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On divisibility in definable groups

by

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Abstract. Let $\mathcal{M}$ be an o-minimal expansion of a real closed field. It is known that a definably connected abelian group is divisible. We show that a definably compact definably connected group is divisible.

Let $\mathcal{M}$ be an o-minimal expansion of a real closed field. A group is said to be *definable* if both the set and the graph of the group operation are definable in $\mathcal{M}$. By results of Pillay in [10], a definable group can be equipped with a definable manifold topology making the group a topological group. Since topological groups are regular spaces, we can suppose that the manifold topology is induced by that of the ambient space (see Theorem 10.1.8 in [6]). In that setting, a definably compact group is a closed and bounded definable group. A definable group is *definably connected* provided it has no definable subgroups of finite index. A definably connected group which is abelian is also divisible, by Strebonski's Theorem on the finiteness of torsion subgroups (see, e.g., the proof of Theorem 2.1 in [9]).

In this note, using the available literature on both definable groups and topological groups, we prove the following.

Theorem 1. *Let G be a definably compact definably connected definable group. Then G is divisible.*

In proving divisibility of the groups we are concerned with, the continuous definable maps $p_k\colon G \to G : a \mapsto a^k$ for $k > 0$ will play an important role (in both the Abelian definable case and the classical topological case).

First, we consider the o-minimal cohomology with coefficients in $\mathbb{Q}$, as defined in Section 3 of [9]. Recall that if X is a definable set then $H^*(X;\mathbb{Q})$ is a finite-dimensional $\mathbb{Q}$-vector space such that $H^m(X;\mathbb{Q}) = 0$ for $m > \dim X$, and moreover $H^0(X;\mathbb{Q}) \cong \mathbb{Q}$ provided X is also definably connected. For an element $x \in H^m(X;\mathbb{Q})$, we say x has *degree* m and write $\deg x = m$. In [9],

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it is observed that $H^*(X;\mathbb{Q}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_*(X),\mathbb{Q})$, where $H_*(X)$ is the o-minimal homology with coefficients in $\mathbb{Z}$. It will be more convenient for our purposes to take o-minimal homology with coefficients in $\mathbb{Q}$. In this case, we also get $H^*(X;\mathbb{Q}) \cong \mathrm{Hom}_{\mathbb{Q}}(H_*(X;\mathbb{Q}),\mathbb{Q})$, because $H^*(X;\mathbb{Q}) \cong H_*(X) \otimes \mathbb{Q}$ and $\mathrm{Hom}_{\mathbb{Q}}(H_*(X) \otimes \mathbb{Q},\mathbb{Q}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_*(X),\mathbb{Q})$.

Notice that $H^*(X;\mathbb{Q})$ is a $\mathbb{Q}$-algebra with product defined as follows: let $d\colon X \to X \times X : x \mapsto (x,x)$ be the diagonal map, identify $H^*(X \times X;\mathbb{Q})$ with $H^*(X;\mathbb{Q}) \otimes H^*(X;\mathbb{Q})$ via the o-minimal Künneth formula for cohomology, and let $x \cdot y := d^*(x \otimes y)$ (see [9] for details).

Moreover, we have the following result.

Lemma 2. *Let G be a nontrivial definably connected definably compact definable group. Then there is a unique integer $r > 0$ and elements $y_1,\dots,y_r$ in the o-minimal $\mathbb{Q}$-cohomology algebra of G such that*

(i) $\deg y_i$ *is odd* $(i = 1,\dots,r)$,
(ii) $H^*(G;\mathbb{Q})$ *is freely generated, as a $\mathbb{Q}$-vector space, by* 1 $(\in H^0(G;\mathbb{Q}))$ *and the monomials* $y_{i(1)} \cdot \ldots \cdot y_{i(l)}$ *with* $1 \le i(1) < \cdots < i(l) \le r$.

Proof. By Corollary 3.6 in [9], there is a unique $r \ge 0$ and $y_1,\dots,y_r$ satisfying the requirements. Now, since G is definably connected, by Theorem 5.2 in [3], the top o-minimal homology group $H_n(G)$ is nontrivial, where $n = \dim G > 0$. Therefore, $r > 0$. ■

We write $\operatorname{len} x = l$ if x is a monomial of length l, i.e., $x = y_{i(1)} \cdot \ldots \cdot y_{i(l)}$ with $1 \le i(1) < \cdots < i(l) \le r$ (with the notation of the above lemma). In the following, we are going to consider the maps p_k, $k > 0$, mentioned above. The computations in [4], for such maps, apply to our o-minimal context and yield the following.

Lemma 3. *Let G be a definably connected definable group. For each $k > 0$, consider the definable continuous map $p_k\colon G \to G : a \mapsto a^k$ for $a \in G$. Then the map $p_k^*\colon H^*(G;\mathbb{Q}) \to H^*(G;\mathbb{Q})$ sends each monomial x to $k^{\operatorname{len} x} x$.*

Proof. See Lemma 5.2 in [9]. ■

Let X be a definable set of dimension n and let $f\colon X \to X$ be a continuous definable map. The *Lefschetz number* of f is defined as follows:

$$L(f) = \sum_{m=0}^{n} (-1)^m \operatorname{trace}(f_*\colon H_m(X;\mathbb{Q}) \to H_m(X;\mathbb{Q})).$$

(See [5] for the semialgebraic case, and compare with the definition of the Lefschetz number as an intersection number in the o-minimal differentiable case given in [2].)

Note that the matrix of $f^*\colon H^m(X;\mathbb{Q}) \to H^m(X;\mathbb{Q})$ is the transpose of the matrix of $f_*\colon H_m(X;\mathbb{Q}) \to H_m(X;\mathbb{Q})$ ($f^* = \mathrm{Hom}_{\mathbb{Q}}(f_*)$). Hence, we also get $L(f) = \sum_{m=0}^{n}(-1)^m \operatorname{trace}(f^*\colon H^m(X;\mathbb{Q}) \to H^m(X;\mathbb{Q}))$.

The next step is to make use of the definable version of the Lefschetz fixed point theorem (see Theorem 1.1 in [8], and Proposition 2 in [5] for the semialgebraic case).

THEOREM 4. *Let X be a closed and bounded definably connected set. If $f\colon X \to X$ is a continuous definable map and $L(f) \neq 0$, then f has a fixed point.*

COROLLARY 5. *Let G be a definably compact definable group. If $f\colon G \to G$ is a continuous definable map and $L(f) \neq 0$, then there is an element $b \in G$ such that $f(b) = b$.*

Finally, we shall follow Brown [4] to compute $L(p_k)$ for each $k \geq 2$, and prove Theorem 1.

LEMMA 6. *Let G be a definably connected definably compact definable group. Then, for each $k \geq 2$, $L(p_k) = (1-k)^r$, where r is as in Lemma 2.*

Proof. Let $\{1, x_1, \ldots, x_s\}$ be a basis of the $\mathbb{Q}$-vector space $H^*(G;\mathbb{Q})$, where the x_i's are monomials. By Lemma 3, $p_k^*(x_i) = k^{\operatorname{len} x_i} x_i$ ($i = 1, \ldots, s$) and $p_k^*(1) = 1$. Then the matrix of $p_k^*\colon H^m(G;\mathbb{Q}) \to H^m(G;\mathbb{Q})$ is either 0 (if $H^m(G;\mathbb{Q}) = 0$) or a diagonal matrix with entry $k^{\operatorname{len} x_i}$ corresponding to each x_i in $H^m(G;\mathbb{Q})$. Therefore, $L(p_k) = \sum_{i=1}^{s}(-1)^{\deg x_i} k^{\operatorname{len} x_i} + 1$. On the other hand, the x_i's are monomials (products of the y_j's of Lemma 2) and the y_j's are of odd degree, so that $\deg x_i \equiv \operatorname{len} x_i \pmod 2$, and hence $L(p_k) = \sum_{i=1}^{s}(-1)^{\operatorname{len} x_i} k^{\operatorname{len} x_i} + 1$. Since there are $\binom{r}{l}$ monomials of length l, we get $L(p_k) = \sum_{l=1}^{r} \binom{r}{l}(-1)^l k^l + 1 = (1-k)^r$. ∎

Proof of Theorem 1. Fix $a \in G$ and k (≥ 2). We shall prove the existence of an element $b \in G$ such that $b^k = a$. Let $f\colon G \to G : c \mapsto c^{k+1}a^{-1}$ for $c \in G$. Since G is definably connected, there is a definable path $\gamma\colon [0,1] \to G$ such that $\gamma(0) = a^{-1}$ and $\gamma(1) = e$, where e is the neutral element of G. Let

$$F\colon [0,1] \times G \to G : (t,c) \mapsto F(t,c) := c^{k+1}\gamma(t).$$

Clearly, F is a definable homotopy between the maps $F(0,-) = f$ and $F(1,-) = p_{k+1}$. Hence, the induced cohomology morphisms f^* and $(p_{k+1})^*$ (both from $H^*(G;\mathbb{Q})$ to $H^*(G;\mathbb{Q})$)) coincide. Therefore, $L(f) = L(p_{k+1})$, and by Lemma 6, $L(p_{k+1}) = (-k)^r$ ($\neq 0$). By Corollary 5, there is an element b in G such that $b^{k+1}a^{-1} = b$, as required. ∎

After this note was written, two alternative proofs of Theorem 1 have been given in [7] and [1], both papers making reference to a preprint version of this note.

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