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Metric spaces admitting only trivial weak contractions

by

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Abstract. If (X, d) is a metric space then a map $f\colon X \to X$ is defined to be a weak contraction if $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$, $x \neq y$. We determine the simplest non-closed sets $X \subseteq \mathbb{R}^n$ in the sense of descriptive set-theoretic complexity such that every weak contraction $f\colon X \to X$ is constant. In order to do so, we prove that there exists a non-closed F_σ set $F \subseteq \mathbb{R}$ such that every weak contraction $f\colon F \to F$ is constant. Similarly, there exists a non-closed G_δ set $G \subseteq \mathbb{R}$ such that every weak contraction $f\colon G \to G$ is constant. These answer questions of M. Elekes.

We use measure-theoretic methods, first of all the concept of generalized Hausdorff measure.

1. Introduction. We use the following descriptive set-theoretical notation.

NOTATION 1.1. The classes of open, closed, F_σ, and G_δ sets are denoted by Σ_1^0, Π_1^0, Σ_2^0, and Π_2^0, respectively. The simultaneously F_σ and G_δ sets are denoted by Δ_2^0.

M. Elekes [E] introduced the definition below.

DEFINITION 1.2. We say that the metric space X has the *Banach Fixed Point Property* (BFPP) if every contraction $f\colon X \to X$ has a fixed point.

The Banach Fixed Point Theorem implies that every complete metric space has the BFPP. E. Behrends [Be] pointed out that the converse implication does not hold. He presented the following example, which he referred to as 'folklore'.

THEOREM 1.3. *Let* $X = \operatorname{graph}(\sin(1/x)|_{(0,1]})$. *Then* $X \subseteq \mathbb{R}^2$ *is a non-closed simultaneously* F_σ *and* G_δ *set with the Banach Fixed Point Property.*

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M. Elekes [E] described the simplest non-closed sets having the BFPP in the sense of descriptive set-theoretic complexity. He proved the following theorems.

THEOREM 1.4 (M. Elekes). *Every open subset of $\mathbb{R}^n$ with the Banach Fixed Point Property is closed. Every simultaneously F_σ and G_δ subset of $\mathbb{R}$ with the Banach Fixed Point Property is closed.*

THEOREM 1.5 (M. Elekes). *There exist non-closed F_σ and non-closed G_δ subsets of $\mathbb{R}$ with the Banach Fixed Point Property.*

The above three theorems answer the question about the lowest possible Borel classes of $\mathbb{R}^n$ having a non-closed element with the BFPP. In the language of descriptive set theory, if $n \geq 2$ then Δ^0_2 is the best possible class, since there are no Σ^0_1 and Π^0_1 examples. If $n = 1$ then Σ^0_2 and Π^0_2 are possible, but Δ^0_2 is not.

Note that if every weak contraction $f: X \to X$ is constant then X has the BFPP. There are infinite complete metric spaces that admit only trivial weak contractions, for example the metric spaces $X = \mathbb{Z} \times \{0\}^{n-1} \subseteq \mathbb{R}^n$ clearly have this property (there is a non-degenerate connected compact example in $\mathbb{R}^n$ for every $n \geq 2$, see later). Therefore it is natural to ask the following question.

QUESTION 1.6 (M. Elekes). *What are the lowest possible Borel classes of $\mathbb{R}^n$ having a non-closed element X such that every weak contraction $f: X \to X$ is constant?*

The main goal of our paper is to answer Question 1.6.

On the one hand, Theorem 1.4 shows that there are no Σ^0_1 and Π^0_1 examples in the cases $n \geq 2$.

On the other hand, T. Dobrowolski [D] pointed out a connection between our question and the so called *Cook continua*, non-degenerate connected compact topological spaces C such that every continuous map $f: C \to C$ is either constant or the identity. They were named after H. Cook [C], who first constructed such an object. Cook's example cannot be embedded in $\mathbb{R}^2$, only in $\mathbb{R}^3$. Later T. Maćkowiak [M, Cor. 32] has shown that there exists an arc-like (snake-like) Cook continuum, and arc-like continua are embeddable in the plane by [Bi, Thm. 4].

The next theorem is straightforward; it follows that the answer to Question 1.6 is Δ^0_2 if $n \geq 2$.

THEOREM 1.7 (Maćkowiak, Dobrowolski). *Let $X = C \setminus \{c_0\}$, where $C \subseteq \mathbb{R}^2$ is a Cook continuum and $c_0 \in C$ is arbitrary. Then $X \subseteq \mathbb{R}^2$ is non-closed, simultaneously F_σ and G_δ, and every weak contraction $f: X \to X$ is constant.*

If $n = 1$ then Theorem 1.4 implies that there is no Δ^0_2 example for Question 1.6. In the positive direction M. Elekes obtained the following partial result.

THEOREM 1.8 (M. Elekes). *There exists a non-closed G_δ set $G \subseteq \mathbb{R}$ such that every contraction $f\colon G \to G$ is constant.*

The proof of Theorem 1.8 is based on the following theorem, interesting in its own right.

THEOREM 1.9 (M. Elekes). *For the generic compact set $K \subseteq \mathbb{R}$ (in the sense of Baire category) for any contraction $f\colon K \to \mathbb{R}$ the set $f(K)$ does not contain a non-empty relatively open subset of K.*

In order to answer Question 1.6 it is enough to show that there are non-closed Σ^0_2 and Π^0_2 subsets of $\mathbb{R}$ that admit only trivial weak contractions. Therefore we prove the following theorems.

THEOREM 6.1 (Main Theorem, F_σ case). *There exists a non-closed F_σ set $F \subseteq \mathbb{R}$ such that every weak contraction $f\colon F \to F$ is constant.*

THEOREM 6.2 (Main Theorem, G_δ case). *There exists a non-closed G_δ set $G \subseteq \mathbb{R}$ such that every weak contraction $f\colon G \to G$ is constant.*

The heart of the proof is the following theorem, a partial measure-theoretic analogue of Theorem 1.9. For a gauge function h let us denote by $\mathcal{H}^h$ the h-Hausdorff measure.

THEOREM 5.1 (simplified version). *There exists a compact set $K \subseteq \mathbb{R}$ and a continuous gauge function h such that $0 < \mathcal{H}^h(K) < \infty$, and for every weak contraction $f\colon K \to \mathbb{R}$ we have $\mathcal{H}^h(K \cap f(K)) = 0$.*

Based on the present paper, A. Máthé and the author show in [BM] the following more general theorem. If X is a Polish space, then the generic compact set $K \subseteq X$ is either finite or there is a continuous gauge function h such that $0 < \mathcal{H}^h(K) < \infty$, and for every weak contraction $f\colon K \to X$ we have $\mathcal{H}^h(K \cap f(K)) = 0$. If X is perfect, then the generic compact set $K \subseteq X$ is infinite, so the first case does not occur. This is the measure-theoretic analogue of Theorem 1.9, which also answers a question of C. Cabrelli, U. B. Darji, and U. M. Molter. This is the reason why we will work in Polish spaces instead of $\mathbb{R}$.

The structure of the paper is as follows. In the Preliminaries section we introduce some notation and definitions. In Section 3 we define balanced compact sets in a Polish space X, and we prove their existence if X is uncountable. In Section 4 we show that every balanced compact set

$K \subseteq X$ has a continuous gauge function h such that $0 < \mathcal{H}^h(K) < \infty$. In Section 5 we show that $\mathcal{H}^h(K \cap f(K)) = 0$ for every weak contraction $f\colon K \to X$, which completes the proof of Theorem 5.1. In Section 6 we prove our Main Theorems, making use of Theorem 5.1 and of ideas from [E].

2. Preliminaries. Let (X,d) be a metric space, and let $A, B \subseteq X$ be arbitrary sets. We denote by $\operatorname{int} A$ and $\operatorname{diam} A$ the interior and the diameter of A, respectively. We use the convention $\operatorname{diam} \emptyset = 0$. The *distance* of the sets A and B is $\operatorname{dist}(A,B) = \inf\{d(x,y) : x \in A,\ y \in B\}$.

The function $h\colon [0,\infty) \to [0,\infty)$ is defined to be a *gauge function* if it is non-decreasing, right-continuous, and $h(x) = 0$ iff $x = 0$.

For all $A \subseteq X$ and $\delta > 0$ consider

$$\mathcal{H}^h_\delta(A) = \inf\Big\{\sum_{i=1}^{\infty} h(\operatorname{diam} A_i) : A \subseteq \bigcup_{i=1}^{\infty} A_i,\ \forall i\ \operatorname{diam} A_i \le \delta\Big\},$$
$$\mathcal{H}^h(A) = \lim_{\delta \to 0+} \mathcal{H}^h_\delta(A).$$

We call $\mathcal{H}^h$ the *h-Hausdorff measure*. For more information on these concepts see [R].

A metric space X is *perfect* if it has no isolated points. A metric space X is *Polish* if it is complete and separable.

Given two metric spaces (X,d_X) and (Y,d_Y), a function $f\colon X \to Y$ is called *Lipschitz* if there is a constant $C \in \mathbb{R}$ such that $d_Y(f(x_1), f(x_2)) \le C \cdot d_X(x_1,x_2)$ for all $x_1, x_2 \in X$. The smallest such constant C is the *Lipschitz constant* of f and denoted by $\operatorname{Lip}(f)$. If $\operatorname{Lip}(f) \le 1$ then f is a 1-*Lipschitz map*; if $\operatorname{Lip}(f) < 1$ then f is a *contraction*. We say that f is a *weak contraction* if $d_Y(f(x_1), f(x_2)) < d_X(x_1,x_2)$ for all $x_1, x_2 \in X$, $x_1 \ne x_2$.

We write λ for the Lebesgue measure of $\mathbb{R}$, and $2\mathbb{N}+1$ for the odd positive integers.

3. The definition and existence of balanced compact sets

Definition 3.1. If a_n $(n \in \mathbb{N}^+)$ are positive integers then set, for all $n \in \mathbb{N}^+$,

$$\mathcal{I}_n = \prod_{k=1}^{n} \{1,\dots,a_k\} \quad \text{and} \quad \mathcal{I} = \bigcup_{n=1}^{\infty} \mathcal{I}_n.$$

We say that a map $\Phi\colon 2\mathbb{N}+1 \to \mathcal{I}$ is an *index function according to the sequence* $\langle a_n\rangle$ if it is surjective and $\Phi(n) \in \bigcup_{k=1}^{n} \mathcal{I}_k$ for every odd n.

Definition 3.2. Let X be a Polish space. A compact set $K \subseteq X$ is *balanced* if it is of the form

$$K = \bigcap_{n=1}^{\infty} \Big(\bigcup_{i_1=1}^{a_1} \cdots \bigcup_{i_n=1}^{a_n} C_{i_1 \ldots i_n} \Big), \tag{3.1}$$

where a_n are positive integers and $C_{i_1 \ldots i_n} \subseteq X$ are non-empty closed sets with the following properties. There are positive reals b_n and an index function Φ: $2\mathbb{N}+1 \to \mathcal{I}$ according to the sequence $\langle a_n \rangle$ such that for all $n \in \mathbb{N}^+$ and $(i_1, \ldots, i_n), (j_1, \ldots, j_n) \in \mathcal{I}_n$,

(i) $a_1 \geq 2$ and $a_{n+1} \geq n a_1 \cdots a_n$,
(ii) $C_{i_1 \ldots i_{n+1}} \subseteq C_{i_1 \ldots i_n}$,
(iii) $\operatorname{diam} C_{i_1 \ldots i_n} \leq b_n$,
(iv) $\operatorname{dist}(C_{i_1 \ldots i_n}, C_{j_1 \ldots j_n}) > 2b_n$ if $(i_1, \ldots, i_n) \neq (j_1, \ldots, j_n)$,
(v) if n is odd, $C_{i_1 \ldots i_n} \subseteq C_{\Phi(n)}$ and $C_{j_1 \ldots j_n} \not\subseteq C_{\Phi(n)}$, then for all $s, t \in \{1, \ldots, a_{n+1}\}$, $s \neq t$, we have

$$\operatorname{dist}(C_{i_1 \ldots i_n s}, C_{i_1 \ldots i_n t}) > \operatorname{diam}\Big(\bigcup_{j_{n+1}=1}^{a_{n+1}} C_{j_1 \ldots j_n j_{n+1}} \Big).$$

Remark 3.3. The only reason why the domain of Φ is $2\mathbb{N}+1$ instead of $\mathbb{N}^+$ is that we refer to this construction in [BM], where this is important.

Remark 3.4. In a countable Polish space X there is no balanced compact set $K \subseteq X$, since every balanced compact set has cardinality $2^{\aleph_0}$.

Theorem 3.5. *If X is an uncountable Polish space, then there exists a balanced compact set $K \subseteq X$.*

Proof. Every uncountable Polish space contains a non-empty perfect subset (see [K, (6.4) Thm.]), so we may assume by shrinking that X is also perfect. Fix positive integers a_n according to (i) and an index function Φ according to $\langle a_n \rangle$. We need to construct non-empty closed sets $C_{i_1 \ldots i_n}$ and positive reals b_n that satisfy (ii)–(v); then the set

$$K = \bigcap_{n=1}^{\infty} \Big(\bigcup_{i_1=1}^{a_1} \cdots \bigcup_{i_n=1}^{a_n} C_{i_1 \ldots i_n} \Big)$$

will be a balanced compact set.

Let $n \in \mathbb{N}$ and assume that b_k and $C_{i_1 \ldots i_k}$ with $\operatorname{int} C_{i_1 \ldots i_k} \neq \emptyset$ are already defined for all $k \leq n$ and $(i_1, \ldots, i_k) \in \mathcal{I}_k$, where we use the convention $\mathcal{I}_0 = \{\emptyset\}$, $C_\emptyset = X$, and $b_0 = \infty$. It is enough to construct b_{n+1} and $C_{i_1 \ldots i_{n+1}}$ such that $\operatorname{int} C_{i_1 \ldots i_{n+1}} \neq \emptyset$ for all $(i_1, \ldots, i_{n+1}) \in \mathcal{I}_{n+1}$.

We define distinct points $x_{i_1\dots i_{n+1}} \in \operatorname{int} C_{i_1\dots i_n}$ for all $(i_1,\dots,i_{n+1}) \in \mathcal{I}_{n+1}$. First assume that n is even. As X is perfect and $\operatorname{int} C_{i_1\dots i_n} \neq \emptyset$, we can fix distinct points $x_{i_1\dots i_{n+1}} \in \operatorname{int} C_{i_1\dots i_n}$ for all $(i_1,\dots,i_{n+1}) \in \mathcal{I}_{n+1}$. Now assume that n is odd. First consider those $(i_1,\dots,i_n)$ for which $C_{i_1\dots i_n} \subseteq C_{\Phi(n)}$, then fix distinct points $x_{i_1\dots i_{n+1}} \in \operatorname{int} C_{i_1\dots i_n}$ for all $i_{n+1} \in \{1,\dots,a_{n+1}\}$. Let δ be the minimum distance between the points $x_{i_1\dots i_{n+1}}$ we have defined so far. Now consider those $(i_1,\dots,i_n)$ for which $C_{i_1\dots i_n} \nsubseteq C_{\Phi(n)}$. For each of them, fix distinct points $x_{i_1\dots i_{n+1}} \in \operatorname{int} C_{i_1\dots i_n}$ for all $i_{n+1} \in \{1,\dots,a_{n+1}\}$ such that

$$\operatorname{diam}\Big(\bigcup_{i_{n+1}=1}^{a_{n+1}} \{x_{i_1\dots i_{n+1}}\}\Big) \leq \frac{\delta}{2}.$$

For $(i_1,\dots,i_{n+1}) \in \mathcal{I}_{n+1}$ consider the non-empty closed sets

$$C_{i_1\dots i_{n+1}} = B(x_{i_1\dots i_{n+1}}, b_{n+1}/2),$$

where $b_{n+1} > 0$ is sufficiently small. Then the sets $C_{i_1\dots i_{n+1}}$ satisfy (ii)–(v), and clearly $\operatorname{int} C_{i_1\dots i_{n+1}} \neq \emptyset$ for all $(i_1,\dots,i_{n+1}) \in \mathcal{I}_{n+1}$. ■

FACT 3.6. *If $K \subseteq \mathbb{R}$ is a balanced compact set, then K has zero Lebesgue measure.*

Proof. For all $n \in \mathbb{N}^+$ and $(i_1,\dots,i_n) \in \mathcal{I}_n$ let $I_{i_1\dots i_n} \subseteq \mathbb{R}$ be compact intervals such that $C_{i_1\dots i_n} \subseteq I_{i_1\dots i_n}$ and $\operatorname{diam} I_{i_1\dots i_n} = \operatorname{diam} C_{i_1\dots i_n}$. Set $I_n^* = \bigcup_{i_1=1}^{a_1} \cdots \bigcup_{i_n=1}^{a_n} I_{i_1\dots i_n}$. Properties (iii) and (iv) imply that $\lambda(I_{n+1}^*) \leq \lambda(I_n^*)/2$ for all $n \in \mathbb{N}^+$, thus $K \subseteq \bigcap_{n=1}^{\infty} I_n^*$ has zero Lebesgue measure. ■

4. Balanced compact sets admit exact continuous gauge functions. The main goal of this section is to prove Theorem 4.2.

Assume that X is a Polish space and $K \subseteq X$ is a fixed balanced compact set. Let a_n, b_n, $C_{i_1\dots i_n}$, Φ witness that K is balanced according to Definition 3.2.

DEFINITION 4.1. Let $K_{i_1\dots i_n} = K \cap C_{i_1\dots i_n}$ for all $(i_1,\dots,i_n) \in \mathcal{I}_n$ and $n \in \mathbb{N}^+$. These sets are called the *nth level elementary pieces* of K. For a set $A \subseteq K$ we define the nth level elementary pieces of A to be the nth level elementary pieces of K that intersect A.

THEOREM 4.2. *There exists a continuous gauge function h with $\mathcal{H}^h(K) = 1$. Moreover,*

$$\mathcal{H}^h(K_{i_1\dots i_n}) = \frac{1}{a_1\cdots a_n}$$

for all $n \in \mathbb{N}^+$ and $(i_1,\dots,i_n) \in \mathcal{I}_n$.

Proof. Consider $h\colon [0,\infty) \to [0,\infty)$,

$$(4.1)\qquad h(x) = \begin{cases} 1 & \text{if } x \geq 2b_1, \\ \dfrac{1}{a_1 \cdots a_n} & \text{if } 2b_{n+1} \leq x \leq b_n \text{ for all } n \in \mathbb{N}^+, \\ \text{linear} & \text{if } b_n \leq x \leq 2b_n \text{ for all } n \in \mathbb{N}^+, \\ 0 & \text{if } x = 0. \end{cases}$$

As $a_n \geq 2$ for all $n \in \mathbb{N}^+$, properties (ii)–(iv) imply that $2b_{n+1} < b_n$ for all $n \in \mathbb{N}^+$. Thus $b_n < b_1/2^{n-1} \to 0$ as $n \to \infty$. Consequently, h is well-defined. Clearly, h is non-decreasing, continuous, and $h(x) = 0$ iff $x = 0$. Therefore h is a continuous gauge function.

It is enough to prove that $\mathcal{H}^h(K) = 1$, because applying the same argument for $K_{i_1\dots i_n}$ yields the more general statement. Then

$$K \subseteq \bigcup_{i_1=1}^{a_1} \cdots \bigcup_{i_n=1}^{a_n} C_{i_1\dots i_n} \quad \text{and} \quad \operatorname{diam} C_{i_1\dots i_n} \leq b_n$$

imply

$$\mathcal{H}^h_{b_n}(K) \leq \sum_{i_1=1}^{a_1} \cdots \sum_{i_n=1}^{a_n} h(\operatorname{diam} C_{i_1\dots i_n}) \leq a_1 \cdots a_n h(b_n) = 1.$$

Since $b_n \to 0$ as $n \to \infty$, we obtain $\mathcal{H}^h(K) = \lim_{n\to\infty} \mathcal{H}^h_{b_n}(K) \leq 1$.

For the opposite inequality assume that $K \subseteq \bigcup_{j=1}^{\infty} U_j$; it is enough to prove that $\sum_{j=1}^{\infty} h(\operatorname{diam} U_j) \geq 1$. By the continuity of h we may assume that the U_j's are non-empty open, and the compactness of K implies that there is a finite subcover, $K \subseteq \bigcup_{j=1}^{k} U_j$. Fix $m \in \mathbb{N}$ such that $2b_m < \min_{1\leq j\leq k} \operatorname{diam} U_j$. For all $j \in \{1, \dots, k\}$ consider

$$s_j = \#\{(i_1, \dots, i_m) \in \mathcal{I}_m : U_j \cap K_{i_1\dots i_m} \neq \emptyset\}.$$

Since $K \subseteq \bigcup_{j=1}^{k} U_j$, we have

$$(4.2)\qquad \sum_{j=1}^{k} s_j \geq a_1 \cdots a_m.$$

Now we show that for all $j \in \{1, \dots, k\}$,

$$(4.3)\qquad h(\operatorname{diam} U_j) \geq \frac{s_j}{a_1 \cdots a_m}.$$

Fix $j \in \{1, \dots, k\}$. If $\operatorname{diam} U_j \geq 2b_1$ then $h(\operatorname{diam} U_j) = 1$ and $s_j \leq a_1 \cdots a_m$ imply (4.3). Thus we may assume that there is an $1 \leq n < m$ such that $2b_{n+1} \leq \operatorname{diam} U_j \leq 2b_n$. On the one hand, (iv) implies that U_j can intersect at most one nth level elementary piece of K, that is, $s_j \leq a_{n+1} \cdots a_m$. On the other hand, the definition of h implies $h(\operatorname{diam} U_j) \geq 1/(a_1 \cdots a_n)$.

Therefore (4.3) holds. Finally, (4.3) and (4.2) yield

$$\sum_{j=1}^{k} h(\operatorname{diam} U_j) \geq \sum_{j=1}^{k} \frac{s_j}{a_1 \cdots a_m} \geq 1,$$

and the proof is complete. ∎

Remark 4.3. Note that property (v) and the notion of an index function Φ are not needed for the proof of Theorem 4.2. We used only the natural condition $a_n \geq 2$ $(n \in \mathbb{N}^+)$ instead of property (i).

Fact 4.4. *Let $K \subseteq \mathbb{R}$ be a balanced compact set, and let h be the gauge function for K according to* (4.1). *Then λ is absolutely continuous for $\mathcal{H}^h$.*

Proof. Let I be a compact interval such that $\bigcup_{i_1=1}^{a_1} C_{i_1} \subseteq I$, and assume $\operatorname{diam} I = c$. Set $g(x) = x/c$. First we prove that $h(x) \geq g(x)$ for all $x \in [0, b_1]$. Let $n \in \mathbb{N}^+$. On the one hand, the definition of h implies $h(b_n) = 1/(a_1 \cdots a_n)$. On the other hand, (iv) yields $2b_n(\#\mathcal{I}_n - 1) \leq \operatorname{diam} I$, so

$$b_n \leq \frac{\operatorname{diam} I}{2(\#\mathcal{I}_n - 1)} \leq \frac{c}{a_1 \cdots a_n}.$$

Thus $h(b_n) \geq b_n/c = g(b_n)$. As h is concave and g is linear on $[b_{n+1}, b_n]$ for all $n \in \mathbb{N}^+$, we have $h(x) \geq g(x)$ for all $x \in [0, b_1]$.

Finally, $h|_{[0,b_1]} \geq g|_{[0,b_1]}$ implies that for all $A \subseteq \mathbb{R}$ we have $\mathcal{H}^h(A) \geq \mathcal{H}^g(A) = \lambda(A)/c$, so λ is absolutely continuous for $\mathcal{H}^h$. ∎

5. The proof of Theorem 5.1. The goal of this section is to prove the following theorem.

Theorem 5.1. *Let X be a Polish space, and let $K \subseteq X$ be a balanced compact set. Then there exists a continuous gauge function h such that $0 < \mathcal{H}^h(K) < \infty$, and for every weak contraction $f\colon K \to X$ we have $\mathcal{H}^h(K \cap f(K)) = 0$.*

Proof. Let a_n, b_n, $C_{i_1 \ldots i_n}$, Φ witness that K is balanced as in Definition 3.2. Let h be the continuous gauge function for K according to (4.1). Theorem 4.2 implies $\mathcal{H}^h(K) = 1$. Let $f\colon K \to X$ be a weak contraction. It is enough to prove that $\mathcal{H}^h(K \cap f(K)) = 0$. For all $n \in \mathbb{N}^+$ let

$$A_n = \bigcup_{i_1=1}^{a_1} \cdots \bigcup_{i_n=1}^{a_n} (K_{i_1 \ldots i_n} \cap f(K \setminus K_{i_1 \ldots i_n})).$$

First we prove

$$K \cap f(K) \subseteq \operatorname{Fix}(f) \cup \bigcup_{n=1}^{\infty} A_n, \tag{5.1}$$

where $\mathrm{Fix}(f) = \{x \in K : f(x) = x\}$. Assume that $y \in K \cap f(K)$ and $y \notin \mathrm{Fix}(f)$; we need to prove that $y \in \bigcup_{n=1}^{\infty} A_n$. There is an $x \in K$ such that $f(x) = y$ and $x \neq y$. Then $\operatorname{diam} K_{i_1 \ldots i_n} \leq b_n$ and $b_n \to 0$ imply that there are $n \in \mathbb{N}^+$ and $(i_1, \ldots, i_n) \in \mathcal{I}_n$ such that $y \in K_{i_1 \ldots i_n}$ and $x \in K \setminus K_{i_1 \ldots i_n}$, so $y \in A_n$. Thus $y \in \bigcup_{n=1}^{\infty} A_n$, hence (5.1) holds.

As f is a weak contraction, $\mathrm{Fix}(f)$ has at most one element. Therefore (5.1) implies that it is enough to prove that $\mathcal{H}^h(\bigcup_{n=1}^{\infty} A_n) = 0$. Property (ii) easily yields $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}^+$, so it is enough to prove that

$$\lim_{n \to \infty} \mathcal{H}^h(A_n) = 0. \tag{5.2}$$

Fix $n \in \mathbb{N}^+$ and $(i_1, \ldots, i_n) \in \mathcal{I}_n$. The definition of Φ implies that there is an odd $m \geq n$ such that $\Phi(m) = (i_1, \ldots, i_n)$. Let Δ_m be the set of mth level elementary pieces of $K \setminus K_{i_1 \ldots i_n}$. Pick $E \in \Delta_m$. As f is a weak contraction, $\operatorname{diam} f(E) \leq \operatorname{diam} E$. Therefore (v) together with (iii) and (iv) implies that $f(E)$ can intersect at most one $(m+1)$st level elementary piece of $K_{i_1 \ldots i_n}$. Thus $f(\bigcup \Delta_m) = f(K \setminus K_{i_1 \ldots i_n})$ can intersect at most $\#\Delta_m \leq a_1 \cdots a_m$ many $(m+1)$st level elementary pieces of $K_{i_1 \ldots i_n}$. Theorem 4.2 shows that every $(m+1)$st level elementary piece of K has $\mathcal{H}^h$ measure $1/(a_1 \cdots a_{m+1})$, and $m \geq n$ implies $a_{m+1} \geq a_{n+1}$. Therefore

$$\mathcal{H}^h(K_{i_1 \ldots i_n} \cap f(K \setminus K_{i_1 \ldots i_n})) \leq \frac{a_1 \cdots a_m}{a_1 \cdots a_{m+1}} = \frac{1}{a_{m+1}} \leq \frac{1}{a_{n+1}}. \tag{5.3}$$

Finally, (5.3), the definition of A_n, the subadditivity of $\mathcal{H}^h$, and property (i) yield

$$\mathcal{H}^h(A_n) \leq \frac{a_1 \cdots a_n}{a_{n+1}} \leq \frac{1}{n}.$$

Thus (5.2) follows, and the proof is complete. ∎

6. The proof of our Main Theorems. Let us recall that the main goal of our paper is to answer the following question.

QUESTION 1.6. *What are the lowest possible Borel classes of $\mathbb{R}^n$ having a non-closed element X such that every weak contraction $f: X \to X$ is constant?*

If $n \geq 2$ then the answer is Δ_2^0, and there is no non-closed Δ_2^0 example in $\mathbb{R}$ (see the Introduction). If $n = 1$ then the following theorems show that Σ_2^0 and Π_2^0 are the lowest possible Borel classes satisfying Question 1.6.

THEOREM 6.1 (Main Theorem, F_σ case). *There exists a non-closed F_σ set $F \subseteq \mathbb{R}$ such that every weak contraction $f: F \to F$ is constant.*

Proof. By Theorem 3.5 there exists a balanced compact set $K \subseteq \mathbb{R}$. Let a_n be the positive integers and let h be the continuous gauge function for K as in Definition 3.2 and (4.1), respectively. Set $\mathbb{Q} = \{q_n : n \in \mathbb{N}^+\}$. Pick

$z_0 \in K$ arbitrarily and for all $n \in \mathbb{N}^+$ let K_n^* be the nth level elementary piece of K containing z_0 (see Definition 4.1). Consider

$$F_0 = \bigcup_{n=1}^{\infty} (K_n^* + q_n). \tag{6.1}$$

Clearly, F_0 is an F_σ set, thus $\mathcal{H}^h$ measurable. The countable subadditivity and translation invariance of $\mathcal{H}^h$, and Theorem 4.2, imply

$$\begin{aligned} \mathcal{H}^h(F_0) &\leq \sum_{n=1}^{\infty} \mathcal{H}^h(K_n^* + q_n) = \sum_{n=1}^{\infty} \mathcal{H}^h(K_n^*) \\ &= \sum_{n=1}^{\infty} \frac{1}{a_1 \cdots a_n} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1. \end{aligned}$$

As F_0 is an $\mathcal{H}^h$-measurable set with finite measure, there is a G_δ set $G_0 \subseteq \mathbb{R}$ such that

$$F_0 \subseteq G_0 \quad \text{and} \quad \mathcal{H}^h(G_0 \setminus F_0) = 0 \tag{6.2}$$

(see [R, Thm. 27] for the proof). Set $F = \mathbb{R} \setminus G_0$. Clearly, F is an F_σ set. First we prove that F is non-closed. Fact 3.6 yields $\lambda(K) = 0$, so the translation invariance and countable subadditivity of the Lebesgue measure imply $\lambda(F_0) = 0$. Fact 4.4 and (6.2) imply $\lambda(G_0 \setminus F_0) = 0$. Hence $\lambda(G_0) = 0$. Therefore $G_0 \neq \emptyset$, hence that G_0 is not open, so $F = \mathbb{R} \setminus G_0$ is non-closed. As F is of full Lebesgue measure, it is dense in $\mathbb{R}$.

Assume to the contrary that there exists a non-constant weak contraction $f\colon F \to F$. As F is dense in $\mathbb{R}$, f has a unique 1-Lipschitz extension $\widehat{f}\colon \mathbb{R} \to \mathbb{R}$. First we prove that $\widehat{f}$ is a weak contraction. Assume to the contrary that there are $a, b \in \mathbb{R}$, $a < b$ such that $|\widehat{f}(b) - \widehat{f}(a)| = |b - a|$. Since $\widehat{f}$ is 1-Lipschitz, for all $x, y \in [a, b]$ we have

$$|\widehat{f}(x) - \widehat{f}(y)| = |x - y|. \tag{6.3}$$

Since F is dense in $\mathbb{R}$, there are $x_0, y_0 \in F \cap [a, b]$, $x_0 \neq y_0$. Applying (6.3) for x_0, y_0 contradicts f being a weak contraction. Thus $\widehat{f}$ is a weak contraction.

As f is non-constant, $I = \widehat{f}(\mathbb{R})$ is a non-degenerate interval. Then $\widehat{f}(F) = f(F) \subseteq F$ and the definition of F implies $F_0 \cap I \subseteq I \setminus F \subseteq \widehat{f}(\mathbb{R} \setminus F) = \widehat{f}(G_0)$, so

$$F_0 \cap I \subseteq F_0 \cap \widehat{f}(G_0). \tag{6.4}$$

Property (iii) and $b_n \to 0$ yield $\operatorname{diam} K_n^* \to 0$ as $n \to \infty$. Thus $z_0 \in K_n^*$ implies that there exists an $n \in \mathbb{N}^+$ such that $K_n^* + q_n \subseteq I$, and Theorem 4.2 implies $\mathcal{H}^h(K_n^*) > 0$. Therefore, by translation invariance,

$$\mathcal{H}^h(F_0 \cap I) \geq \mathcal{H}^h(K_n^* + q_n) = \mathcal{H}^h(K_n^*) > 0. \tag{6.5}$$

Theorem 5.1 implies that for all $p, q \in \mathbb{Q}$ we have $\mathcal{H}^h((K+p)\cap \widehat{f}(K+q)) = 0$, as $\widehat{f}(K+q)$ is a weak contractive image of $K+p$. Therefore $F_0 \subseteq K+\mathbb{Q}$ and the countable subadditivity of $\mathcal{H}^h$ yield

$$\begin{aligned}\mathcal{H}^h(F_0 \cap \widehat{f}(F_0)) &\leq \mathcal{H}^h((K+\mathbb{Q}) \cap \widehat{f}(K+\mathbb{Q})) \\ &\leq \sum_{p,q\in\mathbb{Q}} \mathcal{H}^h((K+p)\cap \widehat{f}(K+q)) = 0.\end{aligned} \tag{6.6}$$

As $\widehat{f}$ is a weak contraction and (6.2) holds, we obtain

$$\mathcal{H}^h(\widehat{f}(G_0 \setminus F_0)) \leq \mathcal{H}^h(G_0 \setminus F_0) = 0. \tag{6.7}$$

Finally, (6.5), (6.4), the subadditivity of $\mathcal{H}^h$, (6.6), and (6.7) imply

$$\begin{aligned}0 < \mathcal{H}^h(F_0 \cap I) &\leq \mathcal{H}^h(F_0 \cap \widehat{f}(G_0)) \\ &\leq \mathcal{H}^h(F_0 \cap \widehat{f}(F_0)) + \mathcal{H}^h(\widehat{f}(G_0 \setminus F_0)) = 0.\end{aligned}$$

This is a contradiction, so the proof is complete. ∎

THEOREM 6.2 (Main Theorem, G_δ case). *There exists a non-closed G_δ set $G \subseteq \mathbb{R}$ such that every weak contraction $f\colon G \to G$ is constant.*

Proof. Let $G = \mathbb{R}\setminus F_0$ (for the definition of F_0, see (6.1)). Clearly, G is a G_δ set. Since $\lambda(F_0) = 0$, G is of full Lebesgue measure, thus it is non-closed and dense in $\mathbb{R}$.

Assume to the contrary that $f\colon G \to G$ is a non-constant weak contraction. Now the argument can be completed by replacing F and G_0 in the proof of Theorem 6.1 by G and F_0, respectively. Notice that F_0 remains unchanged, e.g. $G_0 \setminus F_0$ becomes $F_0 \setminus F_0 = \emptyset$. The reason of this asymmetry is that we do not consider G_δ hulls as in (6.2), which makes things a little easier. ∎

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