

Preservation of the Borel class under open- LC functions

by

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Abstract. Let X be a Borel subset of the Cantor set \mathbf{C} of additive or multiplicative class α , and $f : X \rightarrow Y$ be a continuous function onto $Y \subset \mathbf{C}$ with compact preimages of points. If the image $f(U)$ of every clopen set U is the intersection of an open and a closed set, then Y is a Borel set of the same class α . This result generalizes similar results for open and closed functions.

1. Introduction. Let X be a Borel subset of the Cantor set \mathbf{C} of additive or multiplicative class α , and $f : X \rightarrow Y$ be a continuous function onto $Y \subset \mathbf{C}$ with compact preimages of points.

It is well known that if the image $f(U)$ of every clopen set U is an open subset of Y , then Y is a Borel set of the same class α [9], [8], [1], [7].

Analogously, if the image $f(U)$ of every clopen set U is a closed subset of Y , then Y is a Borel set of the same class.

The aim of this note is to prove (Theorem 2) that if the image $f(U)$ of every clopen set U is the intersection of an open and a closed set, then Y is a Borel set of the same class.

This fact is related to the following problem [6, Problem 3.6]:

Find a class of open-Borel, continuous functions that are as close as possible to open and closed functions and have compact preimages of points and preserve absolute Borel class.

2. Related materials and basic definitions. All spaces in this paper are assumed to be metrizable and separable.

Recall that a subset of a topological space is an LC -set or a *locally closed set* if it is the intersection of an open and a closed set.

Given an arbitrary (not necessarily continuous) function f we say that it is

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- *open* (resp. *closed*) if f takes open (resp. closed) sets into open (resp. closed) sets;
- *open-LC* (resp. *clopen-LC*) if f takes open (resp. clopen) sets into LC-sets.

The following notation will be needed throughout the paper.

We will denote by $S_1(y)$ any sequence converging to y together with its limit point:

$$S_1(y) = \{y\} \cup \{y_i : y_i \rightarrow y\}.$$

It is easy to check that a function f is closed \Leftrightarrow for every $S_1(y)$, every sequence $x_i \in f^{-1}(y_i)$ ($y_i \neq y_j$ for $i \neq j$) has a limit point in $f^{-1}(y)$.

Indeed, if f is closed and, for some $S_1(y)$, there is no limit point in $f^{-1}(y)$ for $x_i \in f^{-1}(y_i)$, then the image $f(T)$ of the closed set $T = \text{cl}_X\{x_i\}$ is not closed in Y .

Conversely, if, for every $S_1(y)$, some sequence $x_i \in f^{-1}(y_i)$ has a limit point in $f^{-1}(y)$ and there is a closed $T \subset X$ for which $f(T)$ is not closed in Y , then there is $S_1(y)$ such that $y \notin f(T)$ and $y_i \in f(T)$. Hence, the sequence of points $x_i \in f^{-1}(y_i) \cap T$ has no limit point in $f^{-1}(y)$.

Analogously, it is easy to check that a function f is open \Leftrightarrow for every $S_1(y)$ and every open ball $O(x)$ with $x \in f^{-1}(y)$, there are only finitely many y_i such that $f^{-1}(y_i) \cap O(x) = \emptyset$.

3. Structure of clopen-LC functions in the Cantor set \mathbf{C} . Let us first prove the following theorem.

THEOREM 1. *Let $f : X \rightarrow Y$ be a clopen-LC function from a subset X of the Cantor set \mathbf{C} onto Y such that the preimage of every point $y \in Y$ is compact. Then Y can be covered by countably many subsets Y_n such that the restrictions $f|f^{-1}(Y_n)$ are open functions ($n = 1, 2, \dots$) and the restriction $f|f^{-1}(Y_0)$ is a closed function.*

Proof. Denote

- (A) $X_n = \bigcup\{f^{-1}(y) : \text{there is } S_1(y) \subset Y \text{ and } \tilde{x}_y \in \mathbf{C} \text{ such that there are } x_k \in f^{-1}(y_k) \text{ with } x_k \rightarrow \tilde{x}_y \text{ and } \text{dist}(\tilde{x}_y, f^{-1}(y)) > 1/n\}.$

LEMMA 1. *The restriction $f|X_n$ is an open function onto $Y_n = f(X_n)$.*

Indeed, suppose the opposite. Then

- (B) for some $y \in f(X_n)$ and $d > 0$, there exist $S_1(y) \subset f(X_n)$ and $x \in f^{-1}(y)$ such that $y_k \rightarrow y$ and $\text{dist}(x, f^{-1}(y_k)) > d$.

Let us consider a countable compact set $S_2(y)$ obtained by replacing (see (B)) the isolated points y_k of $S_1(y)$ by $S_1(y_k) \subset Y$ with isolated points $y_{k_j} \rightarrow y_k$ selected according to (A).

Since X lies in \mathbf{C} there is a limit point $\tilde{x} \in \mathbf{C}$ for \tilde{x}_{y_k} . The proof falls naturally into two parts.

(1) If $\tilde{x} \notin X$, then we can take a clopen (in \mathbf{C}) ball $O_{\delta_1}(\tilde{x})$ with $\delta_1 < 1/n$, and a clopen (in \mathbf{C}) ball $O_{\delta_2}(x)$ with $\delta_2 < d$, according to (B). It is clear that $D = O_{\delta_1}(\tilde{x}) \cup O_{\delta_2}(x)$ is a clopen set in \mathbf{C} , and hence $S_2(y) \cap f(D)$ is the intersection of a closed set F and an open set U in $S_2(y)$. We can suppose that $S_2(y) \cap f(D)$ contains y, y_{k_j} ($j, k = 1, 2, \dots$), and obviously $y_k \notin f(D)$. Since the points y_{k_j} are dense in $S_2(y)$ and $y \in U$, we obtain a contradiction that $y_k \in f(D)$.

(2) If $\tilde{x} \in X$, then we can repeat (1) for $D = O_{\delta_1}(\tilde{x})$. ■

LEMMA 2. *f is a closed function at every point of $Y_0 = Y \setminus \bigcup_n Y_n$. Hence, $f|X_0$ is a closed function onto $Y_0 = f(X_0)$.*

Since preimages of points are compact, the assertion of the lemma follows from the definition of the sets X_n in (A). ■

4. Preservation of the Borel class by clopen-LC functions in the Cantor set \mathbf{C}

THEOREM 2. *Let $f : X \rightarrow Y$ be a continuous, clopen-LC function such that the preimage of every point $y \in Y$ is compact. If X is a Borel set of additive or multiplicative class α in \mathbf{C} , then Y is a Borel set of the same class in \mathbf{C} .*

To begin, we remark that the notation below is as in the proof of Theorem 1.

LEMMA 3. *The restriction $g_n = f|f^{-1}(\text{cl}_Y Y_n)$ is an open function ($n = 1, 2, \dots$).*

Indeed, suppose that $y \in (\text{cl}_Y Y_n) \setminus Y_n$, so there are $y_k \in Y_n$ with $y_k \rightarrow y$. The method of proof of Lemma 1 works for this case too, and a repeated application of this method as in cases (1) and (2) enables us to conclude that g_n is open at every point $x \in f^{-1}(y)$.

We will establish the lemma if we prove the following statement:

Let $y_k \rightarrow y$ and $y_k \in (\text{cl}_Y Y_n) \setminus Y_n$. Then, for every $x \in f^{-1}(y)$ and every open $O_\delta(x)$, the intersection $O_\delta(x) \cap f^{-1}(y_k)$ is nonempty for infinitely many $k = 1, 2, \dots$.

Suppose that this statement is false.

Pick $y \in Y_n$ and a clopen V in X that intersects $f^{-1}(y)$. Let $A = \text{cl}_Y Y_n$ and assume for contradiction that $f(V) \cap A$ is not a neighborhood of y in A . Since (Lemma 1) $f(V) \cap Y_n$ is open in Y_n , there are $y_n \in \text{cl}_A(f(V) \cap Y_n) \setminus A$ with $y_n \rightarrow y$, and then $y_{n_j} \in f(V) \cap Y_n$ with $y_{n_j} \rightarrow y_n$. Thus, one gets a closed set $F \subset A$ with $f(V) \cap F$ not locally closed, which contradicts the assumption that $f(V)$ is locally closed. ■

Now, we turn to the proof of Theorem 2. By Lemma 3, we can assume that every Y_i is closed in Y , and hence Y_0 is G_δ in Y .

Since f is continuous, every preimage $f^{-1}(Y_n)$ ($n = 1, 2, \dots$) is a closed subset of X and $f^{-1}(Y_0)$ is a G_δ -set of X .

Hence, according to the classical theorems on preservation of Borel class by closed and open functions with compact preimages of points [9], [8], [1], [7], we find that every Y_n is an absolute Borel set of the same class.

If X is of additive class α , then Y is of additive class α because it is the countable union of the sets Y_i .

Suppose that X is a Borel set of multiplicative class α . For $\alpha = 1$ (X is an absolute G_δ -set), the conclusion follows from the recent results of S. Gao and V. Kieftenbeld [2] and P. Holický and R. Pol [3], and hence Y is an absolute Borel set of the same class α .

Similar to [5, Theorem 7], we can easily deduce our statement for multiplicative class $\alpha > 1$. ■

5. The case of separable metric spaces. A slight change in the proof of Theorem 1 actually shows the following:

COROLLARY 1. *Let $f : X \rightarrow Y$ be an open-LC function such that the preimage of every point $y \in Y$ is compact. Then Y can be covered by countably many subsets Y_n such that the restrictions $f|f^{-1}(Y_n)$ are open functions ($n = 1, 2, \dots$) and the restriction $f|f^{-1}(Y_0)$ is a closed function.*

The same conclusion can be drawn from the proof of Theorem 2:

COROLLARY 2. *Let $f : X \rightarrow Y$ be a continuous, open-LC function onto Y such that the preimage of every point $y \in Y$ is compact. If X is an absolute Borel set of additive or multiplicative class α , then Y is an absolute Borel set of the same class.*

A function f is a *countable homeomorphism* if X can be partitioned into countably many pairwise disjoint sets X_i such that every restriction $f|X_i$ is a homeomorphism.

COROLLARY 3. *Let $f : X \rightarrow Y$ be a one-to-one function between separable metric spaces X and Y , such that f and $f^{-1} : Y \rightarrow X$ are open-LC functions. Then f is a countable homeomorphism.*

Indeed, let us take the sets $H_n^X \subset X$ and $H_k^Y \subset Y$ such that each of $f|H_n^X$ and $f^{-1}|H_k^Y$ is a one-to-one continuous function. Then every restriction $f|X_n$, where $X_n = H_n^X \cap f^{-1}(H_k^Y)$, is a homeomorphism. ■

If X and Y are absolute Suslin sets, then Corollary 3 follows from the results of J. E. Jayne and C. A. Rogers [4].

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