsilon\| \le 3\epsilon.$$

Thus we obtain

$$\|\mathcal{A}^n G - \mathcal{A}^n \omega_\epsilon\| = \|G - \mathcal{A}^n \omega_\epsilon\| \le 3\epsilon$$

and $G \le 1/2$ for $x \in [0,1)$ by Lemma 2. This contradicts our assumption. ■

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