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On complexification and iteration of quasiregular polynomials which have algebraic degree two

by

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Abstract. We prove that each degree two quasiregular polynomial is conjugate to $Q(z) = z^2 - (p+q)|z|^2 + pq\overline{z}^2 + c$, $|p| < 1$, $|q| < 1$. We also show that the complexification of Q can be extended to a polynomial endomorphism of $\mathbb{CP}^2$ which acts as a Blaschke product $\frac{z-p}{1-\overline{p}z} \cdot \frac{z-q}{1-\overline{q}z}$ on $\mathbb{CP}^2 \setminus \mathbb{C}^2$. Using this fact we study the dynamics of Q under iteration.

1. Introduction. The results of [Li1] imply that quasiregular polynomials of algebraic degree two can live only on $\mathbb{R}^2$. We shall identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Remark 3.6 and Proposition 3.7 of [Li1] show that each quasiregular polynomial mapping $\mathbb{C} \to \mathbb{C}$ of algebraic degree two is conjugate via a holomorphic affine map to

$$Q(z) = z^2 + a|z|^2 + b\overline{z}^2 + c.$$

The quasiregularity is equivalent to the condition

$$1 - |b|^2 > |a|\left|b - \frac{a}{\overline{a}}\right|.$$

(There is a sign error in [Li1].)

In the first section we shall give a detailed proof of the above facts and describe some elementary properties of quasiregular polynomials of degree two. In particular we shall prove that we can always put $a = -(p+q)$, $b = pq$, where $|p| < 1$ and $|q| < 1$.

In Section 2 we shall complexify the mapping $Q(z)$ in the same manner as in [Li2]. We shall prove that the complexified map $f : \mathbb{C}^2 \to \mathbb{C}^2$ extends to a regular mapping $f : \mathbb{CP}^2 \to \mathbb{CP}^2$ of complex projective space. It turns

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out that $f|\mathbb{CP}^2 \setminus \mathbb{C}^2$ is exactly the Blaschke product

$$\phi(\xi) = \frac{\xi - p}{1 - \overline{p}\xi} \cdot \frac{\xi - q}{1 - \overline{q}\xi}.$$

The dynamics of Blaschke products is well known (see [C-G, pp. 79 and 107–108], [H], [Sh-Su]). We shall use results concerning Blaschke products together with the results of Hubbard–Papadopol [H-P] and Bedford–Jonsson [B-J1,2] about polynomial mappings of $\mathbb{CP}^2$ to study the dynamics of quasiregular polynomials of algebraic degree two.

In the homogeneous case we shall describe the dynamics of Q on its Julia set $\mathcal{J}_Q$. We shall also outline the proof of the existence of the Böttcher coordinate near infinity in the case when $Q(z)$ is nonhomogeneous and $\phi(\xi)$ has a fixed point inside the unit disc.

Generally speaking, we hope that the complexification of real-analytic mappings of $\mathbb{C}$ (as in [Li2]) can lead to important applications of holomorphic dynamics in $\mathbb{C}^2$ (or $\mathbb{CP}^2$). We hope that the paper [Li2] and the present study of quasiconformal polynomials of degree two are just the first examples of such approach.

In the present paper we shall use the definitions and notation from [Li1].

2. Preliminaries. Any homogeneous polynomial of algebraic degree two can be written as

$$P(z) = \alpha z^2 + \beta|z|^2 + \gamma\overline{z}^2.$$

If $\alpha = 0$, then $P(z)$ cannot be quasiregular. If $\alpha \neq 0$, then $P(z)$ is conjugate via a linear holomorphic map to

$$Q(z) = z^2 + a|z|^2 + b.$$

We have the following

Proposition 2.1. *The following conditions are equivalent*:

(a) $Q(z)$ *is quasiregular*;
(b) $1 - |b|^2 > |a|\,|b - a/\overline{a}|$;
(c) $Q(z) = (z - p\overline{z})(z - q\overline{z})$, $|p| < 1$, $|q| < 1$. *Hence* $a = -(p + q)$ *and* $b = pq$.

If Q is quasiregular then it has <u>*topological*</u> *degree two.*

Proof. (a)⇔(b). Recall that $Q(z)$ is quasiregular iff

$$\left|\frac{\partial Q/\partial\overline{z}}{\partial Q/\partial z}\right| < k < 1$$

a.e. on $\mathbb{C}$ (see [Li1] for further details). We have

$$\frac{\partial Q/\partial\overline{z}}{\partial Q/\partial z}(z) = \frac{2b\overline{z} + az}{2z + a\overline{z}} = \frac{2b + az/\overline{z}}{2z/\overline{z} + a} = \frac{2b + a\xi}{2\xi + a}$$

where $z \neq 0$, $\xi = z/\bar{z}$. It suffices to prove that

$$\left|\frac{2b + a\xi}{2\xi + a}\right| < 1 \quad \text{if } |\xi| = 1.$$

This is equivalent to

$$\begin{aligned} |2b + a\xi|^2 < |2\xi + a|^2 \;&\Leftrightarrow\; 4|b|^2 + |a|^2 + 4\Re a\overline{b}\xi < 4 + |a|^2 + 4\Re\overline{a}\xi \\ &\Leftrightarrow\; 1 - |b|^2 > \Re(a\overline{b}\xi - \overline{a}\xi). \end{aligned}$$

The right side has maximal value for $\xi = \frac{\overline{a}b - a}{|\overline{a}b - a|}$. This value is equal to

$$|a\overline{b} - \overline{a}| = |\overline{a}b - a|.$$

Hence we get the condition

$$1 - |b|^2 > |\overline{a}b - a| = |a|\left|b - \frac{a}{\overline{a}}\right|$$

for $a \neq 0$, and $|b| < 1$ if $a = 0$.

(a)$\Leftrightarrow$(c). Consider the polynomial $w(s) = s^2 + as + b$. Let p and q be its roots. For $z \neq 0$ we have

$$\begin{aligned} Q(z) &= \bar{z}^2\left(\left(\frac{z}{\bar{z}}\right)^2 + a\,\frac{z}{\bar{z}} + b\right) \\ &= \bar{z}^2 w\left(\frac{z}{\bar{z}}\right) = \bar{z}^2\left(\frac{z}{\bar{z}} - p\right)\left(\frac{z}{\bar{z}} - q\right) = (z - p\bar{z})(z - q\bar{z}). \end{aligned}$$

This implies that if $|p| = 1$ (resp. $|q| = 1$), then $Q(z)$ vanishes along the line $\{z : 2\arg z = \arg p\}$ (resp. $\{z : 2\arg z = \arg q\}$). Hence Q cannot be quasiregular (see [Li1]). If $|p| < 1$, then the argument of $z - p\bar{z}$ increases with $\arg z$. As z makes one turn counterclockwise around zero, the argument of $z - p\bar{z}$ increases by 2π. If $|p| > 1$, the argument of $z - p\bar{z}$ decreases with $\arg z$, and after the whole round counterclockwise it decreases by 2π. If either p or q has a modulus greater than 1 then $\arg(Q(z))$ changes by 0 or -4π after one turn counterclockwise. But $|Q(z)|$ increases with $|z|$ if $|z|$ is sufficiently large. Hence Q is not orientation-preserving and thus cannot be quasiregular.

Finally, if $|p| < 1$ and $|q| < 1$, then $|Q(z)|$ increases with $|z|$ and $\arg Q(z)$ strictly increases with $\arg z$. Hence $Q(z)$ is orientation-preserving, which implies that

$$\psi(z) = \left|\frac{\partial Q/\partial\bar{z}}{\partial Q/\partial z}\right| < 1.$$

Since Q is homogeneous, the function $\psi(z)$ depends only on $\arg z$. Hence

$$\psi(z) \leq \sup_{|z|=1} \psi(z) = k < 1$$

and $Q(z)$ is quasiregular.

The argument of $Q(z)$ increases by 4π if z makes one turn around zero counterclockwise. Thus Q has topological degree two. ∎

Let now $P(z)$ be any quasiregular polynomial of algebraic degree two.

PROPOSITION 2.2. *The polynomial P is conjugate via some holomorphic affine map to a polynomial $Q(z) = z^2 + a|z|^2 + b\overline{z}^2 + c$ where $1 - |b|^2 > |a||b - a/\overline{a}|$ or $a = 0$ and $|b| < 1$. Proposition* 2.1 *implies that $Q(z)$ is also quasiregular.*

Proof. It follows from the results of [Li1] that

$$P(z) = P_2(z) + P_1(z) + c_0$$

where $P_2(z)$ is a quasiregular homogeneous polynomial of algebraic degree 2, $P_1(z)$ is an $\mathbb{R}$-linear quasiconformal mapping and c_0 is a constant.

The Goursat theorem implies that the topological degree of P is the same as that of P_2. By Proposition 2.1, $P_2(z)$ has topological degree two.

Each quasiregular function on $\mathbb{C}$ can be written as $g(z) = f(h(z))$ where f is entire and $h(z)$ is a quasiconformal homeomorphism of $\mathbb{C}$ such that $h(\infty) = \infty$, $h(0) = 0$, $h(1) = 1$. (This is a direct consequence of the solvability of the Beltrami equation.)

Since $P(z)$ has topological degree two, we have

$$P(z) = w(h(z))$$

where h is as above and w is a holomorphic polynomial of degree 2.

The polynomial w is conjugate via some holomorphic affine map ϕ to $w_1(z) = z^2 + c$, $w = \phi^{-1}w_1\phi$. We have

$$P(z) = \phi^{-1}w_1\phi h(z),$$

where conjugating $P(z)$ via ϕ we have

$$\widetilde{P}(z) = \phi P(\phi^{-1}(z)) = w_1(\phi h \phi^{-1}(z)),$$

hence

$$\widetilde{P}(z) = w_1(h_1(z)), \qquad h_1 = \phi h \phi^{-1}.$$

Conjugating (if needed) via translation we can assume that $h_1(0) = 0$. We have

$$\widetilde{P}(z) = (h_1(z))^2 + c, \qquad h_1(0) = 0.$$

The polynomial $\widetilde{P}$ cannot contain terms of order 1. Indeed, the existence of such terms implies that both $\widetilde{P}$ and h_1 are diffeomorphisms in the neighbourhood of zero. However we have

$$\frac{\partial \widetilde{P}}{\partial z}(0) = 2h_1(0)\frac{\partial h_1}{\partial z}(0) = 0.$$

Contradiction.

Hence $\widetilde{P}(z)$ is conjugate to a map $z^2 + a|z|^2 + b\overline{z}^2 + c$. ∎

REMARK 2.3. The mapping $h_1(z)$ can be described explicitly. If

$$P(z) = z^2 - (p+q)|z|^2 + pq\overline{z}^2 + c = (h(z))^2 + c,$$

then $(1^{1/2} = 1)$

$$h(z) = z\left(1 - p\,\frac{\overline{z}}{z}\right)^{1/2}\left(1 - q\,\frac{\overline{z}}{z}\right)^{1/2} = z\left(1 + \sum_{k=1}^{\infty} a_k p^k\,\frac{\overline{z}^k}{z}\right)\left(1 + \sum_{k=1}^{\infty} q^k\,\frac{\overline{z}^k}{z^k}\right)$$

where $(1+x)^{1/2} = 1 + \sum_{k=1}^{\infty} a_k x^k$. Hence

$$h(x) = z\left(1 + \frac{p+q}{2}\,\frac{\overline{z}}{z} + \cdots\right).$$

In order to study the behaviour of

$$Q(z) = z^2 - (p+q)|z|^2 + pq\overline{z}^2 + c, \qquad |p| < 1,\ |q| < 1,$$

in a neighbourhood of ∞ we conjugate it via $\psi(z) = 1/z$. We have

$$\begin{aligned}\psi^{-1} \circ Q \circ \psi(z) &= \frac{z^2}{(1 - pz/\overline{z})(1 - qz/\overline{z}) + cz^2} \\ &= \frac{z^2}{(1 - pz/\overline{z})(1 - qz/\overline{z})} \sum_{k=0}^{\infty}\left(\frac{cz^2}{(1 - pz/\overline{z})(1 - qz/\overline{z})} \cdot (-1)\right)^k.\end{aligned}$$

If we put

$$F(z) = \frac{z^2}{(1 - pz/\overline{z})(1 - qz/\overline{z})},$$

we get

$$\psi^{-1} \circ Q \circ \psi(z) = F(z) + \sum_{k=1}^{\infty} (-1)^k c^k (F(z))^{k+1}.$$

PROBLEM 2.4. *Is $\psi^{-1} \circ Q \circ \psi(z)$ conjugate to $F(z)$ in some neighbourhood of zero?*

In the homogeneous case, when $c = 0$, the following is true.

PROPOSITION 2.5. *There exists a continuous and $\mathbb{R}$-homogeneous function $\phi : \mathbb{C} \to \mathbb{C}$ for which $\phi(F(z)) = (\phi(z))^2$ and*

$$\frac{|z|}{(1+|p|)(1+q)} \le |\phi(z)| \le \frac{|z|}{(1-|p|)(1-|q|)}.$$

Proof. The proof is the same as the proof of Theorem 4.1 in Chapter II of Carleson and Gamelin's book [C-G]. We have

$$F^{\circ n}(z) = \frac{z^{2^n}}{\left[\left(1 - p\frac{z}{\overline{z}}\right)\left(1 - q\frac{z}{\overline{z}}\right)\right]^{2^{n-1}} \cdots \left[\left(1 - p\frac{F^{\circ(n-1)}(z)}{\overline{F^{\circ(n-1)}(z)}}\right)\left(1 - q\frac{F^{\circ(n-1)}(z)}{\overline{F^{\circ(n-1)}(z)}}\right)\right]},$$

hence

$$\frac{|z|^{2^n}}{[(1+|p|)(1+|q|)]^{2^n-1}} \le |F^{\circ n}(z)| \le \frac{|z|^{2^n}}{[(1-|q|)(1-|p|)]^{2^n-1}}.$$

Put

$$\begin{aligned}\phi_n(z) &= (F^{\circ n}(z))^{1/2^n} \\ &= \frac{z}{\left[\left(1-p\frac{z}{\overline{z}}\right)\left(1-q\frac{z}{\overline{z}}\right)\right]^{1/2}\cdots\left[\left(1-p\frac{F^{\circ(n-1)}(z)}{\overline{F^{\circ(n-1)}(z)}}\right)\left(1-q\frac{F^{\circ(n-1)}(z)}{\overline{F^{\circ(n-1)}(z)}}\right)\right]}.\end{aligned}$$

We have

$$\frac{\phi_{n+1}}{\phi_n} = \left(\frac{\phi_1\circ F^{\circ n}}{F^{\circ n}}\right)^{1/2} = \left(\frac{1}{\left(1-p\frac{F^{\circ n}}{\overline{F^{\circ n}}}\right)\left(1-q\frac{F^{\circ n}}{\overline{F^{\circ n}}}\right)}\right)^{1/2^n}$$

and

$$\begin{aligned}\left|1-\frac{\phi_{n+1}}{\phi_n}\right| &= \left|\frac{1-\left[\left(1-p\frac{F^{\circ n}}{\overline{F^{\circ n}}}\right)\left(1-q\frac{F^{\circ n}}{\overline{F^{\circ n}}}\right)\right]^{1/2^{n+1}}}{\left[\left(1-p\frac{F^{\circ n}}{\overline{F^{\circ n}}}\right)\left(1-q\frac{F^{\circ n}}{\overline{F^{\circ n}}}\right)\right]^{1/2^{n+1}}}\right| \\ &\le \frac{1-[(1-|p|)(1-|q|)]^{1/2^{n+1}}}{(1-|p|)(1-|q|)^{1/2^{n+1}}} \\ &\le \frac{1-(1-|p|)(1-|q|)}{1+[(1-|p|)(1-|q|)]^{1/2^{n+1}}+\cdots+[(1-|p|)(1-|q|)]^{(2^{n+1}-1)/2^{n+1}}} \\ &\quad \times \frac{1}{[(1-|q|)(1-|p|)]^{1/2^{n+1}}} \\ &\le \frac{1-(1-|p|)(1-|q|)}{2^{n+1}(1-|p|)(1-|q|)}.\end{aligned}$$

This estimate implies that the infinite product

$$\phi = \prod_{n=1}^{\infty}\frac{\phi_{n+1}}{\phi_n}$$

converges uniformly to a continuous function on $\mathbb{C}$. We have

$$\frac{|z|}{(1+|q|)(1+|p|)} \le |\phi(z)| \le \frac{|z|}{(1-|q|)(1-|p|)}.$$

By the very construction of ϕ,

$$\phi(z) = z\cdot\psi\left(\frac{z}{\overline{z}}\right).$$

Thus for $t\in\mathbb{R}$,

$$\phi(tz) = tz\psi\left(\frac{tz}{t\overline{z}}\right) = tz\psi\left(\frac{z}{\overline{z}}\right) = t\phi(z)$$

and ϕ is 1-homogeneous (over $\mathbb{R}$).

Generally speaking, ϕ need not be univalent. If F has more than one fixed point different from zero, that is, $F(z_1) = z_1$, $F(z_2) = z_2$, $z_1 \neq z_2$, $z_1 \cdot z_2 \neq 0$, then $\phi(z_1) = (\phi(z_1))^2$ and $\phi(z_2) = (\phi(z_2))^2$, which implies $\phi(z_1) = \phi(z_2) = 1$. ∎

Now let again $\psi(z) = 1/z$. The mapping

$$G(z) = \psi^{-1} \circ \phi \circ \psi(z) = \psi \circ \phi \circ \psi(z)$$

is continuous and $\mathbb{R}$-homogeneous on $\mathbb{C}$. Proposition 2.5 implies immediately:

PROPOSITION 2.6. *There exists a continuous and $\mathbb{R}$-homogeneous function $G : \mathbb{C} \to \mathbb{C}$ for which $G(Q(z)) = (G(z))^2$ and*

$$|z|(1 - |p|)(1 - |q|) \leq |G(z)| \leq |z|(1 + |p|)(1 + |q|).$$

PROBLEM 2.7. *Characterize F (or Q) for which ϕ (or G) is a homeomorphism.*

REMARK 2.8. Problem 2.7 seems easy, but it is only an illusion. The main obstacle is that in the nonholomorphic case we have no control over the derivatives of a convergent sequence of mappings. The quasiregularity does not help here because the distortion tends to infinity and no normal family arguments are possible. We think that the complexification is the only thing that can be useful.

We end this section with the following two facts:

PROPOSITION 2.9. *Let $Q(z) = z^2 + a|z|^2 + b\overline{z}^2 + c$ be a quasiregular mapping. There exists $R > 0$ such that $Q(z)$ is uniformly expanding on the set $\{z : |z| > R\}$.*

Proof. We need to prove that for large $|z|$ both eigenvalues of the Jacobi matrix of Q have modulus greater than one. The Jacobi matrix

$$\mathcal{D}Q(z) = \begin{bmatrix} 2z + a\overline{z} & 2b\overline{z} + az \\ 2\overline{b}z + \overline{a}\,\overline{z} & 2\overline{z} + \overline{a}z \end{bmatrix}$$

has characteristic polynomial

$$w(\lambda) = \lambda^2 - 2\Re(2z + a\overline{z}) + |2z + a\overline{z}|^2 - |2\overline{b}z + az|^2.$$

Its roots are

$$\lambda_1(z) = \Re(2z + a\overline{z}) + \sqrt{(\Re(2z + a\overline{z}))^2 - |2z + a\overline{z}|^2 + |2b\overline{z} + az|^2},$$
$$\lambda_2(z) = \Re(2z + a\overline{z}) - \sqrt{(\Re(2z + a\overline{z}))^2 - |2z + a\overline{z}|^2 + |2b\overline{z} + az|^2}.$$

We have

$$\lambda_1(z) = |z|\lambda_1(z/|z|), \qquad \lambda_2(z) = |z|\lambda_2(z/|z|).$$

Since Q is quasiregular, it follows that $0 = \lambda_1(z)\cdot\lambda_2(z)$ iff $z = 0$. Hence

$$k = \inf_{|z|=1} \min(|\lambda_1(z)|, |\lambda_2(z)|) > 0.$$

Thus if $|z| > 1/k$, then $|\lambda_1(z)| > 1$ and $|\lambda_2(z)| > 1$. ■

PROPOSITION 2.10. *Let $Q(z) = z^2-(p+q)|z|^2+pq\overline{z}^2+c$, $|p| < 1$, $|q| < 1$, be a quasiregular polynomial. For every $k > 1$ there exists $R > 0$ such that if $|z| > R$, then $|Q(z)| > k|z|$. This implies that ∞ is a superattracting fixed point for Q.*

Proof. We have

$$Q(z) = (z - p\overline{z})(z - q\overline{z}) + c.$$

Put

$$R = \frac{|c| + k}{(1 - |p|)(1 - |q|)}.$$

If $|z| > R$, then

$$|(z - p\overline{z})(z - q\overline{z})| \geq |z|^2(1 - |p|)(1 - |q|) > |z|(1 - |p|)(1 - |q|) > |c| + k.$$

Hence ($|z| > R > 1$)

$$\begin{aligned}|Q(z)| &\geq |z|^2(1 - |p|)(1 - |q|) - |c| \\ &> |z|(|c| + k) - |c| = k|z| + (|z| - 1)|c| > k|z|. \ ■\end{aligned}$$

3. The complexification. Let $Q(z) = z^2+a|z|^2+b\overline{z}^2+c$ be a quasiregular polynomial. We shall now complexify Q in the same manner as in [Li2].

Define

$$f(z,w) = (z^2 + azw + bw^2 + c, w^2 + \overline{a}zw + \overline{b}z^2 + \overline{c}).$$

Let $H = \{(z,w) \in \mathbb{C}^2 : w = \overline{z}\}$. We have

$$f(H) = H, \qquad f(z,\overline{z}) = (Q(z), \overline{Q(z)}).$$

THEOREM 3.1. *The function $f(z,w)$ extends to a holomorphic endomorphism of $\mathbb{CP}^2$, the complex two-dimensional projective space.*

Proof. The quasiregularity of Q implies (by Proposition 2.1) that the polynomial $\xi^2 + a\xi + b$ has roots with modulus less than one.

The mapping f extends to $\mathbb{CP}^2$ iff

(∗) The conditions $z^2 + azw + bw^2 = 0$ and $\overline{b}z^2 + \overline{a}zw + w^2 = 0$ imply that $z = 0$ and $w = 0$.

Suppose this is not true; then there exist $w \neq 0$ and $z \neq 0$ for which the polynomials in (∗) vanish. Let $\xi = z/w$. Then ξ and $1/\overline{\xi}$ are the roots of the polynomial $s^2 + as + b$, $s \in \mathbb{C}$. Since $|\xi| \neq 1$, we have $\xi \neq 1/\overline{\xi}$. Hence

$$a = -\left(\xi + \frac{1}{\overline{\xi}}\right), \qquad b = \frac{\xi}{\overline{\xi}}.$$

Now the mapping $\overline{\xi}(Q-c)$ is quasiregular and nonconstant. We have

$$\overline{\xi}(Q-c)(z)=\overline{\xi}z^2-(|\xi|^2+1)|z|^2+\xi\overline{z}^2.$$

But this last mapping has only real values. Contradiction. ∎

REMARK 3.2. Theorem 3.1 is not valid for quasiregular polynomials of degree greater than two. We can take for example $Q(z)=z|z|^2$. In this case

$$f(z,w)=(z^2w,w^2z)=0$$

if only $zw=0$.

We shall now describe the behaviour of f on the set $\pi=\mathbb{CP}^2\setminus\mathbb{C}^2$. Let $\widetilde{f}$ denote the homogenization of f. The mapping $\widetilde{f}$ acts on $\mathbb{C}^3\setminus\{0\}$ and is equal to

$$\widetilde{f}(z,w,t)=(z^2+awz+bw^2+ct^2,\overline{b}z^2+\overline{a}zw+w^2+t^2\overline{c},t^2).$$

If $t=0$ and $z/w=\xi$, then we get the mapping

$$\phi(z)=\frac{\xi^2+a\xi+b}{\overline{b}\xi^2+\overline{a}\xi+1}=\frac{(\xi-p)(\xi-q)}{(1-\overline{p}\xi)(1-\overline{q}\xi)}$$

where $a=-(p+q)$, $b=pq$. If $w=\overline{z}$, then $|\xi|=1$. We have

$$\arg\frac{z}{\overline{z}}=\arg\xi=2\arg z.$$

Note that the unit circle is a complete invariant set for ϕ (see the proof of Proposition 2.2). The Julia set $\mathcal{J}_\phi$ is contained in the unit circle. If $Q(z)=z^2+c$, then $f(z,w)=(z^2+c,w^2+\overline{c})$ and $\phi(\xi)=\xi^2$.

This simple example gives us an opportunity to compare our method of complexification to the standard one. We can write

$$Q(z)=Q(x,y)=(x^2-y^2+c_1,2xy+c_2)$$

where $z=x+iy$ and $c=c_1+ic_2$, $c_1,c_2\in\mathbb{R}$. The standard complexification of Q is

$$F(z,w)=(z^2-w^2+c_1,2zw+c_2).$$

The mapping $F(z,w)$ extends to $\mathbb{CP}^2$. Its restriction to $\mathbb{CP}^2\setminus\mathbb{C}^2$ has the form

$$\psi(\xi)=\frac{1}{2}\left(\xi-\frac{1}{\xi}\right).$$

The mappings $\phi(\xi)$ and $\psi(\xi)$ are of course conjugate via the homography

$$h(\xi)=\frac{\xi-i}{\xi+i}.$$

Generally speaking, if $Q(z)$ is some real-analytic polynomial mapping on $\mathbb{R}^2$, $f(z,w)$ is its complexification by our method and $F(z,w)$ is its standard complexification, then $f(z,w)$ and $F(z,w)$ are conjugate via a $\mathbb{C}$-linear isomorphism of $\mathbb{C}^2$, and either both F and f are extendable to $\mathbb{CP}^2$ or neither of them is. In the former case the actions of F and f on $\mathbb{CP}^2 \setminus \mathbb{C}^2$ are conjugate via a homography.

However, our method of complexification seems to be more straightforward and often leads to simpler mappings (as in the above example).

Let us describe the dynamics of $\phi(\xi)$, according to [C-G, p. 79 and Example on pp. 107–108].

There are four cases possible:

Case (1): The mapping $\phi(\xi)$ has an attracting point in the interior of the unit disc. In this case the Julia set of ϕ is equal to the unit circle and the following holds:

Proposition 3.3. *The mapping $\phi(\xi)$ is conjugate via a Möbius map to*

$$\psi(\xi) = \xi \cdot \frac{\xi - p_1}{1 - p_1\xi}, \qquad |p_1| < 1.$$

The map ψ is uniformly expanding on its Julia set $\mathcal{J}_\psi = C(0,1)$, the unit circle. The polynomial $Q(z)$ is conjugate via an $\mathbb{R}$-linear quasiconformal map to $Q_1(z) = z^2 + a|z|^2 + c_1$, $|a| < 1$.

Proof. Let $a \in B(0,1)$, the unit disc, be such that $\phi(a) = a$. Let $h_a(\xi) = (\xi - a)/(1 - \overline{a}\xi)$. Then

$$h_a \circ \phi \circ h_a^{-1}(\xi) = e^{i\theta}\xi - \frac{\xi - p_0}{1 - \overline{p_0}\xi} =: \phi_1(\xi).$$

Put $g(\xi) = e^{i\theta}\xi$. We have

$$\psi(\xi) = g \circ \phi_1 \circ g^{-1}(\xi) = \xi \cdot \frac{\xi - p_1}{1 - p_1\xi}, \qquad |p_1| < 1.$$

The fact that ψ is uniformly expanding in $C(0,1)$ follows from the result of Tischler [T, condition (iii)]. It can also be proved via an immediate calculation of the derivative of ψ at $z/\overline{z}$, $z \neq 0$. If g is an $\mathbb{R}$-linear quasiconformal map, then it can be written as $g(z,\overline{z}) = a(z - b\overline{z})$ where $|b| < 1$, $a \neq 0$.

Let us complexify g as in the proof of Theorem 3.1. We obtain the mapping

$$\widetilde{g}(z,w) = (a(z - bw), \overline{a}(w - \overline{b}z))$$

which extends to $\mathbb{CP}^2$. It acts on $\mathbb{CP}^2 \setminus \mathbb{C}^2$ as

$$h_g(\xi) = \frac{a}{\overline{a}}\,\frac{\xi - b}{1 - \overline{b}\xi}.$$

The mapping $g \mapsto h_g$ is a group homomorphism. (In fact one can prove that the automorphism group of the unit disc is isomorphic to the group of $\mathbb{R}$-linear quasiconformal mappings of $\mathbb{C}$ divided by the equivalence relation $g_1 \sim g_2 \Leftrightarrow \exists t \in \mathbb{R} \setminus \{0\} : g_1 = tg_2$.)

Hence, if ϕ_1 and ϕ_2 are conjugate via a Möbius map, then $Q_1 - c_1$ and $Q_2 - c_2$ are conjugate via a quasiconformal $\mathbb{R}$-linear map. This proves Proposition 3.3. ∎

Case (2): The mapping ϕ has no fixed point in the interior of $B(0,1)$. (The Denjoy–Wolff theorem implies that if ϕ has a fixed point in $B(0,1)$, then it must be attracting.) We also assume that one of the fixed points of ϕ on the unit circle is attracting. In this case the Julia set $\mathcal{J}_\phi$ is a Cantor subset of the unit circle and the basin of attraction of the attracting point is equal to $\widehat{\mathbb{C}} \setminus \mathcal{J}_\phi$.

Case (3): The map ϕ has a triple neutral fixed point in the unit circle. This point has two Leau leafs: $B(0,1)$ and $\widehat{\mathbb{C}} \setminus \overline{B(0,1)}$. It follows from [C-G, Example, pp. 107–108] that the Julia set of ϕ is equal to the unit circle and ϕ must be conjugate via a Möbius map to

$$\psi(\xi) = \frac{\xi - i/\sqrt{3}}{1 + i\xi/\sqrt{3}} \cdot \frac{\xi + i/\sqrt{3}}{1 - i\xi/\sqrt{3}}$$

and $Q - c$ is conjugate via an $\mathbb{R}$-linear quasiconformal map to

$$Q_0(z) = z^2 + \overline{z}^2/3$$

(as in Proposition 3.3).

Case (4): The map ϕ has one double neutral fixed point and one repelling fixed point on the unit circle. Then $\mathcal{J}_\phi$ is a Cantor subset of the unit circle and the neutral fixed point has one Leau leaf which is equal to $\widehat{\mathbb{C}} \setminus \mathcal{J}_\phi$.

For further information concerning the dynamics of Blaschke products see [H] and [Sh-Su].

4. The homogeneous case. In this section we deal with the case of $c = 0$. We have $Q(z) = z^2 + a|z|^2 + b\overline{z}^2$, $a = -(p+q)$, $b = pq$, $|p| < 1$, $|q| < 1$.

The complexified mapping

$$f(z,w) = (z^2 + azw + bw^2, w^2 + \overline{a}zw + \overline{b}z^2)$$

is homogeneous. Hence we can use Proposition 7.1 of [H-P] which says that the basin of attraction of zero for f is equal to

$$\Omega_f = \{(z,w) : h_f(z,w) < 0\}$$

where

$$h_f(z,w) = \lim_{m\to\infty} \frac{1}{2^m} \log \|f^{\circ m}(z,w)\|.$$

The set Ω_f is a complete circular Stein domain in $\mathbb{C}^2$.

The above fact yields immediately

Proposition 4.1. *The basin of attraction of zero for Q in $\mathbb{C}$ is a bounded domain in $\mathbb{C}$, starlike with respect to zero and given by*

$$\omega_Q = \{z \in \mathbb{C} : h_Q(z) < 0\} \quad \textit{where} \quad h_Q(z) = \lim_{m\to\infty} \frac{1}{2^m} \log |Q^{\circ m}(z)|.$$

Proof. We have

$$\|f^{\circ m}(z, \overline{z})\| = \sqrt{2}\,|Q^{\circ m}(z)|.$$

The boundedness of ω_Q follows from Proposition 2.10. ∎

Note that if $h_Q(z) > 0$, then $\lim_{m\to\infty} Q^{\circ m}(z) = \infty$. We have two superattractors: zero and ∞, and the set $\mathcal{J}_Q = \{z \in \mathbb{C} : h_Q(z) = 0\}$ which separates their basins of attraction.

We have

Theorem 4.2. *The set $\mathcal{J}_Q$ is a Jordan curve.*

Proof. The polynomial $Q(z)$ is $\mathbb{R}$-homogeneous. We have

$$h_Q(tz) = \log |t| + h_Q(z) \quad \text{for } z \neq 0.$$

This implies that on each halfline issuing from zero there is exactly one point z_0 for which $h_Q(z_0) = 0$. The function $h_Q(z)$ is continuous on $\mathbb{C}$, since $h_f(z, w)$ is continuous on $\mathbb{C}^2$.

Take $e^{i\theta} \in C(0, 1)$ and define $g(e^{i\theta}) = z(e^{i\theta})$ to be the unique point on the halfline with origin at zero passing through $e^{i\theta}$ for which $h_Q(z(e^{i\theta})) = 0$.

Since h_Q is continuous (see [H-P]), the mapping g is also continuous and univalent. The circle $C(0, 1)$ is compact and thus g is a homeomorphism from $C(0, 1)$ onto $\mathcal{J}_Q$. ∎

In what follows, we shall denote by h the mapping g^{-1}.

Problem 4.3. How to describe the geometric shape of $\mathcal{J}_Q$?

If, for example, $Q(z) = z^2 + |z|^2/2$, then $z_0 = \frac{2}{3}$ is a repelling fixed point. Hence z_0 and its counterimages $-\frac{2}{3}$, $\frac{2}{\sqrt{3}}i$, $-\frac{2}{\sqrt{3}}i$, $\pm\sqrt{3} \pm i$ belong to $\mathcal{J}_Q$. Hence $\mathcal{J}_Q$ cannot be an ellipse with imaginary and real axes. In particular it is not a circle with center at zero.

The dynamics of Q on its Julia set $\mathcal{J}_Q$ is closely related to the dynamics of the Blaschke product $\phi(\xi)$ on the unit circle.

Proposition 4.4. *There is one-to-one correspondence between the fixed points of $\phi(\xi)$ on the unit circle and nonzero fixed points of the homogeneous polynomial $Q(z)$. If the fixed point ξ_0 is repelling, then so is the corresponding point z_0. If ξ_0 is attracting or neutral with one Leau leaf, then the corresponding z_0 is a saddle point. If $\phi(\xi)$ is as in Case* (1) *or* (3) *of Section* 3, *then the dynamics of Q on $\mathcal{J}_Q$ is chaotic. If $\phi(\xi)$ is as in Case* (2) *or* (4), *then $Q|_{\mathcal{J}_Q}$ has an attracting point in $\mathcal{J}_Q$.*

Proof. Let $\xi \in C(0,1)$, $\phi(\xi) = \xi$. Take $z_1 \in C(0,1)$ such that $\arg z_1 = (\arg \xi)/2$. We have

$$\frac{z_1}{\overline{z}_1} = \xi \quad \text{and} \quad \frac{z_1}{\overline{z}_1} = \frac{Q(z_1)}{\overline{Q(z_1)}} = \phi\left(\frac{z_1}{\overline{z}_1}\right).$$

This implies that $Q(z_1)/z_1$ is a nonzero real number.

Put $z_0 = z_1^2/Q(z_1)$. Then

$$Q(z_0) = Q\left[\frac{z_1}{Q(z_1)} \cdot z_1\right] = \frac{z_1^2}{Q^2(z_1)} \cdot Q(z_1) = \frac{z_1^2}{Q(z_1)} = z_0$$

by the homogeneity of Q.

If $Q(z_0) = z_0$, $z_0 \neq 0$, then

$$\phi\left(\frac{z_0}{\overline{z}_0}\right) = \frac{Q(z_0)}{\overline{Q(z_0)}} = \frac{z_0}{\overline{z}_0}$$

and $z_0/\overline{z}_0$ is a fixed point of ϕ. In the neighbourhood of the fixed point z_0 the mapping $Q(z)$ is conjugate to ϕ via the map $z \mapsto z/\overline{z} = \xi$ and the inverse branch of this map which maps ξ_0 to z_0. This permits us to prove the rest of Proposition 4.4. ∎

We shall now try to describe the dynamics of $Q|_{\mathcal{J}_Q}$ in Cases (1)–(4) of Section 3.

Recall that

$$\arg \frac{z}{\overline{z}} = 2 \arg z$$

and hence if $h(z)$ denotes the homeomorphism from $\mathcal{J}_Q$ onto the unit circle $C(0,1)$, constructed in Theorem 4.2, then

$$\frac{h(z)}{\overline{h(z)}} = (h(z))^2, \qquad z \in \mathcal{J}_Q.$$

Note that from the very construction of h it follows that

$$h(-z) = -h(z).$$

Let us consider Case (1). We can assume that

$$Q(z) = z^2 - p|z|^2, \qquad |p| < 1.$$

Hence the complexified map $f(z,w)$ acts on $\mathbb{CP}^2 \setminus \mathbb{C}^2$ as

$$\phi(\xi) = \xi \, \frac{\xi - p}{1 - \overline{p}\xi}.$$

Elementary calculations permit us to show that if $|\xi| = 1$, i.e. $\overline{\xi} = 1/\xi$, then

$$\phi(\xi^2) = \left(\xi^2 \, \frac{|1 - \overline{p}\xi^2|}{1 - \overline{p}\xi^2}\right)^2.$$

Hence
$$\xi^2 \frac{|1-\overline{p}\xi^2|}{1-\overline{p}\xi^2} \quad \text{and} \quad -\xi^2 \frac{|1-\overline{p}\xi^2|}{1-\overline{p}\xi^2}$$
are two continuous branches of $(\phi(\xi^2))^{1/2}$ on $C(0,1)$.

Let $\psi(\xi)$ be the branch for which
$$Q(z) = h^{-1}\psi h(z)$$
for $z \in \mathcal{J}_Q$ (for the other one we have $-Q(z) = h^{-1}\psi h(z)$).

Hence $Q(z)$ has the same dynamics on $\mathcal{J}_Q$ as $\psi(\xi)$ on $C(0,1)$. The dynamics of $\psi(\xi)$ is basically the same as the dynamics of $\phi(\xi)$. It follows from the results of [Sh-Su], [H] and [T] that $\phi(\xi)$ is chaotic, uniformly expanding, ergodic and mixing on $C(0,1)$. Hence we get

Proposition 4.5. *The dynamics of $Q|_{\mathcal{J}_Q}$ in Case* (1) *is the same as the dynamics of the continuous branch $\psi(\xi)$ of $(\phi(\xi^2))^{1/2}$ on $C(0,1)$.*

This dynamics is uniformly expanding, chaotic, mixing and ergodic.

Let us consider Case (3). We can assume that
$$Q(z) = z^2 + \overline{z}^2/3$$
and the complexified mapping $f(z,w)$ acts on $\mathbb{CP}^2 \setminus \mathbb{C}^2$ as
$$\phi(\xi) = \frac{\xi - \frac{i}{\sqrt{3}}}{1+\frac{i}{\sqrt{3}}\xi} \cdot \frac{\xi + \frac{i}{\sqrt{3}}}{1-\frac{i}{\sqrt{3}}\xi}.$$
Elementary calculations show that
$$\phi(\xi^2) = \left(\xi^2 \frac{|1+\xi^4/3|}{1+\xi^4/3}\right)^2$$
if $|\xi| = 1$. We again have two continuous branches of $(\phi(\xi^2))^{1/2}$ on $C(0,1)$ equal to
$$\pm\xi^2 \frac{|1+\xi^4/3|}{1+\xi^4/3}.$$
We can again choose one branch $\psi(\xi)$ for which $Q(z) = h^{-1}\psi h(z)$. The mapping ψ is again chaotic on $C(0,1)$ but it is not uniformly expanding.

It remains to study Cases (2) and (4). In both, the Julia set of the map
$$\phi(\xi) = \frac{\xi - p}{1-\overline{p}\xi} \cdot \frac{\xi - q}{1-\overline{q}\xi}$$
for $Q(z) = (z - p\overline{z})(z - q\overline{z})$ is the Cantor subset of $C(0,1)$. The set
$$U = C(0,1) \setminus \mathcal{J}_Q$$
is the basin of attraction of a fixed point ξ_0 of $\phi(\xi)$, $|\xi_0| = 1$. In Case (2), $\xi_0 \in U$, and in Case (4), $\xi_0 \in \mathcal{J}_\phi \cap \partial U$. By Proposition 4.4 there exists $z_0 \in \mathcal{J}_Q$ such that $Q(z_0) = z_0$ and $(h(z_0))^2 = \xi_0$.

As before, if $|\xi| = 1$, then

$$\phi(\xi^2) = \left(\xi^2 \cdot \frac{|1-\overline{p}\xi^2|}{1-\overline{p}\xi^2} \cdot \frac{|1-\overline{q}\xi^2|}{1-\overline{q}\xi^2}\right)^2$$

and we can again find a continuous branch ψ of $(\phi(\xi^2))^{1/2}$ for which $h^{-1} \circ \psi \circ h(z) = Q(z)$.

Let $W = \{z \in \mathcal{J}_Q : (h(z))^2 \in U\}$ and $C_Q = \{z \in \mathcal{J}_Q : (h(z))^2 \in \mathcal{J}_\phi\}$. We obtain immediately the following

Proposition 4.6. *In Cases* (2) *and* (4), *$\mathcal{J}_Q = W \cup C_Q$, $W \cap C_Q = \emptyset$. The set W is an open set in $\mathcal{J}_Q$ which is attracted to z_0. In Case* (2), *$z_0 \in W$, and in Case* (4), *$z_0 \in C_Q \cap \partial W$. The set C_Q is a Cantor set. The mapping $Q|_{C_Q}$ is conjugate to ψ and also to ϕ. The set C_Q is the support of a chaotic dynamics of Q.*

5. The nonhomogeneous case. Almost nothing is known about the dynamics of regular nonhomogeneous polynomial mappings of $\mathbb{CP}^2$. However Theorem 4.3 (and its proof) from [B-J1] permits us to state the following

Theorem 5.1. *Suppose that $Q(z) = z^2 + a|z|^2 + b\overline{z}^2 + c$ is a quasiregular polynomial conjugate to $z^2 - p|z|^2 + c_1$ (Case* (1) *from Section* 3). *Then there exists a neighbourhood V of ∞ in the Riemann sphere $\widehat{\mathbb{C}}$ and a homeomorphism ψ which maps V onto some neighbourhood of ∞, $\psi(\infty) = \infty$, conjugating Q to $Q - c$. This means that in this case we have a Böttcher coordinate near infinity.*

Outline of the proof. Let $Q(z) = z^2 - p|z|^2 + c$ and $Q_0(z) = z^2 - p|z|^2$. The complexified mapping is $f(z,w) = (z^2 - pzw + c, w^2 - \overline{p}zw + \overline{c})$. We have

$$\phi(\xi) = \xi\frac{z-p}{1-\overline{p}z},$$

$\mathcal{J}_\phi = C(0,1)$ and ϕ is uniformly expanding on $\mathcal{J}_\phi$. Hence the assumptions of Theorem 4.3 of [B-J1] are fulfilled and there exists a homeomorphism

$$\psi : W^s(\mathcal{J}_\phi, Q_0) \cap A_{0,Q_0} \xrightarrow{\text{onto}} W^s(\mathcal{J}_\phi, Q) \cap A_{0,Q}$$

conjugating $f(z,w)$ and

$$f_0(z,w) = (z^2 - pzw, w^2 - \overline{p}zw).$$

Here $W^{\mathrm{s}}(\mathcal{J}_\phi, Q_0)$ and $W^{\mathrm{s}}(\mathcal{J}_\phi, Q)$ denote the sets of points of $\mathbb{CP}^2$ which are attracted to $\mathcal{J}_\phi$ by iterating Q_0 or Q respectively, and

$$A_{0,Q} = \{(z,w) \in \mathbb{CP}^2 : \mathcal{G}_Q(z,w) > R_0\}$$

where $\mathcal{G}_Q$ is the Green function for Q.

Proposition 2.10 implies that there exists $R > 0$ such that

$$\{(z,\overline{z}) : |z| > R\} \subset W^{\mathrm{s}}(\mathcal{J}_\phi, Q_0) \cap A_{0,Q_0} \cap W^{\mathrm{s}}(\mathcal{J}_\phi, Q) \cap A_{0,Q}.$$

Let $\psi(z,w) = (\psi^1(z,w), \psi^2(z,w))$. If we show that

$$\psi^1(z,\overline{z}) = \overline{\psi^2(z,\overline{z})},$$

then $\psi(z) = \psi^1(z,\overline{z})$ will be the needed homeomorphism. Take $z_0 \in \mathbb{C}$ with $|z_0| > R$. Since ϕ is uniformly expanding on $\mathcal{J}_\phi$, there exists $a \in \mathcal{J}_\phi$ such that $(z,\overline{z})$ belongs to the stable disc $W^{\mathrm{s}}(a,Q_0)$. One can check that

$$\psi|_{W^{\mathrm{s}}(a,Q_0)} = \psi_{1,a}$$

where $\psi_{1,a}$ comes from a holomorphic homotopy on $W^{\mathrm{s}}(a,Q_0)$,

$$\psi_{\tau,a,n} = f_\tau^{-nN} \circ f_0^{nN}$$

where $f_\tau(z,w) = (z^2 - pzw + \tau c, w^2 - \overline{p}zw + \tau\overline{c}_1)$, $|\tau| < 2$, namely

$$\psi_{\tau,a} = \lim_{n\to\infty} \psi_{\tau,a,n}.$$

The branches f_τ^{-nN} are chosen such that f_τ^{-nN} is holomorphic in τ, and $\psi_{0,a,n} = \mathrm{Id}$.

Fix $n \in \mathbb{N}$. There exist $\varepsilon > 0$ and $\delta > 0$ such that if $|\tau| < \varepsilon$, then f_τ^{nN} is a nonbranched covering of

$$B((z_0,\overline{z}_0),\delta) \cap W^{\mathrm{s}}(a,Q_0).$$

Thus the counter-images of

$$B((z_0,\overline{z}_0),\delta) \cap W^{\mathrm{s}}(a,Q)$$

with respect to different branches of f_τ^{-nN} are "far apart". This implies that we can find ε_0, $0 < \varepsilon_0 < \varepsilon$, such that

$$\psi^2_{\tau,a,n}(z,\overline{z}) = \overline{\psi^1_{\tau,a,n}(z,\overline{z})}$$

for $\tau \in \mathbb{R}$, $|\tau| < \varepsilon_0$ and $(z,\overline{z}) \in B((z_0,\overline{z}_0),\delta/2) \cap W^{\mathrm{s}}(a,Q_0)$. The function

$$\psi^2_{\tau,a,n}(z,\overline{z}) - \overline{\psi_{\tau,a,n}(z,\overline{z})}$$

is real-analytic on $(-2,2) \subset \mathbb{R}$, vanishes on $(-\varepsilon_0,\varepsilon_0)$, and hence it must be equal to zero for each $\tau \in \mathbb{R}$, $|\tau| < 2$. This implies that $\psi^2_{\tau,a} = \overline{\psi^1_{\tau,a}}$ for $\tau \in \mathbb{R}$ and $\psi^2_{1,a} = \psi^1_{1,a}$. This ends our outline of the proof. ∎

In order to obtain a full proof one must read our outline together with Section 4 of the Bedford–Jonsson paper [B-J1].

Problem 5.2. Is Theorem 5.1 valid for other quasiregular polynomials of degree two?

Remark 5.3. One can feel tempted to act in the standard way: define the filled-in Julia set or something like the Mandelbrot set and try to mimick the theory of the quadratic family $\{z^2 + c\}$. We think this is <u>not</u> the right thing to do <u>now</u>. First we must understand the behaviour of our mappings $Q(z)$ and complexified mappings $f(z,w)$ on $\mathbb{CP}^2$ in the <u>nonhomogeneous</u>

case. At present we do not have good analytic tools to deal with this problem. Proposition 4.6 shows that the dynamical behaviour of $Q(z)$ can be quite different from the behaviour of a quadratic holomorphic polynomial. Moreover, in this case the support of a chaotic dynamics of Q is equal to $C_Q \subsetneq \mathcal{J}_Q$. Hence the important set is C_Q, not the whole Julia set.

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References

[B-J1] E. Bedford and M. Jonsson, *Regular polynomial endomorphisms of* $\mathbb{C}^k$, SUNY Stony Brook, Preprint #1998/3.

[B-J2] —, —, *Dynamics of regular polynomial endomorphisms of* $\mathbb{C}^k$, Amer. J. Math. 122 (2000), 153–212.

[C-G] L. Carleson and T. W. Gamelin, *Complex Dynamics*, Springer, New York, 1993.

[H] D. H. Hamilton, *Absolutely continuous conjugacies of Blaschke products*, *I*, Adv. Math. 121 (1996), 1–20; *II*, J. London Math. Soc. (2) 51 (1995), 279–285, *III*, J. Anal. Math. 63 (1994), 333–343,

[H-P] J. Hubbard and J. Papadopol, *Superattractive fixed points in* $\mathbb{C}^n$, Indiana Univ. Math. J. 43 (1994), 321–365.

[Li1] E. Ligocka, *On quasiregular polynomial mappings*, Colloq. Math. 75 (1998), 79–89.

[Li2] —, *On fixed points of holomorphic type*, ibid. 93 (2002), 61–66.

[Sh-Su] M. Shub and D. P. Sullivan, *Expanding isomorphisms of the circle revisited*, Ergodic Theory Dynam. Systems 5 (1985), 285–289.

[T] D. Tischler, *Blaschke products and expanding maps of the circle*, Proc. Amer. Math. Soc. 128 (2000), 621–622.

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