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Invariant curves and semiconjugacies of rational functions

by

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Abstract. Jordan analytic curves which are invariant under rational functions are studied.

In the last paragraph of his memoir [5] on iteration of rational functions Fatou writes:

> Il nous resterait à étudier les courbes analytiques invariantes par une transformation rationnelle et dont l'étude est intimement liée à celle des fonctions étudiées dans ce Chapitre. Nous espérons y revenir bientôt.

As far as I know, Fatou never returned to this question in his published work. Neither do I know of any systematic study of the question after Fatou. If a Jordan analytic invariant curve is the boundary of a domain of attraction, then it must be a circle, [5].

Which Jordan analytic curves γ in the Riemann sphere can be invariant under rational functions? Of course, γ can be a circle, and it is easy to describe all rational functions which leave a given circle invariant: such functions must commute with the reflection in this circle.

Other examples are obtained as level lines of linearizing functions of Siegel discs or Hermann rings. It is not known whether a Jordan analytic invariant curve, different from a circle, and which is mapped onto itself homeomorphically, can intersect the Julia set.

If f is a polynomial (or an entire function) then the only possible Jordan invariant analytic curves in $\mathbb{C}$ are either circles or preimages of circles under linearizing functions of Siegel discs [1].

Przytycki asked whether a rational function f can have a repeller which is a Jordan analytic curve different from a circle. A *repeller* is a compact set γ which has a neighborhood V such that for $z \in V$, $f^n(z) \in V$ for some integer $n \geq 0$ if and only if $z \in \gamma$.

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The principal result of this paper implies that if f is not a Lattès example, then such Jordan analytic repellers must be algebraic curves. However, the only example of a rational function having such a repeller, different from a circle, that I could produce, is a Lattés example.

THEOREM 1. *Let f be a rational function and γ a Jordan analytic invariant curve such that $f|_\gamma$ is not a homeomorphism and there is a repelling periodic point $a \in \gamma$ with multiplier λ, $|\lambda| > 1$. Assume in addition one of the following:*

(a) *γ does not contain critical or neutral rational fixed points of f, or*
(b) *$\gamma \subset J(f)$.*

Then λ is real, and there exist a non-hyperbolic Riemann surface S with an anti-conformal involution s and an endomorphism $g : S \to S$, and a holomorphic map $h : S \to \overline{\mathbb{C}}$ such that $g \circ s = s \circ g$, $h(X) \subset \gamma$ where $X \subset S$ is the set of points fixed by the involution s, and the following semiconjugacy relation holds:

$$h \circ g = f^n \circ h \tag{1}$$

with some integer $n \geq 0$.

COROLLARY. *The assumptions of Theorem 1 imply that the curve γ is algebraic, or f is a Lattés example.*

If γ is a repeller, then all assumptions of Theorem 1, including both (a) and (b), are satisfied.

Proof of Theorem 1. Replacing f by some iterate, we reduce to the case that the point a is fixed. Let F be the Poincaré function associated with the fixed point a. This means that

$$F(\lambda z) = f \circ F(z), \quad F(0) = a, \quad F'(0) \neq 0. \tag{2}$$

Such a function exists for every repelling fixed point and it is meromorphic in the plane $\mathbb{C}$. Let Γ be the component of $F^{-1}(\gamma)$ that contains 0. Then the intersection of Γ with a neighborhood of 0 is a smooth curve invariant under the map $z \mapsto \lambda^{-1}z$, which implies that λ is real. Replacing f by the second iterate we achieve that $\lambda > 1$. Then it follows that Γ contains a straight line, and without loss of generality we may assume that this is the real line. We have $F(\mathbb{R}) \subset \gamma$.

Consider the restriction mappings $\mathbb{R} \to \gamma$ of all Poincaré maps F_j for all repelling fixed points a_j on γ. There are finitely many of them, as the number of fixed points is finite. We claim that under our assumptions at least one of these maps is not injective.

Indeed, suppose that some F_j is injective on the real line. Then the image $F_j(\mathbb{R})$ is a simple arc $\gamma_j \subset \gamma$, and it is easy to see that the endpoints of this arc must be attracting fixed points with real multipliers.

Indeed these endpoints are fixed because γ_j is invariant and the iterates of f converge to these endpoints. This convergence implies that these endpoints cannot be repelling.

Suppose that assumption (b) in Theorem 1 holds. As γ_j is a subset of a Jordan analytic curve, its endpoints cannot be neutral irrational points. A local description of dynamics near a neutral rational fixed point shows that if such a point w is an endpoint of a smooth invariant curve on which the iterates converge to w, then this curve must intersect the set of normality. This contradiction completes the proof of non-injectivity of $F_j|_{\mathbb{R}}$ under assumption (b).

We continue the non-injectivity proof under assumption (a). The endpoints of γ_j can coincide, in which case $f|_\gamma$ is a homeomorphism. Now let b be an endpoint of γ_j which does not coincide with the other endpoint. Assumption (a) implies that the endpoints are attracting. From the local description of the dynamics near an attracting fixed point b we conclude that there is a simple invariant arc δ_0 disjoint from γ_j such that $\gamma_j \cup \delta_0 \cup \{b\}$ is an arc of γ. For $k \geq 1$, let δ_k be the component of $f^{-1}(\delta_{k-1})$ which contains δ_{k-1}. Then

$$\Delta = \bigcup_{k=1}^{\infty} \delta_k$$

is a simple invariant curve, $\Delta \subset \gamma$, one endpoint of Δ is b and the other is a repelling fixed point a_k on γ distinct from a_j. This means that the curves γ_j and γ_k have a common endpoint b, and their union is an analytic curve near this common point.

We conclude that our curves γ_j cover γ, and $f|_\gamma$ is a homeomorphism. This contradicts the assumptions and proves the claim.

From now on we assume that some Poincaré function F has non-injective restriction to the real line.

We recall a result of [4]. Let $F : \mathbb{C} \to S$ be a non-constant holomorphic map from the complex plane to a Riemann surface S. Consider the following equivalence relation in $\mathbb{C}$: $x \sim y$ iff $F(x) = F(y)$. Let $\Gamma_F \subset \mathbb{C}^2$ be the graph of this equivalence relation. It is easy to see that Γ_F is an analytic variety of pure (complex) dimension 1 (has no isolated points).

Now suppose that an analytic variety $\Gamma \subset \mathbb{C}^2$ of pure dimension 1 which is the graph of an equivalence relation is given. Then there exists a holomorphic map $F : \mathbb{C} \to S$, where S is a non-hyperbolic Riemann surface, such that $F(x) = F(y)$ iff $(x, y) \in \Gamma$. This map F is defined by Γ uniquely up to composition with an automorphism of S.

Now we characterize the graphs $\varGamma$ of equivalence relations corresponding to Poincaré functions.

Lemma 1. *A holomorphic map $F:\mathbb{C}\to S$ to a non-hyperbolic Riemann surface S is a Poincaré function of an endomorphism of S if and only if $\varGamma_F$ is invariant under the map*

$$(x,y)\mapsto(\lambda x,\lambda y) \tag{3}$$

for some $\lambda\in\mathbb{C}$, $|\lambda|>1$.

Proof. That the graph $\varGamma_F$ corresponding to a function F satisfying (2) is invariant with respect to this map is clear.

To prove the converse statement, we recall the result from [4] that the existence of a decomposition $F=f\circ G$, where f and G are maps between non-hyperbolic Riemann surfaces, is equivalent to the inclusion $\varGamma_G\subset\varGamma_F$. Let $\varGamma_1=\lambda\varGamma_F\subset\varGamma_F$. Then $\varGamma_1=\varGamma_G$ where $G(\lambda z)=F(z)$. On the other hand, by the result of [4] just cited, $F=f\circ G$ where $f:S\to S$ is an endomorphism. So $F(\lambda z)=f\circ G(\lambda z)=f\circ F(z)$, that is, F satisfies a Poincaré equation. Putting $z=0$ we see that $F(0)$ is a fixed point of f, and the multiplier of this fixed point is λ by the chain rule.

This completes the proof of the lemma.

A holomorphic map $F:\mathbb{C}\to S$ will be called *real* if there exists an anticonformal involution $s:S\to S$ such that $F(\overline{z})=s\circ F(z)$. The following is clear:

Lemma 2. *A holomorphic map $F:\mathbb{C}\to S$ is real if and only if $\varGamma_F$ is invariant under the map*

$$(x,y)\mapsto(\overline{x},\overline{y}). \tag{4}$$

Now we return to the proof of Theorem 1. As F is not injective on the real line, there is a real analytic germ $\phi\neq\mathrm{id}$ such that $F\circ\phi=F$. This implies that some part $\varGamma_1\neq\{(x,x):x\in\mathbb{C}\}$ of the graph $\varGamma_F$ is parametrized by $(x,\phi(x))$, so this part is invariant under the map (4).

Now let $\varGamma_2\subset\varGamma_F$ be the smallest analytic variety of pure dimension 1 which contains $\varGamma_1$, which is the graph of an equivalence relation and which is invariant under the maps (4) and (3).

Proof of existence of $\varGamma_2$. Let $E\subset\mathbb{C}^2$ be an arbitrary set containing the diagonal $D=\{(x,x):x\in\mathbb{C}\}$. Consider three operations on such sets:

$$\begin{aligned}
&E\mapsto\varLambda E=\{\lambda^n(x,y):(x,y)\in E,\ n\in\mathbb{N}\},\\
&E\mapsto SE=\{(x,y):(y,x)\in E\},\\
&E\mapsto TE=\{(x,z):\exists y,(x,y)\in E,\ (y,z)\in E\}.
\end{aligned}$$

If E does not contain isolated points then $\varLambda E, SE, TE$ do not contain isolated points. Moreover, if E is symmetric with respect to the operation (4) then each ΛE, SE, TE is also symmetric with respect to this operation. Now we apply all finite sequences of the operations Λ, S, T to $\Gamma_1 \cup D$, and take as Γ_2 the union of those irreducible components of Γ_F that have non-isolated intersection with the resulting sets. Then Γ_2 is the minimal analytic set of pure dimension 1 which is the graph of an equivalence relation, is invariant with respect to (3), and by the previous remark, it is symmetric with respect to (4). This completes the construction of the set Γ_2.

If $\Gamma_2 = \Gamma_F$ then F is real and γ is a circle. If $\Gamma_2 \neq \Gamma_F$ then there exists a factorization

$$F = h \circ G, \tag{5}$$

where G is a Poincaré function of some endomorphism $g : S \to S$ of a non-hyperbolic surface S, that is,

$$G(\lambda z) = g \circ G(z), \tag{6}$$

$\Gamma_2 = \Gamma_G$, and $h : S \to \overline{\mathbb{C}}$ is a holomorphic map. Moreover, G is real. Combining (5) and (6) we obtain

$$f \circ h \circ G(z) = f \circ F(z) = F(\lambda z) = h \circ G(\lambda z) = h \circ g \circ G(z),$$

and this implies (1). This completes the proof of Theorem 1.

To prove the Corollary, we discuss the functional equation (1). First of all, S can be a torus, $\mathbb{C}^*, \mathbb{C}$ or $\overline{\mathbb{C}}$. If S is a torus, then f is a Lattés example. If S is $\mathbb{C}$ or $\mathbb{C}^*$, and h has an essential singularity then f also must be a Lattés example. This was proved in [3]. See also [2] for another proof. Otherwise $S = \overline{\mathbb{C}}$, and thus both g and h are rational. This proves the corollary.

Lattés examples indeed have Jordan analytic invariant curves which are not circles. These curves can be algebraic or transcendental.

Example 1. Let $\wp$ be the Weierstrass function with periods $2\omega_1$ and $2i\omega_2$ where we assume that ω_1 and ω_2 are real. Consider the line $L = \{x + 2i\omega_2/3 : x \in \mathbb{R}\}$ and let $\gamma = \wp(L)$. The simplest Lattés function f corresponding to $\wp$ satisfies

$$\wp(2z) = f \circ \wp(z), \tag{7}$$

and we see that γ is invariant under f, because $2L \equiv -L$ modulo periods, and $\wp$ is even. It is easy to see that γ is a Jordan analytic curve which is not a circle, and $f : \gamma \to \gamma$ is a two-sheeted covering map.

Let us show that γ is algebraic. Let $s(z)$ be the reflection in the line L. Define

$$X(z) = (\wp(z) + \overline{\wp(s(z))})/2,$$
$$Y(z) = (\wp(z) + \overline{\wp(s(z))})/(2i).$$

Then X and Y are elliptic functions with the same period lattice as $\wp$. So they are related by an algebraic equation

$$F(X, Y) = 0.$$

When $z \in L$, we have $s(z) = z$, so $X(z) = \operatorname{Re}\wp(z)$ and $Y(z) = \operatorname{Im}\wp(z)$. So points $x + iy$ on our curve $\gamma = \wp(L)$ satisfy the algebraic equation $F(x, y) = 0$.

Example 2. Let $\wp$ be the Weierstrass function with primitive periods $(1, \tau)$, where $\tau = p + i$ and p is real and irrational. Let $\gamma = \wp(\mathbb{R})$, and suppose f is a Lattés function as in (7). It is easy to see that γ is a Jordan analytic curve invariant under f, which is not a circle.

Let us show that γ is transcendental. Consider the function $g(z) = \overline{\wp(\overline{z})}$. It is an elliptic function with periods 1 and $\overline{\tau} = p - i$. Evidently, the period $p - i$ is not a rational combination of 1 and $p + i$. So g is an elliptic function whose lattice is not commensurable with the lattice of $\wp$. Suppose now that γ is algebraic and let $F(x, y) = 0$ be the equation of γ. Then

$$F((\wp + g)/2, (\wp + g)/(2i)) = 0$$

holds on the real line and thus everywhere. We conclude that $\wp$ and g are algebraically dependent, and this is a contradiction, because algebraically dependent elliptic functions must have commensurable lattices. This proves that γ is transcendental.

Equation (1) in rational functions was recently studied by Pakovich [10] (see also [7]). It is closely related to the factorization theory of rational functions with respect to composition, which is due to J. Ritt [11, 13, 12].

The simplest solutions of (1) with $n = 1$ can be constructed as follows [7]. Let u and v be two rational functions. Set

$$f = u \circ v, \quad g = v \circ u \quad \text{and} \quad h = u. \tag{8}$$

Then $h \circ g = f \circ h$, so (1) is satisfied.

Another class of examples is obtained by taking an arbitrary rational function w and setting $f(z) = z^m w^n(z)$, $g(z) = z^m w(z^n)$ and $h(z) = z^n$.

These examples do not exhaust all possibilities which are listed in [10].

It is interesting to know whether (8) leads to Jordan analytic invariant curves which are not circles. Such examples will occur if $v \circ u$ is a real rational function but u is not real and maps the circle $\mathbb{R} \cup \{\infty\}$ onto a Jordan analytic curve γ which is not a circle. Then γ will be an invariant curve for $u \circ v$.

Example 3 (F. Pakovich). Let $P_n(z) = z^n$ and $J(z) = (z + z^{-1})/2$. It is well known that

$$R := J \circ P_n = T_n \circ J,$$

where T_n are real polynomials (they are Chebyshev polynomials normalized so that the leading coefficient is 2^{n-1}). Now, if $\epsilon = \exp(2\pi i/n)$, then the first factorization implies that $R(\epsilon z) = R(z)$. So $R(z) = T_n \circ J(\epsilon z)$. Let $u(z) = J(\epsilon z)$. Then $u(\mathbb{R})$ is a hyperbola γ, it is mapped by T_n onto the real line, and this hyperbola is invariant under the map $f = u \circ T_n$.

However a hyperbola, when considered on the Riemann sphere, is not a Jordan curve. Thus the only examples we have to illustrate Theorem 1 are Examples 1 and 2.

The semiconjugacy equation (1) for rational functions occurs also in [8] and [6] in different contexts.

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