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Cylinders over λ-dendroids have the fixed point property

by

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Abstract. It is proved that the cylinder $X \times I$ over a λ-dendroid X has the fixed point property. The proof uses results of [9] and [10].

1. Introduction. The aim of this paper is to prove the following.

1.1. MAIN THEOREM. *If X is a λ-dendroid then $X \times I$ has the fixed point property.*

REMARKS. 1. It will be clear from the proof that the conclusion still holds if I is replaced by any compact absolute retract. Note that any dendrite, i.e. locally connected λ-dendroid, is a compact absolute retract of dimension 1 (cf. [7, p. 442, Cor. 8 and p. 344, Thm. 16]). Therefore, by a standard argument of fixed point theory, the segment I can be replaced by any λ-dendroid which can be approximated from within by dendrites, for instance, it can be replaced by any smooth dendroid (cf. Fugate [4]).

2. From a result of Cook [3], it follows that every λ-dendroid is a tree-like continuum, i.e. the inverse limit of a sequence of trees, thus, by Lelek [8, 2.2 and 2.3]: *A continuum X is a λ-dendroid if and only if X is tree-like and contains no indecomposable* (*non-degenerate*) *continuum.* Each λ-dendroid has the fixed point property [9]. A first example of a tree-like continuum without the fixed point property was discovered by Bellamy [1] (cf. also [2] and Minc [12]).

3. The proof of the Main Theorem must depend on considerations of arcs because of the following theorem: *The cylinder $X \times I$ has the fixed point property if X has this property and contains no arc.*

Indeed, let $f = (f_1, f_2)$ be a continuous function of $X \times I$ into itself. Fix $y_0 \in I$. Then there is a fixed point $x_0 \in X$ of the function $x' = f_1(x, y_0)$. Consequently, f must carry the arc component $\{x_0\} \times I$ of $X \times I$ into itself, hence f has a fixed point in $\{x_0\} \times I$.

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Here, of course, one can replace I by any arcwise connected space with the fixed point property.

The idea of the proof of the Main Theorem is briefly described in Section 4.

2. Preliminaries. By a *continuum* we mean a non-void compact connected metrizable space. A continuum cotaining only one point is said to be *degenerate*. A continuum is said to be *irreducible* between two of its points if no other subcontinuum of it contains these points. By the Brouwer reduction theorem, for any two points of a continuum X there exists a subcontinuum of X which is irreducible between them (cf. [13, p. 17]). The notion of an irreducible continuum is a generalization of the notion of an *arc*, i.e. homeomorphic image of an interval of real numbers.

A continuum C is said to be *unicoherent* if for every representation $C = A \cup B$, where A and B are continua, the intersection $A \cap B$ is a continuum. A continuum X is said to be *hereditarily unicoherent* if all its subcontinua are unicoherent. This is obviously equivalent to saying that for any two points a and b of X there is a unique subcontinuum of X which is irreducible between a and b; it will be denoted simply by ab. Following Gurevich [5], we denote by $[a]_b$ the set of all points p of X such that ab is irreducible between p and b: $ab = pb$; analogously we define the set of irreducibility $[b]_a$ relative to b in ab.

A continuum is said to be *decomposable* if it can be represented as a union of two proper subcontinua; otherwise it is *indecomposable*. A continuum is *hereditarily decomposable* if all its non-degenerate subcontinua are decomposable.

By a λ-*dendroid* we mean a continuum that is both hereditarily decomposable and hereditarily unicoherent. By a standard result of the theory of irreducible continua (cf. [6, p. 239, Final conclusion]), a hereditarily unicoherent continuum X is a λ-dendroid if and only if for any two points a and b of X the sets of irreducibility $[a]_b$ and $[b]_a$ are continua. They are disjoint (and nowhere dense in ab) whenever ab is non-degenerate, i.e. whenever $a \neq b$.

2.1. Lemma ([9, Props. 13 and 14]). *For any continua ab and bc in a λ-dendroid, if $[b]_c \subsetneq [b]_a$, then $ab \cap bc \subset ac$, and then $ab \cup bc = ac$ and $[a]_c = [a]_b$.*

A space Z has the *fixed point property* if every continuous function f of Z into itself has a *fixed point*, i.e. a point $z \in Z$ such that $z = f(z)$.

Notation. From now on, X will denote an arbitrary non-degenerate λ-dendroid and I will stand for a closed interval of real numbers. Arguing by contradiction, we let f denote a fixed point free continuous function of $X \times I$ into itself. Denoting by π the projection of $X \times I$ onto X, we set

$$F(x) = \pi(f(\{x\} \times I)) \quad \text{for all } x \in X.$$

This formula assigns to every $x \in X$ a (non-void) subcontinuum $F(x)$ of X; F will be called the *c-function induced in X by the function f*. We define the image $F(K)$ of a set $K \subset X$ by the formula

$$F(K) = \bigcup_{x \in K} F(x).$$

Since the c-function F induced by f is *upper semicontinuous*, i.e. the (upper) counter-image $F^{-1}(S) = \{x \in X : F(x) \cap S \neq \emptyset\}$ is closed if S is a closed subset of X, from [9] we obtain

2.2. Lemma. *There exists $a \in X$ such that $a \in F(a)$ (a fixed point of F).*

Since for every arc $A \subset X$ the image $F(A)$ is an arcwise connected (even locally connected) continuum, it follows that the arc component Y of a in X is mapped into itself by $F : F(Y) \subset Y$. Note that Y is *uniquely arcwise connected*, i.e. any two points of Y are joined by a unique arc in Y; this follows directly from the hereditary unicoherence of the λ-dendroid X.

By the upper semicontinuity of F, we also have the equality $F(\bigcap_{n=1}^{\infty} K_n) = \bigcap_{n=1}^{\infty} F(K_n)$ for every decreasing sequence of continua $K_{n+1} \subset K_n \subset X$, $n = 1, 2, \ldots$ Therefore, by the Brouwer reduction theorem, we have

2.3. Lemma. *For every continuum $C \subset X$ such that $C \cap F(C) \neq \emptyset$ there exists a continuum $K \subset C$ which is minimal with respect to the property $K \cap F(K) \neq \emptyset$; it has the property that $K = aa'$ for every $a' \in K \cap F(K)$ and $a \in K$ such that $a' \in F(a)$.*

For any three points $a, b, c \in X$, by the hereditary unicoherence of X, the condition (i) $b \in ac$, is equivalent to each of the following: (ii) $ab \subset ac$, (iii) $bc \subset ac$, (iv) $ab \cup bc = ac$; if both ab and bc are arcs, then each of (i)–(iv) is equivalent to (v) $ab \cap bc = \{b\}$.

3. Families $\mathcal{F}_a$ and $\mathcal{F}$ of subcontinua of X and auxiliary theorems. For $a \in X$ consider the family $\mathcal{F}_a$ of all non-degenerate irreducible continua $ab \subset X$ such that $[b]_a \cap F([b]_a) \neq \emptyset$ and for every $p \in ab - [b]_a$ the irreducible continuum pb is not an arc.

Given $ab \cup bc = ac \subset X$ such that $bc \in \mathcal{F}_b$, we have obviously $ac \in \mathcal{F}_a$, and in the case when $b \in ac - [c]_a$ we have $bc \in \mathcal{F}_b$ if and only if $ac \in \mathcal{F}_a$.

The following three theorems on the families $\mathcal{F}_a$ will play a crucial role in the proof of the Main Theorem 1.1. The first of them summarizes most of the results of [10]. To present it, recall that by a *ray* in a space Y we mean the image $R = \varphi([0, \infty))$ of a continuous injection $\varphi : [0, \infty) \to Y$.

The canonical linear order $\le$ on $[0,\infty)$ can be carried over via φ to R. The point $\varphi(0)$ is independent of the parametrization and it is called the *origin* of R. For $p\in R$, we denote by $R(p)$ the subray $R(p)=\{q\in R: p\le q\}$. By the *limit* of the ray R we mean the set $L(R)=\bigcap\{\operatorname{cl} R(p): p\in R\}$. Clearly, $\operatorname{cl} R = R\cup L(R)$.

It follows directly from [10, Thm. 2.2] that for every $b\in L(R)$ the closure $\operatorname{cl} R$ of a ray R in the λ-dendroid X is irreducible between the origin a of R and b, and $L(R)=[b]_a$ (considering a ray R in the λ-dendroid X, we consider R simultaneously in the arc component Y of a in X).

By a *pursuit ray* of the c-function F we mean a ray $R=\varphi([0,\infty))\subset X$ such that there are arbitrarily large $\tau\in[0,\infty)$ such that $\varphi([0,\tau])$ is contained in an arc $\varphi(0)q'$ for some $q'\in F(\varphi(\tau))$. Obviously, if the pursuit ray R is not contained in an arc in X then for an arbitrary point a of the arc component of R in X we can find an arc ap for some $p\in R$ so that $ap\cup R(p)$ is still a pursuit ray of F in X. Therefore, we have

3.1. THEOREM (cf. [10, Thms. 3.1 and 3.2]). *For every $a\in X$ such that $a\in F(a)$ there exists $ab\in\mathcal{F}_a$ such that $ab-[b]_a$ is a pursuit ray of F.*

Denote by $\mathcal{F}$ the family of all non-degenerate subcontinua C of X such that $C\cap F(C)\ne\emptyset$ and for every $p\in X$ the following condition holds:

$$(*)\qquad \text{for every } pq\in\mathcal{F}_p \text{ such that } [p]_q\subsetneq C \text{ we have } pq\subset C.$$

3.2. THEOREM. *For every continuum $ab\in\mathcal{F}_a$ there exists a continuum $ac\subset X$ which contains ab and is maximal with respect to the property*

$$(**)\qquad ac\in\mathcal{F}_a \quad \textit{and} \quad [a]_c=[a]_b,$$

and then $[c]_a\in\mathcal{F}$.

Proof. Firstly, we prove that for any continuum $ac\subset X$ which is maximal with respect to the property $(**)$ we have $[c]_a\in\mathcal{F}$.

Indeed, $[c]_a\cap F([c]_a)\ne\emptyset$ because $ac\in\mathcal{F}_a$ by $(**)$. By Lemma 2.1, also $(*)$ holds for $C=[c]_a$. It remains to show that $[c]_a$ is non-degenerate.

If, on the contrary, $[c]_a=\{c\}$ then $c\in F(c)$. Hence, by Theorem 3.1, there exists $cd\in\mathcal{F}_c$ such that $cd-[d]_c$ is a ray. Since $ac\in\mathcal{F}_a$, it follows that $ac\cap cd\subset[c]_a$. By Lemma 2.1, $ac\subset ad$ and $[a]_c=[a]_d$. Since $cd\in\mathcal{F}_c$, it follows that $ad\in\mathcal{F}_a$. By the assumed maximality of ac, the inclusion $ac\subset ad$ implies $ac=ad$. Hence $cd\subset[c]_a$ because the last set is a subcontinuum of the λ-dendroid X. But cd is non-degenerate, being a member of $\mathcal{F}_c$, and this is a contradiction.

It remains to prove that for every $ab\in\mathcal{F}_a$ there exists $ac\subset X$ which contains ab and is maximal with respect to $(**)$.

Given a strictly increasing sequence of irreducible continua $ab_n\in\mathcal{F}_a$ such that $[a]_{b_n}=[a]_{b_1}$ and $ab\subset ab_n$ for $n=1,2,\ldots$ it suffices to observe, by the dual to the Brouwer reduction theorem (Milgram [11, Thm. 3]; cf. [9, p. 115]), that there exists $c \in X$ such that $ac = \mathrm{cl}\bigcup_{n=1}^{\infty} ab_n$ (by [9, Remark 2]) and to prove that then $ac \in \mathcal{F}_a$ and $[a]_c = [a]_{b_1}$.

Claim. *The sequence of irreducible continua $b_n c \subset ac$, $n = 1, 2, \ldots$, is decreasing with $\bigcap_{n=1}^{\infty} b_n c = [c]_a$, and $b_n c$ is not an arc for every $n = 1, 2, \ldots$*

Indeed, for every n we have $b_{n+1} \notin ab_n$, and $ab_n \cup b_n c = ac$, thus $b_{n+1} \in b_n c$, i.e. $b_{n+1} c \subset b_n c$ for $n = 1, 2, \ldots$

To prove the equality in the Claim observe that $ab_n \subsetneq ac$ and $ab_n \cup b_n c = ac$ for all n, and hence $[c]_a \subset \bigcap_{n=1}^{\infty} b_n c$ by the definition of $[c]_a$. For the converse inclusion, let $p \in ac - [c]_a$ and assume that, on the contrary, $p \in \bigcap_{n=1}^{\infty} b_n c$. Then $U = ac - (ap \cup [p]_c)$ is an open neighborhood of $[c]_a$ in ac and lies outside of all ab_n: $ab_n \cap U = \emptyset$, $n = 1, 2, \ldots$ Then $\mathrm{cl}\bigcup_{n=1}^{\infty} ab_n$ is disjoint from the non-empty subset U of ac, which contradicts the equality $ac = \mathrm{cl}\bigcup_{n=1}^{\infty} ab_n$.

Finally, we have $b_n b_{n+i} \subsetneq ab_{n+1}$ and $b_n \in ab_{n+i} - [b_{n+1}]_a$ so that, as $ab_{n+1} \in \mathcal{F}_a$, the continua $b_n b_{n+1}$ are not arcs for each n. Since $b_{n+1} \in b_n c$, i.e. $b_n b_{n+1} \subset b_n c$, it follows that $b_n c$ is not an arc for each n, which proves the Claim.

Now, to prove that $ac \in \mathcal{F}_a$ observe that $[b_{n+1}]_a \subset b_n c$ and $[b_{n+1}]_a \cap F([b_{n+1}]_a) \neq \emptyset$, since $ab_{n+1} \in \mathcal{F}_a$, for all n. Hence $b_n c \cap F(b_n c) \neq \emptyset$ for all n, and thus $[c]_a \cap F([c]_a) \neq \emptyset$ by the equality in the Claim.

For every $p \in ac - [c]_a$ we have $ap \cup pc = ac$ and $b_n c \cap ap = \emptyset$ for sufficiently large n. Thus $b_n c \subset pc$ and hence pc is not an arc by the last statement in the Claim.

Finally, if $ab_{n_0} \subset [a]_c$ for some n_0, then $[a]_{b_n} \subsetneq [a]_c$ for all n, and thus $ac \subset [a]_c$ contrary to ac being non-degenerate. Therefore $b_n \in ac - [a]_c$ and then $[a]_c = [a]_{b_n}$ for all $n = 1, 2, \ldots$

3.3. Theorem. *For any two points $a, a' \in X$ such that*

$$a' \in F(a) \quad \textit{and} \quad [a]_{a'} \cap F([a]_{a'}) = \emptyset$$

there exists $ab \in \mathcal{F}_a$ such that $[a]_b = [a]_{a'}$.

The proof of this theorem will occupy Sections 5–7.

4. Theorems 3.1–3.3 imply the Main Theorem 1.1. The auxiliary Theorems 3.1–3.3 yield a procedure of constructing sequences of non-degenerate irreducible continua in the λ-dendroid X, each contained in a set of irreducibility of the previous one. The idea of the proof of the Main Theorem is to show that 1° this procedure must end and 2° it can be continued without end. This contradiction will prove the fixed point property of $X \times I$.

1° *The family $\mathcal{F}$ contains a minimal element, i.e. one contained in no other element of $\mathcal{F}$.*

Indeed, in view of Lemma 2.2, Theorems 3.1 and 3.2 directly imply that $\mathcal{F}$ is non-empty. By the Brouwer reduction theorem, it suffices to show that for any decreasing sequence $C_n \in \mathcal{F}$ we have $\bigcap_{n=1}^{\infty} C_n \in \mathcal{F}$. Obviously, setting $C = \bigcap_{n=1}^{\infty} C_n$, we have $C \cap F(C) \neq \emptyset$ and also $(*)$ is satisfied. It remains to prove that C is non-degenerate.

Assume, to get a contradiction, that $C = \{a\}$ for some $a \in X$. Then $a \in F(a)$ and, by Theorem 3.1, there exists $ab \in \mathcal{F}_a$ such that $[a]_b = \{a\}$. Since all the continua C_n are non-degenerate, we have $[a]_b \subsetneq C_n$ for all n. Therefore $ab \subset C_n$ for all n, and thus $ab \subset C$. Since ab is non-degenerate, being a member of $\mathcal{F}_a$, it follows that C is non-degenerate, a contradiction.

2° *The family $\mathcal{F}$ contains no minimal element, i.e. for every $C \in \mathcal{F}$ there exists $C_0 \in \mathcal{F}$ such that $C_0 \subsetneq C$.*

Namely, take a continuum $K \subset C$ which is minimal with respect to the property $K \cap F(K) \neq \emptyset$ so that $K = aa'$ for some $a, a' \in X$ such that $a' \in F(a)$ (cf. Lemma 2.3). We show that

$$[a]_{a'} \subsetneq C \quad \text{and} \quad [a]_{a'} = [a]_b \quad \text{for some } ab \in \mathcal{F}_a.$$

If $a = a'$ then $[a]_{a'}$ reduces to the point $\{a\}$ and hence, C being non-degenerate as a member of $\mathcal{F}$, we have the proper inclusion needed; the equality follows too because we have $a \in F(a)$ and Theorem 3.1 applies.

If $a \neq a'$ then $[a]_{a'}$, is a proper subcontinuum of $aa' = K$ and thus $[a]_{a'} \cap F([a]_{a'}) = \emptyset$ by the minimality of K. Hence we also have $[a]_{a'} \subsetneq aa' \subset C$, and the equality needed follows by Theorem 3.3.

Now, by Theorem 3.2, there is a continuum $ac \in \mathcal{F}_a$ such that $[a]_c = [a]_b$ and $[c]_a \in \mathcal{F}$. Therefore, from $[a]_c \subsetneq C$ and $C \in \mathcal{F}$, it follows that $ac \subset C$. Since obviously $[c]_a \subsetneq ac$, we can set $C_0 = [c]_a$.

5. A fixed point lemma. In this section, Y is a uniquely arcwise connected space. The unique arc between $p, q \in Y$ will be denoted by pq. Obviously, for any two arcwise connected subsets A and B of Y the intersection $A \cap B$ is arcwise connected.

Let F be any upper semicontinuous continuum-valued function mapping a non-degenerate arc $p_0q_0 \subset Y$ onto Y: $F(p_0q_0) = Y$ (in particular, F can be determined on the arc p_0q_0 only), such that for every proper subarc A of p_0q_0 the image $F(A)$ is an arcwise connected continuum. We are going to prove

5.1. LEMMA. *If there exist $p_0' \in F(p_0)$ and $q_0' \in F(q_0)$ such that $p_0 \in p_0'q_0$ and $q_0 \in q_0'p_0$, then there exists $r_0 \in p_0q_0$ such that $r_0 \in F(r_0)$.*

Proof. We can assume that $p_0 \notin F(p_0)$ and $q_0 \notin F(q_0)$ so that

$$F(p_0) \cap p_0q_0 = \emptyset \quad \text{and} \quad F(q_0) \cap p_0q_0 = \emptyset.$$

Indeed, we have $p_0 \in p_0'q_0$ for some $p_0' \in F(p_0)$ by assumption.

If there were $r \in p_0q_0$ such that $r \in F(p_0)$ then $r \neq p_0$ and thus $p_0 \notin rq_0$. Since $rp_0' \subset F(p_0)$, $F(p_0)$ being arcwise connected by the assumption on F, it follows that $p_0 \notin rp_0'$. Thus $p_0 \notin rq_0 \cup rp_0'$ and hence $p_0 \notin p_0'q_0$, a contradiction proving the first equality needed; the second one follows dually.

Therefore, by the upper semicontinuity of F, there exist $p_1, q_1 \in p_0q_0$ such that

$$F(p_0p_1) \cap p_0q_0 = \emptyset \quad \text{and} \quad F(q_1q_0) \cap p_0q_0 = \emptyset \tag{1}$$

and $p_0 < p_1 < q_1 < q_0$ in the natural order $<$ from p_0 to q_0 in p_0q_0.

Let Φ be the family of all arcs $p_0p - \{p\} \subset p_0q_0$ such that

$$r \in r'q_0 \quad \text{for every } r' \in F(r) \text{ and } p_0 \leq r < p < q_0, \tag{2}$$

and define r_0 to be the end of the union of all arcs in Φ (other than p_0).

The family Φ is non-empty, namely $p_0p_1 - \{p_1\} \in \Phi$.

Indeed, take $r' \in F(r)$ for $p_0 \leq r < p_1$. Then $p_0'r' \subset F(p_0p_1)$, the image $F(p_0p_1)$ being arcwise connected, and obviously $r'p_0 \subset r'p_0' \cup p_0'p_0$. It follows, in view of (1) (first equality), that $r'p_0 \cap p_0q_0 = \{p_0\}$. Then $r'q_0 = r'p_0 \cup p_0q_0$. Since $r \in p_0p_1 \subset p_0q_0$, it follows that $r \in r'q_0$.

Thus Φ is non-empty and we have

$$p_0 < p_1 \leq r_0 < q_0.$$

We need to prove only the last inequality. To this end, assume on the contrary that $r_0 = q_0$. Then for $q_1 < r < p < q_0$, $p_0p - \{p\} \in \Phi$, i.e. (2) holds. But $r'q_0' \subset F(q_1q_0)$, and $r'q_0 \subset r'q_0' \cup q_0'q_0$. It follows, in view of (1) (second equality), that $r'q_0 \cap q_0p_0 = \{q_0\}$. By (2), we have $r \in r'q_0$. Since $r \in p_0q_0$, it follows that $r = q_0$, a contradiction.

Now, we prove that $r_0 \in F(r_0)$. Assume that $r_0 \notin F(r_0)$. By the upper semicontinuity of F, it follows that

$$F(s_0t_0) \cap s_0t_0 = \emptyset \text{ for some } s_0 \text{ and } t_0 \text{ such that } p_0 < s_0 < r_0 < t_0 < q_0. \tag{3}$$

Then we have

$$r^*s_0 \cap s_0q_0 = \{s_0\} \quad \text{for every } r^* \in F(s_0t_0). \tag{4}$$

Indeed, setting $r = s_0$ and taking p such that $s_0 < p < r_0$, we have $p_0p - \{p\} \in \Phi$, i.e. by (2), $s_0 \in s_0'q_0$ for every $s_0' \in F(s_0)$. Hence

$$s_0's_0 \cap s_0q_0 = \{s_0\},$$

and $s_0'r^* \subset F(s_0t_0)$ by assumption on F. It follows by (3) that

$$s_0'r^* \cap s_0t_0 = \emptyset,$$

and obviously $s_0r^* \subset s_0s'_0 \cup s'_0r^*$. Therefore, we have

$$\{s_0\} \subset s_0r^* \cap s_0q_0 \subset s_0s'_0 \cap s_0q_0 \cup s'_0r^* \cap s_0q_0 \subset \{s_0\} \cup t_0q_0.$$

Since the intersection $s_0r^* \cap s_0q_0$ is an arc, (4) follows.

Finally, take any $p_2, r_2 \in p_0q_0$ such that $s_0 < r_0 \leq r_2 < p_2 < t_0$, and let $r'_2 \in F(r_2)$. Then $r'_2s_0 \cap s_0q_0 = \{s_0\}$ by (4) and hence we have $r'_2s_0 \cup s_0q_0 = r'_2q_0$. Since $r_2 \in s_0q_0$, it follows that $r_2 \in r'_2q_0$. But, of course, $p_0r_0 - \{r_0\} \in \Phi$. Hence $p_0p_2 - \{p_2\} \in \Phi$ for any $p_2 \in r_0t_0 - \{t_0\}$, which contradicts the meaning of r_0, concluding the proof of Lemma 5.1.

6. Families $\mathcal{P}_a$, $\mathcal{P}^*_a$ of irreducible continua in the λ-dendroid X. Given irreducible continua ab and ac in the λ-dendroid X, we write $ab \prec ac$ if and only if $[a]_b = [a]_c \neq ab \cap ac$ (thus there is a common portion ap of continua ab and ac). This is equivalent to saying (cf. [9, Prop. 1]) that $a \notin bc$. Thus we then have $a \neq b$ and $a \neq c$, i.e. ab and ac are non-degenerate.

For an arbitrary upper semicontinuous continuum-valued function F mapping X into itself, we say that $ab \in \mathcal{P}_a$ [9, p. 113] if $ab \prec aa'$ for all $a' \in F(a)$ and for every $p \in ab - [b]_a$ there exist $q_0, r \in pb - [p]_b$ such that

$$pq_0 \subset pr \subset pb \quad \text{and} \quad q_0r \prec q_0q'_0 \quad \text{for all } q'_0 \in F(q_0).$$

Note that if the irreducible continuum pb is an arc, the above is equivalent to saying that there exists $q_0 \in pb$ such that (cf. [9, (IV), (V), Prop. 2 and Def. 3])

$$\{p\} \neq pq_0 \subset pb - \{b\} \quad \text{and} \quad q_0b \prec q_0q'_0 \quad \text{for all } q'_0 \in F(q_0).$$

Then it follows that (cf. [9, Prop. 6])

$$pq_0 \subset pq'_0 \quad \text{and} \quad q_0 \notin bq'_0 \quad \text{for all } q'_0 \in F(q_0). \tag{5}$$

We shall write $ab \in \mathcal{P}^*_a$ if $ab \in \mathcal{P}_a$ and $p \notin F(p)$ for all $p \in ab - [b]_a$. As in [9, proof of Lemma 2] we obtain

6.1. Proposition. *If $a' \in F(a)$ and $[a]_{a'} \cap F([a]_{a'}) = \emptyset$, then there exists an irreducible continuum $ab_0 \subset X$ which is maximal in $\mathcal{P}^*_a$, and then $[a]_{b_0} = [a]_{a'}$.*

Also the following proposition has the same proof as in [9].

6.2. Proposition. *If an irreducible continuum $ab_0 \subset X$ is maximal in $\mathcal{P}^*_a$ and $[b_0]_a \cap F([b_0]_a) = \emptyset$, then for every $b'_0 \in F(b_0)$ we have $ab_0 \cap b_0b'_0 \subset [b_0]_a$* (cf. [9, Prop. 13 and Lemma 3]), *and there exists an irreducible continuum $b_0s \subset X$ such that*

(i) $b_0s \subset [b_0]_{b'}$ ([9, (15)]);
(ii) $[b_0]_{b'_0} \cap F(b_0s - [s]_{b_0}) = \emptyset$ ([9, (17)]);
(iii) *there is $d \in [s]_{b_0}$ such that $d \in F(d)$* ([9, pp. 119–120, the (degenerate) continuum K_1]).

7. Proof of Theorem 3.3. Consider the c-function F induced in the λ-dendroid X by the fixed point free continuous function $f : X\times I \to X\times I$. By Proposition 6.1, take an irreducible continuum $ab_0 \subset X$ which is a maximal member of $\mathcal{P}_a^*$. Then $[a]_{b_0} = [a]_{a'}$ for some $a' \in F(a)$.

We are going to prove that there is $ab \in \mathcal{F}_a$ such that $[a]_b = [a]_{b_0}$.

Case 1: $[b_0]_a \cap F([b_0]_a) \neq \emptyset$. We can consider the situation when, for some $p \in ab_0 - [b_0]_a$, the irreducible continuum pb_0 is an arc (otherwise it suffices to set $b = b_0$). Thus we have

$$[b_0]_a = \{b_0\} \quad \text{and} \quad b_0 \in F(b_0). \tag{6}$$

By Theorem 3.1, there exists

$$b_0 b \in \mathcal{F}_{b_0} \tag{7}$$

such that

$$R_0 = b_0 b - [b]_{b_0} \quad \text{is a pursuit ray of } F. \tag{8}$$

Claim. $R_0 \not\subset b_0 a$.

To prove the Claim, assume on the contrary that

$$R_0 \subset b_0 a \text{ and let } b_0 p \text{ be a non-degenerate subarc of } R_0. \tag{9}$$

By (8), there exist $b_0 p_0 \subset R_0$ and $b_0 p_0' \subset X$ such that

$$b_0 p \subset b_0 p_0 \subset b_0 p_0', \qquad b_0 p_0' \text{ is an arc}, \qquad p_0' \in F(p_0). \tag{10}$$

Since $ab_0 \in \mathcal{P}_a^* \subset \mathcal{P}_a$, there exists (cf. (9) and (5))

$$q_0 \in b_0 p - (\{p\} \cup F(q_0)). \tag{11}$$

By (10), $b_0 p_0'$ is an arc, and hence $p_0' p_0 \cap p_0 b_0 = \{p_0\}$. Also, by (10) and (11), we have $q_0 \in b_0 p_0$, i.e. $p_0 q_0 \subset p_0 b_0$. Therefore

$$\{p_0\} \subset p_0' p_0 \cap p_0 q_0 \subset p_0' p_0 \cap p_0 b_0 \subset \{p_0\},$$

and hence $p_0' p_0 \cup p_0 q_0 = p_0' q_0$, i.e.

$$p_0 \in p_0' q_0. \tag{12}$$

By (10), we also have $p_0 p \cap p b_0 = \{p\}$, and $p q_0 \subset p b_0$ by (11). Hence $p_0 p \cap p q_0 = \{p\}$, i.e.

$$p_0 p \cup p q_0 = p_0 q_0 \quad \text{and} \quad p \neq q_0 \tag{13}$$

in view of (11). Hence $q_0 \notin p_0 p$. Take an arbitrary $q_0' \in F(q_0)$. Since $q_0 \in p q_0'$ in view of (5), and obviously $p q_0' \subset p p_0 \cup p_0 q_0'$, it follows that

$$q_0 \in q_0' p_0. \tag{14}$$

Observe that

$$p_0' p_0 \cap p_0 q_0 = \{p_0\} \quad \text{and} \quad q_0' q_0 \cap p_0 q_0 = \{q_0\}. \tag{15}$$

Indeed, the first equality follows by (10) because p'_0q_0 is an arc by (10) and (11). To prove the second equality in (15) (a priori, q'_0q_0 need not be an arc), observe that $q_0b_0 \cap q'_0q_0$ is a subarc q_0t of the arc q_0b_0 (cf. (10) and (11)):

$$q_0t = q_0b_0 \cap q_0q'_0$$

and (cf. (5))

$$q_0 \notin tq'_0.$$

Also we have $q_0 \in b_0p_0$ and thus

$$\{q_0\} \subset p_0q_0 \cap q_0t \subset p_0q_0 \cap q_0b_0 = \{p_0\}.$$

Consequently, $\{q_0\} \subset p_0q_0 \cap q_0q'_0 \subset p_0q_0 \cap (q_0t \cup tq'_0) = (p_0q_0 \cap q_0t) \cup (p_0q_0 \cap tq'_0) = \{q_0\} \cup (p_0q_0 \cap tq'_0)$. Since $q_0 \notin tq'_0$, it follows by the hereditary unicoherence of X that $p_0q_0 \cap q_0q'_0 = \{q_0\}$, which proves (15).

Simultaneously, by the hereditary unicoherence of X, we have $p'_0p_0 \cap q'_0q_0 = \emptyset$, because otherwise there would be a smallest point u of the arc $p_0p'_0$ (directed from p_0 to p'_0) which is common with q'_0q_0, so that $p_0u \cup uq_0$ would be a continuum irreducible between p_0 and q_0 other than the arc p_0q_0.

It follows in view of (15) that $p_0q_0 \subset p'_0q'_0$. Since $p'_0q'_0 \subset F(p_0q_0)$, because the image is arcwise connected, we have $p_0q_0 \subset F(p_0q_0)$.

Thus all assumptions of Lemma 5.1 are satisfied (with $Y = F(p_0q_0)$) by (12)–(14), and therefore there exists $r_0 \in p_0q_0$ such that $r_0 \in F(r_0)$ and $r_0 \neq q_0$ because $q_0 \notin F(q_0)$ by (11). Since $p_0q_0 \subset p_0b_0$ in view of (10) and (11), we have $r_0 \in p_0b_0 - \{b_0\}$. Therefore $r_0 \in ab_0 - [b_0]_a$ in view of (6). Since $ab_0 \in \mathcal{P}^*_a$, this is a contradiction proving the Claim.

By the Claim, there exists an arc b_0e_0 of the ray R_0 (cf. (8)) such that $b_0e_0 = R_0 \cap ab_0$. Then

$$ae_0 \cap e_0b = \{e_0\}$$

so that $ae_0 \cup e_0b = ab$ (cf. Lemma 2.1). Since $b_0b \in \mathcal{F}_{b_0}$ by (7), and $e_0 \in b_0b - [b]_{b_0}$ in view of (8), we have $e_0b \in \mathcal{F}_{e_0}$ and thus $ab \in \mathcal{F}_a$.

It remains to show that $[a]_b = [a]_{b_0}$.

If $e_0 \notin [a]_{b_0}$, i.e. $e_0 \in ab_0 - [a]_{b_0}$, then $[a]_{b_0} = [a]_{e_0}$. Since $ae_0 \cap e_0b \subset [e_0]_a$ and thus $[a]_{e_0} = [a]_b$ by Lemma 2.1, it follows that $[a]_b = [a]_{b_0}$.

If $e_0 \in [a]_{b_0}$, then $[a]_{b_0} = [e_0]_{b_0} = \{e_0\}$. Hence $a = e_0$. Also, $e_0 \in b_0b - [b]_{b_0}$ and $b_0b - [b]_{b_0}$ is a ray by (8). Therefore, $\{e_0\} = [e_0]_b = [a]_b$. It follows that $[a]_b = [a]_{b_0}$.

Case 2: $[b_0]_a \cap F([b_0]_a) = \emptyset$. By Theorem 3.1, in view of Proposition 6.2(iii), there is $db \in \mathcal{F}_d$ such that

(16) $$R_1 = db - [b]_d \quad \text{is a pursuit ray of } F,$$

and we have simultaneously $d \in F(d) \cap [b_0]_{b_0'}$, and

$$ab_0 \cup b_0 d = ad \quad \text{and} \quad [a]_{b_0} = [a]_d$$

in view of Lemma 2.1. Also, $ad \subset ab$ and $[a]_d = [a]_b$ on condition that $ad \cap db \subset [d]_a$. Thus, it remains to prove the last inclusion. Assume that $(ad - [d]_a) \cap db \neq \emptyset$. By (16) and the hereditary unicoherence of X, it follows that there is a point

$$p \in (ad - [d]_a) \cap R_1 \tag{17}$$

and then we have the arc $dp \subset R_1$. Also, for s as in Proposition 6.2, $[d]_a = \{d\} = \{s\}$. Hence there is a point

$$e \in dp \cap db_0 - \{d\} \tag{18}$$

because $ab_0 \subsetneq ad$, since $d \neq b_0$ as $b_0 \notin F(b_0)$ in Case 2 and $d \in F(d)$ by Proposition 6.2. Simultaneously, $b_0 d \subset [b_0]_{b_0'}$ and $[b_0]_{b_0'} \cap F(b_0 e) = \emptyset$ (ibid., (i) and (ii)), and

$$[b_0]_{b_0'} \quad \text{is not a point} \tag{19}$$

and $[b_0]_{b_0'} = [e]_{b_0'}$. Taking any $e' \in F(e)$, we have $e'b_0' \subset F(b_0 e)$. Consequently, $e \notin e'b_0'$ and thus $ee' \prec eb_0'$. Therefore $[e]_{e'} = [e]_{b_0'}$ and thus $[e]_{e'} = [b_0]_{b_0'}$. Hence, by (19),

$$[e]_{e'} \quad \text{is not a point.} \tag{20}$$

On the other hand, by (16) and (17), there exist $dq \subset R_1$ and $dq' \subset X$ such that

$$dp \subset dq \subset dq' \quad \text{and} \quad q' \in F(q) \tag{21}$$

and

$$dq' \quad \text{is an arc.} \tag{22}$$

Hence, $e \in dp \subset dq'$ by (17) and (21), and thus $eq' \subset dq'$. Therefore, by (22), eq' is an arc, and thus $e'e \cap eq' \subset [e]_{e'}$ because $[e]_{e'}$ is not a point by (20). It follows by Lemma 2.1 that $e'e \subset e'q' \subset F(eq)$. But eq is an arc, being contained in the union $de \cup dq$ of arcs dq (cf. (21) and (22)) and de (cf. (17) and (18)). Since $F(eq)$ is arcwise connected because F is induced by f, it follows that ee' is an arc, which contradicts (20).

This proves Theorem 3.3 and completes the proof of the Main Theorem.

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