FUNDAMENTA
MATHEMATICAE
219 (2012)

Separable reduction theorems by the method of elementary submodels

by

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Abstract. We simplify the presentation of the method of elementary submodels and we show that it can be used to simplify proofs of existing separable reduction theorems and to obtain new ones. Given a nonseparable Banach space X and either a subset $A \subset X$ or a function f defined on X, we are able for certain properties to produce a separable subspace of X which determines whether A or f has the property in question. Such results are proved for properties of sets: of being dense, nowhere dense, meager, residual or porous, and for properties of functions: of being continuous, semicontinuous or Fréchet differentiable. Our method of creating separable subspaces enables us to combine results, so we easily get separable reductions of properties such as being continuous on a dense subset, Fréchet differentiable on a residual subset, etc. Finally, we show some applications of separable reduction theorems and demonstrate that some results of Zajíček, Lindenstrauss and Preiss hold in the nonseparable setting as well.

1. Introduction. The method of elementary submodels is a set-theoretical method which can be used in various branches of mathematics. A. Dow [2] illustrated the use of this method in topology, W. Kubiś [5] used it in functional analysis, namely to construct projections on Banach spaces. In the present work we slightly simplify and specify the method of elementary submodels from [5] and we study whether this method can be used to prove separable reduction theorems which have not been proved by other (more standard) methods.

In this way we prove the following two results. First, we show that porosity is a separably determined property. Second, we extend the validity of Zajíček's result [13, Proposition 3.3] from spaces with separable dual to general Asplund spaces.

It seems that the main advantages of the concept of elementary submodels are the following:

2010 *Mathematics Subject Classification*: 46B26, 03C15, 03C30.
Key words and phrases: elementary submodel, separable reduction, Fréchet differentiability, residual set, porous set.

DOI: 10.4064/fm219-3-1

- any finite number of results may be combined,
- the results may be used for more than one space at a time (having two spaces X and Y which are dependent on each other in some way, we can use results for X and Y and combine them).

Thus, the real strength of this method is revealed when we prove enough results to combine them together.

The structure of the paper is as follows: First we introduce elementary submodels and prove some general results. Then we point out how this method is connected with the question of separable subspaces. Next, we collect properties of sets and functions which are separably determined. Finally, we produce two extensions of the results in [13] and [8] using the method of elementary submodels.

Below we recall the most relevant notions, definitions and notations.

We denote by ω the set of all natural numbers (including 0), by $\mathbb{N}$ the set $\omega \setminus \{0\}$, by $\mathbb{R}_+$ the interval $(0, \infty)$, and $\mathbb{Q}_+$ stands for $\mathbb{R}_+ \cap \mathbb{Q}$. Whenever we say that a set is countable, we mean that it is either finite, or infinite and countable. If f is a mapping then we denote by $\operatorname{Rng} f$ its range and by $\operatorname{Dom} f$ its domain. By writing $f : X \to Y$ we mean that f is a mapping with $\operatorname{Dom} f = X$ and $\operatorname{Rng} f \subset Y$. By $f\restriction_Z$ we denote the restriction of f to the set Z. The closure (resp. interior) of a set A are denoted by $\overline{A}$ (resp. $\operatorname{Int}(A)$); the interior relative to a subspace Y is denoted by $\operatorname{Int}_Y(A)$.

If $\langle X, \rho \rangle$ is a metric space, we denote by $B(x, r)$ the open ball $\{y \in X : \rho(x, y) < r\}$. We shall consider normed linear spaces over the field of real numbers (but many results hold for complex spaces as well). If X is a normed linear space and $A \subset X$, we denote by $\operatorname{conv} A$ the convex hull of A, by $\overline{A}^w$ the weak closure of A, and by $\operatorname{span} A$ the linear span of A. Moreover S_X is the unit sphere $\{x \in X : \|x\| = 1\}$, and X^* stands for the (continuous) dual space of X. We denote by $\mathcal{C}(K)$ the space of continuous functions on a compact Hausdorff space K.

2. Elementary submodels. In this section we describe the method of creating countable sets with certain properties using elementary submodels. First, we define what elementary submodels are. Next, we show how countable sets with certain properties can be created using those elementary submodels. This method is based on the set-theoretical Theorem 2.2. This is a combination of the Reflection Theorem and the Löwenheim–Skolem Theorem. We refer the reader to Kunen's book [6], where further details can be found.

The idea to use this method in functional analysis comes from Kubiś's article [5]. Some of the following results are based on this article and slightly modified to our situation (namely Lemma 2.6 and Propositions 2.10, 3.2, 3.6, 3.7).

Let us first recall some definitions:

Let N be a fixed set and ϕ a formula in the language of ZFC. Then the *relativization of ϕ to N* is the formula ϕ^N which is obtained from ϕ by replacing each quantifier of the form "$\forall x$" by "$\forall x \in N$" and each "$\exists x$" by "$\exists x \in N$".

For example, if

$$\phi := (\forall x)(\forall y)(\exists z)(x \in z \wedge y \in z)$$

and $N = \{a, b\}$, then the relativization of ϕ to N is

$$\phi^N = (\forall x \in N)(\forall y \in N)(\exists z \in N)(x \in z \wedge y \in z).$$

It is clear that ϕ is satisfied, but ϕ^N is not.

If $\phi(x_1, \dots, x_n)$ is a formula with all free variables shown (i.e. a formula whose free variables are exactly $x_1, \dots, x_n$) then *ϕ is absolute for N* if and only if

$$(\forall a_1, \dots, a_n \in N)(\phi^N(a_1, \dots, a_n) \leftrightarrow \phi(a_1, \dots, a_n)).$$

A list of formulas, $\phi_1, \dots, \phi_n$, is said to be *subformula closed* if every subformula of a formula in the list is also contained in the list.

Any formula of set theory can be written using the symbols $\in, =, \wedge, \vee, \neg, \rightarrow, \leftrightarrow, \exists, (\,,), [\,,]$ and symbols for variables. Let us assume that a subformula closed list of formulas $\phi_1, \dots, \phi_n$ is written in this way. Then it is not difficult to show that the absoluteness of $\phi_1, \dots, \phi_n$ for N means that those formulas do not create any new sets in N. This result is contained in the following lemma (a proof can be found in [6, Lemma IV.7.3]):

Lemma 2.1. *Let N be a set and $\phi_1, \dots, \phi_n$ a subformula closed list of formulas* (*only containing* $\in, =, \wedge, \vee, \neg, \rightarrow, \leftrightarrow, \exists, (\,,), [\,,]$ *and symbols for variables*). *Then the following are equivalent:*

(i) *$\phi_1, \dots, \phi_n$ are absolute for N.*

(ii) *Whenever ϕ_i is of the form $(\exists x)(\phi_j(x, y_1, \dots, y_l))$* (*with all free variables shown*), *then*

$$(\forall y_1, \dots y_l \in N)[(\exists x)(\phi_j(x, y_1, \dots, y_l)) \rightarrow (\exists x \in N)(\phi_j(x, y_1, \dots, y_l))].$$

The method of elementary submodels is mainly based on the following set-theoretical theorem (a proof can be found in [6, Theorem IV.7.8]).

Theorem 2.2. *Let $\phi_1, \dots, \phi_n$ be any formulas and X any set. Then there exists a set $M \supset X$ such that*

$$(\phi_1, \dots, \phi_n \textit{ are absolute for } M) \wedge (|M| \le \max(\omega, |X|)).$$

Since the set from the previous theorem will often be used, the following definition is useful.

Definition 2.3. Let $\phi_1, \dots, \phi_n$ be any formulas and let X be any countable set. Let $M \supset X$ be a countable set such that $\phi_1, \dots, \phi_n$ are absolute for M. Then we say that M *is an elementary submodel for* $\phi_1, \dots, \phi_n$ *containing* X, and write $M \prec (\phi_1, \dots, \phi_n; X)$. The relation between X, $\phi_1, \dots, \phi_n$ and M is often called the *elementarity of* M.

Using Lemma 2.1 it is easy to see that the countable union of a monotone sequence of elementary submodels is also an elementary submodel.

LEMMA 2.4. *Let* $\varphi_1, \dots, \varphi_n$ *be a subformula closed list of formulas and let* X *be any countable set. Let* $\{M_k\}_{k\in\omega}$ *be a sequence of sets satisfying*

(i) $M_i \subset M_j$, $i \le j$,
(ii) $(\forall k \in \omega)[M_k \prec (\varphi_1, \dots, \varphi_n; X)]$.

Set $M := \bigcup_{k\in\omega} M_k$. *Then also* $M \prec (\varphi_1, \dots, \varphi_n; X)$.

Let $\phi(x_1, \dots, x_n)$ be a formula with all free variables shown and let M be some elementary submodel for ϕ. To use the absoluteness of ϕ for M efficiently, we need to know that many sets are elements of M. The reason is that for $a_1, \dots, a_n \in M$ we have $\phi(a_1, \dots, a_n)$ if and only if $\phi^M(a_1, \dots, a_n)$. Therefore, it is our first aim to force the elementary submodel M to contain many different objects. Let us see a simple example how it can be achieved.

EXAMPLE 2.5. Consider the following formulas:

$$\varphi_1(x, a) := (\forall z)(z \in x \leftrightarrow (z \in a \vee z = a)), \qquad \varphi_2(a) := (\exists x)(\varphi_1(x, a)).$$

Then for any $M \prec (\varphi_1, \varphi_2; \emptyset)$ we have $a \cup \{a\} \in M$ whenever $a \in M$.

Proof. Fix an $a \in M$. Then $\varphi_2(a)$ is satisfied (the set of x satisfying $\varphi_1(x, a)$ is $a \cup \{a\}$). By the absoluteness of φ_2 for M there exists an $x \in M$ satisfying $\varphi_1^M(x, a)$. Fix such an $x \in M$. Then $\varphi_1^M(x, a)$ holds. Therefore, using the absoluteness of φ_1, $\varphi_1(x, a)$ is satisfied as well. But the only possibility for $\varphi_1(x, a)$ to be satisfied is that $x = a \cup \{a\}$; hence $a \cup \{a\} \in M$. ∎

The preceding example can be generalized. Using the following lemma we can force an elementary submodel M to contain all the required objects created (uniquely) from elements of M.

LEMMA 2.6. *Let* $\phi(y, x_1, \dots, x_n)$ *be a formula with all free variables shown and let* X *be a countable set. Let* M *be a fixed set satisfying* $M \prec (\phi, (\exists y)(\phi(y, x_1, \dots, x_n)); X)$ *and let* $a_1, \dots, a_n \in M$ *be such that there exists only one set* u *satisfying* $\phi(u, a_1, \dots, a_n)$. *Then* $u \in M$.

Proof. By the absoluteness of the formula $(\exists y)(\phi(y, x_1, \dots, x_n))$, there exists $y_0 \in M$ satisfying $\phi^M(y_0, a_1, \dots, a_n)$. By the absoluteness of ϕ, for this $y_0 \in M$ the formula $\phi(y_0, a_1, \dots, a_n)$ holds. But such a y_0 is unique and therefore $u = y_0 \in M$. ∎

Let us see how one can force M to contain its finite subsets and natural numbers.

Proposition 2.7. *Consider the following formulas:*

$$\begin{aligned}\varphi_1 &:= (\forall z)(z \in x \leftrightarrow z \neq z),\\ \varphi_{1E} &:= (\exists x)(\varphi_1(x)),\\ \varphi_2 &:= (\forall z)(z \in x \leftrightarrow (z \in u \vee z = v)),\\ \varphi_{2E} &:= (\exists x)(\varphi_2(x, u, v)).\end{aligned}$$

Let X be a nonempty countable set. Then

(i) *if $M \prec (\varphi_1, \varphi_{1E}; X)$, then $\emptyset \in M$;*
(ii) *if $M \prec (\varphi_2, \varphi_{2E}; X)$, then $u \cup \{v\} \in M$ for every $u, v \in M$;*
(iii) *if $M \prec (\varphi_1, \varphi_{1E}, \varphi_2, \varphi_{2E}; X)$, then $\omega \subset M$;*
(iv) *if $M \prec (\varphi_1, \varphi_{1E}, \varphi_2, \varphi_{2E}; X)$, then $s \in M$ for every finite set $s \subset M$.*

Proof. (i) and (ii) follow immediately from Lemma 2.6; (iii) follows from (i) and (ii) by induction on n; (iv) follows from (i) and (ii) by induction on the cardinality of s. ■

It would be laborious and pointless to use only the basic language of set theory. For example, we often write $x < y$ as a shortcut for the formula $\varphi(x, y, <)$ with all free variables shown. Therefore, in the following text we use this extended language of set theory, as is customary. We shall also use the following convention.

Convention. Whenever we say

- *for any suitable elementary submodel M (the following holds...)*,

we mean that

- *there exists a list of formulas $\phi_1, \dots, \phi_n$ and a countable set Y such that for every $M \prec (\phi_1, \dots, \phi_n; Y)$ (the following holds...).*

By using this terminology we hide the information about the formulas $\phi_1, \dots, \phi_n$ and the set Y. This is, however, not important in applications.

Remark 2.8. Suppose have sentences $T_1(a), \dots, T_n(a)$. Assume that whenever we fix an $i \in \{1, \dots, n\}$, then for any suitable elementary submodel M_i the sentence $T_i(M_i)$ is satisfied. Then it is easy to verify that for any suitable elementary submodel M the sentence

$$T_1(M) \wedge \cdots \wedge T_n(M)$$

is satisfied (it suffices to combine all the lists of formulas and all the sets from the convention above). In other words, we are able to combine any finite number of results we have proved using the method of elementary submodels. This includes all the theorems starting with "For any suitable elementary submodel M the following holds:".

Let us give some more results about suitable elementary submodels.

Proposition 2.9. *For any suitable elementary submodel M the following holds: Let f be a function such that $f \in M$. Then*

(i) $\mathrm{Dom}\, f \in M$,
(ii) $\mathrm{Rng}\, f \in M$,
(iii) $(\forall x \in M \cap \mathrm{Dom}\, f)(f(x) \in M)$.

Proof. Fix an elementary submodel M for formulas marked $(*)$ in the proof below and all their subformulas. Let $f \in M$ be a function. Then $\mathrm{Dom}\, f$ is the object uniquely defined by the following formula (the same for all functions f; f is a free variable in this formula):

$$(*) \qquad (\exists D)(\forall x)[x \in D \leftrightarrow (\exists y)(f(x) = y)].$$

By Lemma 2.6, $\mathrm{Dom}\, f \in M$. Similarly, $\mathrm{Rng}\, f \in M$ as it is the object uniquely defined by the formula

$$(*) \qquad (\exists R)(\forall y)[y \in R \leftrightarrow (\exists x)(f(x) = y)].$$

For (iii) we use (i) and the absoluteness of the formula

$$(*) \qquad (\forall x \in \mathrm{Dom}\, f)(\exists y)(f(x) = y). \ \blacksquare$$

The proofs in the following text often begin in the same way. To avoid unnecessary repetitions, by saying "Fix a $(*)$-elementary submodel M [containing $A_1, \ldots, A_n$]" we will understand the following:

"Consider the formulas $\varphi_1, \varphi_{1E}, \varphi_2, \varphi_{2E}$ from Proposition 2.7 and all the formulas marked $(*)$ in all the preceding proofs (and all their subformulas). Add to them formulas marked $(*)$ in the proof below (and all their subformulas). Denote by $\phi_1, \ldots, \phi_n$ the resulting list of formulas. Fix a countable set X containing the sets ω, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}_+$, $\mathbb{R}$, $\mathbb{R}_+$ and all the common operations and relations on real numbers, $(+, -, \cdot, :, <)$. Fix an elementary submodel M for formulas $\phi_1, \ldots, \phi_n$ with $X \in M$ [such that $A_1, \ldots, A_n \in M$]".

Thus, any $(*)$-elementary submodel M is suitable for all the preceding theorems, propositions and lemmas from this paper, making it possible to use all these results for M.

Using this new agreement, let us prove another proposition.

Proposition 2.10. *For any suitable elementary submodel M the following holds:*

(i) *Let S be a finite set. Then*

$$S \in M \leftrightarrow S \subset M.$$

(ii) *Let S be a countable set. Then*

$$S \in M \rightarrow S \subset M.$$

(iii) *For every natural number* $n > 0$ *and for arbitrary sets* $a_0, \dots, a_n$,

$$a_0, \dots, a_n \in M \leftrightarrow \langle a_0, \dots, a_n \rangle \in M.$$

(iv) *If* $A, B \in M$, *then* $A \cap B \in M$, $B \setminus A \in M$ *and* $A \cup B \in M$.

Proof. Fix a $(*)$-elementary submodel M. To prove (ii), let $S \in M$ be a countable set. If $S = \emptyset$, then $S \subset M$. If $S \neq \emptyset$, then

$$(*) \qquad (\exists f)(f \text{ is a function from } \omega \text{ onto } S).$$

Thus, by the absoluteness of the formula above, there exists $f \in M$ satisfying

$$(f \text{ is a function from } \omega \text{ onto } S)^M.$$

Fix one such f. Then, using the absoluteness of the formula "f is a function from ω onto S", f is a function from ω onto S. Because f is a function with $\operatorname{Rng} f = S$ and $\operatorname{Dom} f = \omega \subset M$, by Proposition 2.9, $S \subset M$.

Let us prove that (i) holds. If $S \in M$ is finite, then $S \subset M$ by (ii). If $S \subset M$ is finite, then $S \in M$ by Proposition 2.7.

(iii) follows easily from (i) by induction on $n \in \omega, n \geq 1$. It is enough to realize that $\langle a_0, a_1 \rangle = \{a_0, \{a_0, a_1\}\}$ and $\langle a_0, \dots, a_n \rangle = \langle \langle a_0, \dots, a_{n-1} \rangle, a_n \rangle$.

Suppose we have sets $A, B \in M$. Then, by Lemma 2.6 and the absoluteness of the formulas (and their subformulas)

$$(*) \qquad (\exists C)(\forall x)(x \in C \leftrightarrow x \in A \wedge x \in B),$$

$$(*) \qquad (\exists D)(\forall x)(x \in D \leftrightarrow x \in B \wedge x \notin A),$$

$$(*) \qquad (\exists E)(\forall x)(x \in E \leftrightarrow x \in A \vee x \in B),$$

(iv) holds. ∎

3. Elementary submodels in the context of normed linear spaces. Now we are prepared for some more concrete results concerning mostly metric spaces or normed linear spaces. Before we proceed, let us propose the following agreements.

If $\langle X, \rho \rangle$ is a metric space (resp. $\langle X, +, \cdot, \|\cdot\| \rangle$ is a normed linear space) and M an elementary submodel, then by saying M *contains* X (or by writing $X \in M$) we mean that $\langle X, \rho \rangle \in M$ (resp. $\langle X, +, \cdot, \|\cdot\| \rangle \in M$). If A is a set, then by saying that an elementary submodel M contains A we mean that $A \in M$.

If X is a topological space and M an elementary submodel, then we denote by X_M the set $\overline{X \cap M}$.

PROPOSITION 3.1. *For any suitable elementary submodel M the following holds: Let $\langle X, \rho \rangle$ be a metric space. If M contains X, then $B(x, r) \in M$ whenever $x \in X \cap M$ and $r \in \mathbb{R}_+ \cap M$.*

Proof. Fix a $(*)$-elementary submodel M containing X. Let $x \in X \cap M$ and $r \in \mathbb{R}_+ \cap M$. Then $B(x,r)$ is the object uniquely defined by the formula

$$(*) \qquad (\exists U)(\forall z)(z \in U \leftrightarrow z \in X \wedge \rho(x,z) < r).$$

Thus, by Lemma 2.6, $B(x,r) \in M$. ∎

The idea of the following proposition comes from [5].

PROPOSITION 3.2. *For any suitable elementary submodel M the following holds: Let X be a normed linear space. If M contains X and a set $A \subset X$, then:*

(i) $\overline{\operatorname{span}(A) \cap M}$ *is a closed separable linear subspace of X.*
(ii) $\overline{\operatorname{conv}(A) \cap M}$ *is a convex set.*
(iii) *If A is convex, then $\overline{A \cap M} = \overline{A \cap M}^w$.*

In particular, X_M is a separable subspace of X and $X_M = \overline{X \cap M}^w$.

Proof. Fix a $(*)$-elementary submodel M containing X and A. By Proposition 2.10, $\mathbb{Q} \subset M$ and $\langle \mathbb{R}, +, -, \cdot, :, < \rangle \in M$.

The elementary submodel M contains the functions $+ : X \times X \to X$ and $\cdot : \mathbb{R} \times X \to X$. Consequently (by Proposition 2.9), $X \cap M$ is a $\mathbb{Q}$-linear subspace of X. Therefore (i) and (ii) hold. Assertion (iii) follows easily from (ii) because for convex sets the weak and the norm closures coincide. ∎

Given a Banach space X, a list of formulas $\phi_1, \ldots, \phi_n$ and a countable set Y, we are able to get a family of sets

$$\mathcal{M}(X) := \{X_M : M \prec (\phi_1, \ldots, \phi_n; Y)\}.$$

By choosing $\phi_1, \ldots, \phi_n$ and Y suitably, it is possible to force $\mathcal{M}(X)$ to be a family of closed separable subspaces of X having some specific properties. One can easily join a finite number of arguments (lists of formulas) and get another family of separable subspaces having the same properties as the original family and perhaps even some more.

In [9] similar families of closed separable subspaces are used to get separable reduction theorems. Those families are called rich. This concept has been originally introduced in [1] by Borwein and Moors. For further applications of this method, see for example [10], where more references can be found.

DEFINITION 3.3. Let X be a Banach space. A family $\mathcal{R}$ of separable subspaces of X is called *rich* if

(i) for every increasing sequence R_i in $\mathcal{R}$, $\overline{\bigcup_{i \in \omega} R_i}$ belongs to $\mathcal{R}$, and
(ii) each separable subspace of X is contained in an element of $\mathcal{R}$.

A connection between the notion of rich families and elementary submodels is described in the following lemma.

LEMMA 3.4. *Let X be a Banach space. Then there exists a list of formulas $\phi_1, \ldots, \phi_n$ and a countable set Y such that for every countable set Z and every list of formulas $\varphi_1, \ldots, \varphi_k$ such that $\phi_1, \ldots, \phi_n, \varphi_1, \ldots, \varphi_k$ is subformula closed, the family*

$$\mathcal{M} := \{M : M \prec (\phi_1, \ldots, \phi_n, \varphi_1, \ldots, \varphi_k; Y \cup Z)\}$$

satisfies the following conditions:

(i) $\{X_M : M \in \mathcal{M}\}$ *is a family of closed separable subspaces of X.*
(ii) *For every increasing sequence $\{M_i\}_{i\in\omega} \subset \mathcal{M}$ of elementary submodels,*

$$\bigcup_{i\in\omega} M_i \in \mathcal{M} \quad \textit{and} \quad \overline{\bigcup_{i\in\omega} X_{M_i}} = X_{\bigcup_{i\in\omega} M_i}.$$

(iii) *For every separable subspace V of X there exists $M \in \mathcal{M}$ such that $V \subset X_M$.*

Proof. The existence of $\phi_1, \ldots, \phi_n$ and Y such that $\{X_M : M \in \mathcal{M}\}$ is a family of closed separable subspaces follows from Proposition 3.2 above. For (ii), fix an increasing sequence M_i of elementary submodels from the assumption. Then (by Lemma 2.4) it is enough to show that $\overline{\bigcup_{i\in\omega} X_{M_i}} = X_{\bigcup_{i\in\omega} M_i}$. One inclusion follows from the fact that $\bigcup_{i\in\omega} X_{M_i} \subset \overline{\bigcup_{i\in\omega} X \cap M_i} = X_{\bigcup_{i\in\omega} M_i}$. The opposite one holds, because $\bigcup_{i\in\omega} X \cap M_i \subset \bigcup_{i\in\omega} \overline{X \cap M_i} = \bigcup_{i\in\omega} X_{M_i}$. Thus, $X_{\bigcup_{i\in\omega} M_i} = \overline{\bigcup_{i\in\omega} X \cap M_i} \subset \overline{\bigcup_{i\in\omega} X_{M_i}}$. For (iii), take any separable subspace V of X and a countable set D dense in V. Then taking $M \prec (\phi_1, \ldots, \phi_n, \varphi_1, \ldots, \varphi_k; Y \cup Z \cup D)$, we conclude that $V \subset X_M$. ■

It is not known to the author whether those two approaches (rich families and elementary submodels) to the separable reduction theorems are equivalent in some way. It seems that the method using elementary submodels is slightly stronger, as it is capable of working with more than one space at a time (see Lemmas 3.6, 3.7 and 6.15), while the method of rich families concerns one space.

In [5] a slightly different method of getting the elementary submodels M is introduced. It is proved there that in the case of some classical Banach spaces (namely $\ell_p(\Gamma)$ and $\mathcal{C}(K)$) it is possible to describe the subspace X_M. Slightly modifying the ideas from [5], the same results hold in our case as well.

DEFINITION 3.5. Let Γ be a set. Then we denote by suppt_Γ the function which maps $x \in \mathbb{R}^\Gamma$ to $\mathrm{suppt}_\Gamma(x) = \{\alpha \in \Gamma : x(\alpha) \neq 0\}$.

PROPOSITION 3.6. *For any suitable elementary submodel M the following holds: Let $X = \ell_p(\Gamma)$, where $1 \le p < \infty$ and Γ is an arbitrary set. If M*

contains X, suppt_Γ *and Γ, then*

$$X_M = \{x \in X : \mathrm{suppt}_\Gamma(x) \subset M\}.$$

Consequently, X_M can be identified with $\ell_p(\Gamma \cap M)$.

Proof. Fix a $(*)$-elementary submodel M containing X, suppt_Γ, Γ. Denote by A the set on the right-hand side above. For every $x \in X \cap M$ the set $\mathrm{suppt}_\Gamma(x)$ is countable. Thus, by Propositions 2.9 and 2.10, $\mathrm{suppt}_\Gamma(x) \subset M$ and $x \in A$. We have proved that $X \cap M \subset A$. From the obvious fact that A is a closed set we have $X_M \subset A$. On the other hand, if $x \in A$ then arbitrarily close to x we can find $y \in A$ such that $s = \mathrm{suppt}_\Gamma(y) \subset M$ is finite and $y(\alpha) \in \mathbb{Q}$ for $\alpha \in s$. Thus, using Proposition 2.10, we have $s \in M$ and $y\restriction_s \in M$ (because $y\restriction_s = \bigcup_{\alpha\in s}\{\langle\alpha, y(\alpha)\rangle\}$). Using the absoluteness of the formula

$$(*) \qquad (\exists z \in X)(z\restriction_s = y\restriction_s \wedge z\restriction_{\Gamma\setminus s} = 0),$$

we have $y \in M$. Hence $x \in \overline{X \cap M} = X_M$. ∎

Given a compact space K and an arbitrary elementary submodel M we define the following equivalence relation $\sim_M$ on K:

$$x \sim_M y \;\leftrightarrow\; (\forall f \in \mathcal{C}(K) \cap M))(f(x) = f(y)).$$

We shall write K/M instead of $K/\sim_M$ and we shall denote by q^M the canonical quotient map. It is not hard to check that K/M is a compact Hausdorff space.

Observe that we can identify the spaces $\{\varphi \circ q^M : \varphi \in \mathcal{C}(K/M)\}$ and $\mathcal{C}(K/M)$. Indeed, define

$$F(\varphi) := \varphi \circ q^M, \qquad \varphi \in \mathcal{C}(K/M).$$

It is obvious that F is an isometric mapping from $\mathcal{C}(K/M)$ onto $\{\varphi \circ q^M : \varphi \in \mathcal{C}(K/M)\}$.

Lemma 3.7. *For any suitable elementary submodel M the following holds: Let K be a compact space and $X = \mathcal{C}(K)$. Let $\cdot$ denote the pointwise product of functions in $\mathcal{C}(K)$. If M contains X, $\cdot$ and K, then*

$$X_M = \{\varphi \circ q^M : \varphi \in \mathcal{C}(K/M)\}.$$

Consequently, we can identify X_M with $\mathcal{C}(K/M)$, where K/M is a metrizable compact space.

Proof. Fix a $(*)$-elementary submodel M containing X, $\cdot$ and K. Denote by Y the set on the right-hand side above. For a given function $f \in \mathcal{C}(K) \cap M$ we define

$$\varphi([x]_M) := f(x), \quad x \in K.$$

It is easy to verify that φ is a continuous function. Consequently, $f \in Y$ and $X_M \subset Y$.

For the proof of the reverse inclusion, let us identify X_M with a subspace of $\mathcal{C}(K/M)$. Then, by Propositions 3.2 and 2.9, X_M is a closed subspace closed under the operation $\cdot$. From the definition of $\sim_M$ it follows that X_M separates points in K/M. Using the absoluteness of the formula

$$(*) \qquad (\forall c \in \mathbb{R})(\exists f \in X)(\forall x \in K : f(x) = c),$$

M contains every constant rational function; thus, X_M contains all the constant functions. From the Stone–Weierstrass theorem, $X_M = \mathcal{C}(K/M)$.

Since $X_M = \mathcal{C}(K/M)$ is a separable space, K/M is metrizable. ■

4. Properties of sets. Let us consider the following situation. Let X be a normed linear space. We would like to recognize whether a given set $A \subset X$ has a property (P). For every separable subspace $V_0 \subset X$ we would like to find a closed separable subspace $V \supset V_0$ such that A has property (P) in X if and only if $A \cap V$ has property (P) in the subspace V.

Using the method of elementary submodels, it is enough to show that for any suitable elementary submodel M (dependent only on the space X and perhaps also on the set A), the set A has property (P) if and only if $A \cap X_M$ has property (P) in X_M.

Let us prove some results for the properties of being dense and having empty interior.

PROPOSITION 4.1. *For any suitable elementary submodel M the following holds: Let $\langle X, \rho\rangle$ be a metric space and $A, S \subset X$. If M contains X, A and S, then*

$$\mathrm{Int}_S(A \cap S) \neq \emptyset \;\leftrightarrow\; \mathrm{Int}_{S\cap X_M}(A \cap S \cap X_M) \neq \emptyset,$$
$$A \cap S \text{ is dense in } S \;\leftrightarrow\; A \cap S \cap X_M \text{ is dense in } S \cap X_M.$$

Proof. Fix a $(*)$-elementary submodel M containing X, A and S. By Proposition 2.10, $A^C \in M$ whenever $A \in M$. Since A is dense in X if and only if A^C has empty interior in X, it is enough to prove the first equivalence.

If $A \cap S$ has nonempty interior in S, then there exists a ball in S which is a subset of $A \cap S$. Thus,

$$(*) \qquad (\exists x \in S)(\exists r \in \mathbb{R}_+)(\forall y \in S)(y \in B(x,r) \to y \in A).$$

In the preceding formula we use the abbreviation $y \in B(x,r)$ for $y \in X \wedge \rho(y,x) <_{\mathbb{R}} r$. Free variables in the preceding formula are $\mathbb{R}_+, X, \rho, <_{\mathbb{R}}, A, S$. Those are contained in M. This allows us to use the elementarity of M (i.e. the absoluteness of the preceding formula for M). Thus, we find $x \in S \cap M$ and $r \in \mathbb{R}_+ \cap M$ such that $((\forall y \in S)(y \in B(x,r) \to y \in A))^M$. By elementarity again, $B(x,r) \cap S$ is a subset of $A \cap S$. Consequently, $B(x,r) \cap S \cap X_M \subset A \cap S \cap X_M$. Since $x \in B(x,r) \cap S \cap X_M$, we have proved that $A \cap S \cap X_M$ contains a nonempty open set in $S \cap X_M$.

Conversely, assume that $\mathrm{Int}_{S\cap X_M}(A\cap S\cap X_M)\neq\emptyset$. Then

$$(\exists x\in S\cap X_M)(\exists r\in\mathbb{R}_+)(B(x,r)\cap S\cap X_M\subset A\cap S).$$

Take $q\in(0,r/2)\cap\mathbb{Q}_+$ and $x_0\in X\cap M$ such that $\rho(x,x_0)<q$. Then

$$B(x_0,q)\cap S\cap X_M\subset B(x,r)\cap S\cap X_M\subset A\cap S.$$

The statement $B(x_0,q)\cap S\cap M\subset A\cap S$ can be written in the following way:

$$(\forall y\in S\cap M)(\rho(y,x_0)<q\to y\in A\cap S).$$

Therefore, using the absoluteness of

$$(*)\qquad (\forall y\in S)(\rho(y,x_0)<q\to y\in A\cap S),$$

we can see that $B(x_0,q)\cap S\subset A\cap S$. But the point x is in $B(x_0,q)\cap S$. Consequently, $\mathrm{Int}_S(A\cap S)\neq\emptyset$. ∎

Another set property which is separably determined is that of being nowhere dense.

Proposition 4.2. *For any suitable elementary submodel M the following holds: Let $\langle X,\rho\rangle$ be a metric space, $G\subset X$ an open set and $A\subset X$. If M contains X, A and G, then*

$$A\cap G \text{ is nowhere dense in } G \;\leftrightarrow\; A\cap G\cap X_M \text{ is nowhere dense in } G\cap X_M.$$

Proof. Fix a $(*)$-elementary submodel M containing X, A and G. By Proposition 2.10, $C\cap B\in M$ whenever $C,B\in M$. It is well known that $E\subset G$ is nowhere dense in G if and only if it is nowhere dense in X (see [7, p. 71]). Consequently, it is enough to prove the proposition for $G=X$.

It is well known that a set A is nowhere dense in a metric space X if and only if the following formula holds:

$$(\forall x\in X)(\forall r\in\mathbb{R}_+)(\exists y\in X)(\exists s\in\mathbb{R}_+)(B(y,s)\subset B(x,r)\setminus A).$$

It is easy to check that this is equivalent to

$$(4.1)(*)\qquad (\forall x\in X)(\forall r\in\mathbb{Q}_+)(\exists y\in X)(\exists s\in\mathbb{Q}_+)(B(y,s)\subset B(x,r)\setminus A).$$

All the free variables in the last formula are elements of M.

Let us prove the implication from right to left first. If A is not nowhere dense in X, then

$$(*)\qquad (\exists x\in X)(\exists r\in\mathbb{Q}_+)(\forall y\in X)(\forall s\in\mathbb{Q}_+)(B(y,s)\not\subset B(x,r)\setminus A).$$

Using the elementarity of M, there exist $x\in X\cap M$ and $r\in\mathbb{Q}_+$ such that

$$(4.2)\qquad (\forall y\in X)(\forall s\in\mathbb{Q}_+)(B(y,s)\not\subset B(x,r)\setminus A).$$

Choose an arbitrary $y\in X_M$, $s\in\mathbb{Q}_+$ and find $y_0\in X\cap M$ such that $\rho(y,y_0)<s/2$. Then $B(y_0,s/2)\subset B(y,s)$. From (4.2),

$$(*)\qquad (\exists z\in X)(z\in B(y_0,s/2)\setminus(B(x,r)\setminus A)).$$

Using the elementarity of M, we may fix $z \in X \cap M$ satisfying the formula above. Thus, for given $y \in X_M$ and $s \in \mathbb{Q}_+$ we have found $z \in X \cap M$ satisfying

$$z \in B(y_0, 1/2s) \setminus (B(x,r) \setminus A) \subset B(y,s) \setminus (B(x,r) \cap X_M \setminus A).$$

Consequently,

$$B(y,s) \cap X_M \not\subset (B(x,r) \cap X_M) \setminus A.$$

The negation of (4.1)($*$) holds in X_M; thus, $A \cap X_M$ is not nowhere dense in X_M.

For the proof of the converse, let A be nowhere dense in X. Choose any $x \in X_M$ and $r \in \mathbb{Q}_+$. Pick $x_0 \in X \cap M$ satisfying $\rho(x, x_0) < r/2$. Then $B(x_0, r/2) \subset B(x,r)$. For the point x_0 and the number $r/2$ choose $y \in X$ and $s \in \mathbb{Q}_+$ as in formula (4.1)($*$). Using the elementarity of M, we may assume that $y \in X \cap M$. Consequently,

$$B(y,s) \subset B(x_0, r/2) \setminus A \subset B(x,r) \setminus A.$$

Formula (4.1)($*$) is satisfied in X_M; thus, $A \cap X_M$ is nowhere dense in X_M. ∎

It is natural to ask whether the property of being meager is separably determined. One implication is easy:

Proposition 4.3. *For any suitable elementary submodel M the following holds: Let X be a metric space. If M contains X and a set $A \subset X$, then*

$$A \text{ is meager in } X \ \rightarrow\ A \cap X_M \text{ is meager in } X_M.$$

Proof. Fix a ($*$)-elementary submodel M containing X and A. Let $\{R_n\}_{n\in\omega}$ be a family of nowhere dense sets such that $A \subset \bigcup_{n\in\omega} R_n$. Then

$$(*)\qquad (\exists\varphi)\Big(\varphi \text{ is a function with } \operatorname{Dom}\varphi = \omega,\ \varphi(n) \text{ is a nowhere dense subset of } X \text{ for every } n \in \omega, \text{ and } A \subset \bigcup_{n\in\omega} \varphi(n)\Big).$$

Using the elementarity of M, we find $\varphi \in M$ satisfying the formula above. Consequently, by Proposition 2.9, $\varphi(n) \in M$ for every $n \in \omega$.

By Proposition 4.2, the set $\varphi(n) \cap X_M$ is nowhere dense in X_M for every $n \in \omega$. Moreover, $A \cap X_M \subset \bigcup_{n\in\omega}(\varphi(n) \cap X_M)$. Therefore, $A \cap X_M$ is meager in X_M. ∎

For the converse to the implication of the preceding proposition, we need to add some assumptions. Let us first recall what it means to be somewhere meager.

Definition 4.4. Let X be a metric space and $A \subset X$. If there are $x \in X$ and $r > 0$ such that $B(x,r) \cap A$ is meager in X, we say that A *is somewhere meager in X*.

We will need the following easy and well-known fact.

Lemma 4.5. *Let X be a complete metric space and let $A \subset X$ have the Baire property. Then*

$$X \setminus A \text{ is not meager} \quad \leftrightarrow \quad A \text{ is somewhere meager in } X.$$

With the help of this lemma we can prove a converse to the implication of Proposition 4.3. First, we need a result for the properties of having the Baire property and being somewhere meager.

Proposition 4.6. *For any suitable elementary submodel M the following holds: Let X be a metric space. If M contains X and a set $A \subset X$, then*

$$A \text{ is somewhere meager in } X \;\rightarrow\; A \cap X_M \text{ is somewhere meager in } X_M.$$

Proof. Fix a $(*)$-elementary submodel M containing X and A, and assume that A is somewhere meager. By Propositions 2.10 and 3.1, $B(x,r) \in M$ whenever $x \in X \cap M$ and $r \in \mathbb{R}_+ \cap M$, and $C \cap B \in M$ whenever $C, B \in M$.

Because A is somewhere meager, the following formula holds:

$$(*) \qquad (\exists x \in X)(\exists r \in \mathbb{R}_+)(B(x,r) \cap A \text{ is meager in } X).$$

Using the elementarity of M, we find $x \in X \cap M$ and $r \in \mathbb{R}_+ \cap M$ such that $B(x,r) \cap A$ is meager in X. Since $B(x,r) \cap A \in M$, by Proposition 4.3, $B(x,r) \cap A \cap X_M$ is meager in X_M. ∎

Proposition 4.7. *For any suitable elementary submodel M the following holds: Let X be a metric space. If M contains X and a set $A \subset X$, then*

$$A \text{ has the Baire property in } X \;\rightarrow\; A \cap X_M \text{ has the Baire property in } X_M.$$

Proof. Fix a $(*)$-elementary submodel M containing X and A and assume that A has the Baire property. Then

$$(*) \qquad (\exists D)(\exists P)(D \text{ is } G_\delta \text{ in } X,\ P \text{ is meager in } X, \text{ and } A = D \cup P).$$

Using the elementarity of M, we find $D, P \in M$ satisfying the formula above. By Proposition 4.3, $P \cap X_M$ is meager in X_M. Consequently, $A \cap X_M$ is the union of the G_δ set $D \cap X_M$ and the meager set $P \cap X_M$. ∎

Finally, we prove a converse of Proposition 4.3 under additional assumptions.

Theorem 4.8. *For any suitable elementary submodel M the following holds: Let X be a complete metric space, $G \subset X$ an open set and $A \subset X$ a set with the Baire property. If M contains X, G and A, then*

$$A \cap G \textit{ is meager in } G \ \leftrightarrow \ A \cap G \cap X_M \textit{ is meager in } G \cap X_M,$$
$$A \cap G \textit{ is residual in } G \ \leftrightarrow \ A \cap G \cap X_M \textit{ is residual in } G \cap X_M.$$

Proof. Fix a $(*)$-elementary submodel M containing X, A and G. By Proposition 2.10, $B \cap C \in M$ and $B^C \in M$ whenever $B, C \in M$. It is well known that a set $D \subset G$ is meager in X if and only if it is meager in G (see [7, p. 83]). Thus, it is sufficient to prove the first equivalence for $G = X$.

The implication from left to right follows from Proposition 4.3. For the converse, assume that A is not meager in X. Then, by Lemma 4.5, A^C is somewhere meager in X. Thus, by Proposition 4.6, $A^C \cap X_M$ is somewhere meager in X_M. Hence, by Propositions 4.7 and 4.5, $A \cap X_M$ is not meager in X_M. ∎

Let us find out whether the property of sets of being porous is separably determined. We use the following definition from [11].

Definition 4.9. Let X be a metric space, $A \subset X$, $x \in X$ and $R > 0$. Then we define $\gamma(x, R, A)$ as the supremum of all $r \geq 0$ for which there exists $z \in X$ such that $B(z, r) \subset B(x, R) \setminus A$.

Further, we define the *upper porosity of A at x in X* as

$$\overline{p}_X(A, x) := 2 \limsup_{R \to 0^+} \frac{\gamma(x, R, A)}{R},$$

and the *lower porosity of A at x in X* as

$$\underline{p}_X(A, x) := 2 \liminf_{R \to 0^+} \frac{\gamma(x, R, A)}{R}.$$

When it is clear which space X we mean, we often say *upper* (resp. *lower*) *porosity of A at x* and write $\overline{p}(A, x)$ (resp. $\underline{p}(A, x)$).

We say that A is *upper porous* (resp. *lower porous*, *c-upper porous*, *c-lower porous*) at x if $\overline{p}(A, x) > 0$ (resp. $\underline{p}(A, x) > 0$, $\overline{p}(A, x) \geq c$, $\underline{p}(A, x) \geq c$).

We say that A is *upper porous* (resp. *lower porous*, *c-upper porous*, *c-lower porous*) if A is upper porous (resp. lower porous, c-upper porous, c-lower porous) at each $y \in A$. We say that A is *σ-upper porous* (resp. *σ-lower porous*) if it is a countable union of upper porous (resp. lower porous) sets.

Definition 4.10. Let $\langle X, \rho \rangle$ be a metric space and $A \subset X$. Then $\mathrm{d}(x, A) := \inf\{\rho(x, a) : a \in A\}$ for $x \in X$.

The following lemma is probably well known, but I have not found any reference.

Lemma 4.11. *Let $\langle X, \rho \rangle$ be a metric space, $A \subset X$ and $x \in A$. Set*

$$p_1(A, x) := \limsup_{R \to 0^+} \sup_{u \in B(x,R)} \frac{\mathrm{d}(u, A)}{R}, \qquad p_2(A, x) := \liminf_{R \to 0^+} \sup_{u \in B(x,R)} \frac{\mathrm{d}(u, A)}{R}.$$

Then $p_1(A, x) \leq \overline{p}(A, x) \leq 2p_1(A, x)$ and $p_2(A, x) \leq \underline{p}(A, x) \leq 2p_2(A, x)$.

Proof. To show $\overline{p}(A,x) \le 2p_1(A,x)$ and $\underline{p}(A,x) \le 2p_2(A,x)$, it is sufficient to prove that $\gamma(x,R,A) \le \sup_{u\in B(x,R)} \mathrm{d}(u,A)$ for every $R>0$. Choose $R>0$, $r\ge 0$ and $z\in X$ satisfying $B(z,r)\subset B(x,R)\setminus A$. We would like to find $u\in B(x,R)$ such that $r\le \mathrm{d}(u,A)$. But it is easy to check that $u=z$ satisfies those conditions.

Now we will prove that $\overline{p}(A,x)\ge p_1(A,x)$ and $\underline{p}(A,x)\ge p_2(A,x)$. Take any $R>0$ and $u\in B(x,R)$ and notice that then $\mathrm{d}(u,A)\le\gamma(x,2R,A)$.

Indeed, put $r=\mathrm{d}(u,A)$ and $z=u$. Then for every $y\in B(z,r)$ we have

$$\rho(u,y)=\rho(z,y)<r=\mathrm{d}(u,A),$$

so $y\notin A$. Moreover (using the fact that $r=\mathrm{d}(u,A)<R$, since $x\in A$ and so $u\in B(x,R)$),

$$\rho(y,x)\le\rho(y,z)+\rho(z,x)<r+R<2R.$$

Thus, $B(z,r)\subset B(x,2R)\setminus A$ and $\mathrm{d}(u,A)\le\gamma(x,2R,A)$.

As an immediate consequence we get

$$2\limsup_{R\to 0^+}\frac{\gamma(x,2R,A)}{2R}\ge p_1(A,x),\qquad 2\liminf_{R\to 0^+}\frac{\gamma(x,2R,A)}{2R}\ge p_2(A,x).$$

Now it is easy to check that also $\overline{p}(A,x)\ge p_1(A,x)$ and $\underline{p}(A,x)\ge p_2(A,x)$. ∎

The following two propositions show that the first implication about porous sets holds.

Proposition 4.12. *For any suitable elementary submodel M the following holds: Let $\langle X,\rho\rangle$ be a metric space. If M contains X and a set $A\subset X$, then*

A is not upper porous in X $\;\to\;$ $A\cap X_M$ is not upper porous in X_M.

Proof. Fix a $(*)$-elementary submodel M containing X and A. The set A is upper porous in X if and only if the following formula holds:

$$(\forall x\in A)(\exists m\in\mathbb{Q}_+)(\forall R_0>0)(\exists R\in(0,R_0))(\gamma(x,R,A)>Rm).$$

This formula is equivalent to

$$(\forall x\in A)(\exists m\in\mathbb{Q}_+)(\forall R_0>0)(\exists R\in(0,R_0))(\exists r>Rm)(\exists z\in X)\\ (B(z,r)\subset B(x,R)\setminus A).$$

Notice that this last formula is equivalent to one where we take only rational numbers R_0,R and r. Indeed, it is obvious that we may consider only rational numbers R_0. Take any $x\in A$ choose $m\in\mathbb{Q}_+$ as in the formula above, and pick $R_0\in\mathbb{Q}_+$. Then

$$(\exists R\in(0,R_0))(\exists r>Rm)(\exists z\in X)(B(z,r)\subset B(x,R)\setminus A).$$

Fix $R\in(0,R_0)$, $r>Rm$ and $z\in X$ as in the formula above. If we take a rational number $R_q\in(R,\min\{R_0,r/m\})$, then $B(z,r)\subset B(x,R_q)\setminus A$.

Thus, R may be without loss of generality considered to be rational. Having now the rational $R \in (0, R_0)$, real $r > Rm$ and $z \in X$ such that $B(z,r) \subset B(x,R) \setminus A$, take a rational $r_q \in (Rm, r)$. Then $B(z, r_q) \subset B(x,R) \setminus A$. Consequently, r may be without loss of generality considered to be rational.

We have seen that A is not upper porous in X if and only if the following formula holds:

(4.3)(∗)

$$(\exists x \in A)(\forall m \in \mathbb{Q}_+)(\exists R_0 \in \mathbb{Q}_+)(\forall R \in (0,R_0) \cap \mathbb{Q}_+)(\forall r \in (Rm,\infty) \cap \mathbb{Q}_+) \\ (\forall z \in X)(B(z,r) \not\subset B(x,R) \setminus A).$$

Thus, when A is not upper porous in X we can choose $x \in A$ as in (4.3). Using the elementarity of M, we may assume that $x \in M$. Now, fix $m \in \mathbb{Q}_+$ and pick $R_0 \in \mathbb{Q}_+$ as in (4.3). Fix $R \in (0, R_0) \cap \mathbb{Q}_+$, $r \in (Rm, \infty) \cap \mathbb{Q}_+$ and $z \in X_M$. Then take $r' \in (Rm, r) \cap \mathbb{Q}$ and $z_0 \in X \cap M$ such that $\rho(z, z_0) < r - r'$. Thus, $B(z_0, r') \subset B(z, r)$. Then

$$(*) \qquad (\exists y \in X)(y \in B(z_0, r') \setminus (B(x,R) \setminus A)).$$

For r' and z_0 we can find (using the elementarity of M) a point $y \in M$ such that

$$y \in B(z_0, r') \setminus (B(x,R) \setminus A) \subset B(z,r) \setminus (B(x,R) \setminus A).$$

Consequently, (4.3) is satisfied in X_M so $A \cap X_M$ is not upper porous in X_M. ∎

Proposition 4.13. *For any suitable elementary submodel M the following holds: Let X be a metric space. If M contains X and a set $A \subset X$, then*

A is not lower porous in X $\rightarrow$ $A \cap X_M$ is not lower porous in X_M.

Proof. Fix a (∗)-elementary submodel M containing X and A. If A is not lower porous, then as in the proof of Proposition 4.12,

(4.4)(∗)

$$(\exists x \in A)(\forall m \in \mathbb{Q}_+)(\forall R_0 \in \mathbb{Q}_+)(\exists R \in (0,R_0))(\forall r \in (Rm,\infty) \cap \mathbb{Q}_+) \\ (\forall z \in X)(B(z,r) \not\subset B(x,R) \setminus A).$$

Using the elementarity of M, choose $x \in A \cap M$ as in the formula above. Then fix $m, R_0 \in \mathbb{Q}_+$ and find $R \in (0, R_0)$ such that

$$(\forall r \in (Rm,\infty) \cap \mathbb{Q}_+)(\forall z \in X)(B(z,r) \not\subset B(x,R) \setminus A).$$

Using the elementarity of M we may assume that $R \in M$. Now choose any $r \in (Rm, \infty) \cap \mathbb{Q}_+$ and $z \in X_M$. Then find $r' \in (Rm, r) \cap \mathbb{Q}$ and $z_0 \in B(z, r - r') \cap M$. Thus, $B(z_0, r') \subset B(z, r)$. Then

$$(*) \qquad (\exists y \in X)(y \in B(z_0, r') \setminus (B(x,R) \setminus A)).$$

For r' and z_0 we can find (using the elementarity of M) a point $y \in M$ such that $y \in B(z_0, r') \setminus (B(x,R) \setminus A)$. Consequently,

$$X_M \cap B(z,r) \not\subseteq B(x,R) \setminus A.$$

Thus, (4.4) is satisfied in X_M and so $A \cap X_M$ is not lower porous in X_M. ∎

To see that the converse holds we will follow the ideas of [9, p. 42]. The following result is proved there for a rich family of subspaces (in the case where X is a Banach space). We give the proof for spaces constructed from elementary submodels (which holds even in the case of metric spaces).

Lemma 4.14. *For any suitable elementary submodel M the following holds: Let $\langle X, \rho\rangle$ be a metric space and $f : X \to \mathbb{R}$ a function. If M contains X and f, then for every $R > 0$ and $x \in X_M$,*

$$\sup_{u \in B(x,R)} f(u) = \sup_{u \in B(x,R) \cap X_M} f(u).$$

Proof. Fix a $(*)$-elementary submodel M containing X and f. Fix $x \in X_M$ and $R > 0$. To verify that $\sup_{u \in B(x,R)} f(u) \le \sup_{u \in B(x,R)\cap X_M} f(u)$ (the other inequality is obvious), take any $S \in \mathbb{Q}_+$ satisfying $S < \sup_{u \in B(x,R)} f(u)$. Then there exists $u \in B(x,R)$ such that $S < f(u)$. Now, find $R_q, \varepsilon \in \mathbb{Q}_+$ such that $R_q < R$ and $\rho(u,x) < R_q - \varepsilon$. Pick some $x_0 \in B(x, \varepsilon/2) \cap M$. Then $u \in B(x_0, R_q - \varepsilon/2)$ and by the absoluteness of the formula

$$(\exists u \in X)(\rho(u, x_0) < R_q - \varepsilon/2 \wedge S < f(u)), \tag{$*$}$$

there exists $u \in B(x_0, R_q - \varepsilon/2) \cap M \subset B(x,R) \cap M$ such that $S < f(u)$. Consequently, $S < \sup_{u \in B(x,R)\cap X_M} f(u)$. ∎

Proposition 4.15. *For any suitable elementary submodel M the following holds: Let X be a metric space. If M contains X, $A \subset X$ and $\mathrm{d}(\cdot, A)$, then for every $x \in A \cap X_M$,*

A is lower porous at x $\to$ $A \cap X_M$ is lower porous at x in X_M,

A is upper porous at x $\to$ $A \cap X_M$ is upper porous at x in X_M.

Proof. Fix a $(*)$-elementary submodel M containing X, A and $\mathrm{d}(\cdot, A)$, and fix some $x \in A \cap X_M$ such that A is c-upper porous at x for some rational $c > 0$. Thus, by Lemmas 4.11 and 4.14,

$$\begin{aligned} c \le \overline{p}_X(A,x) &\le 2 \limsup_{R \to 0^+} \sup_{u \in B(x,R)} \frac{\mathrm{d}(u,A)}{R} = 2 \limsup_{R \to 0^+} \sup_{u \in B(x,R) \cap X_M} \frac{\mathrm{d}(u,A)}{R} \\ &\le 2 \limsup_{R \to 0^+} \sup_{u \in B(x,R)\cap X_M} \frac{\mathrm{d}(u, A \cap X_M)}{R} \le 2\overline{p}_{X_M}(A \cap X_M, x). \end{aligned}$$

Consequently, $A \cap X_M$ is $c/2$-upper porous at x in X_M. The result for lower porosity follows similarly. ∎

Corollary 4.16. *For any suitable elementary submodel M the following holds: Let X be a metric space. If M contains X, $A \subset X$ and* $\mathrm{d}(\cdot, A)$, *then*

$$\begin{aligned} A \text{ is lower porous in } X &\leftrightarrow A \cap X_M \text{ is lower porous in } X_M,\\ A \text{ is upper porous in } X &\leftrightarrow A \cap X_M \text{ is upper porous in } X_M,\\ A \text{ is } \sigma\text{-lower porous in } X &\rightarrow A \cap X_M \text{ is } \sigma\text{-lower porous in } X_M,\\ A \text{ is } \sigma\text{-upper porous in } X &\rightarrow A \cap X_M \text{ is } \sigma\text{-upper porous in } X_M. \end{aligned}$$

Proof. Fix a $(*)$-elementary submodel M containing X, A and $\mathrm{d}(\cdot, A)$. Then the porosity results follow from Propositions 4.12, 4.13 and 4.15. The σ-porosity results are then obtained as in the proof of Proposition 4.3 using the absoluteness of the following two formulas:

$$(*) \qquad (\exists \varphi)\Big(\varphi \text{ is a function with } \operatorname{Dom}\varphi = \omega,\ \varphi(n) \text{ is a lower porous subset of } X \text{ for every } n \in \omega, \text{ and } A \subset \bigcup_{n\in\omega} \varphi(n)\Big).$$

$$(*) \qquad (\exists \varphi)\Big(\varphi \text{ is a function with } \operatorname{Dom}\varphi = \omega,\ \varphi(n) \text{ is an upper porous subset of } X \text{ for every } n \in \omega, \text{ and } A \subset \bigcup_{n\in\omega} \varphi(n)\Big). \blacksquare$$

The author does not know whether the converse implications of the preceding result about σ-porosity hold as well.

5. Properties of functions. Suppose X is a normed linear space and f a function defined on X. The aim of this section is to study the properties (P) of f which are "separably determined". To be more concrete, we want to find a closed separable subspace X_M such that for every $x \in X_M$,

$$f \text{ has property } (P) \text{ at } x \leftrightarrow f{\restriction}_{X_M} \text{ has property } (P) \text{ at } x.$$

Using the method of elementary submodels it is possible to combine the results about functions with those about sets.

The first property we are interested in is continuity.

Definition 5.1. Let $\langle X, \rho\rangle$ and $\langle Y, \sigma\rangle$ be metric spaces, $G \subset X$ an open subset and $f : G \to Y$ a function. Then we denote by $C(f)$ the set of points where f is continuous.

Theorem 5.2. *For any suitable elementary submodel M the following holds: Let $\langle X, \rho\rangle$ and $\langle Y, \sigma\rangle$ be metric spaces, $G \subset X$ an open subset and $f : G \to Y$ a function. If M contains X, f and Y, then $C(f) \in M$ and for every $x \in X_M \cap G$,*

$$f \text{ is continuous at } x \leftrightarrow f{\restriction}_{X_M} \text{ is continuous at } x.$$

Proof. Fix a $(*)$-elementary submodel M containing X, Y and f. Then $G \in M$, since $G = \operatorname{Dom} f$. Now, $C(f)$ is uniquely defined by the formula

$$(*) \qquad (\exists C)(\forall z)(z \in C \leftrightarrow z \in G \wedge f \text{ is continuous at } z);$$

hence $C(f) \in M$. Let us prove the desired equivalence. The left-to-right implication holds for every subspace of X. Conversely, suppose that f is not continuous at $x \in X_M \cap G$. Then we can find $k \in \mathbb{N}$ such that

$$(5.1) \qquad (\forall n \in \mathbb{N})(\exists y, z \in G)[y, z \in B(x, 1/n) \wedge \sigma(f(y), f(z)) > 1/k].$$

Fix $n \in \mathbb{N}$ and $x_0 \in B(x, 1/2n) \cap M$. As $B(x_0, 1/2n)$ is an open set containing x, there exists $l \in \mathbb{N}$ such that $B(x, 1/l) \subset B(x_0, 1/2n)$. By (5.1), there are $y, z \in G$ satisfying

$$y, z \in B(x, 1/l) \wedge \sigma(f(y), f(z)) > 1/k.$$

Consequently,

$$(*) \qquad (\exists y, z \in G)(y, z \in B(x_0, 1/2n) \wedge \sigma(f(y), f(z)) > 1/k).$$

All the free variables in this formula are in M, so by the elementarity of M and the fact that $B(x_0, 1/2n) \subset B(x, 1/n)$, there are $y, z \in G \cap M$ such that

$$(5.2) \qquad y, z \in B(x, 1/n) \wedge \sigma(f(y), f(z)) > 1/k.$$

We have just shown that for each $n \in \mathbb{N}$ we can find $y, z \in G \cap M$ satisfying (5.2). Consequently, $f\restriction_{X_M}$ is not continuous at x. ∎

Having proved that property (P) of f (continuity in this case) is separably determined, for the set $A := \{x : f \text{ has property } (P) \text{ at } x\}$ the following holds:

$$A \cap X_M = \{x : f\restriction_{X_M} \text{ has property } (P) \text{ at } x\}.$$

If $A \in M$, we can combine results about set properties and function properties. In particular, if $A \in M$, by Proposition 4.1 there exists a closed separable subspace X_M such that

$$\{x : f \text{ has property } (P) \text{ at } x\} \text{ is dense in } X \\ \leftrightarrow \ \{x : f\restriction_{X_M} \text{ has property } (P) \text{ at } x\} \text{ is dense in } X_M.$$

Thus, an immediate consequence of the preceding theorem and results about separably determined set properties is the following.

Corollary 5.3. *For any suitable elementary submodel M the following holds: Let X and Y be metric spaces, $G \subset X$ an open subset and $f : G \to Y$*

a function. Suppose that X is complete. If M contains X, Y and f, then

$$
\begin{aligned}
C(f) \text{ is dense in } G &\leftrightarrow C(f\restriction_{X_M}) \text{ is dense in } G\cap X_M,\\
C(f) \text{ is nowhere dense in } G &\leftrightarrow C(f\restriction_{X_M}) \text{ is nowhere dense in } G\cap X_M,\\
C(f) \text{ is meager in } G &\leftrightarrow C(f\restriction_{X_M}) \text{ is meager in } G\cap X_M,\\
C(f) \text{ is residual } G &\leftrightarrow C(f\restriction_{X_M}) \text{ is residual in } G\cap X_M,\\
C(f)^C \text{ is upper porous in } X &\leftrightarrow C(f\restriction_{X_M})^C \text{ is upper porous in } X_M,\\
C(f)^C \text{ is lower porous in } X &\leftrightarrow C(f\restriction_{X_M})^C \text{ is lower porous in } X_M.
\end{aligned}
$$

Proof. Fix a $(*)$-elementary submodel M containing X, Y and f. Then $G \in M$, because $G = \operatorname{Dom} f$. It is well known that $C(f)$ is a G_δ set [7, pp. 207–208]. From the preceding theorem, $C(f) \cap X_M = C(f\restriction_{X_M})$. Therefore, the result is an immediate consequence of Propositions 4.1, 4.2, 4.12, 4.13 and Theorems 4.8, 5.2. ∎

The next property we examine is lower (or upper) semicontinuity. Let us recall the definition in metric spaces.

DEFINITION 5.4. Let X be a metric space, $G \subset X$ an open subset, $f : G \to [-\infty, \infty]$ a function and $x \in G$. If for every sequence $\{x_n\}_{n\in\omega} \subset G$,

$$x_n \to x \quad \text{implies} \quad \liminf_{n\to\infty} f(x_n) \ge f(x),$$

then we say that f is *lower semicontinuous* (*lsc*) at x.

If $-f$ is lsc at x, we say that f is *upper semicontinuous* (*usc*) at x.

The following lemma will be used to prove that the lower (and upper) semicontinuity is a separably determined property.

LEMMA 5.5. *Let X be a metric space, $G \subset X$ an open subset, $f : G \to [-\infty, \infty]$ a function and $x \in G$. Then f is lsc at x if and only if for every $c \in \mathbb{Q} \cap (-\infty, f(x))$ there exists $n \in \mathbb{N}$ such that $f[B(x, 1/n) \cap G] \subset (c, \infty]$.*

Proof. We may assume that $f(x) > -\infty$ (if $f(x) = -\infty$, then the conclusion is obvious).

"⇒" Suppose there exists $c \in \mathbb{Q} \cap (-\infty, f(x))$ and $\{x_n\}_{n\in\omega} \subset G$ such that $x_n \in B(x, 1/n)$, but $f(x_n) \le c$. Then $x_n \to x$, but $\liminf_{n\to\infty} f(x_n) \le c < f(x)$. Thus, f is not lsc at x.

"⇐" First, assume that $f(x) < \infty$. Fix $\varepsilon > 0$, $c \in \mathbb{Q}\cap(f(x)-\varepsilon, f(x))$ and a sequence $\{x_n\}_{n\in\omega} \subset G$ with $x_n \to x$. Then there exists $k \in \mathbb{N}$ such that $f[B(x, 1/k) \cap G] \subset (c, \infty]$. Next, there exists n_0 such that $x_n \in B(x, 1/k)$ for every $n \ge n_0$. Consequently, $f(x_n) > c > f(x) - \varepsilon$ for every $n \ge n_0$; hence, $\liminf_{n\to\infty} f(x_n) \ge f(x)-\varepsilon$. As ε could be arbitrarily small, we have $\liminf_{n\to\infty} f(x_n) \ge f(x)$.

In the case that $f(x) = \infty$, we fix $K \in \mathbb{N}$, $c \in \mathbb{Q}\cap(K, \infty)$ and a sequence $\{x_n\}_{n\in\omega} \subset G$ with $x_n \to x$. As above it follows that $\liminf_{n\to\infty} f(x_n) \ge K$. ∎

Proposition 5.6. *For any suitable elementary submodel M the following holds: Let X be a metric space, $G \subset X$ an open subset and $f : G \to [-\infty, \infty]$ a function. If M contains X and f, then for every $x \in X_M \cap G$,*

$$f \text{ is lsc at } x \;\leftrightarrow\; f\restriction_{X_M} \text{ is lsc at } x.$$

Proof. Immediately from the definition it is obvious that the left-to-right implication holds for any subspace of X. Fix a $(*)$-elementary submodel M containing $X$and f and assume that f is not lsc at $x \in X_M \cap G$. Then, by Lemma 5.5, there exists $c \in \mathbb{Q} \cap (-\infty, f(x))$ such that for every $n \in \mathbb{N}$ there exists $y \in B(x, 1/n) \cap G$ such that $f(y) \le c$. Choose any $n \in \mathbb{N}$ and $x_0 \in B(x, 1/2n) \cap M$. Then $B(x_0, 1/2n) \subset B(x, 1/n)$ is an open set containing x, so there exists $l \in \mathbb{N}$ such that $B(x, 1/l) \subset B(x_0, 1/2n)$. For such an $l \in \mathbb{N}$ there exists $y \in B(x, 1/l) \cap G$ such that $f(y) \le c$. Consequently,

$$(\exists y \in B(x_0, 1/2n) \cap G)(f(y) \le c). \tag{$*$}$$

Using the elementarity of M, we find $y \in B(x_0, 1/2n) \cap G \cap M \subset B(x, 1/n) \cap G \cap M$ such that $f(y) \le c$. For any $n \in \mathbb{N}$ we have found $y \in B(x, 1/n) \cap G \cap X_M$ such that $f(y) \le c$. By Lemma 5.5, $f\restriction_{X_M}$ is not lsc at x. ∎

Corollary 5.7. *For any suitable elementary submodel M the following holds: Let X be a metric space, $G \subset X$ an open subset and $f : G \to [-\infty, \infty]$ a function. Let $-$ denote the operation which maps every function $h : G \to [-\infty, \infty]$ to $-h$. If M contains X, f and $-$, then for every $x \in X_M \cap G$,*

$$f \text{ is usc at } x \;\leftrightarrow\; f\restriction_{X_M} \text{ is usc at } x.$$

Proof. Fix a $(*)$-elementary submodel M containing X, f and $-$. Then $-f \in M$, thus it is enough to use the preceding proposition. ∎

The last function property examined in this article is Fréchet differentiability. We use the following definition.

Definition 5.8. Let X and Y be normed linear spaces, $G \subset X$ an open subset, $f : G \to Y$ a function and $x \in G$.

(i) If there exists a continuous linear operator $A : X \to Y$ such that

$$\lim_{u \to x} \frac{f(u) - f(x) - A(u - x)}{\|u - x\|} = 0,$$

then we say that f is *Fréchet differentiable at* x. We denote by $D(f)$ the set of points at which f is Fréchet differentiable.

(ii) For $c, \varepsilon, \delta > 0$ we define $D(f, c, \varepsilon, \delta)$ as the set of all $x \in G$ satisfying

$$\left\| \frac{f(y + tv) - f(y)}{t} - \frac{f(y) - f(y - hv)}{h} \right\| \le \varepsilon$$

whenever

$$v \in X, \quad \|v\| = 1, \quad t > 0, \quad h > 0, \quad y \in B(x,\delta), \quad y - hv \in B(x,\delta),$$
$$y + tv \in B(x,\delta) \quad \text{and} \quad \min(t,h) > c\|y - x\|.$$

The following relationship between sets $D(f,c,\varepsilon,\delta)$ and Fréchet differentiability is shown in [12].

LEMMA 5.9. *Let X be a normed linear space, $G \subset X$ an open subset and Y a Banach space. Let $f : G \to Y$ be a function. Then f is Fréchet differentiable at a point $x \in G$ if and only if f is continuous at x and $x \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} D(f,1/n,1/n,1/k)$.*

Using this lemma, it is shown in [12] that the property of being Fréchet differentiable is separably determined. Let us prove a similar result using the method of elementary submodels.

THEOREM 5.10. *For any suitable elementary submodel M the following holds: Let X be a normed linear space, $G \subset X$ an open subset and Y a Banach space. Let $f : G \to Y$ be a function. If M contains X, f and Y, then $D(f) \in M$ and for every $x \in X_M \cap G$,*

$$f \text{ is Fréchet differentiable at } x \;\leftrightarrow\; f{\restriction}_{X_M} \text{ is Fréchet differentiable at } x.$$

Proof. Fix a $(*)$-elementary submodel M containing X, Y and f. Now, $D(f)$ is the object uniquely defined by the formula

$$(*) \qquad (\exists D)(\forall z)(z \in D \leftrightarrow z \in D \wedge f \text{ is Fréchet differentiable at } z),$$

hence $D(f) \in M$. Fix $x \in X_M \cap G$. Then, by Theorem 5.2, f is continuous at x if and only if $f{\restriction}_{X_M}$ is continuous at x. Thus, using Lemma 5.9, it is sufficient to check that

$$x \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} D(f,1/n,1/n,1/k) \;\leftrightarrow\; x \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} D(f{\restriction}_{X_M},1/n,1/n,1/k).$$

The left-to-right implication is obvious (it holds for every subspace of X). Conversely, assume that $x \notin \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} D(f,1/n,1/n,1/k)$. Fix $n \in \mathbb{N}$ satisfying $x \notin \bigcup_{k\in\mathbb{N}} D(f,1/n,1/n,1/k)$. Then for every $k \in \mathbb{N}$,

$$(*) \qquad (\exists v \in X,\, \|v\| = 1)(\exists t,h > 0)(\exists y \in X)$$
$$\left(\begin{array}{l} y \in B(x,1/k), \quad y - hv \in B(x,1/k), \quad y + tv \in B(x,1/k), \\ \min(t,h) > \dfrac{1}{n}(\|y-x\|+0), \;\; \left\| \dfrac{f(y+tv)-f(y)}{t} - \dfrac{f(y)-f(y-hv)}{h} \right\| > \dfrac{1}{n} \end{array}\right).$$

Pick some v, t, h and y as in the formula above and find $\eta \in \mathbb{Q}_+$ such that

$$\|y-x\| < \frac{1}{k} - 2\eta, \quad \|y-hv-x\| < \frac{1}{k} - 2\eta,$$
$$\|y+tv-x\| < \frac{1}{k} - 2\eta, \quad \min(t,h) > \frac{1}{n}(\|y-x\| + 2\eta).$$

Further, take $x_0 \in B(x,\eta) \cap M$. Then

$$\|y-x_0\| \le \|y-x\| + \|x-x_0\| < \frac{1}{k} - \eta, \quad \|y-hv-x_0\| < \frac{1}{k} - \eta,$$
$$\|y+tv-x_0\| < \frac{1}{k} - \eta, \quad \frac{1}{n}(\|y-x_0\| + \eta) \le \frac{1}{n}(\|y-x\| + 2\eta) < \min(t,h).$$

Using the elementarity of M we get the existence of $v \in X \cap M$ with $\|v\| = 1$, $t, h \in \mathbb{R}_+ \cap M$ and $y \in X \cap M$ such that:

$$y \in B(x_0, 1/k - \eta) \subset B(x, 1/k), \quad y - hv \in B(x_0, 1/k - \eta) \subset B(x, 1/k),$$
$$y + tv \in B(x_0, 1/k - \eta) \subset B(x, 1/k),$$
$$\min(t,h) > \frac{1}{n}(\|y-x_0\| + \eta) > \frac{1}{n}\|y-x\|,$$
$$\left\| \frac{f(y+kv) - f(y)}{k} - \frac{f(y) - f(y-hv)}{h} \right\| > \frac{1}{n}.$$

Consequently, $x \notin \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} D(f\restriction_{X_M}, 1/n, 1/n, 1/k)$. ∎

We would like to combine this result with Theorem 4.8, stating that being a residual subset is a separably determined property for sets with the Baire property in complete metric spaces. The following result comes from [12].

THEOREM 5.11. *Let X be a normed linear space, $G \subset X$ an open subset and Y a Banach space. Let $f : G \to Y$ be a function. Then $D(f)$ is an $F_{\sigma\delta}$ set.*

Using this result we immediately get the following corollary (obviously, even more is true, as in the case of continuity).

COROLLARY 5.12. *For any suitable elementary submodel M the following holds: Let X, Y be Banach spaces, $G \subset X$ an open subset and $f : G \to Y$ a function. If M contains X, Y and f, then*

$$D(f) \text{ is dense in } G \ \leftrightarrow\ D(f\restriction_{X_M}) \text{ is dense in } G \cap X_M,$$
$$D(f) \text{ is residual in } G \ \leftrightarrow\ D(f\restriction_{X_M}) \text{ is residual in } G \cap X_M.$$

6. Applications. In this last section we give two applications of the theorems proved above. Both extend the validity of already known theorems to the nonseparable setting. In the first case we take up the result proved in [13, Proposition 3.3] for spaces with separable dual. The method of elementary submodels will allow us to prove that the same theorem holds in general Asplund spaces. The second application will extend the result proved in [8, Theorem 4.8] for $\mathcal{C}(K)$ spaces with K countable compact and for subspaces of c_0 to the case of $\mathcal{C}(K)$ spaces with K a general scattered compact space and to subspaces of $c_0(\Gamma)$ with Γ possibly uncountable.

Separable reductions of the results mentioned above have already been examined using the method of rich families (for the concept of rich families see Section 3). In the first case Zajíček [13, Theorem 5.2] only achieved to prove a weaker variant of the theorem in Asplund spaces. In the second case, the separable reduction to subspaces of $c_0(\Gamma)$ easily follows using the work of J. Lindenstrauss, D. Preiss and J. Tišer [9, Corollary 5.6.2] and the result of Zajíček [13, Theorem 4.7]. The extension to $\mathcal{C}(K)$ spaces with K scattered compact can be achieved using the result of Górak [4, proof of Theorem 2.1] and the above mentioned results by Lindenstrauss, Preiss, Tišer and Zajíček.

Remark 6.1. Various combinations of the above proved theorems may be considered to be applications as well. For example, by Corollary 5.12 the following holds: Let X, Y be Banach spaces, $f : X \to Y$ a function. Then for every separable subspace $V \subset X$ there exists a closed separable subspace $W \subset X$ with $V \subset W$ such that f is Fréchet differentiable on a residual set if and only if $f\restriction_W$ is Fréchet differentiable on a residual set in W.

Let us now discuss the first application.

L. Zajíček proved in [13] a result included as Theorem 6.5 below. This theorem was proved for spaces with separable dual. We will use the method of elementary submodels to get the same result for Asplund spaces.

In the following, unless stated otherwise, X will be a Banach space. The equality $X = X_1 \oplus \cdots \oplus X_n$ means that X is the direct sum of nontrivial closed linear subspaces $X_1, \ldots, X_n$ and the corresponding projections $P_i : X \to X_i$ are continuous.

Recall that X is an *Asplund space* if each continuous convex real-valued function on X is Fréchet differentiable at each point of X except on a first category set; it is known that X is an Asplund space if and only if Y^* is separable for every separable subspace $Y \subset X$.

We will need the following well-known fact (see [13]).

Lemma 6.2. *Let X be a Banach space, $0 \neq u \in X$, and suppose $X = W \oplus \operatorname{span}\{u\}$. Then the mapping $w \in W \mapsto w + \mathbb{R}u \in X/\operatorname{span}\{u\}$ is a linear homeomorphism.*

The following definition is used in the theorem from [13].

Definition 6.3. Let f be a real-valued function defined on an open subset G of a Banach space X.

(i) We say that f is *generically Fréchet differentiable* on G if the set $D(f)$ of points where f is Fréchet differentiable is residual in G.

(ii) We say that f is *strictly differentiable* at $a \in G$ if there exists $x^* \in X^*$ such that

$$\lim_{(x,y)\to(a,a),\, x\neq y} \frac{f(y)-f(x)-x^*(y-x)}{\|y-x\|} = 0.$$

(iii) We say that f is *essentially smooth* (*esm* for short) *on the line* $L = a + \mathbb{R}v$ (where $a \in X$, $0 \neq v \in X$) if the function $\phi(t) := f(a+tv)$ is strictly differentiable at a.e. point of its domain. (Obviously, the definition is correct: it does not depend on the choice of a or v.)

(iv) We say that a line L is *parallel* to v (where $0 \neq v \in X$) if there exists $a \in X$ such that $L = a + \mathbb{R}v$.

(v) We say that f is *essentially smooth on a generic line parallel to* $0 \neq v \in X$ if f is essentially smooth on all lines parallel to v, except for a first category set of lines in the factor space $X/\operatorname{span}\{v\}$.

Remark 6.4. Let X be a normed linear space, $G \subset X$ an open subset, $f : G \to \mathbb{R}$ function, Y a subspace of X and $a, v \in Y$, $v \neq 0$. Consider the line $L = a + \mathbb{R}v$. Then it follows immediately from the definition above that $L \subset Y$ and that f is essentially smooth on L if and only if $f\restriction_Y$ is.

The theorem proved in [13, Proposition 3.3] reads as follows.

Theorem 6.5. *Let $X = X_1 \oplus \cdots \oplus X_n$ be a Banach space with separable dual X^*. Let $G \subset X$ be an open set and $f : G \to \mathbb{R}$ a locally Lipschitz function. Suppose that, for each $1 \leq i \leq n$, there exists a dense set $D_i \subset S_{X_i}$ such that, for each $v \in D_i$, f is essentially smooth on a generic line parallel to v. Then f is generically Fréchet differentiable on G.*

Using the concept of rich families, it is proved in [13, Theorem 5.2] that this result holds under slightly stronger assumptions even in the case of nonseparable Asplund spaces. Using the method of elementary submodels we will prove that the conclusion of Theorem 6.5 holds in exactly the same form in nonseparable Asplund spaces.

Let us start with the following lemma.

Lemma 6.6. *For any suitable elementary submodel M the following holds: Let X be a normed linear space and $X = X_1 \oplus \cdots \oplus X_n$. Let $P_1, \dots, P_n$ be the corresponding projections onto $X_1, \dots, X_n$. If M contains X and $P_1, \dots, P_n$, then*

$$X_M = P_1(X_M) \oplus \cdots \oplus P_n(X_M).$$

Proof. Fix a $(*)$-elementary submodel M containing X and $P_1, \dots, P_n$. Then, by Proposition 2.9, $P_i(X \cap M) \subset X \cap M$ for each $i \in \{1, \dots, n\}$. From the continuity of the projections $P_1, \dots, P_n$ it follows that $P_i(X_M) \subset X_M$ for each $i \in \{1, \dots, n\}$. Consequently, $X_M = P_1(X_M) \oplus \cdots \oplus P_n(X_M)$. ∎

THEOREM 6.7. *Let $X = X_1 \oplus \cdots \oplus X_n$ be an Asplund space. Let $G \subset X$ be an open set and $f : G \to \mathbb{R}$ a locally Lipschitz function. Suppose that, for each $1 \le i \le n$, there exists a dense set $D_i \subset S_{X_i}$ such that, for each $v \in D_i$, f is essentially smooth on a generic line parallel to v. Then f is generically Fréchet differentiable on G.*

Proof. Let $P_1, \ldots, P_n$ be the continuous projections onto the subspaces $X_1, \ldots, X_n$. By Corollary 5.12, Propositions 4.1, 2.10, 3.2 and Lemma 6.6, there exist formulas $\varphi_1, \ldots, \varphi_l$ and a countable set Y such that for the set

$$Z := \{X, f, P_1, \ldots, P_n, D_1, \ldots, D_n, S_{X_1}, \ldots, S_{X_n}, Y\}$$

and for every elementary submodel $M \prec (\varphi_1, \ldots, \varphi_l; Z)$ it is true that:

(P1) Every countable set $S \in M$ is a subset of M.
(P2) $X_M = P_1(X_M) \oplus \cdots \oplus P_n(X_M)$.
(P3) Whenever sets $A, S \subset X$ are in M, then

$$A \cap S \text{ is dense in } S \ \leftrightarrow\ A \cap S \cap X_M \text{ is dense in } S \cap X_M.$$

(P4) $D(f)$ is residual in $G \leftrightarrow D(f\restriction_{X_M})$ is residual in $G \cap X_M$.
(P5) X_M is a separable subspace of X.

Without loss of generality we may assume that the list $\varphi_1, \ldots, \varphi_l$ is subformula closed. Notice that for every subspace N of X satisfying $N = P_1(N) \oplus \cdots \oplus P_n(N)$ we have $S_{X_i} \cap N = S_{P_i(N)}$. Indeed,

$$S_{X_i} \cap N = S_X \cap X_i \cap N = S_X \cap X_i \cap P_i(N) = S_X \cap P_i(N) = S_{P_i(N)}.$$

Let us define inductively a sequence $\{M_k\}_{k\in\omega}$ of elementary submodels:

- For $k = 0$ choose any elementary submodel $M_0 \prec (\varphi_1, \ldots, \varphi_n; Z)$.
- Whenever M_k is defined, we pick for every $i \in \{1, \ldots, n\}$ a countable subset $C_{k,i}$ of $D_i \cap X_{M_k}$ dense in $S_{P_i(X_{M_k})} = S_{X_i} \cap X_{M_k}$. Then, for every $v \in C_{k,i}$, it follows from the assumptions and Lemma 6.2 that the set $\{a \in G : f \text{ is esm on the line } a + \mathbb{R}v\}$ is residual. Consequently, there exists a G_δ dense subset $G_{k,v}$ such that f is esm on each line parallel to v, intersecting $G_{k,v}$. Now we let M_{k+1} be an elementary submodel for the formulas $\varphi_1, \ldots, \varphi_l$ containing $\{Z, C_{k,1}, \ldots, C_{k,n}, M_k, \{G_{k,v}\}_{v\in\bigcup_{i=1}^n C_{k,i}}\}$.

Finally, we define $M := \bigcup_{k\in\omega} M_k$. Then, by Lemma 2.4, $M \prec (\varphi_1, \ldots, \varphi_n; Z)$. Therefore, (P1)–(P5) hold for M.

We need to verify that for the space X_M and the function $f\restriction_{X_M}$ the conditions of Theorem 6.5 are satisfied. Then, by (P4), f is generically Fréchet differentiable on G.

Since X is an Asplund space, $(X_M)^*$ is separable. Obviously, $f\restriction_{X_M}$ is locally Lipschitz. By (P2), $X_M = P_1(X_M) \oplus \cdots \oplus P_n(X_M)$. For $i \in \{1, \ldots, n\}$

we define $C_i := \bigcup_{k\in\omega} C_{k,i}$. Let us verify that this set is dense in $S_{P_i(X_M)} = S_{X_i} \cap X_M$.

Fix $\varepsilon > 0$ and $y \in S_{X_i} \cap X_M = S_{X_i} \cap \overline{\bigcup_{k\in\omega}(X \cap M_k)}$. Then find some $y_0 \in B(y, \varepsilon/3) \cap \bigcup_{k\in\omega}(X \cap M_k)$ and take $k \in \omega$ such that $y_0 \in X \cap M_k$. Then $y_0/\|y_0\| \in X_{M_k} \cap S_{X_i}$. Furthermore,

$$\begin{aligned}\left\|\frac{y_0}{\|y_0\|} - y\right\| &\le \left\|\frac{y_0}{\|y_0\|} - y_0\right\| + \|y_0 - y\| = \big|1 - \|y_0\|\big| + \|y_0 - y\| \\ &= \big|\|y\| - \|y_0\|\big| + \|y_0 - y\| \le 2\|y_0 - y\| < 2\varepsilon/3.\end{aligned}$$

Since $C_{k,i}$ is dense in $S_{X_i} \cap X_{M_k}$, there exists $c_{k,i} \in C_{k,i} \subset C_i$ such that $\|c_{k,i} - y_0/\|y_0\|\,\| < \varepsilon/3$. Consequently,

$$\|c_{k,i} - y\| \le \left\|c_{k,i} - \frac{y_0}{\|y_0\|}\right\| + \left\|\frac{y_0}{\|y_0\|} - y\right\| < \varepsilon.$$

Notice that, by (P1), $C_i \subset M$ for every $i \in \{1, \dots, n\}$. It remains to show that for every $i \in \{1, \dots, n\}$ and $v \in C_i$ the set

$$R_v := \{a \in G \cap X_M : f\restriction_{X_M} \text{ is esm on the line } a + \mathbb{R}v\}$$

is residual in X_M.

Fix $v \in C_i$ and find $k \in \omega$ such that $v \in C_{k,i}$. Then $R_v \supset G_{k,v} \cap X_M$. As $G_{k,v} \in M$, using (P3), $G_{k,v} \cap X_M$ is a dense G_δ set in X_M. Consequently, R_v is residual in X_M. ∎

The second application extends [8, Theorem 4.8], recalled here as Theorem 6.11 below. This theorem was proved for $\mathcal{C}(K)$ spaces where K is a countable compact space, and for subspaces of c_0. We will use the method of elementary submodels to get the same result for $\mathcal{C}(K)$ spaces where K is a scattered compact space, and for subspaces of $c_0(\Gamma)$ for Γ possibly uncountable.

Recall that a set $A \subset T$ (where T is an arbitrary topological space) is called *scattered* if every nonempty subset of A has an isolated point. It is well known that a continuous image of a scattered compact space is scattered and that a metrizable scattered compact space is countable (see [3, Lemmas 14.20 and 14.21]). Using those two well-known facts we easily get the following.

Lemma 6.8. *Let K, L be compact spaces, with K scattered and L metrizable, and let $f : K \to L$ be a continuous mapping onto L. Then L is a countable set.*

Recall that a Banach space Y is said to have *the Radon–Nikodým property* (RNP) if every Lipschitz function $f : \mathbb{R} \to Y$ is differentiable almost everywhere (or equivalently every such f has a point of differentiability; see [8]).

The result of J. Lindenstrauss and D. Preiss uses the notion of Γ-null sets. Therefore, let us give some basic notations. For further information about this notion see [9, Chapter 5].

Let X be a Banach space and let $T := [0,1]^{\mathbb{N}}$ be endowed with the product topology and product Lebesgue measure $\mathscr{L}^{\mathbb{N}}$. We denote by $\Gamma(X)$ the space of continuous mappings

$$\gamma : T \to X$$

having continuous partial derivatives $D_j\gamma$ (we consider one-sided derivatives at points where the jth coordinate is 0 or 1). We equip $\Gamma(X)$ with the topology generated by the seminorms

$$\|\gamma\|_\infty = \sup_{t\in T} \|\gamma(t)\| \quad \text{and} \quad \|\gamma\|_k = \sup_{t\in T} \|D_k\gamma(t)\|, \quad k \geq 1.$$

Equivalently, this topology may be defined by the seminorms

$$\|\gamma\|_{\leq k} = \max\{\|\gamma\|_\infty, \|\gamma\|_1, \dots, \|\gamma\|_k\}.$$

The space $\Gamma(X)$ with this topology is a Fréchet space; in particular it is a Polish space whenever X is separable.

We also define $\Gamma_n(X) = \mathcal{C}^1([0,1]^n, X)$ and consider the norm $\|\cdot\|_{\leq n}$ on this space. Notice that $\Gamma_n(X)$ is a subspace of $\Gamma(X)$ in the sense that functions depending on the first n coordinates only are naturally identified with functions from $\Gamma_n(X)$.

A Borel subset $A \subset X$ is called *Γ-null* if the set $\{\gamma \in \Gamma(X) : \mathscr{L}^{\mathbb{N}}\gamma^{-1}(A) = 0\}$ is residual in $\Gamma(X)$.

The following two lemmas come from [9, Lemmas 5.3.2 and 5.4.1].

Lemma 6.9. *Whenever (X_n) is an increasing sequence of subspaces of X whose union is dense in X, then $\bigcup_{n=1}^\infty \Gamma_n(X_n)$ is dense in $\Gamma(X)$.*

Lemma 6.10. *Let A be a Borel subset of a Banach space X. Then the set $\{\gamma \in \Gamma(X) : \mathscr{L}^{\mathbb{N}}\gamma^{-1}(A) = 0\}$ is Borel.*

The result from [8, Theorem 4.8] is as follows.

Theorem 6.11. *The following spaces have the property that every Lipschitz mapping of them into a space with the RNP is Fréchet differentiable everywhere except on a Γ-null set: $\mathcal{C}(K)$ for countable compact K and subspaces of c_0.*

Let us first focus on the set property of being Γ-null. To see that it is separably determined, we give the following lemmas.

Lemma 6.12. *Let X be a finite-dimensional Banach space and let $\{x_1, \dots, x_n\}$ be a basis of X. Then for every $k \in \omega$,*

$$\Gamma_k(X) = \Big\{ \sum_{i=1}^n \gamma_i x_i : \gamma_i \in \Gamma_k(\mathbb{R}) \Big\}.$$

Proof. For every $k \in \omega$, $\gamma \in \Gamma_k(X)$ and $t \in [0,1]^k$ there are unique numbers $\gamma_1(t), \dots, \gamma_n(t)$ such that $\gamma(t) = \sum_{i=1}^n \gamma_i(t) x_i$. It is easy to verify that for every $i \in \{1, \dots, n\}$ the mapping γ_i is an element of $\Gamma_k(\mathbb{R})$ and that $D_j \gamma(t) = \sum_{i=1}^n D_j \gamma_i(t) x_i$ whenever $j \in \{1, \dots, k\}$ and $t \in [0,1]^k$. Thus, $\Gamma_k(X) = \{\sum_{i=1}^n \gamma_i x_i : \gamma_i \in \Gamma_k(\mathbb{R})\}$. ∎

LEMMA 6.13. *Let X be a separable Banach space with a countable dense set D. Then*

$$\Gamma(X) = \overline{\{\textstyle\sum_{i=1}^n \gamma_i x_i : \gamma_i \in \Gamma_n(\mathbb{R}),\ x_i \in D,\ n \in \mathbb{N}\}}.$$

Proof. Let N be either the dimension of X if it is finite, or $N = \mathbb{N}$ if X is infinite-dimensional. Then take a countable linearly dense set $\{x_n\}_{n \in N} \subset D$ which is linearly independent. Denote by X_n the subspace $\operatorname{span}\{x_i : i \le n\}$. Then, by the preceding lemma and Lemma 6.9, the set $\{\sum_{i=1}^n \gamma_i x_i : \gamma_i \in \Gamma_n(\mathbb{R}), n \in \mathbb{N}\}$ is dense in $\Gamma(X)$. ∎

REMARK 6.14. The preceding lemma holds even in the case when X is nonseparable (with uncountable set $D := X$). This is because the range of every $\gamma \in \Gamma(X)$ is separable. Thus, considering that $\gamma \in \Gamma(\overline{\operatorname{span}}\{\operatorname{Rng} \gamma\})$, we may use the result for separable spaces.

LEMMA 6.15. *For any suitable elementary submodel M the following holds: Let X be a Banach space. If M contains X and $\{\Gamma_n(X)\}_{n=1}^\infty$, then*

$$\overline{\Gamma(X) \cap M} = \Gamma(X_M).$$

Proof. Fix a $(*)$-elementary submodel M containing X, $\{\Gamma_n(\mathbb{R})\}_{n=1}^\infty$ and $\{\Gamma_n(X)\}_{n=1}^\infty$ (it is not necessary to mention the set $\{\Gamma_n(\mathbb{R})\}_{n=1}^\infty$ in the assumptions of the lemma as it does not depend on the space X—see Convention on page 195). Then, by Proposition 2.9, $\Gamma(X) \cap M \subset \Gamma(X_M)$; consequently, $\overline{\Gamma(X) \cap M} \subset \Gamma(X_M)$.

For the other inclusion, denote, for every $n \in \mathbb{N}$,

$$A_n := \Big\{ \sum_{i=1}^n \gamma_i x_i : \gamma_i \in \Gamma_n(\mathbb{R}),\ x_i \in X \cap M \Big\}.$$

Using the preceding lemma, it is sufficient to show that for every $n \in \mathbb{N}$, $A_n \subset \overline{\Gamma(X) \cap M}$. Fix $n \in \mathbb{N}$. Using the absoluteness of the formula (for every $n \in \mathbb{N}$ the formula is the same—what does change is the free variable $\Gamma_n(\mathbb{R})$ in it)

$$(*) \qquad (\exists D)(D \text{ is countable and dense in } \Gamma_n(\mathbb{R})),$$

we may find a countable set $D \in M$ such that D is dense in $\Gamma_n(\mathbb{R})$. Moreover, whenever $\gamma_0 \in \Gamma(\mathbb{R}) \cap M$ and $x_0 \in X \cap M$, then $\gamma_0 x_0$ is the function uniquely defined by the formula

$$(*) \qquad (\exists f \in \Gamma_n(X))(\forall t \in [0,1]^n)(f(t) = \gamma_0(t) x_0);$$

consequently, $\gamma_0 x_0 \in M$. As the space $\overline{\Gamma(X) \cap M}$ is $\mathbb{Q}$-linear, it follows that $\{\sum_{i=1}^n \gamma_i x_i : \gamma_i \in D, x_i \in X \cap M\} \subset \overline{\Gamma(X) \cap M}$. It is easy to verify that this subset of $\overline{\Gamma(X) \cap M}$ is dense in A_n. ∎

Remark 6.16. The preceding lemma is of independent interest. Observe that combining it with the results from the previous sections one finds that, for every suitable elementary submodel and for every set $A \subset \Gamma(X)$ contained in M, the set A is dense (resp. nowhere dense) in $\Gamma(X)$ if and only if $A \cap \Gamma(X_M)$ is dense (resp. nowhere dense) in $\Gamma(X_M)$. When A has the Baire property, then the same equivalence holds for the residuality of A. This result gives us separable subspaces with properties that were not achieved in [9] using the method of rich families (see [9, Lemma 5.6.1]).

Corollary 6.17. *For any suitable elementary submodel M the following holds: Let X be a Banach space. If M contains X, $\{\Gamma_n(X)\}_{n=1}^\infty$ and a Borel set A, then*

$$A \text{ is } \Gamma\text{-null in } X \;\leftrightarrow\; A \cap X_M \text{ is } \Gamma\text{-null in } X_M.$$

Proof. Fix a $(*)$-elementary submodel M containing X, $\{\Gamma_n(X)\}_{n=1}^\infty$ and a Borel set A. Then, in view of Lemmas 6.10 and 6.15, the set $\{\gamma \in \Gamma(X) : \mathscr{L}^{\mathbb{N}}\gamma^{-1}(A) = 0\}$ is residual in $\Gamma(X)$ if and only if $\{\gamma \in \Gamma(X_M) : \mathscr{L}^{\mathbb{N}}\gamma^{-1}(A \cap X_M) = 0\}$ is residual in $\Gamma(X_M)$. ∎

Using the preceding results, we can give the promised extension of Theorem 6.11.

Theorem 6.18. *The following spaces have the property that every Lipschitz function on them into a space with the RNP is Fréchet differentiable everywhere except on a Γ-null set: $\mathcal{C}(K)$ for K scattered compact and subspaces of $c_0(\Gamma)$, where Γ is an arbitrary set.*

Proof. Suppose we have a space X as in the assumptions (either $X = \mathcal{C}(K)$ for K scattered compact, or $X \subset c_0(\Gamma)$), a Banach space Y with RNP and a Lipschitz function $f : X \to Y$. Using the preceding corollary and Theorem 5.10, choose an elementary submodel M satisfying:

- X_M is a separable subspace of X.
- f is Fréchet differentiable everywhere except on a Γ-null set in X if and only if $f\restriction_{X_M}$ is Fréchet differentiable everywhere except on a Γ-null set in X_M.

If $X = \mathcal{C}(K)$, then (using Lemma 3.7) choose M such that in addition $X_M = \mathcal{C}(K/M)$, where K/M is metrizable compact and a continuous image of K. By Lemma 6.8, K/M is a countable compact space. Hence, by Theorem 6.11, $f\restriction_{X_M}$ is Fréchet differentiable everywhere except on a Γ-null set in X_M. Therefore, f is Fréchet differentiable everywhere except on a Γ-null set.

If $X = c_0(\Gamma)$, then X_M is a separable subspace of X, so X_M is a subspace of c_0. Then, by the same arguments as above, f is Fréchet differentiable everywhere except on a Γ-null set. ∎

Acknowledgements. This work was supported by the grant SVV-2010-261316.

The author would like to thank Ondřej Kalenda for suggesting the topic and for many useful remarks. My gratitude also goes to J. Tišer for a useful remark which helped to prove Lemma 6.15.

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Received 4 April 2011;
in revised form 19 July 2012