FUNDAMENTA
MATHEMATICAE
192 (2006)

$C(K)$ spaces which cannot be uniformly embedded into $c_0(\Gamma)$

by

Jan Pelant, **Petr Holický** and **Ondřej F. K. Kalenda** (Praha)

Abstract. We give two examples of scattered compact spaces K such that $C(K)$ is not uniformly homeomorphic to any subset of $c_0(\Gamma)$ for any set Γ. The first one is $[0, \omega_1]$ and hence it has the smallest possible cardinality, the other one has the smallest possible height $\omega_0 + 1$.

Foreword. We present two results of Jan Pelant. He presented them at seminars in the Mathematical Institute of Czech Academy of Sciences during the last two years. The example described in Theorem 4.1 below is quite recent. Unfortunately, Jan Pelant died before the results were prepared for publication. We decided to reconstruct them using his hand-written notes.

1. Introduction. We give two examples of scattered compact spaces K such that $C(K)$ is not uniformly homeomorphic to any subset of $c_0(\Gamma)$ for any set Γ. This contributes to the study of nonlinear embeddings of (real) Banach spaces into other ones. This investigation is related to the study of the question which topological (or metric) properties enable one to reconstruct the linear structure of a Banach space.

The well known Mazur–Ulam theorem says that the existence of an isometry of two Banach spaces implies their linear isometry. On the other hand, the result of H. Toruńczyk [17] (of M. I. Kadec [10] for separable spaces) shows that homeomorphism of two infinite-dimensional Banach spaces gives no information about their linear structure. This is the reason why uniform

---

2000 *Mathematics Subject Classification*: Primary 46B26; Secondary 54E15, 54D20.

Key words and phrases: Banach spaces of continuous functions, uniform embedding, uniform Stone property, scattered compact space.

The work is a part of the research project MSM 0021620839 financed by MSMT. Our research was further partly supported by the research grant GAČR 201/03/0933. We are also grateful to the Erwin Schrödinger Institute in Vienna for the hospitality during the work on this paper.

homeomorphisms and Lipschitz homeomorphisms are intensively studied in this context rather than isometries or topological homeomorphisms. It is still an open question whether two Lipschitz equivalent separable Banach spaces are linearly isomorphic. For nonseparable Banach spaces this is not true (see [2]). There are some positive results for particular separable Banach spaces (see e.g. [6]). Uniform homeomorphism does not imply linear isomorphism even for separable spaces by [15]. A positive result in this area asserts that a Banach space uniformly homeomorphic to ℓ_2 is isomorphic to ℓ_2 (see [5]).

As a related question the possibility of Lipschitz or uniform embeddings of one Banach space into another is studied. I. Aharoni [1] proved that any separable metric space is Lipschitz equivalent to a subset of c_0 (in fact, to a subset of the positive cone of c_0). We concentrate our attention on the possibility of Lipschitz or uniform embeddings of spaces of continuous functions on a compact space into the space $c_0(\Gamma)$ for sufficiently large Γ, i.e., equivalently, to $C(\alpha(\Gamma))$, where $\alpha(L)$ denotes the Aleksandrov one-point compactification for any locally compact space L, and Γ is endowed with the discrete topology.

The main results of the present paper are the following two examples.

THEOREM 1.1. *The space $C([0,\omega_1])$ is not uniformly homeomorphic to any subset of the space $c_0(\Gamma)$ for any set Γ.*

THEOREM 1.2. *There is a compact space K with $K^{(\omega_0+1)} = \emptyset$ such that $C(K)$ is not uniformly homeomorphic to any subspace of $c_0(\Gamma)$ for any set Γ.*

The first one gives a compact space K of the smallest possible cardinality such that $C(K)$ cannot be uniformly embedded into $c_0(\Gamma)$. Indeed, if K is countable, then it is metrizable and hence $C(K)$ is separable. Therefore the above mentioned result of I. Aharoni implies that $C(K)$ can be embedded into c_0 by a Lipschitz homeomorphism.

The fact that $C([0,\omega_1])$ cannot be uniformly embedded onto a *linear subspace* of any $c_0(\Gamma)$ also follows from [7, Theorem 5.6(i)]. (Indeed, suppose that Φ is such a uniform embedding. Set $X_\alpha = \{f \in C([0,\omega_1]) : f(\gamma) = 0$ for $\gamma > \alpha\}$ for $\alpha < \omega_1$. It is easy to see that there is a separable Banach space $S_\alpha \supset X_\alpha$ such that $\Phi(S_\alpha)$ is linear. By [7, Theorem 5.6(i)] the Szlenk index of S_α is at most ω_0. However, it is not difficult to observe that the Szlenk index of S_α is at least the height of $[0,\alpha]$ which is greater than ω_0 for $\alpha < \omega_1$ sufficiently large.)

The second theorem gives an example of a scattered compact space K with minimal height such that $C(K)$ cannot be uniformly embedded into $c_0(\Gamma)$. Recall that the *height* of a scattered compact space is the smallest ordinal α such that the Cantor–Bendixson derivative $K^{(\alpha)}$ is empty. Indeed, by [4, Theorem 3] the space $C(K)$ is Lipschitz equivalent to some $c_0(\Gamma)$ whenever $K^{(\omega_0)} = \emptyset$ (i.e., whenever K is a scattered compact with finite height). (In fact, by a result of Godefroy [9, Theorem 6.3] the height of K is finite whenever $C(K)$ is uniformly homeomorphic to some $c_0(\Gamma)$.)

The second example is also minimal in the following sense. The construction is motivated by the result of W. Marciszewski [11] who showed that $C(K)$ is linearly isomorphic to some $c_0(\Gamma)$ if and only if K is homeomorphic to a subset of $[\Lambda]^{\leq n}$ for a set Λ and some $n \in \mathbb{N}$. Here $[\Lambda]^{\leq n}$ denotes the set of all subsets of Λ of cardinality at most n. We consider this set endowed with the natural topology inherited from $\{0,1\}^{\Lambda}$ (identifying each set with its characteristic function). The example is the one-point compactification of the topological sum of $[\Lambda]^{\leq n}$, $n \in \mathbb{N}$, for a sufficiently large set Λ. Note also that this space is an Eberlein compact space, hence $C(K)$ is weakly compactly generated, and thus it admits an injective continuous linear mapping into some $c_0(\Gamma)$.

2. Uniform Stone property. The basic tool for the study of uniform embeddings in this paper is a property of uniform covers of metric spaces (*uniform Stone property*) which is a modification of paracompactness. It turns out that a metric space can be uniformly embedded into some $c_0(\Gamma)$ if and only if it has the uniform Stone property ([13], see Theorem 2.1 below). The classical Stone theorem asserts that any open cover of a metric space has a locally finite open refinement. It is natural to ask whether the following uniform version is true. Does every uniform cover of a metric space have a locally finite uniform refinement? Recall that a cover $\mathcal{C}$ of a metric space M is *uniform* if there is $r > 0$ such that the family $\{B(x,r) : x \in M\}$ refines $\mathcal{C}$. We then say that $\mathcal{C}$ is *r-uniform.* The question was answered in the negative by E. V. Shchepin [16] and J. Pelant [12]. This led to the question which spaces do have this property. Some equivalent conditions, in particular the existence of uniform embeddings into $c_0(\Gamma)$, are stated in the following theorem.

THEOREM 2.1. *Let (M,ϱ) be a metric space. The following assertions are equivalent.*

(1) *Any uniform cover of M has a locally finite uniform refinement.*
(2) *Any uniform cover of M has a point-finite uniform refinement.*
(3) *The metric uniformity of the space (M,ϱ) has a basis of uniform covers consisting of point-finite covers.*
(4) *M is uniformly homeomorphic to a subset of $c_0(\Gamma)$ for a set Γ.*

The equivalence (2)⇔(3) follows from the definition of a basis of uniform covers. The equivalence (3)⇔(4) is contained in [13, Corollary 2.4]. The implication (1)⇒(2) is trivial, and its converse is easy; its proof may be found in [8, Lemma 3 in Chapter VII].

3. Proof of Theorem 1.1. Let us consider

$$M = \{f \in \mathcal{C}([0,\omega_1]) : f(0) = 1,\ f(\omega_1) = 0,\ f \text{ nonincreasing}\}$$

as a metric subspace of $C([0,\omega_1])$. We are going to show that M cannot be uniformly embedded into $c_0(\Gamma)$. Due to Theorem 2.1 the latter is equivalent to the fact that M does not have the uniform Stone property.

For $\iota < \omega_1$ we set

$$T_\iota = \{f \in M : f(\iota+1) < 1\}.$$

Then $\mathcal{T} = \{T_\iota : \iota < \omega_1\}$ is clearly an open cover of M. Let us show it is uniform. For any $f \in M$ there is $\iota < \omega_1$ such that $f = 0$ on $(\iota, \omega_1]$. Then clearly $B_M(f,1) \subset T_\iota$.

We claim that $\mathcal{T}$ has no point-finite uniform refinement, which shows that M does not have the uniform Stone property. Suppose the contrary: let $\mathcal{P}$ be a point-finite refinement of $\mathcal{T}$ and $\delta > 0$ be such that $\{B_M(f,\delta) : f \in M\}$ refines $\mathcal{P}$. We choose a mapping $\iota : \mathcal{P} \to [0,\omega_1)$ satisfying $P \subset T_{\iota(P)}$ and an $n \in \mathbb{N}$ such that $1/n < \delta$. If $\gamma_n \le \gamma_{n-1} \le \cdots \le \gamma_1 < \omega_1$, we define

$$f_{\gamma_n,\ldots,\gamma_1} = \chi_{[0,\gamma_n]} + \sum_{k=1}^{n-1} \frac{k}{n}\, \chi_{(\gamma_{k+1},\gamma_k]}.$$

Then clearly $f_{\gamma_n,\ldots,\gamma_1} \in M$. We define

$$\sigma(\gamma_n,\ldots,\gamma_1) = \{\iota(P) : P \in \mathcal{P} \ \&\ B_M(f_{\gamma_n,\ldots,\gamma_1},\delta) \subset P\}.$$

We also define $\tau(\gamma_n,\ldots,\gamma_m) \in [0,\omega_1]$ for nonincreasing sequences of $n-m+1$ countable ordinals, where $m = 1,\ldots,n+1$, by the following inductive procedure:

$$\begin{aligned} \tau(\gamma_n,\ldots,\gamma_1) &= \min \sigma(\gamma_n,\ldots,\gamma_1),\\ \tau(\gamma_n,\ldots,\gamma_{m+1}) &= \liminf_{\gamma\to\omega_1} \tau(\gamma_n,\ldots,\gamma_{m+1},\gamma) \quad \text{for } m \in \{1,\ldots,n\}. \end{aligned}$$

By convention, $\tau(\gamma_n,\ldots,\gamma_{n+1}) = \tau(\emptyset)$ and $\tau(\gamma_n,\ldots,\gamma_{n+1},\gamma) = \tau(\gamma)$.

Note also that

$$\liminf_{\gamma\to\omega_1} \tau(\gamma_n,\ldots,\gamma_{m+1},\gamma) = \sup_{\iota\in[0,\omega_1)} \min\{\tau(\gamma_n,\ldots,\gamma_{m+1},\gamma) : \gamma > \iota\}, \tag{1}$$

as the values are ordinals.

If $\gamma_n \le \gamma_{n-1} \le \cdots \le \gamma_1 < \omega_1$, then $\tau(\gamma_n,\ldots,\gamma_1) < \omega_1$ by our definition.

We claim that $\tau(\emptyset) = \omega_1$. Indeed, suppose that $\tau(\emptyset) < \omega_1$. As $\tau(\emptyset) = \liminf_{\gamma\to\omega_1} \tau(\gamma)$, using (1) we find $\gamma_n > \tau(\emptyset)$ with $\tau(\gamma_n) \le \tau(\emptyset)$. Repeating this argument, we get $\gamma_{n-1} > \gamma_n$ with $\tau(\gamma_n,\gamma_{n-1}) \le \tau(\emptyset)$. Continuing inductively we get $\gamma_n < \cdots < \gamma_1$ with $\iota = \tau(\gamma_n,\ldots,\gamma_1) \le \tau(\emptyset) < \gamma_n$. So $\iota \in \sigma(\gamma_n,\ldots,\gamma_1)$, and therefore $f_{\gamma_n,\ldots,\gamma_1} \in T_\iota$, i.e. $f_{\gamma_n,\ldots,\gamma_1}(\iota+1) < 1$. However, $\gamma_n > \iota$ implies that $f_{\gamma_n,\ldots,\gamma_1}(\iota+1) = 1$, a contradiction.

Let $j \in \{0, \ldots, n\}$ be minimal such that there are $\gamma_n \le \cdots \le \gamma_{j+1} < \omega_1$ with $\tau(\gamma_n, \ldots, \gamma_{j+1}) = \omega_1$, and let $\gamma_n, \ldots, \gamma_{j+1}$ be a choice of such ordinals. Due to our previous observation, such a j exists and it is positive.

As $\tau(\gamma_n, \ldots, \gamma_{j+1}) = \liminf_{\gamma \to \omega_1} \tau(\gamma_n, \ldots, \gamma_{j+1}, \gamma) = \omega_1$, we can find countable ordinals $\gamma_j^k > \gamma_{j+1}$, $k \in \mathbb{N}$, such that

$$\tau(\gamma_n, \ldots, \gamma_{j+1}, \gamma_j^1) < \tau(\gamma_n, \ldots, \gamma_{j+1}, \gamma_j^2) < \cdots$$

and we set $\gamma_j = \sup\{\gamma_j^k : k \in \mathbb{N}\}$.

We proceed inductively for $m = j, \ldots, 1$ to choose sequences of ordinals γ_m^k as follows. Let γ_m, γ_m^k, $k \in \mathbb{N}$, be already found for some $m \in \{j, \ldots, 2\}$ such that

$$\tau(\gamma_n, \ldots, \gamma_{j+1}, \gamma_j^1, \ldots, \gamma_m^1) < \tau(\gamma_n, \ldots, \gamma_{j+1}, \gamma_j^2, \ldots, \gamma_m^2) < \cdots.$$

As

$$\begin{aligned}\tau(\gamma_n, \ldots, \gamma_{j+1}, \gamma_j^{k+1}, \ldots, \gamma_m^{k+1}) &= \liminf_{\gamma \to \omega_1} \tau(\gamma_n, \ldots, \gamma_{j+1}, \gamma_j^{k+1}, \ldots, \gamma_m^{k+1}, \gamma) \\ &> \tau(\gamma_n, \ldots, \gamma_{j+1}, \gamma_j^k, \ldots, \gamma_m^k),\end{aligned}$$

the complement of the set

$$\begin{aligned}S_{k+1} = \{\gamma \in (\gamma_m, \omega_1) : \tau(\gamma_n, \ldots, \gamma_{j+1}, \gamma_j^{k+1}, \ldots, \gamma_m^{k+1}, \gamma) \\ > \tau(\gamma_n, \ldots, \gamma_{j+1}, \gamma_j^k, \ldots, \gamma_m^k)\}\end{aligned}$$

is countable for $k \in \mathbb{N}$.

Using (1) we see that the set

$$\begin{aligned}T_k = \{\gamma \in (\gamma_m, \omega_1) : \tau(\gamma_n, \ldots, \gamma_{j+1}, \gamma_j^k, \ldots, \gamma_m^k, \gamma) \\ \le \tau(\gamma_n, \ldots, \gamma_{j+1}, \gamma_j^k, \ldots, \gamma_m^k)\}\end{aligned}$$

does not have a countable upper bound for $k \in \mathbb{N}$.

We choose a $\gamma_{m-1}^1 \in T_1$ and $\gamma_{m-1}^k \in T_k \cap S_k$ for $k \ge 2$. We put $\gamma_{m-1} = \sup\{\gamma_{m-1}^k : k \in \mathbb{N}\}$. In this way we get

$$\tau(\gamma_n, \ldots, \gamma_{j+1}, \gamma_j^1, \ldots, \gamma_{m-1}^1) < \tau(\gamma_n, \ldots, \gamma_{j+1}, \gamma_j^2, \ldots, \gamma_{m-1}^2) < \cdots.$$

Summarizing, the ordinals γ_m^k, $k \in \mathbb{N}$, $m \in \{j, \ldots, 1\}$, satisfy

$$\gamma_j^1, \gamma_j^2, \ldots \le \gamma_j < \gamma_{j-1}^1, \gamma_{j-1}^2, \ldots \le \gamma_{j-1} < \cdots \le \gamma_2 < \gamma_1^1, \gamma_1^2, \ldots \le \gamma_1 \tag{2}$$

and

$$\tau(\gamma_n, \ldots, \gamma_{j+1}, \gamma_j^1, \ldots, \gamma_1^1) < \tau(\gamma_n, \ldots, \gamma_{j+1}, \gamma_j^2, \ldots, \gamma_1^2) < \cdots. \tag{3}$$

We define $g_k = f_{\gamma_n, \ldots, \gamma_{j+1}, \gamma_j^k, \ldots, \gamma_1^k}$ for $k \in \mathbb{N}$ and $g = f_{\gamma_n, \ldots, \gamma_{j+1}, \gamma_j, \ldots, \gamma_1}$. By (2) it is easy to check that $\|g_k - g\| \le 1/n < \delta$ for each $k \in \mathbb{N}$. Therefore $g \in \bigcap_{k \in \mathbb{N}} B_M(g_k, \delta)$. We claim that $\bigcup_{k \in \mathbb{N}} \sigma(\gamma_n, \ldots, \gamma_{j+1}, \gamma_j^k, \ldots, \gamma_1^k)$ is finite. Indeed, if $\iota = \iota(P)$ belongs to this union, then $B_M(g_k, \delta) \subset P$ for some $k \in \mathbb{N}$. Hence $g \in P$, and there are only finitely many such P's as $\mathcal{P}$ is point-finite.

By (3) the ordinals

$$\tau(\gamma_n,\dots,\gamma_{j+1},\gamma_j^k,\dots,\gamma_1^k)=\min\sigma(\gamma_n,\dots,\gamma_{j+1},\gamma_j^k,\dots,\gamma_1^k),\qquad k\in\mathbb{N},$$

are pairwise distinct, and therefore $\bigcup_{k\in\mathbb{N}}\sigma(\gamma_n,\dots,\gamma_{j+1},\gamma_j^k,\dots,\gamma_1^k)$ is infinite. This contradiction completes the proof.

4. Proof of Theorem 1.2. For a set Λ we denote by $[\Lambda]^n$ the set of all subsets of Λ of cardinality n, and by $[\Lambda]^{\le n}$ those of cardinality at most n. We consider these sets endowed with the natural topology inherited from $\{0,1\}^\Lambda$ (identifying each set with its characteristic function). The topological sum of topological spaces K_n, $n\in\mathbb{N}$, is denoted by $\bigoplus_{n\in\mathbb{N}}K_n$. The symbol α is used to denote the Aleksandrov one-point compactification as mentioned above.

Theorem 1.2 is an immediate consequence of the following theorem.

THEOREM 4.1. *If Λ is a set of sufficiently large cardinality, then the space $C(K)$, where $K=\alpha(\bigoplus_{n\in\mathbb{N}}[\Lambda]^{\le n})$, cannot be uniformly embedded into any $c_0(\Gamma)$.*

Before proving this theorem, we need some preparatory observations.

LEMMA 4.2. *Let M be a metric space, S be a subset of M, and ε and δ be positive numbers. Suppose that the cover of M by $\varepsilon/2$-balls admits a δ-uniform point-finite refinement. Then the cover of S by ε-balls also admits such a refinement.*

Proof. Let $\mathcal{P}$ be a point-finite δ-uniform refinement of the cover of M by $\varepsilon/2$-balls. The family $\mathcal{P}_S=\{P\cap S:P\in\mathcal{P}\}$ is clearly a point-finite cover of S. As $\mathcal{P}$ is δ-uniform, so obviously is $\mathcal{P}_S$. Further, if $P\cap S\in\mathcal{P}_S$, then there is $x\in M$ with $P\cap S\subset B(x,\varepsilon/2)$. Choose $y\in P\cap S$. Then clearly $B(x,\varepsilon/2)\subset B(y,\varepsilon)$, hence $P\cap S\subset B(y,\varepsilon)$, which completes the proof. ∎

LEMMA 4.3. *If the cardinal κ is sufficiently large, then for every set L, every $n\in\mathbb{N}$, and every mapping $\sigma:[[0,\kappa)]^n\to L$, either*

(a) *there are pairwise disjoint $w_k\in[[0,\kappa)]^n$ such that σ is constant on $\{w_k:k\in\mathbb{N}\}$, or*
(b) *there is $u\in[[0,\kappa)]^{n-1}$ such that $\{\sigma(u\cup\{e\}):e\in[0,\kappa)\setminus u\}$ is infinite.*

Proof. It follows in a straightforward way from Baumgartner's theorem [3, Theorem 1] that there is a cardinal number κ_0 such that whenever L, n and σ are as in the statement of the lemma, and $\kappa\ge\kappa_0$, then there is an uncountable set $A\subset[0,\kappa)$ and $\Delta\subset\{1,\dots,n\}$ such that

$$\forall\{a_1 < \cdots < a_n\} \in [A]^n \ \forall\{b_1 < \cdots < b_n\} \in [A]^n : \\ \sigma(\{a_1,\ldots,a_n\}) = \sigma(\{b_1,\ldots,b_n\}) \Leftrightarrow \forall i \in \Delta : a_i = b_i.$$

If $\Delta = \emptyset$, then σ is constant on $[A]^n$ and hence (a) is satisfied.

Suppose now that $\Delta \neq \emptyset$. As A is uncountable, we can find $a_1 < \cdots < a_n$ where $a_i \in A$ such that all the sets $[0,a_1) \cap A$, $(a_1,a_2) \cap A, \ldots, (a_n,\kappa) \cap A$ are infinite. Pick $i \in \Delta$ and set $u = \{a_1,\ldots,a_n\} \setminus \{a_i\}$. Then u clearly witnesses that (b) is satisfied. ■

From now on, we use Λ for a fixed set of cardinality κ from Lemma 4.3. Let $n \in \mathbb{N}$. Given $w \in [\Lambda]^n$ we define a function $f_w \in C([\Lambda]^{\leq n})$ by $f_w(s) = \operatorname{card}(s \cap w)/n$. Consider the metric subspace $M_n = \{f_w : w \in [\Lambda]^n\}$ of $C([\Lambda]^{\leq n})$.

Lemma 4.4. *Let $n \in \mathbb{N}$ and let*

$$\mathcal{T}_n = \{T^n_A : A \text{ a clopen subset of } [\Lambda]^{\leq n}\},$$

where

$$T^n_A = \{f \in M_n : f^{-1}(\{1\}) \subset A \subset f^{-1}((0,1])\}.$$

Then

(a) *$\mathcal{T}_n$ is an ε-uniform cover of M_n for every $\varepsilon \in (0,1/2)$.*
(b) $\sup\{\delta > 0 :$ *there exists a point-finite δ-uniform refinement $\mathcal{P}$ of $\mathcal{T}_n$*$\} \leq 1/n$.

Proof. (a) Choose an $\varepsilon \in (0,1/2)$. Given $f \in M_n$, we define $A_1 = \{s \in [\Lambda]^{\leq n} : f(s) \geq 1-\varepsilon\}$ and $A_0 = \{s \in [\Lambda]^{\leq n} : f(s) \leq \varepsilon\}$. Clearly, A_1 and A_0 are disjoint compact subsets of $[\Lambda]^{\leq n}$. As the range of f is a finite subset of $[0,1]$, the set $A = f^{-1}([1/2,1])$ is a clopen set such that $A_1 \subset A \subset [\Lambda]^{\leq n} \setminus A_0$. By a straightforward calculation we see that $B(f,\varepsilon) \subset T^n_A$.

(b) Suppose that $\mathcal{P}$ is a point-finite refinement of $\mathcal{T}_n$ and that $\{B(f,\delta) : f \in M_n\}$ refines $\mathcal{P}$ for some $\delta > 1/n$. For every $P \in \mathcal{P}$ we choose a clopen subset $A(P) \subset [\Lambda]^{\leq n}$ such that $P \subset T^n_{A(P)}$. Given $w \in [\Lambda]^n$ we define the family of sets $\sigma(w) = \{A(P) : B(f_w,\delta) \subset P,\ P \in \mathcal{P}\}$. As $\mathcal{P}$ is a point-finite δ-uniform cover of M_n, each $\sigma(w)$ is finite and nonempty.

By Lemma 4.3 one of the following possibilities holds.

(A) There are pairwise disjoint $w_k \in [\Lambda]^n$ with $\sigma(w_1) = \sigma(w_2) = \cdots$.
(B) There is a $u \in [\Lambda]^{n-1}$ and singletons $v_1, v_2, \ldots$ in $\Lambda \setminus u$ such that the families $\sigma(u \cup \{v_k\})$, $k \in \mathbb{N}$, are pairwise distinct.

Suppose that (A) holds true. Then $\lim_{k\to\infty} w_k = \emptyset \in [\Lambda]^{\leq n}$. We may choose an $A \in \sigma(w_1) = \bigcap_{k\in\mathbb{N}} \sigma(w_k)$. As $f_{w_k}(w_k) = 1$ the sets w_k belong to A. Since $\lim_{k\to\infty} w_k = \emptyset$ and A is closed, $\emptyset \in A$. As $A \in \sigma(w_1)$, $B(f_{w_1},\delta) \subset T^n_A$. In particular, $f_{w_1}(\emptyset) > 0$, which contradicts the definition of f_{w_k}.

Suppose that (B) occurs. We define $g_k = f_{u \cup \{v_k\}}$. As the sets $\sigma(u \cup \{v_k\})$ are pairwise distinct, the family $\bigcup_{k \in \mathbb{N}} \sigma(u \cup \{v_k\})$ is infinite. For every $A \in \bigcup_{k \in \mathbb{N}} \sigma(u \cup \{v_k\})$ there is a $P \in \mathcal{P}$ with $A = A(P)$ and $B(g_k, \delta) \subset P$. Note that

$$|g_k(s) - g_{k'}(s)| = \frac{|\mathrm{card}(s \cap (u \cup \{v_k\})) - \mathrm{card}(s \cap (u \cup \{v_{k'}\})|}{n} = \frac{|\mathrm{card}(s \cap \{v_k\}) - \mathrm{card}(s \cap \{v_{k'}\})|}{n} \le \frac{1}{n}.$$

Hence $g_1 \in P$. However, there are only finitely many such P's, which contradicts the observation that $\bigcup_{k \in \mathbb{N}} \sigma(u \cup \{v_k\})$ is infinite. ∎

Now we can prove Theorem 4.1.

Proof of Theorem 4.1. Suppose that $X = C(\alpha(\bigoplus_{n \in \mathbb{N}} [\Lambda]^{\le n}))$ can be uniformly embedded into $c_0(\Gamma)$. By Theorem 2.1, X has the uniform Stone property. Let $\delta > 0$ be such that the cover of X by 1/8-balls has a point-finite δ-uniform refinement. Note that each $X_n = C([\Lambda]^{\le n})$, and so also each M_n, is isometric to a subset of X. Hence, by Lemma 4.2, the cover of M_n by 1/4-balls admits a point-finite δ-uniform refinement. Thus, by Lemma 4.4(a), $\mathcal{T}_n$ has a point-finite δ-uniform refinement and therefore $\delta \le 1/n$ by Lemma 4.4(b). As $n \in \mathbb{N}$ is arbitrary, this is a contradiction. ∎

5. Final remarks and open problems. The following problem is open.

QUESTION 1. *Is there a compact space K such that $K^{(\omega_0+1)} = \emptyset$, the cardinality of K is ω_1, and $C(K)$ is not uniformly homeomorphic to any subset of $c_0(\Gamma)$?*

By the results mentioned in the introduction it is the best one can expect. Note that the space $[0, \omega_1]$ from Theorem 1.1 has cardinality ω_1 but its height is $\omega_1 + 1$. On the other hand, the space K from Theorem 1.2 (Theorem 4.1) has the smallest possible height $\omega_0 + 1$ but its cardinality is quite large due to the use of Baumgartner's theorem. However, we do not know the answer to the following question.

QUESTION 2. *Let $K = \alpha(\bigoplus_{n \in \mathbb{N}} [[0, \omega_1)]^{\le n})$. Is $C(K)$ uniformly homeomorphic to a subset of $c_0(\Gamma)$?*

A negative answer to this question would yield a positive answer to the first one. As the only place where the largeness of the cardinality of Λ is used is Lemma 4.3, a negative answer to the previous question would follow from a positive answer to the following one.

QUESTION 3. *Let Λ be an uncountable set, $n \in \mathbb{N}$, L an arbitrary set and $\sigma : [\Lambda]^n \to L$ a mapping. Must one of the following conditions hold?*

(a) *There are pairwise disjoint $w_k \in [\Lambda]^n$ such that σ is constant on $\{w_k : k \in \mathbb{N}\}$.*
(b) *There is $u \in [\Lambda]^{n-1}$ such that $\{\sigma(u \cup \{e\}) : e \in \Lambda \setminus u\}$ is infinite.*

Our proof of Lemma 4.3 uses Baumgartner's theorem, which requires a large cardinality (see [3, Corollary 2 and following remarks]). However, it seems that we do not use the whole strength of Baumgartner's theorem. This suggests the question whether the result can be proved in an elementary way. It is not hard to prove that the answer is positive for $n = 2$. But we do not know how to attack the general case.

References

[1] I. Aharoni, *Every separable metric space is Lipschitz equivalent to a subset of c_0^+*, Israel J. Math. 19 (1974), 284–291.

[2] I. Aharoni and J. Lindenstrauss, *Uniform equivalence between Banach spaces*, Bull. Amer. Math. Soc. 84 (1978), 281–283.

[3] J. E. Baumgartner, *Canonical partition relations*, J. Symbolic Logic 40 (1975), 541–554.

[4] R. Deville, G. Godefroy and V. E. Zizler, *The three space problem for smooth partitions of unity and $C(K)$ spaces*, Math. Ann. 288 (1990), 613–625.

[5] P. Enflo, *Uniform structures and square roots in topological groups II*, Israel J. Math. 8 (1970), 253–272.

[6] G. Godefroy, N. Kalton, and G. Lancien, *Subspaces of $c_0(\mathbb{N})$ and Lipschitz isomorphisms*, Geom. Funct. Anal. 10 (2000), 798–820.

[7] —, —, —, *Szlenk indices and uniform homeomorphisms*, Trans. Amer. Math. Soc. 353 (2001), 3895–3918.

[8] J. R. Isbell, *Uniform Spaces*, Math. Surveys 12, Amer. Math. Soc., Providence, 1964.

[9] W. B. Johnson, J. Lindenstrauss, and G. Schechtman, *Banach spaces determined by their uniform structures*, Geom. Funct. Anal. 6 (1996), 430–470.

[10] M. I. Kadec, *A proof of the topological equivalence of all separable infinite-dimensional Banach spaces*, Funktsional. Anal. i Prilozhen. (1967), no. 1, 61–70 (in Russian).

[11] W. Marciszewski, *On Banach spaces $C(K)$ isomorphic to $c_0(\Gamma)$*, Studia Math. 156 (2003), 295–302.

[12] J. Pelant, *Cardinal reflections and point-character of uniformities—counterexamples*, in: Seminar Uniform Spaces (Prague, 1973–1974), Math. Inst. Czechoslovak Acad. Sci., Prague, 1975, 149–158.

[13] —, *Embeddings into c_0*, Topology Appl. 57 (1994), 259–269.

[14] J. Pelant and V. Rödl, *On coverings of infinite-dimensional metric spaces*, Discrete Math. 108 (1992), 75–81.

[15] M. Ribe, *Existence of separable uniformly homeomorphic nonisomorphic Banach spaces*, Israel J. Math. 48 (1984), 139–147.

[16] E. V. Shchepin, *On a problem of Isbell*, Dokl. Akad. Nauk SSSR 222 (1976), 541–543 (in Russian).

[17] H. Toruńczyk, *Characterizing Hilbert space topology*, Fund. Math. 111 (1981), 247–262.

Department of Mathematical Analysis
Faculty of Mathematics and Physics
Charles University
Sokolovská 83
186 75 Praha 8, Czech Republic
E-mail: holicky@karlin.mff.cuni.cz
kalenda@karlin.mff.cuni.cz

Received 3 August 2005;
in revised form 5 April 2006