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Realization of fixed point sets of relative maps

by

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Abstract. Given a relative map $f\colon (X,A) \to (X,A)$ on a pair (X,A) of compact polyhedra and a closed subset Y of X, we shall give some criteria for Y to be the fixed point set of some map relatively homotopic to f.

1. Introduction. The paper is motivated by the following natural question:

PROBLEM 1.1. *Given a self-map $f\colon X \to X$ and a closed subset Y of X, can we find a map $g\colon X \to X$ homotopic to f with* $\mathrm{Fix}(g) = Y$*?*

The question is related to early work in fixed point theory. For example, the existence of some g in the case of $Y = \emptyset$ means that the space X does not have the so-called *fixed point property*. If each nonempty closed subset Y of X is the fixed point set of a self-map on X, then X is said to have the *complete invariance property* (see [7]).

When Y is a finite set, the question was answered affirmatively in [8] for a fairly general type of compact polyhedra. Problem 1.1 was treated in [14]. Unfortunately, a mistake was found by H. Schirmer (cf. [11], [15]), and a corrected condition was given. Some similar results are obtained in [20].

Relative Nielsen fixed point theory ([9], [3], [21]) deals with the estimation of the number of fixed points of maps of the form $f\colon (X,A) \to (X,A)$. Similarly, one can ask the above question in a relative version:

PROBLEM 1.2. *Given a relative self-map $f\colon (X,A) \to (X,A)$ and a closed subset Y of X, can we find a map $g\colon (X,A) \to (X,A)$ relatively homotopic to f with* $\mathrm{Fix}(g) = Y$*?*

Clearly, if there is such a map g, the restriction of the relative homotopy to Y is a homotopy from the restriction $f|_Y$ of f to Y to $g|_Y$, which is just the inclusion $i^Y\colon Y \to X$. This is a natural necessary condition. It is a typical

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problem in homotopy theory to decide if there is such a homotopy. However, the existence of a homotopy is not enough to guarantee the existence of a map g with $\mathrm{Fix}(g) = Y$. Some criteria were obtained in [6] (cf. [13]) by generalizing the conditions on H in [11].

In this paper, we focus on necessary conditions that an arbitrary relative homotopy from $f|_Y$ to the inclusion i^Y must satisfy. Using the generalized Jiang group, we obtain a necessary condition, which we call the key property. This property is defined for a given homotopy from $f|_Y$ to i^Y, but it is actually independent of the choice of the homotopy. We are therefore able to check whether a subset Y can be realized as the fixed point set of a map relatively homotopic to the given map f.

Our paper is organized as follows. Section 2 presents a review of fixed point classes on subsets, and makes some conventions on notation. The Jiang group (1-dimensional Gottlieb group) is generalized in Section 3. Some properties of these groups are discussed. Section 4 is devoted to the key property which is satisfied by any homotopy from the restriction of f to Y to the inclusion $i^Y: Y \to X$ if Y can be realized as the fixed point set of a map relatively homotopic to f. In Section 5, we prove that under some mild assumptions on X, A and Y, the key property is sufficient for Y to be the fixed point set of a map relatively homotopic to f.

2. Fixed point classes on subsets. We shall give a brief review of fixed point classes on subsets. This concept is a natural generalization of fixed point class, was used in [12] and treated systematically in [2], where these classes are called local fixed point classes. Readers may find more details in [2] or [21].

Let $\phi\colon Z \to B$ be a map from a subset Z of B into B, where B is a compact polyhedron; both Z and B may be disconnected. For the universal covering $p_B\colon \tilde{B} \to B$, we say that a map $\tilde{\phi}\colon \tilde{Z} \to \tilde{B}$ is a *lifting* of ϕ if there is a commutative diagram

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\tilde{\phi}} & \tilde{B} \\ {\scriptstyle p_B}\downarrow & & \downarrow{\scriptstyle p_B} \\ Z & \xrightarrow{\phi} & B \end{array}$$

where $\tilde{Z}$ is a union of some components of $p_B^{-1}(Z)$ such that $\tilde{Z}$ contains exactly one component of $p_B^{-1}(Z_j)$ for each component Z_j of Z. Clearly, the number of components of $\tilde{Z}$ is the same as that of Z.

Consider a homotopy $H\colon Z \times I \to B$ from $\phi_0\colon Z \to B$ to $\phi_1\colon Z \to B$. We also say that $\tilde{H}\colon \tilde{Z} \times I \to \tilde{B}$ is a lifting of H if the following diagram commutes:

$$\begin{array}{ccc} \tilde{Z} \times I & \xrightarrow{\tilde{H}} & \tilde{B} \\ {\scriptstyle p_B \times \mathrm{id}} \big\downarrow & & \big\downarrow {\scriptstyle p_B} \\ Z \times I & \xrightarrow{H} & B \end{array}$$

where $\tilde{Z}$ is as before.

Let $\tilde{Z}_C$ be a component of $\tilde{Z}$. The pair $(\tilde{\phi}, \tilde{Z}_C)$ is said to be a *lifting data pair*, and we write $\mathrm{Fix}(\tilde{\phi}, \tilde{Z}_C)$ for the fixed point set $\tilde{Z}_C \cap \mathrm{Fix}(\tilde{\phi})$ of $\tilde{\phi}$ on $\tilde{Z}_C$. Two lifting data pairs $(\tilde{\phi}, \tilde{Z}_C)$ and $(\tilde{\phi}', \tilde{Z}_{C'})$ are said to be *conjugate* if there is an element γ in the covering transformation group $T(\tilde{B})$ of $\tilde{B}$ such that $\tilde{\phi}'|_{\tilde{Z}_{C'}} = \gamma \circ \tilde{\phi}|_{\tilde{Z}_C} \circ \gamma^{-1}$ and $\tilde{Z}_{C'} = \gamma(\tilde{Z}_C)$. Notice that B may be disconnected. The set $T(\tilde{B})$ is regarded as the disjoint union $\bigsqcup T(\tilde{B}_C)$, where B_C ranges over all components of B.

PROPOSITION 2.1. *The fixed point set* $\mathrm{Fix}(\phi)$ *equals* $\bigcup p_B(\mathrm{Fix}(\tilde{\phi}, \tilde{Z}_C))$, *the union of the projections of the fixed point sets of all its lifting data pairs. For any two lifting data pairs* $(\tilde{\phi}, \tilde{Z}_C)$ *and* $(\tilde{\phi}', \tilde{Z}_{C'})$, $p_B(\mathrm{Fix}(\tilde{\phi}, \tilde{Z}_C)) = p_B(\mathrm{Fix}(\tilde{\phi}', \tilde{Z}_{C'}))$ *if they are conjugate;* $p_B(\mathrm{Fix}(\tilde{\phi}, \tilde{Z}_C)) \cap p_B(\mathrm{Fix}(\tilde{\phi}', \tilde{Z}_{C'})) = \emptyset$ *if they are not conjugate.*

Thus, the fixed point set $\mathrm{Fix}(\phi)$ is split into a disjoint union of subsets of the form $p_B(\mathrm{Fix}(\tilde{\phi}, \tilde{Z}_C))$, which is said to be the fixed point class of ϕ determined by the lifting data pair $(\tilde{\phi}, \tilde{Z}_C)$. By definition, fixed points in different components lie in different fixed point classes.

EXAMPLE 2.2. Let $i^Z \colon Z \to B$ be the inclusion. The set of all lifting data pairs of i^Z is

$$\{(\gamma|_{\tilde{Z}_C}, \tilde{Z}_C) \mid \tilde{Z}_C \text{ is a component of } p_B^{-1}(Z),\ \gamma \in T(\tilde{B})\}.$$

Clearly, the subset $\{(\mathrm{id}_{\tilde{B}}|_{\tilde{Z}_C}, \tilde{Z}_C) \mid \tilde{Z}_C \text{ is a component of } p_B^{-1}(Z)\}$ consists of a complete conjugacy class, where the identity map $\mathrm{id}_{\tilde{B}}$ on $\tilde{B}$ is regarded as the trivial element in $T(\tilde{B})$. The fixed point class determined by this conjugacy class is just the unique non-empty fixed point class of i^Z. The fixed point classes determined by the other lifting data pairs will be empty. Thus, there do exist some empty fixed point classes as long as B is not simply-connected.

Let W be a non-empty subset of B with $W \subset Z$. Clearly, any fixed point class of $\phi|_W \colon W \to B$, which is said to be a fixed point class of ϕ on W, is contained in a unique fixed point class of $\phi \colon Z \to B$. More precisely, the fixed point class $p_B(\mathrm{Fix}(\widetilde{\phi|_W}, \tilde{W}_C)$ is contained in the fixed point class $p_B(\mathrm{Fix}(\tilde{\phi}, \tilde{Z}_C))$, where $\tilde{Z}_C$ is the unique component of $p_B^{-1}(Z)$ containing $\tilde{W}_C$, and $\tilde{\phi}$ is the unique lifting of $\phi \colon Z \to B$ such that $\tilde{\phi}|_{\tilde{W}_C} = \widetilde{\phi|_W}$.

Given a lifting $\tilde{\phi}_0\colon \tilde{Z} \to \tilde{B}$ of ϕ_0, there is a unique lifting $\tilde{H}\colon \tilde{Z}\times I \to \tilde{B}$ of H such that $\tilde{H}(\tilde{z},0)=\tilde{\phi}_0(\tilde{z})$ for all $\tilde{z}\in\tilde{Z}$. Hence, $\tilde{\phi}_0$ determines a unique lifting $\tilde{\phi}_1\colon \tilde{Z}\to\tilde{B}$, given by $\tilde{\phi}_1(\tilde{z})=\tilde{H}(\tilde{z},1)$ for all $\tilde{z}\in\tilde{Z}$. The converse is the same. Thus, any homotopy from ϕ_0 to ϕ_1 induces a bijection between the set of liftings of ϕ_0 and that of ϕ_1. It is not hard to check that such a bijection induces a bijection between the sets of fixed point classes of ϕ_0 and of ϕ_1. A fixed point class of ϕ_0 and a fixed point class of ϕ_1 are said to be *H-related* if they correspond to each other under the above bijection.

Throughout this paper, we shall use the following notation:

- (X,A): a pair of compact polyhedra, where X is connected;
- $\phi|_Z$: the restriction of the map ϕ to Z;
- ϕ_W: the restriction of the map ϕ to W which is considered to map into the invariant subspace, i.e. $\phi_W\colon W\to A$ if $\phi\colon (Y,Y\cap A)\to(X,A)$ is a relative map and $W\subset Y\cap A$;
- i^Z: the inclusion map of a subset Z into A or X;
- $\sharp^c S$: the number of components of S;
- $\mathrm{Fix}(\phi,Z)$: the fixed point set of a self-map ϕ on Z, where Z is a subset of the domain of ϕ;
- $\mathrm{FPC}(\phi)$: the set of fixed point classes of a self-map ϕ;
- $\mathrm{FPC}_e(\phi)$: the set of essential fixed point classes of a self-map ϕ;
- $\mathrm{FPC}_d(\phi)$: the set of fixed point classes of a relative map ϕ which do not assume their indices in the invariant subspace of ϕ;
- $p\colon \tilde{X}\to X$: the universal covering of X;
- $p_k\colon \tilde{A}_k\to A_k$: the universal covering of a component A_k of A;
- $p_A\colon \tilde{A}\to A$: the universal covering of A;
- $\tilde{W}$: a union of components of $p^{-1}(W)$ (or $p_k^{-1}(W)$) such that $\tilde{W}$ contains exactly one component of $p^{-1}(W_j)$ (or $p_{A_k}^{-1}(W_j)$) for each component W_j of W if $W\subset X$ (or $W\subset A_k$);
- $T(\tilde{V})$: the covering transformation group of the universal covering of V.

3. Generalized Jiang groups. In this section, we shall introduce some generalized Jiang groups.

Definition 3.1. Let $\phi\colon Z\to B$ be a map, $H\colon Z\times I\to B$ be a *cyclic homotopy* of ϕ (i.e. a homotopy from ϕ to itself), and $\tilde{\phi}\colon \tilde{Z}\to\tilde{B}$ be a lifting of ϕ with respect to the universal covering $p_B\colon \tilde{B}\to B$. When Z is a connected subset of B we define $\ell(\tilde{\phi},H)$ to be the element β in $T(\tilde{B}')$, where B' is the complement of B containing $\phi(Z)$, such that $\tilde{\phi}$ and $\beta\circ\tilde{\phi}$ are respectively the 0- and 1-slices of a lifting $\tilde{H}\colon \tilde{Z}\times I\to\tilde{B}$ of H, i.e. $\tilde{H}(\tilde{z},0)=\tilde{\phi}(\tilde{z})$ and $\tilde{H}(\tilde{z},1)=\beta\circ\tilde{\phi}(\tilde{z})$ for all $\tilde{z}\in\tilde{Z}$.

If Z has finitely many components, say $Z_1,\dots,Z_v$, we define $\ell(\tilde\phi,H)$ to be the array

$$(\ell(\tilde\phi|_{\tilde Z_1},H|_{Z_1\times I}),\ \dots,\ \ell(\tilde\phi|_{\tilde Z_v},H|_{Z_v\times I})),$$

where $\tilde Z=\bigsqcup_{j=1}^{v}\tilde Z_j$ and $\tilde\phi|_{\tilde Z_j}\colon \tilde Z_j\to\tilde B$ is a lifting of $\phi|_{Z_j}$, for $j=1,\dots,v$.

Furthermore, we define

$$J_{B:Z}(\tilde\phi)=\{\ell(\tilde\phi,H)\mid H \text{ is a cyclic homotopy of } \phi\}.$$

It is easy to prove that if Z is connected, then $J_{B:Z}(\tilde\phi)$ is a subgroup of $T(\tilde B')$, where B' is the complement of B containing $\phi(Z)$. In particular, if $Z=B$ is connected, $J_{B:B}(\tilde\phi)$ is just the Jiang group $J(\tilde\phi)$ (see [5, Ch. II, 3.1]). If Z has finitely many components, we have

$$J_{B:Z}(\tilde\phi)=J_{B:Z_1}(\tilde\phi|_{\tilde Z_1})\times\cdots\times J_{B:Z_v}(\tilde\phi|_{\tilde Z_v}),$$

where $v=\sharp^c Z$.

The generalized Jiang groups do depend on the choices of liftings of ϕ in the following way.

Proposition 3.2. *Let $\tilde\phi\colon \tilde Z\to\tilde X$ and $\tilde\phi'\colon \tilde Z'\to\tilde X$ be two liftings of $\phi\colon Z\to X$, where Z is a connected subset of X. Then $J_{X:Z}(\tilde\phi')=\gamma J_{X:Z}(\tilde\phi)\gamma^{-1}$ for some $\gamma\in T(\tilde X)$.*

Proof. Since both $\tilde Z$ and $\tilde Z'$ are components of $p^{-1}(Z)$, there is a unique element $\delta\in T(\tilde X)$ such that $\tilde Z'=\delta(\tilde Z)$. From uniqueness of lifting, there is also a unique γ such that $\tilde\phi'=\gamma\circ\tilde\phi\circ\delta^{-1}$. For any $\alpha\in J_{X:Z}(\tilde\phi)$, there is a lifting $\tilde H$ of a cyclic homotopy H of ϕ such that $\tilde H(\tilde z,0)=\tilde\phi(\tilde z)$ and $\tilde H(\tilde z,1)=\alpha\circ\tilde\phi$ for all $\tilde z\in\tilde Z$. Hence, $\gamma\circ\tilde H(\delta^{-1}(\tilde z'),0)=\gamma\circ\tilde\phi\circ\delta^{-1}(\tilde z')=\tilde\phi'(\tilde z')$, and $\gamma\circ\tilde H(\delta^{-1}(\tilde z'),1)=\gamma\circ\alpha\circ\tilde\phi\circ\delta^{-1}(\tilde z')=\gamma\circ\alpha\circ\gamma^{-1}\circ\tilde\phi'(\tilde z')$ for all $\tilde z'\in\tilde Z'$. It follows that $\gamma\alpha\gamma^{-1}\in J_{X:Z}(\tilde\phi')$. ■

Consider a lifting $\tilde\phi$ of a map $\phi\colon Z\to X$. If an element γ in $T(\tilde X)$ satisfies $\gamma(\tilde Z)\subset\tilde Z$, then $\tilde\phi\circ\gamma$ is still a lifting of ϕ with domain $\tilde Z$. Thus, there is a unique $\gamma'\in T(\tilde X)$ such that $\gamma'\circ\tilde\phi=\phi\circ\gamma$. We write $\tilde\phi_\pi(\gamma)$ for γ'. Clearly, $\{\gamma\in T(\tilde X)\mid\gamma(\tilde Z)\subset\tilde Z\}$ is a subgroup of $T(\tilde X)$, and $\tilde\phi_\pi\colon\{\gamma\in T(\tilde X)\mid\gamma(\tilde Z)\subset\tilde Z\}\to T(\tilde X)$ is a homomorphism. Unlike the case $Z=X$, such a homomorphism may not extend onto the whole $T(\tilde X)$ if Z is a just a connected subset of X.

Proposition 3.3. *Let $\phi\colon Z\to X$ be a map, where Z is a connected subset of X. Let $\tilde\phi\colon\tilde Z\to\tilde X$ be a lifting of ϕ. If γ is an element in $T(\tilde X)$ such that $\gamma(\tilde Z)\subset\tilde Z$, then $\tilde\phi_\pi(\gamma)\alpha=\alpha\tilde\phi_\pi(\gamma)$ for any $\alpha\in J_{X:Z}(\tilde\phi)$, i.e. $J_{X:Z}(\tilde\phi)\subset Z(\operatorname{Im}\tilde\phi_\pi,T(\tilde X))$, the center of $\operatorname{Im}\tilde\phi_\pi$ in $T(\tilde X)$.*

Proof. Let $\alpha\in J_{X:Z}(\tilde\phi)$. Then, by definition, there is a lifting $\tilde H$ of a cyclic homotopy H of ϕ such that $\tilde H(\tilde z,0)=\tilde\phi(\tilde z)$ and $\tilde H(\tilde z,1)=\alpha\circ\tilde\phi(\tilde z)$ for all $\tilde z\in\tilde Z$. Thus, $\tilde\phi_\pi(\gamma)\circ\tilde H$ is a homotopy from $\tilde\phi_\pi(\gamma)\circ\tilde\phi$ to $\tilde\phi_\pi(\gamma)\circ\alpha\circ\tilde\phi$. On the other hand, $\tilde{H}(\gamma(\tilde{z}),t)$ is a homotopy from $\tilde{\phi}\circ\gamma$ to $\alpha\circ\tilde{\phi}\circ\gamma$, i.e. from $\tilde{\phi}_\pi(\gamma)\circ\tilde{\phi}$ to $\alpha\circ\tilde{\phi}_\pi(\gamma)\circ\tilde{\phi}$. Note that both $\tilde{\phi}_\pi(\gamma)\circ\tilde{H}$ and $\tilde{H}(\gamma(\tilde{z}),t)$ are liftings of H starting from the same lifting of ϕ. Hence they are the same lifting of H. Thus, $\tilde{\phi}_\pi(\gamma)\circ\alpha\circ\tilde{\phi}=\alpha\circ\tilde{\phi}_\pi(\gamma)\circ\tilde{\phi}$. It follows that $\tilde{\phi}_\pi(\gamma)\alpha=\alpha\tilde{\phi}_\pi(\gamma)$. ∎

Applying this result to the case that ϕ is the inclusion $i\colon Z\to X$, we have

Corollary 3.4. *Let Z be a connected subset of X, and let $\tilde{Z}$ be a component of $p^{-1}(Z)$. If γ is an element in $T(\tilde{X})$ such that $\gamma(\tilde{Z})\subset\tilde{Z}$, then $\gamma\alpha=\alpha\gamma$ for any $\alpha\in J_{X:Z}(i^{\tilde{Z}}\colon\tilde{Z}\to\tilde{X})$.*

Fix a point $\tilde{x}_0\in p^{-1}(x_0)$. It is well-known that there is a canonical isomorphism $\psi_{\tilde{x}_0}\colon T(\tilde{X})\to\pi_1(X,x_0)$, defined as follows. Given an element $\alpha\in T(\tilde{X})$, pick a path $\tilde{c}$ from $\tilde{x}_0$ to $\alpha\tilde{x}_0$. Then $\psi_{\tilde{x}_0}(\alpha)$ is defined to be the element $\langle p\circ\tilde{c}\rangle\in\pi_1(X,x_0)$. Clearly, $\psi_{\tilde{x}_0}$ depends on the choice of $\tilde{x}_0$, but does not depend on the choice of the path $\tilde{c}$ because $\tilde{X}$ is simply-connected.

Lemma 3.5. *Let $\tilde{\phi}\colon\tilde{Z}\to\tilde{X}$ be a lifting of $\phi\colon Z\to X$, where Z is a connected subset of X with $x_0\in Z$. Then the isomorphic image $\psi_{\tilde{\phi}(\tilde{x}_0)}(J_{X:Z}(\tilde{\phi}))$ of $J_{X:Z}(\tilde{\phi})$ under $\psi_{\tilde{\phi}(\tilde{x}_0)}\colon T(\tilde{X})\to\pi_1(X,\phi(x_0))$ is*

$$\{\langle H(x_0,\cdot)\rangle\in\pi_1(X,\phi(x_0))\mid H\textit{ is a cyclic homotopy of }\phi\},$$

and hence is independent of the choice of the point $\tilde{x}_0$ in $\tilde{Z}\cap p^{-1}(x_0)$.

Proof. Let $\alpha\in J_{X:Z}(\tilde{\phi})$. By definition, there is a lifting $\tilde{H}$ of a cyclic homotopy of ϕ such that $\tilde{H}(\tilde{z},0)=\tilde{\phi}(\tilde{z})$ and $\tilde{H}(\tilde{z},1)=\alpha\circ\tilde{\phi}(\tilde{z})$ for all $\tilde{z}\in\tilde{Z}$. Pick a point $\tilde{x}_0\in\tilde{Z}\cap p^{-1}(x_0)$; the path $\tilde{H}(\tilde{x}_0,\cdot)$ starts at $\tilde{\phi}(\tilde{x}_0)$ and ends at $\alpha\tilde{\phi}(\tilde{x}_0)$. Projecting it down to X, we get a loop at $\phi(x_0)$, given by $H(x_0,\cdot)$.

Suppose that there is another cyclic homotopy H' of ϕ with a lifting $\tilde{H}'$ starting from $\tilde{\phi}$ and ending with $\alpha\tilde{\phi}$. Notice that the lifting path $\tilde{H}(\tilde{x}_0,\cdot)$ and $\tilde{H}'(\tilde{x}_0,\cdot)$ have the same end points $\tilde{\phi}(\tilde{x}_0)$ and $\alpha\tilde{\phi}(\tilde{x}_0)$. It follows that $H(x_0,\cdot)$ and $H'(x_0,\cdot)$ are homotopic relative to the end points. Thus, $\langle H(x_0,\cdot)\rangle$ is a well-defined element in $\pi_1(X,\phi(x_0))$, which is independent of the choice of the point $\tilde{x}_0$ and the cyclic homotopy H.

Clearly, any element of the form $\langle H(x_0,\cdot)\rangle$ is in the image of $\psi_{\tilde{\phi}(\tilde{x}_0)}$. ∎

By this lemma, we can write this isomorphic image $\psi_{\tilde{\phi}(\tilde{x}_0)}(J_{X:Z}(\tilde{\phi}))$ as $J_{X:Z}(\phi,x_0)$, which is a subgroup of the fundamental group of $\pi_1(X,\phi(x_0))$. From this lemma, we have

$$J_{X:Z}(\phi,x_0)=\{\langle H(x_0,\cdot)\rangle\in\pi_1(X,\phi(x_0))\mid H\text{ is a cyclic homotopy of }\phi\}.$$

When ϕ is the inclusion $i^Z\colon Z\to X$, $J_{X:Z}(i^Z,x_0)$ is just the 1-dimensional general Gottlieb group $G_1(X,Z,x_0)$ introduced in [18].

We can generalize the concept of Jiang group (1-dimensional general Gottlieb group) to the following:

DEFINITION 3.6. Let $\phi\colon (Z, Z\cap A)\to (X,A)$ be a relative map, where Z is a connected and compact subset of X such that $Z\cap A$ has finitely many components. Let $\tilde\phi\colon \tilde Z\to\tilde X$ be a lifting of ϕ, and let $\tilde\phi_{Z\cap A}\colon \widetilde{Z\cap A}\to\tilde A$ be a lifting of $\phi_{Z\cap A}\colon Z\cap A\to A$. We define

$$J_{(X,A):Z}(\tilde\phi,\tilde\phi_{Z\cap A})=\bigcup_H\{(\ell(\tilde\phi,H),\,\ell(\tilde\phi_{Z\cap A},H_{(Z\cap A)\times I})\}$$

where $H\colon (Z\times I,(Z\cap A)\times I)\to (X,A)$ ranges over all cyclic relative homotopies of ϕ.

From this definition, we immediately get

COROLLARY 3.7. *The set $J_{(X,A):Z}(\tilde\phi,\tilde\phi_{Z\cap A})$ is a subgroup of*

$$T(\tilde X)\times T(\tilde A_1)^{r_1}\times\cdots\times T(\tilde A_u)^{r_u},$$

where r_k is the number of components of $Z\cap A$ which are mapped by ϕ into A_k, for $k=1,\ldots,u=\sharp^c A$.

It should be noticed that if $Z\cap A$ is empty, then $J_{(X,A):Z}(\tilde\phi,\tilde\phi_{Z\cap A})$ will be the same as $J_{X:Z}(\tilde\phi)$.

4. Necessity. In this section, we consider a relative map $f\colon (X,A)\to (X,A)$ and a closed subset Y in X. We shall provide some necessary conditions on Y for it to be the fixed point set of some map relatively homotopic to the given map f.

Suppose that f is relatively homotopic to a map g with $\mathrm{Fix}(g)=Y$. The restriction to $Y\times I$ of a relative homotopy from f to g gives us a relative homotopy from the restriction $f|_Y$ to the inclusion i^Y. Here, we are going to ask for common features of all relative homotopies from $f|_Y$ to i^Y. It is obvious that for any two such homotopies H' and H'', the homotopy product $(H')^{-1}*H''$ is a cyclic relative homotopy of the inclusion i^Y, which is a kind of "difference" between the two homotopies. Any homotopy induces a one-to-one correspondence, which may not be the identity, between the sets of fixed point classes of homotopic maps. We will find that all possible differences in the sense of fixed point classes are coded in the elements (key elements) of generalized Jiang groups, which are defined in the last section. So, our approach is different from the work of [11] and [6] (cf. [13]), where the existence of special homotopies was assumed.

In relative fixed point theory, because of the invariant subset A, the location of fixed points of a relative map is no longer arbitrary. In [10], Schirmer considered the fixed point classes which do not assume their indices in A,

$$\mathrm{FPC}_d(f) = \{F \in \mathrm{FPC}(f) \mid \mathrm{ind}(f, F) \neq \mathrm{ind}(f_A, A \cap F)\}.$$

Its cardinality, $N(f; \overline{X - A})$, is a lower bound for the number of fixed points on $\mathrm{cl}(X - A)$ of all maps in the relative homotopy class of f. It is evident that Y has at least $N(f; \overline{X - A})$ components meeting $\mathrm{cl}(X - A)$ if f is relatively homotopic to a map g with $\mathrm{Fix}(g) = Y$. More necessary conditions on Y appear in our main theorem:

THEOREM 4.1. *Let Y be a closed subset of X such that Y and $Y \cap A$ have finitely many components. Suppose a relative map $f\colon (X, A) \to (X, A)$ is relatively homotopic to a map $g\colon (X, A) \to (X, A)$ with* $\mathrm{Fix}(g) = Y$. *Then the restriction $f|_Y\colon Y \to X$ is relatively homotopic to the inclusion $i^Y\colon Y \to X$, and there are injective functions*

$$\eta\colon \mathrm{FPC}_d(f) \to \{\textit{components of } Y \textit{ meeting } \mathrm{cl}(X - A)\},$$
$$\bar\eta\colon \mathrm{FPC}_e(f_A) \to \{\textit{components of } Y \cap A\}$$

such that any relative homotopy $H\colon (Y \times I, (Y \cap A) \times I) \to (X, A)$ from $f|_Y$ to i^Y has the following key property:

To each component Y' of Y one can assign a key element

$$\alpha' \in J_{(X,A):Y'}(i^{\tilde Y'}, i^{\widetilde{Y'\cap A}}),$$

for any lifting $i^{\tilde Y'}$ of $i^{Y'}$ and any lifting $i^{\widetilde{Y'\cap A}}\colon \widetilde{Y' \cap A} \to \tilde A$ of the inclusion $i^{Y'\cap A}\colon Y' \cap A \to A$, such that

- (K1) *if $\eta(F) = Y'$, then F contains a fixed point class of f on Y' which is H-related to the fixed point class $p(\mathrm{Fix}(\alpha'_0, \tilde Y'))$ of the inclusion $i^{Y'}\colon Y' \to X$, where α_0 is the first entry in α', i.e. the one corresponding to Y';*
- (K2) *if $\bar\eta(\bar F) = U$ is contained in a (unique) component A' of A, then $\bar F$ contains a fixed point class of $f_{A'}$ on U which is $H_{A'\times I}$-related to the fixed point class $p_A(\mathrm{Fix}(\alpha'_U, \widetilde{Y' \cap A'} \cap p_A^{-1}(U)))$ of the inclusion $i^U\colon U \to A'$, where α'_U is the entry of α' corresponding to U.*

Proof. Let $H'\colon (X \times I, A \times I) \to (X, A)$ be a relative homotopy from f to g. Its restriction $H'|_{Y\times I}$ is obviously a relative homotopy from the restriction $f|_Y\colon (Y, Y \cap A) \to (X, A)$ to the inclusion $i^Y\colon Y \to X$.

Firstly, we construct an injective function

$$\eta\colon \mathrm{FPC}_d(f) \to \{\text{components of } Y \text{ meeting } \mathrm{cl}(X - A)\}.$$

Pick a fixed point class $F = p(\mathrm{Fix}(\tilde f)) \in \mathrm{FPC}_d(f)$. Then F is H'-related to a fixed point class F' of g which does not assume its index in A, and hence F' meets $\mathrm{cl}(X - A)$ (see [10]). Since $\mathrm{Fix}(g) = Y$, F' contains a component, say Y', of Y which meets $\mathrm{cl}(X - A)$. We set $\eta(F) = Y'$. Note that "H'-related" is a one-to-one correspondence between $\mathrm{FPC}_d(f)$ and $\mathrm{FPC}_d(g)$. It follows that η is injective. By definition of "H'-related", there is a lifting $\tilde{H}'$ of H' such that $\tilde{H}'(\tilde{x}, 0) = \tilde{f}(\tilde{x})$ for all $\tilde{x} \in \tilde{X}$, and such that F' is determined by the 1-slice of $\tilde{H}'$. Hence, there is a component $\tilde{Y}'$ of $p^{-1}(Y')$ lying in the fixed point set of the 1-slice of $\tilde{H}'$, i.e. $\tilde{H}'(\tilde{y}, 1) = \tilde{y} = i^{\tilde{Y}'}(\tilde{y})$ for all $\tilde{y} \in \tilde{Y}'$. Clearly, the fixed point class $p(\mathrm{Fix}(\tilde{f}, \tilde{Y}'))$ of f on Y' is contained in F and is $H'|_{Y\times I}$-related to the fixed point class $p(\mathrm{Fix}(\tilde{H}'(\cdot, 1), \tilde{Y}')) = p(\mathrm{Fix}(i^{\tilde{Y}'}, \tilde{Y}'))$ of the inclusion $i^{Y'}\colon Y' \to X$.

Secondly, we construct an injective function

$$\bar{\eta}\colon \mathrm{FPC}_e(f_A) \to \{\text{components of } Y \cap A\}.$$

Let $\bar{F} \in \mathrm{FPC}_e(f_{A_k})$. By essentiality, $\bar{F}$ is $H'_{A\times I}$-related to a non-empty fixed point class $\bar{F}'$ of g_A. There is a component U of $Y \cap A$ such that $U \subset \bar{F}'$. We can pick a component $\tilde{U}$ of $p_A^{-1}(U)$ such that the 0-slice of a lifting $\tilde{H}'_{A\times I}$ of $H'_{A\times I}$ determines the fixed point class $\bar{F}$, and such that $\tilde{U}$ is contained in the fixed point set of the 1-slice of $\tilde{H}'_{A\times I}$. We define the component U as $\bar{\eta}(\bar{F})$. Clearly, the fixed point class $p_A(\mathrm{Fix}(\tilde{H}'_{A\times I}(\cdot, 0), \tilde{U}))$ of f_A on U is contained in $\bar{F}$ and is $H'_{(Y\cap A)\times I}$-related to the fixed point class $p_A(\mathrm{Fix}(\tilde{H}'_{A\times I}(\cdot, 1), \tilde{U})) = p_A(\mathrm{Fix}(i^{\tilde{U}}, \tilde{U}))$ of the inclusion $i^U : U \to A$.

Up to now, each of the components of Y or $Y \cap A$ in the image of η or $\bar{\eta}$ has been chosen a component of $p^{-1}(Y)$ or $p_A^{-1}(Y \cap A)$. Now, choose a component of $p^{-1}(Y)$ or $p_A^{-1}(Y \cap A)$ for each component of Y or $Y \cap A$ which does not lie in the image of η or $\bar{\eta}$. Then for the chosen components of $p^{-1}(Y)$ or $p_A^{-1}(Y \cap A)$ as above, the key element for each component Y' of Y is just the trivial element in the group $J_{(X,A):Y'}(i^{\tilde{Y}'}, i^{\widetilde{Y'\cap A}})$. Conditions (K1) and (K2) are satisfied automatically from the method of choosing the components of $p^{-1}(Y)$ or $p_A^{-1}(Y \cap A)$.

Fix a component Y' of Y. Any component of $p^{-1}(Y)$ can be written as $\gamma(\tilde{Y}')$, where $\tilde{Y}'$ is the chosen component of $p^{-1}(Y)$ and $\gamma \in T(\tilde{X})$. The fixed point class $F = p(\mathrm{Fix}(\tilde{f})) = p(\mathrm{Fix}(\gamma\tilde{f}\gamma^{-1}))$ will contain a fixed point class $p(\mathrm{Fix}(\gamma\tilde{f}\gamma^{-1}, \gamma(\tilde{Y})))$ of f on Y' which is $H'|_{Y\times I}$-related to the fixed point class $p(\mathrm{Fix}(i^{\gamma(\tilde{Y}')}, \gamma(\tilde{Y}')))$ of the inclusion $i^{Y'} : Y' \to X$. There is an analogy for an arbitrary component of $p_A^{-1}(Y \cap A)$. It follows that the key elements for the homotopy $H'|_{Y\times I}$ with respect to the chosen η and $\bar{\eta}$ are all trivial elements for any component of $p^{-1}(Y)$ and any component of $p_A^{-1}(Y \cap A)$.

Now, we will show that any relative homotopy $H\colon (Y \times I, (Y \cap A) \times I) \to (X, A)$ from $f|_Y$ to i^Y has the key property with respect to the chosen η and $\bar{\eta}$.

Define $H^c\colon (Y\times I,(Y\cap A)\times I)\to (X,A)$ by

$$H^c(y,t)=\begin{cases} H'(y,1-2t) & \text{if } 0\le t\le 1/2,\\ H(y,2t-1) & \text{if } 1/2\le t\le 1.\end{cases}$$

It is a cyclic relative homotopy of the inclusion $i^Y: Y\to X$. Fix a component Y' of Y. Given any component $\tilde{Y}'$ of $p^{-1}(Y')$, let $\widetilde{Y'\cap A}$ be a disjoint union of components of $p_A^{-1}(Y'\cap A)$ which contains exactly one component of $p_A^{-1}(U)$ for each component U of $Y'\cap A$. We have an element

$$(\ell(i^{\tilde{Y}'},H^c),\ell(i^{\widetilde{Y'\cap A}},H^c_{Y'\cap A}))\in J_{(X,A):Y'}(i^{\tilde{Y}'},i^{\widetilde{Y'\cap A}}).$$

It is sufficient to show that it is a key element, i.e. to check conditions (K1) and (K2). If $\eta(F)=Y'$, then by the above argument, we may assume that $F=p(\mathrm{Fix}(\tilde{f}))$ and that there is a unique lifting $\tilde{H}'$ (depending on the component Y') of H' such that $\tilde{H}'(\tilde{y},0)=\tilde{f}(\tilde{y})$ and $\tilde{H}'(\tilde{y},1)=\tilde{y}$ for all $\tilde{y}$ in the given component $\tilde{Y}'$ of $p^{-1}(Y')$. By definition of $\ell(i^{\tilde{Y}'},H^c)$, there is a lifting $\tilde{H}^c$ of H^c such that $\tilde{H}^c(\tilde{y},0)=i^{\tilde{Y}'}(\tilde{y})=\tilde{y}$ and $\tilde{H}^c(\tilde{y},1)=\ell(i^{\tilde{Y}'},H^c)(\tilde{y})$ for all $\tilde{y}\in\tilde{Y}'$. Define $\tilde{H}''\colon \tilde{Y}'\times I\to\tilde{X}$ by

$$\tilde{H}''(\tilde{y},t)=\begin{cases} \tilde{H}'(\tilde{y},2t) & \text{if } 0\le t\le 1/2,\\ \tilde{H}^c(\tilde{y},2t-1) & \text{if } 1/2\le t\le 1.\end{cases}$$

It is a lifting of the restriction of $H''\colon (Y\times I,(Y\cap A)\times I)\to (X,A)$ to $Y'\times I$, which is a relative homotopy from $f|_Y$ to the inclusion $i^Y: Y\to X$ given by

$$\begin{aligned} H''(y,t)&=\begin{cases} H'(y,2t) & \text{if } 0\le t\le 1/2,\\ H^c(y,2t-1) & \text{if } 1/2\le t\le 1,\end{cases}\\ &=\begin{cases} H'(y,2t) & \text{if } 0\le t\le 1/2,\\ H'(y,3-4t) & \text{if } 1/2\le t\le 3/4,\\ H(y,4t-3) & \text{if } 3/4\le t\le 1.\end{cases}\end{aligned}$$

Note that $\tilde{H}''(\tilde{y},0)=\tilde{H}'(\tilde{y},0)=\tilde{f}(\tilde{y})$ and $\tilde{H}''(\tilde{y},1)=\tilde{H}^c(\tilde{y},1)=\ell(i^{\tilde{Y}},H^c)(\tilde{y})$ for all $\tilde{y}\in\tilde{Y}'$. Thus, F contains a fixed point class $p(\mathrm{Fix}(\tilde{f},\tilde{Y}'))$ of f on $Y'=\eta(F)$, which is H''-related to the fixed point class $p(\mathrm{Fix}(\tilde{H}''(\cdot,1),\tilde{Y}'))=p(\mathrm{Fix}(\ell(i^{\tilde{Y}'},H^c),\tilde{Y}'))$ of the inclusion $i^{Y'}$. Notice that H'' and H are deformed into each other [5, 2.11 Definition]. By [5, 2.13 Theorem] and [21, Theorem 2.8], "H''-related" and "H-related" are the same correspondence between sets of fixed point classes. Thus, the fixed point class $p(\mathrm{Fix}(\tilde{f},\tilde{Y}'))$ of f on Y' is also H-related to the fixed point class $p(\mathrm{Fix}(\ell(i^{\tilde{Y}'},H^c),\tilde{Y}'))$ of the inclusion $i^{Y'}$.

Let $\bar{F}$ be an essential fixed point class of f_A with $\bar{\eta}(\bar{F}) = U$, which is a component of $Y' \cap A'$ for some component Y' of Y and some component A' of A. Similarly, we can prove that $\bar{F}$ contains a fixed point class of f_A on U which is $H_{(Y'\cap A')\times I}$-related to the fixed point class $p_A(\mathrm{Fix}(\beta, \tilde{U}))$ of the inclusion $i^U \colon U \to A'$, where $\beta = \ell(i^{\tilde{U}}, H^c_{(Y'\cap A')\times I})$. This finishes the proof. ∎

The key property does depend on the choice of the injective functions η and $\bar{\eta}$. The existence of η obviously implies that Y has at least $\sharp\mathrm{FPC}_d(f) = N(f; \overline{X - A})$ components meeting $\mathrm{cl}(X - A)$. If $N(f; \overline{X - A}) = 0$, then condition (K1) is satisfied automatically. Similarly, (K2) is satisfied automatically when $N(f_A) = 0$.

Notice that $\mathrm{FPC}_d(f)$ may contain inessential fixed point classes. For example, the identity map $\mathrm{id}\colon (S^1 \times D^2, C \times D^2) \to (S^1 \times D^2, C \times D^2)$, where C is an arc in S^1, has no essential fixed point class on the total space. The unique non-empty fixed point class does not assume its index in the subspace $C \times D^2$. Hence no set Y in the interior of $C \times D^2$ can be a fixed point set of any map in the relative homotopy class of id, although the conditions in [13, Theorem 4.6] are satisfied if we take Y to be a singleton.

The proof of Theorem 4.1 suggests that the existence of key elements satisfying conditions (K1) and (K2) is independent of the choices of components of $p^{-1}(Y)$'s and $p_A^{-1}(Y \cap A)$'s. Such an independence relies on a more general statement:

Proposition 4.2. *Suppose that $(\alpha|_{\tilde{Z}}, \tilde{Z})$ and $(\alpha'|_{\tilde{Z}'}, \tilde{Z}')$ are conjugate lifting data pairs of the inclusion $i^Z \colon Z \to X$, where Z is a connected subset of X. Then $\alpha \in J_{X:Z}(i^{\tilde{Z}})$ if and only if $\alpha' \in J_{X:Z}(i^{\tilde{Z}'})$.*

Proof. Since $(\alpha|_{\tilde{Z}}, \tilde{Z})$ and $(\alpha'|_{\tilde{Z}'}, \tilde{Z}')$ are conjugate, by definition, there is an element $\gamma \in T(\tilde{X})$ such that $\alpha' = \gamma\alpha\gamma^{-1}$ and $\tilde{Z}' = \gamma(\tilde{Z})$. By Proposition 3.2, $\alpha \in J_{X:Z}(i^{\tilde{Z}})$ if and only if $\alpha' = \gamma\alpha\gamma^{-1} \in J_{X:Z}(\gamma \circ i^{\tilde{Z}})$. Note that $\gamma \circ i^{\tilde{Z}} = i^{\tilde{Z}'}$. We are done. ∎

Compared with the necessary conditions (C1) and (C2) in [11], our conditions have an obvious advantage when proving that a certain set Y cannot be realized as the fixed point set of a map in a given relative homotopy class. It is sufficient to find one homotopy from $f|_Y$ to i^Y which does not have the key property. The following examples illustrate the way of checking our key property, as well as of computing generalized Jiang groups.

Example 4.3. Let $X = S^1 \times D^2 = \{(e^{\theta i}, re^{\phi i}) \mid 0 \le r \le 1\}$ be the solid torus, and $A = S^1 \times \{0\}$ be the core circle in X. A relative map $f\colon (X, A) \to (X, A)$ is defined by $f(e^{\theta i}, re^{\phi i}) = (e^{-\theta i}, re^{\phi i})$. Consider the compact set $Y = Y_1 \cup \{y_0\}$ where Y_1 is the arc $\{(e^{\pi t i}, \sin(\pi t)e^{0i})\}_{0\le t\le 1}$ and

$y_0 = (e^{0i}, e^{\pi i})$. Then Y touches A at end points. We shall show that Y cannot be relatively deformed to a map g with $\mathrm{Fix}(g) = Y$.

Let $p\colon \tilde{X} = \mathbb{R}^1 \times D^2 \to X$ denote the universal covering map which is defined by $p(u, re^{\phi i}) = (e^{2\pi ui}, re^{\phi i})$. Then the universal covering $p_A\colon \tilde{A} \to A$ can be regarded as the restriction of p to $\tilde{A} = \mathbb{R}^1 \times \{0\}$. Hence,

$$T(\tilde{X}) = \{\gamma_k \mid \gamma_k(u, re^{\phi i}) = (u+k, re^{\phi i}),\ k = 0, \pm1, \pm2, \ldots\},$$
$$T(\tilde{A}) = \{\bar{\gamma}_k \mid \bar{\gamma}_k = \gamma_k|_{\tilde{A}},\ k = 0, \pm1, \pm2, \ldots\}.$$

Pick the lifting $\tilde{f}\colon \tilde{X} \to \tilde{X}$ given by $\tilde{f}(u, re^{\phi i}) = (-u, re^{\phi i})$. Then the restriction $\tilde{f}_{\tilde{A}}$ is a lifting of f_A with respect to the universal covering $p_A\colon \tilde{A} \to A$.

Clearly, f_A has two fixed point classes $F_1 = \{(e^{0i}, 0)\} = p_A(\mathrm{Fix}(\tilde{f}_{\tilde{A}}))$ and $F_2 = \{(e^{\pi i}, 0)\} = p_A(\mathrm{Fix}(\bar{\gamma}_1 \tilde{f}_{\tilde{A}}))$, and f also has two fixed point classes which contain F_1 and F_2 respectively. Each of these four fixed point classes has fixed point index 1. It follows that $\mathrm{FPC}_d(F) = \emptyset$ and $\mathrm{FPC}_e(f_A) = \{F_1, F_2\}$. Consider the components $Y_1 = \{(e^{\pi ti}, \sin(\pi t)e^{0i})\}_{0\le t\le 1}$ and $Y_2 = \{(e^{0i}, e^{\pi i})\}$ of Y, and the components $U_1 = \{(e^{0i}, 0)\}$ and $U_2 = \{(e^{\pi i}, 0)\}$ of $Y_1 \cap A$.

Pick the component $\tilde{Y}_1 = \{(t/2, \sin(\pi t)e^{0i})\}_{0\le t\le 1}$ of $p^{-1}(Y_1)$, the component $\tilde{U}_1 = \{(0,0)\}$ of $p_A^{-1}(U_1)$ and the component $\tilde{U}_2 = \{(1/2, 0)\}$ of $p_A^{-1}(U_2)$. Let us compute the generalized Jiang group $J_{(X,A):Y_1}(i^{\tilde{Y}_1}, i^{\tilde{U}})$, where $\tilde{U} = \tilde{U}_1 \sqcup \tilde{U}_2$. Let $H_1\colon (Y_1 \times I, (Y_1 \cap A) \times I) \to (Y_1, A)$ be an arbitrary cyclic relative homotopy of the inclusion $i^{Y_1}\colon Y_1 \to X$. We have $\ell(i^{\tilde{Y}_1}, H_1) = \gamma_k$ for some $\gamma_k \in T(\tilde{X})$, i.e. there is a lifting $\tilde{H}_1$ of H_1 such that $\tilde{H}_1(\tilde{y}, 0) = \tilde{y}$ and $\tilde{H}_1(\tilde{y}, 1) = \tilde{y} + k$ for any $\tilde{y} \in \tilde{Y}_1$. It is obvious from Definition 3.6 that $\ell(i^{\tilde{U}}, H_{1,(A\cap Y)\times I}) = (\bar{\gamma}_k, \bar{\gamma}_k)$. Conversely, for any integer m, the cyclic homotopy $H^{(m)}\colon (X \times I, A \times I) \to (X, A)$ given by $H^{(m)}((e^{\theta i}, re^{\phi i}), t) = (e^{(\theta + 2\pi mt)i}, re^{\phi i})$ yields $\ell(i^{\tilde{Y}_1}, H^{(m)}|_{Y_1\times I}) = \gamma_m$. Thus, we obtain

$$J_{(X,A):Y_1}(i^{\tilde{Y}_1}, i^{\tilde{U}}) = \{(\gamma_k, \bar{\gamma}_k, \bar{\gamma}_k) \mid k = 0, \pm1, \pm2, \ldots\} \cong \mathbb{Z}.$$

Define a relative homotopy $H\colon (X \times I, A \times I) \to (X, A)$ starting from f by

$$H((e^{\theta i}, re^{\phi i}), t) = \begin{cases} (e^{\theta i}, re^{\phi i}) & \text{if } 0 \le \theta \le \pi t, \\ (e^{(\frac{t+2}{t-2}(\theta - \pi t) + \pi t)i}, re^{\phi i}) & \text{if } \pi t < \theta < 2\pi. \end{cases}$$

Its restriction to $Y_1 \times I$ is a homotopy from $f|_{Y_1}$ to the inclusion i^{Y_1}.

Let us check the key property. Since $\mathrm{FPC}_d(f) = \emptyset$ and Y_2 does not intersect A, it is sufficient to consider the possibility of an injective function $\bar{\eta}\colon \{F_1, F_2\} \to \{U_1, U_2\}$.

Suppose that $\bar{\eta}(F_1) = U_1$ and $\bar{\eta}(F_2) = U_2$. Note that any fixed point class of f_A on U_1 which is contained in $F_1 = p_A(\mathrm{Fix}(\tilde{f}_{\tilde{A}}))$ must have the form $p_A(\mathrm{Fix}(\bar{\gamma}_m \tilde{f}_{\tilde{A}} \bar{\gamma}_m^{-1}, \tilde{U}_1))$. Since $\bar{\eta}(F_1) = U_1$, condition (K2) implies that

$p_A(\mathrm{Fix}(\bar{\gamma}_m \tilde{f}_{\tilde{A}} \bar{\gamma}_m^{-1}, \tilde{U}_1))$ is $H_{A\times I}$-related to a fixed point class $p_A(\mathrm{Fix}(\bar{\gamma}_k, \tilde{U}_1))$ of $i^{U_1} \colon U_1 \to A$. Since H is constant on U_1, we have $\bar{\gamma}_m \tilde{f}_{\tilde{A}} \bar{\gamma}_m^{-1}(0,0) = \bar{\gamma}_k(0,0)$. It follows that $k = 2m$ is an even number.

Similarly, any fixed point class of f_A on U_2 in $F_2 = p_A(\mathrm{Fix}(\bar{\gamma}_1 \tilde{f}_{\tilde{A}}))$ must have the form $p_A(\mathrm{Fix}(\bar{\gamma}_n \bar{\gamma}_1 \tilde{f}_{\tilde{A}} \bar{\gamma}_n^{-1}, \tilde{U}_2))$. As $\bar{\eta}(F_2) = U_2$, condition (K2) forces that the fixed point class $p_A(\mathrm{Fix}(\bar{\gamma}_n \bar{\gamma}_1 \tilde{f}_{\tilde{A}} \bar{\gamma}_n^{-1}, \tilde{U}_2))$ is $H_{A\times I}$-related to some fixed point class $p_A(\mathrm{Fix}(\bar{\gamma}_{k'}, \tilde{U}_2))$, i.e. $\bar{\gamma}_n \bar{\gamma}_1 \tilde{f}_{\tilde{A}} \bar{\gamma}_n^{-1}$ and $\mathrm{Fix}(\bar{\gamma}_{k'})$ are respectively the 0- and 1-slices of the restriction to $\tilde{U}_2 \times I$ of some lifting of $H_{A\times I}$. Let us denote by $\tilde{H}$ the lifting of H such that $\tilde{H}((0,0),0) = \tilde{f}(0,0) = (0,0)$. Note that $\tilde{H}((1/2,0),t) = \big(\frac{3t-2}{4-2t}, 0\big)$. The restrictions of its 0- and 1-slices to $\tilde{U}_2$ are respectively $\tilde{f}|_{\tilde{U}_2}$ and $\mathrm{id}_{\tilde{U}_2}$. The 1-slice of the lifting $\bar{\gamma}_{2n+1} \tilde{H}|_{\tilde{U}_2\times I}$ of $H|_{U_2\times I}$ is $\bar{\gamma}_{2n+1}|_{\tilde{U}_2}$ since its 0-slice is $\bar{\gamma}_{2n+1} \tilde{f}_{\tilde{A}}|_{\tilde{U}_2} = \bar{\gamma}_n \bar{\gamma}_1 \tilde{f}_{\tilde{A}} \bar{\gamma}_n^{-1}|_{\tilde{U}_2}$. It follows by uniqueness of lifting that $k' = 2n+1$. Thus, $p_A(\mathrm{Fix}(\bar{\gamma}_n \bar{\gamma}_1 \tilde{f}_{\tilde{A}} \bar{\gamma}_n^{-1}, \tilde{U}_2))$ is $H_{A\times I}$-related to the fixed point class $p_A(\mathrm{Fix}(\bar{\gamma}_{2n+1}, \tilde{U}_2))$.

Since $(\beta, \bar{\gamma}_{2m}, \bar{\gamma}_{2n+1}) \notin J_{(X,A):Y_1}(i^{\tilde{Y}_1}, i^{\tilde{U}})$ for any $\beta \in T(\tilde{X})$, this homotopy $H|_{Y\times I}$ will not have the key property with respect to the injective function $\bar{\eta}$ given by $\bar{\eta}(F_1) = U_1$ and $\bar{\eta}(F_2) = U_2$.

The situation is the same if $\bar{\eta}(F_1) = U_2$ and $\bar{\eta}(F_2) = U_1$. Thus, by Theorem 4.1, Y cannot be the fixed point set of any map in the relative homotopy class of f, although $Y \cap A = \mathrm{Fix}(f) \cap A$.

EXAMPLE 4.4. Let $X = S_0^1 \cup D_1^2 \cup S_2^1 \cup D_3^2$ be a subset of the complex plane, where

$$S_0^1 = \{e^{\theta i} - 3 \mid 0 \le \theta < 2\pi\}, \quad D_1^2 = \{re^{\theta i} - 1 \mid 0 \le \theta < 2\pi,\ 0 \le r \le 1\},$$
$$S_2^1 = \{e^{\theta i} + 1 \mid 0 \le \theta < 2\pi\}, \quad D_3^2 = \{re^{\theta i} + 3 \mid 0 \le \theta < 2\pi,\ 0 \le r \le 1\}.$$

Let $A = S_1^1 \cup S_2^1 \cup S_3^1$ with $S_1^1 = \partial D_1^2$ and $S_3^1 = \partial D_3^2$. A relative map $f \colon (X,A) \to (X,A)$ is defined by

$$f(z) = \begin{cases} \bar{z} & \text{if } z \in S_2^1, \\ z & \text{otherwise,} \end{cases}$$

where $\bar{z}$ is the conjugate complex number of z. Take the compact set $Y = S_1^1$.

Pick $x_0 = 0$ as the base point of both A and X. We have the following loop classes at x_0:

$$\alpha_0 = \langle \{e^{\pi t i} - 1\}_{0\le t\le 1} \cdot \{e^{2\pi t i} - 3\}_{0\le t\le 1} \cdot \{e^{(\pi - \pi t)i} - 1\}_{0\le t\le 1} \rangle,$$
$$\alpha_1 = \langle \{e^{2\pi t i} - 1\}_{0\le t\le 1} \rangle,$$
$$\alpha_2 = \langle \{e^{(2\pi t - \pi)i} + 1\}_{0\le t\le 1} \rangle,$$
$$\alpha_3 = \langle \{e^{(\pi t - \pi)i} + 1\}_{0\le t\le 1} \cdot \{e^{(2\pi t - \pi)i} + 3\}_{0\le t\le 1} \cdot \{e^{\pi t i} + 1\}_{0\le t\le 1} \rangle.$$

Thus, $\pi_1(A,x_0)$ is a free group of rank 3 with generators $\alpha_1,\alpha_2,\alpha_3$, and $\pi_1(X,x_0)$ is a free group of rank 2 with generators α_0,α_2.

Take a point $\tilde{a}_0\in p_A^{-1}(x_0)$, and a point $\tilde{x}_0\in p^{-1}(x_0)$. We have two canonical isomorphisms, $\psi_{\tilde{a}_0}\colon T(\tilde{A})\to\pi_1(A,x_0)$ and $\psi_{\tilde{x}_0}\colon T(\tilde{X})\to\pi_1(X,x_0)$. Let $\tilde{Y}_A$ be the component of $p_A^{-1}(Y)$ containing $\tilde{a}_0$, and $\tilde{Y}$ the component of $p^{-1}(Y)$ containing $\tilde{x}_0$.

We shall compute $J_{(X,A):Y}(i^{\tilde{Y}},i^{\tilde{Y}_A})$ as follows. Suppose that $(\beta,\bar{\beta})\in J_{(X,A):Y}(i^{\tilde{Y}},i^{\tilde{Y}_A})$. Then $\bar{\beta}\in J_A(i^{\tilde{Y}_A})$, and by Lemma 3.5 and its corollary, we have $\bar{\beta}\gamma=\gamma\bar{\beta}$ for all $\gamma\in T(\tilde{A})$ satisfying $\gamma(\tilde{Y}_A)\subset\tilde{Y}_A$. Since $T(\tilde{A})$ is a free group generated by $\psi_{\tilde{a}_0}^{-1}(\alpha_1)$, $\psi_{\tilde{a}_0}^{-1}(\alpha_2)$ and $\psi_{\tilde{a}_0}^{-1}(\alpha_3)$, and since its subgroup $\{\gamma\in T(\tilde{A})\mid\gamma(\tilde{Y}_A)\subset\tilde{Y}_A\}$ is the infinite cyclic group generated by $\psi_{\tilde{a}_0}^{-1}(\alpha_1)$, the element $\bar{\beta}$ must have the form $\psi_{\tilde{a}_0}^{-1}(\alpha_1^k)$. Notice that for any integer m, the map $H_m\colon Y\times I\to A$ given by $(e^{\theta i},t)\mapsto e^{(\theta+2\pi mt)i}$ is a cyclic homotopy such that $\psi_{\tilde{a}_0}^{-1}(\alpha_1^m)=\ell(i^{\tilde{Y}_A},H_m)$. Let $H\colon Y\times I\to A$ be an arbitrary cyclic homotopy of i^Y with $\ell(i^{\tilde{Y}_A},H)=\psi_{\tilde{a}_0}^{-1}(\alpha_1^m)\in T(\tilde{A})$. Then $\{H(x_0,t)\}_{0\le t\le 1}\simeq\{H_m(x_0,t)\}_{0\le t\le 1}$ rel $\{0,1\}$, i.e. they are homotopic in A relative to their end points. Since the loop $\{H_m(x_0,t)\}_{0\le t\le 1}$ is trivial in X, so is $\{H(x_0,t)\}_{0\le t\le 1}$. Thus, the lifting of the path $\{H(x_0,t)\}_{0\le t\le 1}$ with respect to $p\colon\tilde{X}\to X$ starting at $\tilde{x}_0$ will end at $\tilde{x}_0$. It follows that the lifting of the cyclic homotopy H starting with $i^{\tilde{Y}}$ will end also with $i^{\tilde{Y}}$. Thus, $\ell(i^{\tilde{Y}},H)$ is the trivial element 1 in $T(\tilde{X})$. We obtain

$$J_{(X,A):Y}(i^{\tilde{Y}},i^{\tilde{Y}_A})=\{(1,\psi_{\tilde{a}_0}^{-1}(\alpha_1^k))\mid k\in\mathbb{Z}\}\cong\mathbb{Z}.$$

The symmetric behavior of f_A tells us that the two components S_1^1 and S_3^1 of the fixed point set of f_A must have the same indices. Hence both have zero indices because $L(f_A)=0$. Thus, $\mathrm{FPC}_e(f_A)=\emptyset$.

Let us consider the fixed point classes of f. Note that the fixed point set of f consists of two components: $S_0^1\cup D_1^2$ and D_3^2. Clearly, $\mathrm{ind}(f,D_3^2)=1$. The fact that $L(f)=1$ implies $\mathrm{ind}(f,S_0^1\cup D_1^2)=0$. Let $\tilde{b}\colon I\to\tilde{X}$ be the lifting of $\{e^{(2\pi t-\pi)i}+1\}_{0\le t\le 1}$ with $\tilde{b}(0)=\tilde{x}_0$. Since $\{e^{(2\pi t-\pi)i}+1\}_{t=1/2}$ is contained in the fixed point set of f, there is a lifting $\tilde{f}$ of f such that $\tilde{f}(\tilde{b}(1/2))=\tilde{b}(1/2)$. Thus, the unique fixed point class of f which does not assume its index in A is just $p(\mathrm{Fix}(\tilde{f}))$. By the definition of f, we have that $\tilde{f}(\tilde{b}(1/2-\varepsilon))=\tilde{b}(1/2+\varepsilon)$ for $-1/2<\varepsilon<1/2$. This implies that $\tilde{f}(\tilde{b}(0))=\tilde{b}(1)$. Since $\alpha_2=\langle b\rangle\in\pi_1(X,x_0)$, from the argument preceding Lemma 3.5, we have $\tilde{b}(1)=\psi_{\tilde{x}_0}^{-1}(\alpha_2)\tilde{b}(0)$. It follows that $\tilde{f}(\tilde{x}_0)=\psi_{\tilde{x}_0}^{-1}(\alpha_2)\tilde{x}_0$. This also implies that the two components of the fixed point set of f lie in different fixed point classes of f, and therefore $\mathrm{FPC}_d(f)=\{D_3^2\}$.

Notice that $f(y)=y$ for all $y\in Y$. We can choose the constant homotopy from $f|_Y$ to i^Y. Notice that any element in $J_{(X,A):Y}(i^{\tilde{Y}},i^{\tilde{Y}_A})$ has the form $(1,*)$. If the key property were satisfied, condition (K1) would imply that $\tilde{f}'|_{\tilde{Y}} = i^{\tilde{Y}}$ for some lifting $\tilde{f}'$ which also determines $p(\mathrm{Fix}(\tilde{f}))$. Clearly, $\tilde{f}' = \gamma\tilde{f}\gamma^{-1}$ for some $\gamma \in T(\tilde{X})$. In particular, we would have $\gamma\tilde{f}\gamma^{-1}(\tilde{x}_0) = \tilde{x}_0$. It would follow that $x_0 \in p(\mathrm{Fix}(\tilde{f}))$. This is a contradiction. Thus, by Theorem 4.1, Y cannot be the fixed point set of any map in the relative homotopy class of f.

If we consider the map $f \times \mathrm{id}_{D^2} \colon (X \times D^2, A \times D^2) \to (X \times D^2, A \times D^2)$, with the same argument we can show that $Y \times \{0\}$ is not the fixed point set of any map in the relative homotopy class of $f \times \mathrm{id}_{D^2}$. But, since it is contained in the contractible set $D^2_1 \times \{0\}$, $Y \times \{0\}$ can be realized as the fixed point set of a map which is homotopic (not relatively homotopic) to $f \times \mathrm{id}_{D^2}$ by using [11, Theorem 4.3] directly.

5. Sufficiency. In this section, we shall provide some sufficient conditions under which a compact subset Y of X can be realized as the fixed point set of a map relatively homotopic to a given relative map f. The main idea is the same as in the proof of the minimum theorem [9, Theorem 6.2] for the relative Nielsen number $N(f; X, A)$.

THEOREM 5.1. *Let $f \colon (X, A) \to (X, A)$ be a relative map of a pair (X, A) of compact polyhedra. Suppose that Y is a compact subpolyhedron of X satisfying:*

1. *$Y \cup A$ can be bypassed in X,*
2. *$Y \cap A'$ can be bypassed in A' for each component A' of A,*
3. *$X - (Y \cup A)$ has no local separating point and is not a 2-manifold,*
4. *$A' - Y$ has no local separating point and is not a 2-manifold for each component A' of A.*

If there is a relative homotopy $H \colon (Y \times I, (Y \cap A) \times I) \to (X, A)$ from the restriction of f to Y to the inclusion of Y into X having the key property with respect to injective functions η and $\bar{\eta}$ as in Theorem 4.1, then f is relatively homotopic to a map $g \colon (X, A) \to (X, A)$ with $\mathrm{Fix}(g) = Y$.

Proof. Let us outline the scheme of our proof. We shall construct a relative homotopy from $f \colon (X, A) \to (X, A)$ to: f' satisfying $Y \subset \mathrm{Fix}\, f'$, f'' satisfying $Y \subset \mathrm{Fix}(f'')$ and $Y \cap A = \mathrm{Fix}(f'') \cap A$, f''' satisfying $Y \subset \mathrm{Fix}(f''')$, $Y \cap A = \mathrm{Fix}(f''') \cap A$ and $\mathrm{Fix}(f''') - Y$ are fixed points in distinct classes.

Since both Y and A are compact subpolyhedra of X, Y and $Y \cap A$ must have finitely many components. Consider a component Y' of Y which meets the images $\mathrm{Im}(\eta) \cup \mathrm{Im}(\bar{\eta})$. By definition of the key property, there is a key element α' in the group $J_{(X,A):Y'}(i^{\tilde{Y}}, i^{\widetilde{Y' \cap A}})$, where $\tilde{Y}'$ is a component of $p^{-1}(Y')$, and $\widetilde{Y' \cap A}$ is a union of components of $p_A^{-1}(Y' \cap A)$ such that

$\widetilde{Y' \cap A}$ contains exactly one component for each component of $Y' \cap A$. By definition of the above generalized Jiang group, there is a cyclic homotopy $H^c_{Y'}\colon (Y' \times I, (Y' \cap A) \times I) \to (X, A)$ of the inclusion of Y' in X such that $i^{\tilde{Y}'}\colon \tilde{Y}' \to \tilde{X}$ and $\alpha'_0 \circ i^{\tilde{Y}'}\colon \tilde{Y}' \to \tilde{X}$ are respectively the 0- and 1-slices of a lifting of the homotopy $H^c_{Y'}$, and such that $i^{\tilde{U}}\colon \tilde{U} \to \tilde{A}$ and $\alpha'_U \circ i^{\tilde{U}}\colon \tilde{U} \to \tilde{A}$ are respectively the 0- and 1-slices of a lifting of the homotopy $(H^c_{Y'})_{(Y' \cap A) \times I}\colon (Y' \cap A) \times I \to A$ for any component U of $Y' \cap A$, where α'_0 is the first entry of α', α'_U is the entry of α' corresponding to the component U, and $\tilde{U}$ is a component of $p_A^{-1}(U)$.

Having the $H^c_{Y'}$ for each component Y' of Y meeting $\mathrm{Im}(\eta) \cup \mathrm{Im}(\bar{\eta})$, we can define

$$H'(y,t) = \begin{cases} H(y, 2t) & \text{if } 0 \le t \le \frac{1}{2},\ y \in Y',\ Y' \cap (\mathrm{Im}(\eta) \cup \mathrm{Im}(\bar{\eta})) \neq \emptyset, \\ H^c_{Y'}(y, 2-2t) & \text{if } \frac{1}{2} \le t \le 1,\ y \in Y',\ Y' \cap (\mathrm{Im}(\eta) \cup \mathrm{Im}(\bar{\eta})) \neq \emptyset, \\ H(y,t) & \text{if } y \in Y'',\ Y'' \cap (\mathrm{Im}(\eta) \cup \mathrm{Im}(\bar{\eta})) = \emptyset. \end{cases}$$

It is a relative homotopy from $f|_Y$ to the inclusion of Y into X. By condition (K1), each fixed point class F of f that does not assume its index in A will contain a fixed point class of f on the component $\eta(F)$ of Y, which is H'-related to the unique non-empty fixed point class of the inclusion of $\eta(F)$ into X; from condition (K2), each essential fixed point class $\bar{F}$ of f_A will contain a fixed point class of f_A on the component $\bar{\eta}(\bar{F})$ of $Y \cap A$, which is $H'_{(Y \cap A) \times I}$-related to the unique non-empty fixed point class of the inclusion of $\bar{\eta}(\bar{F})$ into A.

Using the homotopy extension property for pairs $(A, Y \cap A)$ and then on the pair (X, Y), we extend H' to a relative homotopy $H'\colon (X \times I, A \times I) \to (X, A)$. Let f' denote the relative map given by $f'(x) = H'(x, 1)$. Then $Y \subset \mathrm{Fix}(f')$. Each fixed point class F of f that does not assume its index in A is H'-related to a fixed point class of f' containing the component $\eta(F)$ of Y; and each essential fixed point class $\bar{F}$ of f_A is $H'_{A \times I}$-related to a fixed point class of f'_A containing the component $\bar{\eta}(\bar{F})$ of $Y \cap A$.

Let A' be a component of A. The restriction of f' to A' can be regarded as a relative map $f'_{A'}\colon (A', Y \cap A') \to (A', Y \cap A')$. Using the method of [9, Lemma 4.1], we may assume that $f'_{A'}$ has finitely many fixed points on $A' - Y$. Since $Y \cap A'$ can be bypassed in A' and since $A' - Y$ has no local separating point and is not a 2-manifold, we can combine the fixed points of $f'_{A'}$ in $A' - Y$ lying in the same fixed point class of it into one point. The isolated fixed points with zero indices can be removed in the usual way (cf. [1, p. 123, Theorem 4]). The resulting map is still denoted f'. Now, the fixed points on $A' - Y$ lie in distinct fixed point classes of $f'_{A'}$ and have non-zero indices. Let a be a fixed point on $A' - Y$. If a lies in an essential fixed point class, this class must be H'-related to an essential fixed point class $\bar{F}$ of $f_{A'}$. Thus, the point a and the component $\bar{\eta}(\bar{F})$ of $Y \cap A'$ are in the same fixed point class of $f'_{A'}$. We can then move a into the component $\bar{\eta}(\bar{F})$. If a lies in an inessential fixed point class of $f'_{A'}$, since a has non-zero fixed point index, there must be a component of $Y \cap A'$ lying in the same fixed point class as a. Hence, we can move a into this component.

Repeating this procedure in each component of A, we get a new map $f''_A\colon A \to A$ which is homotopic to f'_A relative to $Y \cap A$ such that $\mathrm{Fix}(f''_A) = Y \cap A$.

Let $H''\colon A \times I \to A$ denote the homotopy from f'_A to f''_A. Clearly, $H''(y,t) = y$ for all $y \in Y \cap A$. We can extend H'' to $H''\colon ((Y \cup A) \times I, A \times I) \to (X, A)$ by defining $H''(y,t) = y$ for all $y \in Y$. Using the homotopy extension property again, we can extend H'' to a relative homotopy $H''\colon (X \times I, A \times I) \to (X, A)$ starting from $f'\colon (X,A) \to (X,A)$. We write $f''\colon (X,A) \to (X,A)$ for the 1-slice of H'', i.e. $f''(x) = H''(x,1)$ for all $x \in X$.

Now, we regard f'' as a relative map of the pair $(X, Y \cup A)$. Using the technique of the proof of [9, Lemma 4.1] again, we may homotope f'' relative to $Y \cup A$ so that f'' has finitely many fixed points on $X - (Y \cup A)$. The fixed points on $X - (Y \cup A)$ in the same fixed point classes will be combined into one point. After that, the fixed points on $X - (Y \cup A)$ with zero indices will be removed as in [1, p. 123, Theorem 4]. Thus, we have a new map f''' which is homotopic to f'' relative to $Y \cup A$ so that the fixed points on $X - (Y \cup A)$ lie in distinct fixed point classes and have non-zero indices.

Let x be a fixed point of f''' on $X - (Y \cup A)$, lying in a fixed point class F_x. If F_x does not assume its index in A, then x is homotopy related to a fixed point class F of f which does not assume its index in A. Item (K1) in the key property ensures that the component $\eta(F)$ of Y meets $\mathrm{cl}(X - A)$ and is contained in F_x. Therefore, we can move x to become a point in $\eta(F) \cap \mathrm{cl}(X - A)$. If F_x assumes its index in A, there must be a component Y' of Y such that $Y' \cap \mathrm{cl}(X - A) \neq \emptyset$. Otherwise, F_x would be a disjoint union of $\{x\}$ with several components of Y lying in $\mathrm{int}(A)$. Hence, $\mathrm{ind}(f''', F_x) = \mathrm{ind}(f''', x) + \sum_{Y' \subset F_x} \mathrm{ind}(f''', Y') = \mathrm{ind}(f''', x) + \sum_{Y' \subset F_x} \mathrm{ind}(f'''_A, Y') = \mathrm{ind}(f''', x) + \mathrm{ind}(f'''_A, F_x \cap A)$. This contradicts the fact that F_x assumes its index in A, because $\mathrm{ind}(f''', x)$ is non-zero. Thus, in this case, we can still find a fixed point in $Y \cap \mathrm{cl}(X - A)$ lying in F_x, and therefore we can move x into $Y \cap \mathrm{cl}(X - A)$.

Since each fixed point on $X - (Y \cup A)$ has already been moved into Y, we obtain the desired relative map. ∎

By using the idea in [11, Theorem 3.2] or in [13], the assumption that Y is a subpolyhedron can be weakened. We omit the details.

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