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Conformal actions with prescribed periods on Riemann surfaces

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Abstract. It is a natural question what is the set of minimal periods of a holomorphic maps on a Riemann surface of negative Euler characteristic. Sierakowski studied ordinary holomorphic periods on classical Riemann surfaces. Here we study orientation reversing automorphisms acting on classical Riemann surfaces, and also automorphisms of non-orientable unbordered Klein surfaces to which, following Singerman, we shall refer to as non-orientable Riemann surfaces. We get a complete set of conditions for the existence of conformal actions with a prescribed order and a prescribed set of periods together with multiplicities. This lets us determine the minimal genus of a surface which admits such an action.

1. Introduction. It is a natural problem to describe all possible minimal periods of homeomorphisms f of finite order of a closed manifold X, e.g. to estimate from above the orders of such homeomorphisms provided they exist. In terms of actions of finite groups, this question reads: what is the maximal order N of a finite group G acting effectively on X, and for given N, what are the isotropy subgroups for all possible actions? In general these questions are difficult and one can hardly expect an answer in terms of the topology of X only. The problem of classification of all actions of finite cyclic groups on a surface was completely solved by Yokoyama [20] (see also his earlier papers). He gave sufficient and necessary conditions for the equivalence of two such actions, up to conjugacy by a homeomorphism, but did not study the problem of realization of periods.

However, if X is a two-dimensional closed manifold of negative Euler characteristic, orientable or not, then the above questions can be successfully handled, since due to the Hurwitz [8] and Kerekjarto [10] theorems they reduce to the study of the related questions for a conjugate (by a

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homeomorphism $h : X' \to X$) automorphism $f' = h^{-1}fh$ of a Klein surface X'. Consequently, by considering conformal maps of Klein surfaces we do not restrict the generality of the problem. Moreover, due to the algebraic representation of a Klein surface X it is possible to answer finer and more special versions of the above questions, still referring to the topology of X determined by its genus g.

The classical Hurwitz theorem [8] says that a group G of automorphisms of a compact unbordered Klein surface X of topological genus g has at most $84(g-\varepsilon)$ elements, where $\varepsilon = 1$ or 2 depending on whether X is orientable or not, and correspondingly $g \geq 2$ or $g \geq 3$. If, in the orientable case, we allow orientation reversing automorphisms then this bound doubles up to $168(g-1)$. It is known that in all three cases these bounds are both attained and non-attained for infinitely many values of g. Furthermore Wiman [19] gave this bound for a cyclic group, obtaining $2(2g+1)$ as the maximum possible order for an orientation preserving periodic homeomorphism of a Riemann surface.

The starting point for our work was Sierakowski's paper [15] where he studied holomorphic periods on classical compact Riemann surfaces. It is well known that given a sequence $N_1, \overset{s_1}{\ldots}, N_1, \ldots, N_r, \overset{s_r}{\ldots}, N_r$ of proper divisors of N there are Riemann surfaces for which $\{N_1, \ldots, N_r\}$ is the set of all singular periods, though it may happen that these periods cannot be realized with given multiplicities $s_1, \ldots, s_r$ which are the numbers of orbits of length N_i. Sierakowski has found formulas for the minimal genus of such surfaces. However these formulas involve the prime decomposition of N and so they are highly non-closed. Also he does not analyze the problem of multiplicities.

In this work we study periods of orientation reversing automorphisms of classical Riemann surfaces and of automorphisms of non-orientable unbordered compact Klein surfaces which, following Singerman, we call non-orientable Riemann surfaces. Such a surface is a compact topological surface with a dianalytic structure, which roughly speaking differs from the classical analytic structure by the fact that complex conjugation is allowed for transition maps between charts [1]. The ordinary holomorphy of a map between classical Riemann surfaces means that such a map is angle and sense preserving. Here we relax the requirements by considering only angle preserving mappings and calling, just in this paper, the maps in question conformal (in the theory of classical Riemann surfaces, "conformal" and "holomorphic" are usually synonymous). This is reasonable for maps between non-orientable Riemann surfaces and for orientation reversing homeomorphisms between classical Riemann surfaces. So with this convention, complex conjugation is not holomorphic but it is conformal.

By definition, for an automorphism φ of order N, a *period* of φ is the minimal period of a point $x \in X$, i.e. the length of the orbit of x. Such an orbit may be isolated or not in the set of orbits of length $< N$, and we distinguish between these cases. Orbits of the second type are characterized by their cardinality, which is $N/2$, and by the fact that all their elements are fixed by certain symmetries of X if X is a classical Riemann surface, and by involutions with properties similar to those of symmetries in the case of non-orientatable Riemann surfaces. More precisely, the set of all such orbits is a disjoint union of continua each of which is homeomorphic to the circle, or simply is an oval of some of the above mentioned involutions, in Hilbert's nineteenth century terminology. Two such orbits having elements on the same continua are said to be equivalent and we call the corresponding classes *reflexive periods.*

In this paper, we give necessary and sufficient conditions for the existence of orientable Riemann surfaces with given isolated periods counted with multiplicities and with a given number of reflexive periods for orientation reversing automorphisms (Section 5). Next we give formulas for the minimal genera of such surfaces (Section 6), minimizing the genus of the orbit space. This simplifies the formulas, and surprisingly, compared with the orientation preserving cyclic action on a classical Riemann surface, our formulas are rather explicit in the sense that they depend only on N and $N_1, \ldots, N_r$, their 2-adic parts and multiplicities $s_1, \ldots, s_r$ and not on complete prime decompositions. We also consider these problems for non-orientable Riemann surfaces, obtaining similar results. Finally we mention that not only minimal genera, but the spectra of all genera of such surfaces can be found, though we do not formulate the results explicitly to avoid technicalities.

We leave unsolved the problem of finding the conformal dynamics of a single automorphism of a bordered Klein surface, which seems to be tractable by using similar methods to those employed in this paper. We mention, however, that some attempts in this direction have been made by Sierakowski in his thesis [16]. Finally, we emphasize that to find the whole conformal dynamics on Klein surfaces seems to be a difficult problem, though its special form, consisting in finding, in terms of the group of automorphisms of the surface and the topological type of the action, the set of fixed points of a single automorphism has been solved in [4, 6] for bordered and non-orientable unbordered Klein surfaces, respectively, and in [5, 9, 13] for classical Riemann surfaces in the case of symmetries and non-involutionary automorphisms respectively.

2. Some preliminaries. We shall use a combinatorial approach based on Fuchsian and non-euclidean crystallographic groups (NEC-groups for short); we refer the reader to the monographs [1] and [3] for a detailed exposition of the whole theory.

2.1. Fuchsian and non-euclidean crystallographic groups. An *NEC-group* is a discrete and cocompact subgroup of the group $\mathcal{G}$ of isometries of the hyperbolic plane $\mathcal{H}$, including those which reverse orientation, and such a subgroup containing only orientation preserving isometries is a *Fuchsian group.* It is worth mentioning that $\mathcal{G}$ is known to coincide with the group of all conformal (in the above generalized sense) automorphisms of $\mathcal{H}$.

Macbeath and Wilkie [12, 18] associated to every NEC-group Λ a *signature* which determines its algebraic structure. It has the form

$$(g;\pm;[m_1,\dots,m_r];\{C_1,\dots,C_k\}). \tag{1}$$

The numbers $m_i \geq 2$ are called the *proper periods*, the sequences $C_i = (n_{i1},\dots,n_{is_i})$ the *period cycles*, the numbers $n_{ij} \geq 2$ the *link periods*, and $g \geq 0$ is said to be the *orbit genus* of Λ. The orbit space $\mathcal{H}/\Lambda$ is a surface having k boundary components, orientable or not according to the sign being $+$ or $-$, and having topological genus g. A Fuchsian group can be regarded as an NEC-group with signature

$$(g;+;[m_1,\dots,m_r];\{-\})$$

which will be briefly denoted by $(g;m_1,\dots,m_r)$; a Fuchsian group without periods will be denoted by $(g;-)$ and will be called a *Fuchsian surface group.* The group with signature (1) has a presentation given by the generators

(a) $x_i,\ i=1,\dots,r$ (hyperbolic rotations),
(b) $c_{ij},\ i=1,\dots,k; j=0,\dots,s_i$ (hyperbolic reflections),
(c) $e_i,\ i=1,\dots,k$ (connecting generators),
(d) $a_i,b_i,\ i=1,\dots,g,$ if the sign is $+$ (hyperbolic translations),
$d_i,\ i=1,\dots,g,$ if the sign is $-$ (hyperbolic glide reflections),

and relations

(i) $x_i^{m_i}=1,\ i=1,\dots,r,$
(ii) $c_{is_i}=e_i^{-1}c_{i0}e_i,\ i=1,\dots,k,$
(iii) $c_{i,j-1}^2=c_{ij}^2=(c_{i,j-1}c_{ij})^{n_{ij}}=1,\ i=1,\dots,k; j=1,\dots,s_i,$
(iv) $x_1\dots x_re_1\dots e_ka_1b_1a_1^{-1}b_1^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}=1$ if the sign is $+$,
$x_1\dots x_re_1\dots e_kd_i^2\dots d_g^2=1$ if the sign is $-$.

Any set of generators of an NEC-group satisfying the above relations will be called a *canonical set of generators* and the reflections $c_{i,j-1},c_{ij}$ will be said to be *consecutive.* For convenience we shall call the products $c_{i,j-1}c_{ij}$ the *canonical decomposable elliptic elements* of Λ. Connecting generators are usually hyperbolic translations but when the orbit genus is zero and the signature has only one period cycle and only one proper period then the corresponding connecting generator is an elliptic element, i.e. a hyperbolic rotation. Finally it is known that an element of an NEC group has finite order if and only if it is conjugate either to a canonical reflection, to a power of a canonical elliptic element, or to a power of a canonical decomposable elliptic element.

Every NEC-group has an associated fundamental region, whose hyperbolic area $\mu(\Lambda)$ for an NEC-group Λ with signature (1) is given by

$$2\pi\left(\varepsilon g + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{i=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right)\right), \tag{2}$$

where $\varepsilon = 2$ if the sign is $+$ and $\varepsilon = 1$ otherwise. It is known that an abstract group with the presentation given by the generators (a)–(d) and relations (i)–(iv) can be realized as an NEC-group with signature (1) if and only if the above quantity (2) is positive. If Γ is a subgroup of finite index in an NEC-group Λ then it is an NEC-group itself, and the *Hurwitz–Riemann formula* states that

$$[\Lambda : \Gamma] = \mu(\Gamma)/\mu(\Lambda). \tag{3}$$

2.2. Riemann and Klein surfaces and their groups of automorphisms. A *Klein surface* is a compact surface (in this paper unbordered) with dianalytic structure, which roughly speaking differs from the classical analytic structure by the fact that the complex involution $z \mapsto -\bar{z}$ is allowed for transition maps [1]. Non-orientable surfaces allow such structures indeed, and following Singerman, we shall call them *non-norientable Riemann surfaces.* The ordinary holomorphy of a map between classical Riemann surfaces means that the map is angle and sense preserving. Here we relax this condition by considering only angle preserving mappings and calling, just in this paper, such maps *conformal* (in the theory of classical Riemann surfaces "conformal" and "holomorphic" are usually synonymous). This concept is reasonable for maps between non-orientable Riemann surfaces and for orientation reversing homeomorphisms between classical Riemann surfaces.

Now, by the Riemann uniformization theorem, a classical compact Riemann surface of genus $g \geq 2$ can be represented as the orbit space $\mathcal{H}/\Gamma$ for some Fuchsian surface group Γ with signature $(g; -)$ with the structure inherited from $\mathcal{H}$. Such a group is characterized among all Fuchsian groups as being one which is torsion free.

Furthermore a group G of conformal automorphisms of a surface thus given can be represented as the factor group Λ/Γ, where Λ is a proper NEC or a Fuchsian group according to whether G contains orientation reversing automorphisms or not. A combinatorial study of groups of automorphisms of non-orientable Riemann surfaces is possible essentially due to the same facts. This time however a group Γ uniformizing X is an NEC group with signature $(g; -; [-]; \{-\})$ and so Λ is a proper NEC group. The group Γ is again characterized among NEC groups as one which is torsion free.

Summing up, a Riemann surface (orientable or not) with a conformal action of G on it is determined by an exact sequence

$$1 \to \Gamma \to \Lambda \xrightarrow{\theta} G \to 1,$$

where Λ is an NEC or Fuchsian group and Γ is torsion free.

Finally as an arbitrary element of finite order in Λ, say with signature (1), is conjugate either to a canonical reflection or to a power of a canonical elliptic element, or else to a power of a canonical decomposable elliptic element, and Γ is torsion free, the above epimorphism θ is characterized as one which preserves the orders of canonical elements of finite order in the canonical presentation corresponding to this signature, and we shall refer to it as a *smooth epimorphism.*

To avoid a misunderstanding, from now on, for a pair of groups $\Gamma \subset \Lambda$ as above we denote by $\tilde{g}$ the genus of $X = \mathcal{H}/\Gamma$, i.e. the genus of the Riemann or Klein surface on which G acts (the orbit genus of Γ), and by g the genus of $\mathcal{H}/\Lambda = X/G$ (the orbit genus of Λ). The letter g will also be used to denote elements of G but this should not lead to confusion.

3. Singular orbits at large. Let X be a Riemann surface of genus $\tilde{g} \geq 2$ with a group G of automorphisms. The orbit Gx of a point $x \in X$ is said to be *singular* if $|Gx| < |G|$, or equivalently, x is a fixed point of some element $g \neq e$ of G.

Let $X = \mathcal{H}/\Gamma$ be an unbordered Riemann surface, orientable or not, let $G = \Lambda/\Gamma$ be its group of automorphisms and let $\pi : \mathcal{H} \to X$ and $\theta : \Lambda \to G$ be the canonical projections. Given a canonical system of generators, let $h_1, \ldots, h_s$ be the set of fixed points of all canonical elliptic generators and canonical decomposable elliptic elements $x_1, \ldots, x_s$, and let $\ell_1, \ldots, \ell_t$ be the axes of the canonical reflections $c_1, \ldots, c_t$ of Λ. Observe that $s = r + s_1 + \cdots + s_k$ if Λ has signature (1).

The following lemma follows from the properties of the canonical projection $X \to X/G$ (see also an alternative proof by Sierakowski based on the Macbeath formula for fixed points [13] in the orientable case); for completeness we give an alternative, direct, algebraic proof.

Lemma 3.1. *Each singular orbit of the action G equals either $\{\pi(\lambda h_i) \mid \lambda \in \Lambda\}$ and has $|G|/2m_i$ or $|G|/m_i$ elements depending on whether x_i is decomposable or not, or $\{\pi(\lambda y_i) \mid \lambda \in \Lambda\}$ for some $y_i \in \ell_i \setminus \{h_1, \ldots, h_r\}$ and has $|G|/2$ elements.*

Proof. Observe that given $x \in X$ and $g \in G$ we have

$$gx = \pi(\lambda h) \quad \text{if } g = \theta(\lambda) \text{ and } x = \pi(h).$$

Now let Gx be a singular orbit of $x = \pi(h)$. Then x is a fixed point for some $g = \theta(\lambda)$. But this means that $\gamma\lambda h = h$ for some $\gamma \in \Gamma$. So $\gamma\lambda$

is either elliptic or a reflection and therefore is conjugate to a power of some canonical elliptic generator or canonical decomposable element, say $\gamma\lambda = \lambda_i x_i^{k_i} \lambda_i^{-1}$, or to some c_i, say $\gamma\lambda = \lambda_i c_i \lambda_i^{-1}$. So $h = \lambda_i h_i$ or $h = \lambda_i y_i$ for some $y_i \in \ell_i \setminus \{h_1, \ldots, h_s\}$. In the first case $\pi(h_i) = \theta(\lambda_i)^{-1}x \in Gx$ and in the second case $\pi(y_i) = \theta(\lambda_i)^{-1}x \in Gx$.

Finally, $\pi(\lambda h_i) = \pi(\lambda' h_i)$ if and only if $\lambda^{-1}\gamma\lambda' = x_i^{k_i}$ for some $\gamma \in \Gamma$ and so if and only if $\lambda^{-1}\lambda'$ belongs to $\Gamma\langle x_i\rangle$ if x_i is not a product of two reflections or to $\Gamma\langle c, c'\rangle$ if $x_i = cc'$. Hence Gx has respectively

$$[\Lambda : \Gamma\langle x_i\rangle] = \frac{[\Lambda : \Gamma]}{[\Gamma\langle x_i\rangle : \Gamma]} = \frac{|G|}{m_i}$$

or

$$[\Lambda : \Gamma\langle c, c'\rangle] = \frac{[\Lambda : \Gamma]}{[\Gamma\langle c, c'\rangle : \Gamma]} = \frac{|G|}{2m_i}$$

elements.

Similarly $\pi(\lambda y_i) = \pi(\lambda' y_i)$ if and only if $\lambda^{-1}\gamma\lambda' = c_i$ for some $\gamma \in \Gamma$ and so if and only if $\lambda^{-1}\lambda'$ belongs to $\Gamma\langle c_i\rangle$ and hence Gx has

$$[\Lambda : \Gamma\langle c_i\rangle] = \frac{[\Lambda : \Gamma]}{[\Gamma\langle c_i\rangle : \Gamma]} = \frac{|G|}{2}$$

elements. ∎

In other words we have obtained

Corollary 3.2. *All singular orbits divide into three classes: the orbits corresponding bijectively to proper periods m_i, each orbit with $|G|/m_i$ elements, the orbits corresponding bijectively to link periods n_{ij}, each with $|G|/2n_{ij}$ elements, and finally the orbits with $|G|/2$ elements, infinitely many for each canonical reflection.* ∎

Definition 3.3. The three types of singular orbits from the above corollary will be called respectively *isolated elliptic*, *non-isolated elliptic* and *reflexive orbits* and the lengths of the first two orbits will be called respectively *isolated elliptic* and *non-isolated elliptic periods* of G.

Remark 3.4. (1) If, in the above notation, c is a reflection of the group Λ then for any $h \in \mathrm{Fix}(c)$, Gx is a reflexive orbit for $x = \pi(h)$. So it is senseless to consider all such orbits. Therefore we define two reflexive orbits to be *equivalent* if they come from conjugate reflections, that is, they are orbits of two elements fixed by the images of conjugate reflections. The numbers $|G|/2$ corresponding to these classes will be called *reflexive periods.*

(2) If

$$\mathrm{Fix}(X) = \bigcup_{1\neq\varphi\in\mathrm{Aut}(X)} \mathrm{Fix}(\varphi),$$

then the elliptic isolated orbits are characterized as the singular orbits consisting of points which are isolated in $\mathrm{Fix}(X)$, while the non-isolated elliptic and reflexive ones are those which are not isolated.

(3) The fact that, in the above terminology, an element of a singular orbit is fixed by $\theta(c)$ does not guarantee that the orbit is reflexive. This is, however, guaranteed by the additional assumption that the order of the orbit is $|G|/2$. ∎

4. On singular orbits for cyclic actions on orientable and non-orientable Riemann surfaces. Here we shall specify the results of the previous section to the case of cyclic actions.

PROPOSITION 4.1. *A cyclic orientation preserving conformal action on an orientable Riemann surface has no reflexive orbits.*

Proof. Let the action of a cyclic group $G = \mathbb{Z}_N = \langle \varphi \rangle$ on a Riemann surface X be given by an epimorphism $\theta : \Lambda \to G$. Clearly Λ has no reflections since for a reflection c, $\theta(c)$ would be an orientation reversing conformal involution X belonging to G, while the latter is assumed to be generated by an orientation preserving automorphism φ. ∎

PROPOSITION 4.2. *A cyclic conformal action on a Riemann surface* (*orientable or not*) *has no non-isolated singular elliptic orbits.*

Proof. Let the action of a cyclic group $G = \mathbb{Z}_N = \langle x \rangle$ on a Riemann surface X be given by an epimorphism $\theta : \Lambda \to G$. If G has a non-isolated elliptic orbit, then Λ has two reflections c, c' whose product is an elliptic element. Now N is even since otherwise $c = c^N \in \ker\theta = \Gamma$. But in that case $\theta(c) = \theta(c') = x^{N/2}$, which means that $cc' \in \Gamma$, a contradiction since the last group is torsion free. ∎

REMARK 4.3. By Proposition 4.2 a cyclic action has no non-isolated elliptic orbits. So the isolated elliptic orbits will be briefly called *elliptic orbits* and their lengths *elliptic periods.*

PROPOSITION 4.4. *If a cyclic, conformal and orientation reversing action of order N on an orientable Riemann surface has reflexive orbits, then N is even and $N/2$ is odd. If in addition the action has elliptic periods $N_1, \ldots, N_r$, then $N/N_1, \ldots, N/N_r$ are all odd and*

$$\mathrm{lcm}(N/N_1, \ldots, N/N_r) \neq N.$$

Proof. An action having reflexive singular orbits is determined by an epimorphism $\theta : \Lambda \to \mathbb{Z}_N = \langle x \rangle$ with a Fuchsian surface group as kernel for an NEC group Λ with signature $(g; \pm; [m_1, \ldots, m_r]; \{(-), \overset{k}{\ldots}, (-)\})$ for some $k > 0$. If N is odd then $c_1 \in \Gamma = \ker\theta$, which is impossible since the latter is a Fuchsian group. Now $x = \theta(\lambda)$ for some orientation reversing isometry

λ in Λ. But then $\lambda^{N/2}c_1 \in \Gamma = \ker\theta$ and it is orientation reversing if $N/2$ is even. Now $N_i = N/m_i$, by Lemma 3.1, and if some m_i were even then $x_i^{m_i/2}c_1$ would be an orientation reversing isometry in $\Gamma = \ker\theta$, contrary to the latter being a Fuchsian group. Finally, if $\mathrm{lcm}(N/N_1, \ldots, N/N_r) = N$, then $\theta(x_1), \ldots, \theta(x_r)$ generate $\mathbb{Z}_N$ and therefore $\theta(c_1)$ would be an orientation preserving involution of X, which is impossible. ■

PROPOSITION 4.5. *The number of reflexive periods of a cyclic action given by an epimorphism $\theta : \Lambda \to \mathbb{Z}_N$ equals the number of empty period cycles of Λ.*

Proof. We already know, from the proof of Proposition 4.2, that the above Λ has no non-empty period cycles. On the other hand, empty period cycles are in bijective correspondence with the conjugacy classes of reflections of Λ, and hence the result. ■

5. On smooth epimorphisms from proper NEC groups onto finite cyclic groups. In [7] Harvey has found his famous necessary and sufficient conditions on the signature $(g; m_1, \ldots, m_r)$ of a Fuchsian group Δ for the existence of a Fuchsian surface kernel epimorphism $\theta : \Delta \to \mathbb{Z}_N = \langle x \rangle$. Here we deal with such epimorphisms with torsion free kernels for proper NEC groups.

From Proposition 4.2 we know that if such an epimorphism from an NEC group Λ onto $\mathbb{Z}_N$ exists then Λ has signature $(g; \pm; [m_1, \ldots, m_r]; \{(-), \overset{k}{\ldots}, (-)\})$; here we shall find necessary and sufficient conditions for such epimorphisms to exist. Given a positive integer N and a prime p, let $\exp_p(N)$ be defined by $N = p^{\exp_p(N)}M$, where $(M, p) = 1$.

In this section we assume that the set of singular orbits of the action is non-empty, which means that $r + k > 0$.

5.1. Fuchsian surface kernel epimorphisms from groups without reflections. We start with the study of epimorphisms which produce anticonformal cyclic actions with only elliptic orbits.

LEMMA 5.1. *Given integers $N, m_1, \ldots, m_r \geq 2$, there exists a Fuchsian surface kernel epimorphism $\theta : \Lambda \to \mathbb{Z}_N$ for an NEC group with signature $(g; -; [m_1, \ldots, m_r]; \{-\})$ for some $g > 0$ if and only if all m_i divide N, all N/m_i are even and* $\mathrm{lcm}(m_1, \ldots, m_r) \neq N$.

Proof. Given a canonical set of generators, let $x_1, \ldots, x_r$ be the elliptic generators. Let x be a generator of $\mathbb{Z}_N$. Since $\Gamma = \ker\theta$ is torsion free, all m_i divide N. If N were odd, then d_1^N would be an orientation reversing isometry in Γ. If now N/m_i is odd, then for a desired epimorphism θ, $\theta(x_i) = x^{l_i(N/m_i)}$ for some l_i coprime to m_i, which in particular means that l_i is odd. But if now

$\theta(d_1) = x^m$, then $d_1^{l_i(N/m_i)} x_i^{-m} \in \Gamma$, which is impossible since Γ is a Fuchsian group. Finally, if $\mathrm{lcm}(m_1, \ldots, m_r) = N$ then $\theta(d_1) \in \langle \theta(x_1), \ldots, \theta(x_r) \rangle$ and so we have an orientation reversing isometry in $\Gamma = \ker \theta$.

Now assume that our conditions are satisfied. Let $g = m/2$, where $m = N/m_1 + \cdots + N/m_r$. Then the epimorphism $\theta : \Lambda \to \mathbb{Z}_N = \langle x \rangle$ defined by $\theta(d_i) = x^{-1}$ and by $\theta(x_i) = x^{N/m_i}$ is one we are looking for. ∎

Lemma 5.2. *Let Λ be an NEC group with signature $(1; -; [m_1, \ldots, m_r]; \{-\})$. Then there exists a Fuchsian surface kernel epimorphism $\theta : \Lambda \to \mathbb{Z}_N$ if and only if all m_i divide N, all N/m_i are even,* $\mathrm{lcm}(m_1, \ldots, m_r) = N/2$ *and the number of i with* $\exp_2(N/m_i) = 1$ *is odd.*

Proof. Assume that the desired epimorphism exists. Then the first two conditions are given by Lemma 5.1. Observe that $\theta(d_1) \notin \langle \theta(x_1), \ldots, \theta(x_r) \rangle$ since otherwise an orientation reversing isometry belongs to $\Gamma = \ker \theta$. But $\theta(d_1)$ together with $\theta(x_1), \ldots, \theta(x_r)$ generate $\mathbb{Z}_N = \langle x \rangle$ and $\theta(d_1)^2 \in \langle \theta(x_1), \ldots, \theta(x_r) \rangle$. So $\theta(x_1), \ldots, \theta(x_r)$ generate the subgroup of $\mathbb{Z}_N$ of order $N/2$, which means that $\mathrm{lcm}(m_1, \ldots, m_r) = N/2$. If now the number of i with $\exp_2(N/m_i) = 1$ is even then $\theta(x_1 \ldots x_r) = x^{4s}$ for some integer s, and so $\theta(d_1) = x^{-2s}$, which is impossible since x^2 represents an orientation preserving automorphism while d_1 is a glide-reflection.

Now assume that our conditions are satisfied. Then $N/m_1 + \cdots + N/m_r = 2k$, where k is odd. Then the mapping $\theta : \Lambda \to \mathbb{Z}_N = \langle x \rangle$ defined by $\theta(x_i) = x^{N/m_i}$ and $\theta(d_1) = x^{-k}$ defines an epimorphism we are looking for. Indeed, $\Gamma = \ker \theta$ is torsion free. Furthermore it contains no orientation reversing isometry since otherwise $\theta(d_1) \in \langle \theta(x_1), \ldots, \theta(x_r) \rangle$ and so we would have $\mathrm{lcm}(m_1, \ldots, m_r) = N$. ∎

Lemma 5.3. *Given an NEC group Λ with signature $(2; -; [m_1, \ldots, m_r]; \{-\})$, there exists a Fuchsian surface kernel epimorphism $\theta : \Lambda \to \mathbb{Z}_N$ if and only if all m_i divide N, all N/m_i are even,* $\mathrm{lcm}(m_1, \ldots, m_r) \neq N$ *and either $N/2$ is odd or the number of i with* $\exp_2(N/m_i) = 1$ *is even.*

Proof. Assume that such an epimorphism exists. Then the first three conditions follow from Lemma 5.1. Now suppose that $N/2$ is even and the number of i with $\exp_2(N/m_i) = 1$ is odd. Then m_i is even and so $\theta(x_i) = x^{l_i N/m_i}$ for some odd l_i for those i. Therefore $\theta(x_1 \ldots x_r) = x^{2s}$ for some odd integer s. But then $\theta(d_1 d_2) = x^{-s}$, which is impossible since $d_1 d_2$ preserves orientation whilst x^{-s} does not.

Now let $N/2$ be odd. Then all m_i are odd and so for $\varepsilon = 1$ or $\varepsilon = 2$, $N/m_1 + \cdots + \varepsilon N/m_r = 2k$ where k is even. Defining $\theta : \Lambda \to \mathbb{Z}_N = \langle x \rangle$ by $\theta(x_i) = x^{N/m_i}$ for $i \leq r-1$, $\theta(x_r) = x^{\varepsilon N/m_r}$ and $\theta(d_1) = x, \theta(d_2) = x^{-(k+1)}$ we obtain an epimorphism we are looking for. Finally, assume that $N/2$ is even and the second condition holds. Then $N/m_1 + \cdots + N/m_r = 2k$ where

k is even, and the above formula for $\varepsilon = 1$ defines an epimorphism we are looking for. ■

LEMMA 5.4. *Let $N, m_1, \ldots, m_r \geq 2$ be integers satisfying the assumptions of Lemma* 5.1 *but not those of Lemma* 5.3. *Then there is a Fuchsian surface kernel epimorphism $\theta : \Lambda \to \mathbb{Z}_N$ for an NEC group Λ with signature $(3; -; [m_1, \ldots, m_r]; \{-\})$.*

Proof. Since our numbers do not satisfy the assumptions of Lemma 5.3, the number of i for which $\exp_2(N/m_i) = 1$ is odd. Then $N/m_1 + \cdots + N/m_r = 2k$ for some odd k and therefore the mapping $\theta : \Lambda \to \mathbb{Z}_N = \langle x \rangle$ defined by $\theta(x_i) = x^{N/m_i}$, $\theta(d_1) = \theta(d_2) = x$ and $\theta(d_3) = x^{-(k+2)}$ is an epimorphism we are looking for. ■

5.2. Fuchsian surface kernel epimorphisms from groups with reflections. Now we shall study Fuchsian surface kernel epimorphisms $\theta : \Lambda \to \mathbb{Z}_N$ for an NEC group Λ having period cycles. These will produce anticonformal cyclic actions with some reflexive periods on orientable Riemann surfaces.

LEMMA 5.5. *Given integers $k > 0$, $N, m_1, \ldots, m_r \geq 2$, there exists a Fuchsian surface kernel epimorphism $\theta : \Lambda \to \mathbb{Z}_N$ for an NEC group Λ with signature $(g; -; [m_1, \ldots, m_r]; \{(-), \overset{k}{\ldots}, (-)\})$ for some $g \geq 0$ if and only if $N/2$ is odd, all m_i are odd divisors of N and* $\mathrm{lcm}(m_1, \ldots, m_r) \neq N$.

Proof. Assume that the desired epimorphism exists and let x be a generator of $\mathbb{Z}_N$. Then N is even since otherwise $c_1 = c_1^N \in \Gamma = \ker\theta$. Now $x = \theta(\lambda)$ for some orientation reversing isometry λ in Λ. But then $\lambda^{N/2}c_1 \in \Gamma$ and it is an orientation reversing element if $N/2$ is even. So $N/2$ is odd. Furthermore m_i are odd since otherwise $x_i^{m_i/2}c_1$ would again be an orientation reversing isometry in Γ. Now the m_i divide N since otherwise Γ would contain elliptic elements. Finally, if $\mathrm{lcm}(m_1, \ldots, m_r) = N$, then the images of canonical elliptic elements of Λ would generate $\mathbb{Z}_N$ and so there would exist $\lambda \in \langle x_1, \ldots, x_r \rangle$ such that $\theta(\lambda)$ would generate $\mathbb{Z}_N$, and therefore $\lambda^{N/2}c_1$ would be an orientation reversing isometry in the Fuchsian group Γ once again.

Conversely, assume that the integers $k > 0$, $N, m_1, \ldots, m_r \geq 2$ satisfy the assumptions. Then we define $\theta(x_i) = x^{N/m_i}$, $\theta(c_i) = x^{N/2}$, $\theta(e_i) = x^2$ and $g = 0$ if $N/m_1 + \ldots + N/m_r + 2k$ is a multiple of N, and $g = 1$ with $\theta(d_1) = x^m$ where $m = -(N/m_1 + \cdots + N/m_r + 2k)/2$ in the other case. ■

COROLLARY 5.6. *If $k, N, m_1, \ldots, m_r$ satisfy the conditions of Lemma* 5.5 *then*

(i) *for* $\mathrm{lcm}(m_1, \ldots, m_r) = N/2$ *an epimorphism as above exists for all $k > 0$ and g,*

(ii) *for* $\operatorname{lcm}(m_1, \ldots, m_r) \neq N/2$ *and* $k = 1$, *a desired epimorphism exists for all* $g > 1$ *and does not exist for* $g = 0$,

(iii) *for* $\operatorname{lcm}(m_1, \ldots, m_r) \neq N/2$ *and* $k \geq 2$, *a desired epimorphism exists for all* $g \geq 0$,

(iv) *for* $r = 0$ *and any* $g + k > 2$, *a desired epimorphism exists.*

Proof. We define θ on canonical elliptic generators and on reflections as in the above lemma.

In case (i), we have $\theta(x_1) \ldots \theta(x_r) = x^{2t}$. We take $\theta(d_i) = x$, $\theta(e_1) = x^{-2(t+g)}$ and $\theta(e_i) = 1$ for $i > 1$.

In case (ii) for $g \geq 1$ we define $\theta(d_i) = x$ and $\theta(e_1) = x^m$, where $m = -(N/m_1 + \cdots + N/m_r + 2g)$ to obtain an epimorphism as desired. For $g = 0$ the epimorphism obviously cannot exist since we would have $\theta(e_1) = \theta(x_1 \ldots x_r)^{-1}$, so $\mathbb{Z}_N = \langle \theta(x_1) \ldots, \theta(x_r), \theta(c_1) \rangle$, which is impossible since $\operatorname{lcm}(m_1, \ldots, m_r) \neq N, N/2$.

In case (iii) we can define $\theta(d_i) = x$ and $\theta(e_1) = \cdots = \theta(e_{k-1}) = x^2$, $\theta(e_k) = x^m$, where $m = -(N/m_1 + \cdots + N/m_r + 2(g + k - 1))$.

Finally in case (iv) we can define $\theta(d_i) = x$ and $\theta(e_1) = \cdots = \theta(e_{k-1}) = x^2$ and $\theta(e_k) = x^{-2(g+k-1)}$. ∎

5.3. Non-Fuchsian surface kernel epimorphisms from groups without reflections. Here we shall deal with the cases needed in the study of cyclic conformal actions on non-orientable Riemann surfaces with only elliptic singular orbits.

LEMMA 5.7. *Given integers* $N, m_1, \ldots, m_r \geq 2$, *there is* $g \geq 1$ *such that for an NEC group* Λ *with signature* $(g; -; [m_1, \ldots, m_r]; \{-\})$, *there exists a non-Fuchsian surface kernel epimorphism* $\theta : \Lambda \to \mathbb{Z}_N$ *if and only if all* m_i *divide* N *and either* N *is odd or the number of* i *with* N/m_i *odd is even.*

Proof. Given a canonical set of generators, let $x_1, \ldots, x_r$ be the elliptic generators. Since $\Gamma = \ker \theta$ is torsion free, all m_i divide N. Furthermore if N is even and the number of i with N/m_i odd is odd, then $\theta(x_1) \ldots \theta(x_r) = x^m$ where m is odd, whilst $\theta(d_1^2 \ldots d_g^2) = x^n$ where n is even, and therefore $n + m$ cannot be a multiple of N.

Conversely, let $\theta(x_i) = x^{N/m_i}$, $i = 1, \ldots, r$. If N is odd, then $rN + N/m_1 + \cdots + N/m_r = 2g$ and defining $\theta(d_i) = x^{-1}$ for $i = 1, \ldots, g$ we obtain an epimorphism as desired. If N is even and the number of i with N/m_i odd is even then $rN + N/m_1 + \cdots + N/m_r = 2g$ for some g and we define $\theta(d_1) = 1$, $\theta(d_2) = x^{-2}$ and $\theta(d_i) = x^{-1}$ for $i = 3, \ldots, g$. ∎

LEMMA 5.8. *Given integers* $N, m_1, \ldots, m_r \geq 2$ *and an NEC group* Λ *with signature* $(1; -; [m_1, \ldots, m_r]; \{-\})$, *there exists a non-Fuchsian surface kernel epimorphism* $\theta : \Lambda \to \mathbb{Z}_N$ *if and only if* $\operatorname{lcm}(m_1, \ldots, m_r) = N$, *all* m_i *divide* N *and either* N *is odd or the number of* i *with* N/m_i *odd is even.*

Proof. Since Γ contains an orientation reversing isometry and since $d_1^2 \in \langle x_1, \dots, x_r \rangle$ there exists such an isometry of the form $d_1\omega(x_1, \dots, x_r)$, where ω is a word in variables $x_1, \dots, x_r$. So $\theta(x_1), \dots, \theta(x_r)$ generate $\mathbb{Z}_N$ and therefore $\operatorname{lcm}(m_1, \dots, m_r) = N$. The remaining conditions follow from Lemma 5.7.

Conversely, let $\theta(x_i) = x^{N/m_i}$ for $i = 1, \dots, r$. If N is odd or N is even and the number of i with N/m_i odd is even then $rN - (N/m_1 + \dots + N/m_r) = 2n$ for some n and so defining $\theta(d_1) = x^n$ we obtain a epimorphism as desired. ∎

LEMMA 5.9. *Given integers $N, m_1, \dots, m_r \geq 2$ and an NEC group Λ with signature $(2; -; [m_1, \dots, m_r]; \{-\})$, there exists a non-Fuchsian surface kernel epimorphism $\theta : \Lambda \to \mathbb{Z}_N$ if and only if all m_i divide N and either N is odd or the number of i with N/m_i odd is even, and either* $\operatorname{lcm}(m_1, \dots, m_r) = N$*, or some of the N/m_i is odd, or else all N/m_i are even but the number of i with* $\exp_2(N/m_i) = 1$ *is odd.*

Proof. Given such an epimorphism, all m_i divide N, N is odd or the number of i with N/m_i odd is even by Lemma 5.7. Assume that $\operatorname{lcm}(m_1, \dots, m_r) \neq N$ and all N/m_i are even but the number of i with $\exp_2(N/m_i) = 1$ is even. Then $\theta(x_1 \dots x_r) = x^{4n}$. Thus $\theta(d_1)\theta(d_2) = x^{-2n}$ and therefore $\theta(d_1) = x^p$ and $\theta(d_2) = x^q$, where p and q have the same parity. But if both are even, then θ cannot be surjective, while in the other case no word in $\theta(d_1), \theta(d_2)$ of odd length can belong to $\langle \theta(x_1), \dots, \theta(x_r) \rangle$ and so $\ker\theta$ has no orientation reversing isometries.

Conversely, we define $\theta(x_i) = x^{N/m_i}$ for every i. Assume first that $N = \operatorname{lcm}(m_1, \dots, m_r)$. Then $\langle \theta(x_1), \dots, \theta(x_r) \rangle = \mathbb{Z}_N$. Let $n = N/m_1 + \dots + N/m_r$. If N is even then $n = 2k$, while if N is odd then either $n = 2k$ or $N - n = 2k$. So taking $\theta(d_1) = 1$ and $\theta(d_2) = x^{-k}$ we obtain an epimorphism as desired. Now let $\operatorname{lcm}(m_1, \dots, m_r) \neq N$. If some N/m_i is odd, then this is so for an even number of i, by Lemma 5.7. Therefore $n = 2k$ and we define $\theta(d_1) = x^{-1}$ and $\theta(d_2) = x^{-k+1}$. Then $\theta(x_1 \dots x_r d_1^2 d_2^2) = 1$ and $\theta(d_1)$ is a generator of $\mathbb{Z}_N$. Now observe that since N/m_i is odd, $\langle \theta(x_1), \dots, \theta(x_r) \rangle$ has an odd index s in $\mathbb{Z}_N$. So $\theta(d_1)^s \in \langle \theta(x_1), \dots, \theta(x_r) \rangle$ and therefore $\ker\theta$ is a proper NEC surface group. Finally, if none of the N/m_i is odd, then $n = 2k$ for odd k. Therefore the same formula as before defines an epimorphism with torsion free kernel for which $d_1^{1-k} d_2 \in \ker\theta = \Gamma$ and so Γ is a proper NEC surface group again. ∎

LEMMA 5.10. *Given integers $N, m_1, \dots, m_r \geq 2$ and an NEC group Λ with signature $(3; -; [m_1, \dots, m_r]; \{-\})$, there exists a non-Fuchsian surface kernel epimorphism $\theta : \Lambda \to \mathbb{Z}_N$ if and only if all m_i divide N and either N is odd or the number of i with N/m_i odd is even.*

Proof. Let $\theta(x_i) = x^{N/m_i}$. If N is odd, or N is even and the number of i with N/m_i odd is even, then $rN - (N/m_1 + \dots + N/m_r + 2) = 2n$ for some n. So defining $\theta(d_1) = 1$, $\theta(d_2) = x^{-1}$ and $\theta(d_3) = x^{-n+1}$ we obtain an epimorphism as desired. ∎

Remark 5.11. Comparing the conditions in the above Lemmas 5.8–5.10 shows that if there exists an epimorphism as desired for $g = 1$ or $g = 2$ then there exists such an epimorphism for $g + 1$. Observe however that this may not be the case for Lemmas 5.2–5.4 from Section 5.1.

5.4. Non-Fuchsian surface kernel epimorphisms from groups with reflections. Finally, the results of this subsection will serve for the study of dynamics with some reflexive periods on non-orientable Riemann surfaces.

Lemma 5.12. *Given integers $N, m_1, \dots, m_r \geq 2$, $k \geq 1$, there is an epimorphism $\theta : \Lambda \to \mathbb{Z}_N$ with a non-Fuchsian surface kernel where Λ is an NEC group with signature $(g\ -; [m_1, \dots, m_r]; \{(-), \overset{k}{\dots}, (-)\})$ for some $g \geq 0$ if and only if N is even and all m_i are divisors of N.*

Proof. Given such an epimorphism, N must be even since otherwise c_1 would belong to $\ker\theta$. Now all m_i must divide N since otherwise $\ker\theta$ would have non-trivial elliptic elements.

Conversely, assume that N is even and all m_i divide it. Let Λ be an NEC group with signature $(3; -; [m_1, \dots, m_r]; \{(-), \overset{k}{\dots}, (-)\})$ and let $n = -(N/m_1 + \dots + N/m_r)$. If $k \geq 2$, then the mapping $\theta(x_i) = x^{N/m_i}$, $\theta(c_i) = x^{N/2}$, $\theta(e_1) = \dots = \theta(e_{k-1}) = x$, $\theta(e_k) = x^{n-k+1}$, $\theta(d_1) = x$, $\theta(d_2) = x^{-1}$ and $\theta(d_3) = 1$ is as desired. Indeed, we see that θ preserves the orders of the canonical generators of Λ. So $\Gamma = \ker\theta$ is torsion free. But $d_3 \in \Gamma$ and so it is not Fuchsian, which completes the proof. ∎

Lemma 5.13. *Given integers $N, m_1, \dots, m_r \geq 2$, $r > 0$ and $k \geq 2$, there exists a non-Fuchsian surface kernel epimorphism $\theta : \Lambda \to \mathbb{Z}_N$ where Λ is an NEC group with signature $(0; -; [m_1, \dots, m_r]; \{(-), \overset{k}{\dots}, (-)\})$ if and only if N is even and all m_i are divisors of N.*

Proof. One can define θ by $\theta(x_i) = x^{N/m_i}$, $\theta(c_i) = x^{N/2}$ and $\theta(e_1) = \dots = \theta(e_{k-1}) = x$ and $\theta(e_k) = x^n$, where $n = -(N/m_1 + \dots + N/m_r + k - 1)$. Indeed, θ preserves the orders of the canonical elliptic elements and so $\Gamma = \ker\theta$ is torsion free. Furthermore $e_1^{N/2} c_1$ is an orientation reversing isometry in Γ. So Γ is an NEC surface group. ∎

Lemma 5.14. *Given integers $N, m_1, \dots, m_r \geq 2$, there is a non-Fuchsian surface kernel epimorphism $\theta : \Lambda \to \mathbb{Z}_N$ where Λ is an NEC group with signature $(1; -; [m_1, \dots, m_r]; \{(-)\})$ if and only if N is even, all m_i are divisors of N and some of $N/2, m_1, \dots, m_r$ is even.*

Proof. The first two conditions are satisfied due to Lemma 5.12. So assume that $N/2, m_1, \dots, m_r$ are odd. Since $\theta(e_1) \in \langle \theta(x_1), \dots, \theta(x_r), \theta(d_1)^2 \rangle$, we see that the images under θ of orientation preserving isometries do not generate G and therefore $\ker\theta$ is a Fuchsian group.

Conversely, let $n = -(N/m_1 + \cdots + N/m_r)$, $\theta(c_1) = x^{N/2}$ and $\theta(x_i) = x^{N/m_i}$ for all i. For $N/2$ even, we define $\theta(d_1) = x$ and $\theta(e_1) = x^{n-2}$. This yields an epimorphism as desired since $d_1^{N/2}c_1$ is an orientation reversing isometry in $\Gamma = \ker\theta$. Now assume that $N/2$ is odd and say m_i is even. If $n = 2k+1$ then we set $\theta(e_1) = x$ and $\theta(d_1) = x^k$. If $n = 2k$ then for $\theta(e_1) = x^2$ and $\theta(d_1) = x^{k-1}$ we also obtain an epimorphism as desired since $\theta(x_i)$ and $\theta(e_1)$ generate G. ∎

LEMMA 5.15. *There is a non-Fuchsian surface kernel epimorphism $\theta : \Lambda \to \mathbb{Z}_N$ where Λ is an NEC group with signature $(2; -; [m_1, \ldots, m_r]; \{(-)\})$ if and only if N is even and all m_i are its divisors.*

Proof. The conditions follow from Lemma 5.12. We define $\theta(x_i) = x^{N/m_i}$ and $\theta(c_1) = x^{N/2}$. Let $N/m_1 + \cdots + N/m_r = s$. For $s = 2m+1$ we define $\theta(e_1) = x^{-1}$, $\theta(d_1) = 1$ and $\theta(d_2) = x^{-m}$ to obtain an epimorphism as desired. So assume that $s = 2m$. If m is even then we take $\theta(e_1) = x^2$, $\theta(d_1) = 1$ and $\theta(d_2) = x^{-(m+1)}$. Let $n = \exp_2(N)$. Then $\theta(d_2)^{N/2^n}$ has order 2^n while $\theta(e_1)^{2^n}$ has order $N/2^n$ and therefore this mapping defines an epimorphism as desired. If m is odd then we define $\theta(e_1) = x^4$, $\theta(d_1) = 1$ and $\theta(d_2) = x^{-(m+2)}$. Then again $\theta(d_2)^{N/2^n}$ has order 2^n and $\theta(e_1)^{2^n}$ has order $N/2^n$ and therefore the above map defines an epimorphism as desired. ∎

LEMMA 5.16. *Given integers $N, m_1, \ldots, m_r \geq 2$, there is a non-Fuchsian surface kernel epimorphism $\theta : \Lambda \to \mathbb{Z}_N$ where Λ is an NEC group with signature $(0; -; [m_1, \ldots, m_r]; \{(-)\})$ if and only if N is even, all m_i are divisors of N and $\operatorname{lcm}(m_1, \ldots m_r) = N$.*

Proof. For such an epimorphism θ, $\theta(e_1) \in \langle\theta(x_1), \ldots, \theta(x_r)\rangle$. But since $\ker\theta$ has orientation reversing isometries, we have $\theta(c_1) \in \langle\theta(e_1), \theta(x_1), \ldots, \theta(x_r)\rangle = \langle\theta(x_1), \ldots, \theta(x_r)\rangle$. So $\langle\theta(x_1), \ldots, \theta(x_r)\rangle = \mathbb{Z}_N$ and therefore $\operatorname{lcm}(m_1, \ldots, m_r) = N$.

Conversely, for $N, m_1, \ldots, m_r$ satisfying these conditions, the mapping $\theta(x_i) = x^{N/m_i}$, $\theta(c_1) = x^{N/2}$ and $\theta(e_1) = x^n$, where $n = -(N/m_1 + \cdots + N/m_r)$, defines an epimorphism as desired. ∎

The last lemma of this subsection is rather easy and can be proved using the same methods as before.

LEMMA 5.17. *Given $g \geq 0$ and $k > 0$ such that $g + k > 2$ there is a non-Fuchsian surface kernel epimorphism $\theta : \Lambda \to \mathbb{Z}_N$ where Λ is an NEC group with signature $(g; -; [-]; \{(-), \overset{k}{\ldots}, (-)\})$ if and only if N is even.* ∎

6. Conformal dynamics on orientable and non-orientable compact Riemann surfaces. The possible periods of orientation preserving automorphisms of orientable Riemann surfaces and their minimal genera have been studied by Sierakowski [15]. Here we deal with orientation reversing automorphisms and conformal automorphisms of non-orientable Riemann surfaces. We must emphasize that, in contrast to [15], we take into account the multiplicities of elliptic periods and so, somewhat curiously, our approach is actually simpler than that of [15], because we just have to minimize the orbit genus g of the quotient orbifold $X/G = \mathcal{H}/\Lambda$. Then from the expression for the area of the fundamental region (2) and from the Hurwitz–Riemann formula (3) one can derive the minimal value of $\tilde{g}$ for a minimal g.

Let, as always, $\theta : \Lambda \to \mathbb{Z}_N$, where Λ is an NEC group, be a smooth epimorphism defining an automorphism x of order N having elliptic periods $N_1, \ldots, N_r$ and k reflexive periods. In view of the Hurwitz–Riemann formula, in order to find the minimum genus of a Riemann surface allowing such periods, we have to find an NEC group with the minimal possible area for which $N/N_1, \ldots, N/N_r$ is the system of proper periods and which has exactly k empty period cycles. We can do this using the results of the previous section.

6.1. Classical Riemann surfaces. Using the results of Subsection 5.1, we obtain the following result on the existence and minimal genus of an orientable Riemann surface admitting a cyclic conformal orientation reversing action of a given order without reflexive periods and with a given set of elliptic periods.

THEOREM 6.1. *There exists an orientable Riemann surface X with an orientation reversing automorphism of order N having elliptic periods $N_1, \ldots, N_r$ possibly with repetitions and not having reflexive periods if and only if all N_i are even, they divide N and* $\operatorname{lcm}(N/N_1, \ldots, N/N_r) \neq N$. *Furthermore:*

(i) *For $r = 1$ the minimal genus of such a surface equals $(\eta N - N_1)/2 + 1$ where $\eta = 1$ if N_1 is a multiple of 4 or N is not a multiple of 4, and $\eta = 2$ otherwise.*

(ii) *For $r = 2$ and $N_1 = N_2 = N/2$ the minimal genus equals $N/2 + 1$.*

(iii) *For the remaining cases the minimal genus is*

$$\frac{N}{2}(\alpha + r - 2) - \frac{1}{2}(N_1 + \cdots + N_r) + 1$$

where $\alpha = 1$, 2 or 3 according as $N, N/m_1, \ldots, N/m_r$ satisfy exclusively the conditions of Lemmas 5.2, 5.3 *or* 5.4 *respectively, that is, either* 5.2, *or* 5.3 *but not* 5.2, *or* 5.4 *but not* 5.2 *or* 5.3. ■

Proof. Indeed, there exists a surface with such an automorphism if and only if there is an epimorphism $\theta : \Lambda \to \mathbb{Z}_N$ with kernel being a Fuchsian surface group, where Λ has no reflections, since our automorphism φ has no reflexive periods. Therefore Λ has a signature $(g; -; [m_1, \ldots, m_r]; \{-\})$, where the sign is $-$ since φ reverses orientation and where $N_i = N/m_i$, by Corollary 3.2 and Proposition 4.2. So the first part of the theorem follows from Lemma 5.1 which gives necessary and sufficient conditions for the existence of such an epimorphism. To find the minimum genus of such a surface we have, in view of the Hurwitz–Riemann formula, to find Λ with the smallest possible area and admitting such an epimorphism.

For $r = 1$, the orbit genus g of Λ must be ≥ 2 since for $g \leq 1$, the expression (2) is negative. But, by Lemma 5.3, $g = 2$ if and only if either N is not a multiple of 4 or N_1 is a multiple of 4, and by the Hurwitz–Riemann formula in such cases the corresponding genus equals $(N - N_1)/2 + 1$, which is minimal. If now N is a multiple of 4 and N_1 is not, then there exists an epimorphism as desired for Λ with orbit genus 3, by Lemma 5.4, and the corresponding genus is equal to $(2N - N_1)/2 + 1$.

For $r = 2$ and $m_1 = m_2 = 2$, the orbit genus g of Λ must be ≥ 2 since otherwise the quantity (2) is negative. But $r = 2$ and $m_1 = m_2$ satisfy the last condition of Lemma 5.3, which gives the minimal genus $N/2 + 1$.

Finally, for the remaining cases, (2) is positive for all $g \geq 1$, and so by the Hurwitz–Riemann formula, the minimum genus is

$$\frac{N}{2}(\alpha + r - 2) - \frac{1}{2}(N_1 + \cdots + N_r) + 1$$

where $\alpha = 1, 2$ or 3 according as the orbit genus of Λ with minimal area is $1, 2$ or 3, which means that $\alpha = 1$ if $N, N/m_1, \ldots, N/m_r$ satisfy the conditions of Lemma 5.2; $\alpha = 2$ if they satisfy the conditions Lemma 5.3 but not those of Lemma 5.2; and $\alpha = 3$ if they satisfy the conditions of Lemma 5.4 but neither those of Lemma 5.2 nor those of Lemma 5.3. ■

Similarly, using Subsection 5.2, we obtain the following result on the existence and minimal genus of an orientable Riemann surface admitting a cyclic orientation reversing conformal action of a given order with given sets of reflexive and elliptic periods, provided the first set is non-empty.

THEOREM 6.2. *There exists an orientable Riemann surface X with an orientation reversing automorphism of order N having elliptic periods $N_1, \ldots, N_r$, possibly with repetitions, and $k > 0$ reflexive periods if and only if $N/2$ is odd and all N_i are even divisors of N. Furthermore:*

(i) *For $r = 0$ the minimal genus of such a surface equals $N(\alpha + k - 2)/2 + 1$ where $\alpha = 2, 1, 0$ if respectively $k = 1, 2, \geq 3$.*

(ii) *For $r = 1$ the minimal genus equals $(N(\alpha+k-1)-N_1)/2+1$ where $\alpha = 1$ if $k = 1$, and $\alpha = 0$ otherwise.*
(iii) *For $r = 2$, $N_1 = N_2 = N/2$ and $k = 1$ the minimal genus is $N - 1$.*
(iv) *For the remaining cases the minimal genus equals*

$$\frac{1}{2}N(\alpha + k + r - 2) - \frac{1}{2}(N_1 + \cdots + N_r) + 1$$

where $\alpha = 1$ if $k = 1$ and $\mathrm{lcm}(N/N_1, \ldots, N/N_r) \neq N/2$, and $\alpha = 0$ otherwise. ∎

Note that $\mathrm{lcm}(N/N_1, \ldots, N/N_r) \neq N$ under the assumptions of Theorem 6.2.

6.2. Non-orientable Riemann surfaces. For configurations of elliptic periods for a cyclic conformal action without reflexive periods on a non-orientable Riemann surface we have the following theorem, which can be proved similarly to Theorem 6.1 using the results of Subsection 5.3.

THEOREM 6.3. *There exists a non-orientable Riemann surface X with an automorphism of order N having elliptic periods $N_1, \ldots, N_r$ possibly with repetitions and with no reflexive periods if and only if each N_i divides N and either N is odd or the number of i with N_i odd is even. Furthermore:*

(i) *For $r = 1$ the minimal genus of such a surface equals $(\eta N - N_1)/2+1$ where $\eta = 1$ if either N is odd or N_1 is even and $N_1/2$ is odd, and $\eta = 2$ if N is even and N_1 is a multiple of 4.*
(ii) *For $r = 2$ and $N_1 = N_2 = N/2$ the minimal genus of such a surface equals $\eta N/2 + 1$ where $\eta = 1$ or 2 according as $N/2$ is odd or not.*
(iii) *For the remaining cases the minimal genus is*

$$\frac{N}{2}(\alpha + r - 2) - \frac{1}{2}(N_1 + \cdots + N_r) + 1$$

where $\alpha = 1, 2, 3$ according as $N, N/m_1, \ldots, N/m_r$ satisfy exclusively the assumptions of Lemmas 5.8, 5.9 or 5.10 respectively; that is, either 5.8, or 5.9 but not 5.8, or 5.10 but not 5.9 respectively. ∎

Finally, using the results of Subsection 5.4, we can prove the following result on configurations of reflexive and elliptic periods for a cyclic conformal action on a non-orientable Riemann surface.

THEOREM 6.4. *There exists an orientable Riemann surface X with an orientation reversing automorphism of order N having elliptic periods $N_1, \ldots, N_r$, possibly with repetitions, and $k > 0$ reflexive periods if and only if N is even and all N_i are divisors of N. Furthermore:*

(i) *For $r=0$ the minimal genus of such a surface equals $N(\alpha + k - 2)/2 + 1$, where $\alpha = 2, 1, 0$ if respectively $k = 1, 2, \geq 3$.*

(ii) *For $r = 1$ and $k = 1$ the minimal genus is $(\alpha N - N_1)/2 + 1$ where $\alpha = 1$ if $N/2$ is even or N_1 is odd, and $\alpha = 2$ otherwise,*

(iii) *For $r = 2$, $N_1 = N_2 = N/2$ the minimal genus is*

$$\frac{1}{2}N(\alpha + k - 1) + 1$$

where $\alpha = 1$ if $k = 1$, and $\alpha = 0$ otherwise.

(iv) *For $k = 1$ and $k, r, N_1, \ldots, N_r$ not satisfying any of the above conditions the minimal genus equals*

$$\frac{1}{2}N(\alpha + r - 1) - \frac{1}{2}(N_1 + \cdots + N_r) + 1$$

where $\alpha = 0$ if $\mathrm{lcm}(N/N_1, \ldots, N/N_r) = N$, $\alpha = 1$ if some of $N/N_1, \ldots, N/N_r$, $N/2$ is even, and $\alpha = 2$ otherwise.

(v) *In the remaining cases the minimal genus equals*

$$\frac{1}{2}N(k + r - 2) - \frac{1}{2}(N_1 + \cdots + N_r) + 1. \ ■$$

6.3. Free actions of cyclic groups. We end this section with a complete description of free conformal actions of cyclic groups on non-orientable surfaces and free antiholomorphic actions of cyclic groups on orientable Riemann surfaces. In other words, this is the case when the set of singular orbits is empty, i.e. $r = 0$ and $k = 0$. The considerations of this case illustrate in an easy way how the minimal value of the orbit genus g leads to the minimal value of the genus of the surface with such an action.

THEOREM 6.5. *There is a free conformal action of a cyclic group $\mathbb{Z}_N$ on a non-orientable Riemann surface of topological genus $\tilde{g} \geq 3$ if and only if $\tilde{g} \equiv 2 \bmod N$. Consequently, for a given $N \geq 2$ the minimal genus of such a surface is $N + 2$.*

Proof. If G acts on such a surface X, say of genus $\tilde{g}$, then $G = \Lambda/\Gamma$ where $\mu(\Gamma) = 2\pi(\tilde{g} - 2)$ and $\mu(\Lambda) = 2n\pi$ with $n > 0$, as Λ is a torsion free NEC group (in fact $n = g - 2$). So $\tilde{g} = nN + 2$. Conversely, for $\tilde{g} - 2 = nN$ with some $n > 0$ take an NEC group Λ with signature $(n + 2; -; [-]; \{-\})$ and consider the epimorphism $\theta : \Lambda \to \mathbb{Z}_N = \langle a \rangle$ given by

$$\theta(d_i) = \begin{cases} 1 & \text{for } i = 1, \\ a & \text{for } i = 2, \ldots, n+1, \\ a^{-n} & \text{for } i = n+2. \end{cases}$$

Then for $\Gamma = \ker\theta$, $X = \mathcal{H}/\Gamma$ is a non-orientable Riemann surface of topological genus g with a free action of $\mathbb{Z}_N$. ■

THEOREM 6.6. *There is a free antiholomorphic action of a cyclic group $\mathbb{Z}_N$ on an orientable Riemann surface of topological genus $\tilde{g} \geq 2$ if and only if N is even, $\tilde{g} \equiv 1 \bmod (N/2)$ and in addition $\tilde{g} \equiv 1 \bmod N$ if $N/2$ is even.*

Consequently, for a given $N \geq 2$ the minimal genus of such a surface is $N/2+1$ if N is not multiple of 4, and $N+1$ otherwise.

Proof. For a generator a of $\mathbb{Z}_N$, we have on the one hand $a^N = \mathrm{id}$, and a^N preserves orientation on the other hand. So N must be even. If G acts on such a surface X say of genus g, then $G = \Lambda/\Gamma$, where $\mu(\Gamma) = 4\pi(\tilde{g}-1)$ and $\mu(\Lambda) = 2n\pi$, with $n > 0$, as the last is a torsion free NEC group (in fact $n = 2(g-1)$). So $\tilde{g} = nN/2+1$, by the Hurwitz–Riemann formula. Observe that Λ has signature $(n+2; -; [-]; \{-\})$. Now for the canonical epimorphism $\theta : \Lambda \to \mathbb{Z}_N = \langle a \rangle$ we have $\theta(d_i) = a^{\alpha_i}$ for some odd α_i. So $2(\alpha_1 + \cdots + \alpha_n)$ is a multiple of N, which means that for n odd, $N/2$ must also be odd. So if $N/2$ is even then $n = 2(\tilde{g}-1)/N$ is also even, which means that $\tilde{g} \equiv 1 \bmod N$.

Conversely, for $n = (\tilde{g}-1)/N$, take an NEC group Λ with signature $(n+2; -; [-]; \{-\})$. For n even consider the epimorphism $\theta : \Lambda \to \mathbb{Z}_N = \langle a \rangle$ given by

$$\theta(d_i) = \begin{cases} a & \text{for } i = 2s+1,\ s = 0, \ldots, n/2, \\ a^{-1} & \text{for } i = 2s,\ s = 1, \ldots, (n+2)/2. \end{cases}$$

For n odd, $N/2$ is odd and so the mapping

$$\theta(d_i) = \begin{cases} a & \text{for } i = 2s+1,\ s = 0, \ldots, (n-1)/2, \\ a^{-1} & \text{for } i = 2s,\ s = 1, \ldots, (n+1)/2, \\ a^{N/2} & \text{for } i = n+2, \end{cases}$$

induces an epimorphism $\theta : \Lambda \to \mathbb{Z}_N$ such that for $\Gamma = \ker\theta$, $X = \mathcal{H}/\Gamma$ is an orientable Riemann surface of topological genus $\tilde{g}$ with a free antiholomorphic action of $\mathbb{Z}_N$. ∎

6.4. Concluding remarks

REMARK 6.7. Using the results of Section 5, we have found in this section the minimal genera of surfaces admitting an automorphism of a prescribed order and with a prescribed family of periods. However, using these results, slightly generalized, one can go further, finding the spectra of all genera of surfaces with such prescribed data.

REMARK 6.8. As we mentioned in the introduction, the case of conformal automorphisms completely covers the topological generality, i.e. gives an answer to the same problem for any homeomorphism of a closed, connected, two-dimensional manifold X of genus $g \geq 2$. But such a manifold is a $K(\pi, 1)$ space, with $\pi = \pi_1(X)$ equal to the fundamental group of X, i.e. to Γ if $X = \mathcal{H}/\Gamma$ is represented as a Klein surface. This implies that the homotopy classes $[X, X]$ of self-maps of X are in one-one correspondence with $\mathrm{End}(\Gamma, \Gamma)$. Then the use of the Dehn–Nielsen and Nielsen–Kerckhoff theorems allows one to extend the homotopical theory of periodic points of periodic homeomorphisms to the case of homotopically periodic self-maps of such a surface, i.e. to maps f such that $f^N \sim \mathrm{id}$ [14]. Combining that approach with the results of this work one can answer similar questions for the homotopy minimal periods of homotopically periodic maps of X, or equivalently of automorphisms of Γ of finite order N. In particular it seems to be possible to answer, in an effective way, the question of Boyland [2] about the existence of a homeomorphism h of X realizing all prime Nielsen–Jiang periodic numbers $NP_n(f)$ for n dividing N, of a given self-map f of X (see [14] for the definition of $NP_n(f)$).

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