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Multifractal dimensions for invariant subsets of piecewise monotonic interval maps

by

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Abstract. The multifractal generalizations of Hausdorff dimension and packing dimension are investigated for an invariant subset A of a piecewise monotonic map on the interval. Formulae for the multifractal dimension of an ergodic invariant measure, the essential multifractal dimension of A, and the multifractal Hausdorff dimension of A are derived.

Introduction. Consider a piecewise monotonic map T on the interval (exact definitions will be given later), and denote by $\mathcal{Z}$ the collection of its intervals of monotonicity. Let U be an open interval and consider the set A of all points x whose orbit $\{T^n x : n \in \mathbb{N}_0\}$ omits U, that is,

$$A := \bigcap_{n=0}^{\infty} \overline{[0,1] \setminus T^{-n}U}$$

(of course this is only of interest if $A \neq \emptyset$). Usually this set is a "fractal". We wish to investigate the size of A.

In order to motivate the problems investigated in this paper we make some simplifications (in this simpler case the results presented in this paper are known). Suppose that T is a piecewise monotonic map, that $\mathcal{Z}$ consists of three intervals, U is the second of these intervals (which is assumed to be open), and the first and third intervals are mapped onto $[0,1]$ by T. Moreover, we assume that the restrictions of T to the first and third interval of monotonicity are C^1 and $\sup |T'| > 1$. This situation is known as "*cookie-cutter*" in the literature (see e.g. [6] and [10]).

Let us consider two concrete examples (of cookie-cutters). The first one is the map

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$$T_1x := \begin{cases} 3x & \text{if } x \in [0,1/3], \\ x/2+1/4 & \text{if } x \in (1/3,2/3), \\ 3x-2 & \text{if } x \in [2/3,1], \end{cases}$$

$U_1 := (1/3,2/3)$, and $A_1 := \bigcap_{n=0}^{\infty} \overline{[0,1] \setminus T_1^{-n}U_1}$. In this case A_1 is the usual Cantor set. Note that A_1 is exactly the set of all points satisfying $\lim_{n\to\infty} T_1^n x \neq 1/2$. Our second map is defined by

$$T_2x := \begin{cases} 3x & \text{if } x \in [0,1/3], \\ x/2+1/4 & \text{if } x \in (1/3,1-10^{-8}), \\ 10^8x-10^8+1 & \text{if } x \in [1-10^{-8},1]. \end{cases}$$

Set $U_2 := (1/3, 1-10^{-8})$ and $A_2 := \bigcap_{n=0}^{\infty} \overline{[0,1] \setminus T_2^{-n}U_2}$. Again A_2 is a Cantor set, but it is obviously "much less regular". Note that also in this case A_2 is exactly the set of all points satisfying $\lim_{n\to\infty} T_2^n x \neq 1/2$.

One possibility of measuring the size of A is to calculate its Hausdorff dimension $\mathrm{HD}(A)$. In the above examples we get $\mathrm{HD}(A_1) = \log 2/\log 3 = 0.63093\ldots$ and $\mathrm{HD}(A_2) = 0.11548577\ldots$ It is not surprising that the Hausdorff dimension of A_2 is smaller than that of A_1. However, these numbers do not give us any information that A_1 is "very symmetric" and A_2 is "very asymmetric". In order to obtain a better understanding of the "size" of a set we should not assign only one number to it.

Assume that T is a piecewise monotonic map and that $n \in \mathbb{N}$. We call a nonempty set Z an *n-cylinder* if there exist $Z_0, Z_1, \ldots, Z_{n-1} \in \mathcal{Z}$ such that $Z = \bigcap_{j=0}^{n-1} T^{-j}Z_j$. In the case of cookie-cutters there are exactly 2^n different n-cylinders having nonempty intersection with A. Fix an $n \in \mathbb{N}$. Denote by $\mathcal{Z}_n$ the collection of all n-cylinders having nonempty intersection with A. Whereas in the first example all n-cylinders have the same length, in the second example the lengths of the n-cylinders differ significantly.

The phenomenon described above gives a motivation to define *local dimensions*. We assume that we are in the situation of cookie-cutters. First we define $m(Z) := 1/2^n$ (the weight should be equally distributed to the n-cylinders). This gives rise to a Borel probability measure m on $[0,1]$. For $x \in A$ define the *local dimension* $\mathrm{ld}(x)$ by

$$\mathrm{ld}(x) := \lim_{r\to 0^+} \frac{\log m((x-r,x+r))}{\log(2r)} \,.$$

Note that $2r$ is the length of the interval $(x-r,x+r)$. In the first example we have $\mathrm{ld}(x) = \log 2/\log 3$ for all $x \in A_1 \setminus C$, where C is a countable set. This means that in this case the local dimension is essentially independent of x. On the other hand, it turns out that in our second example the local dimension depends very much on x (which is not surprising because of the "asymmetry" of A_2).

To describe the "asymmetry" the facts discussed above motivate the following definition of "multifractal spectrum". The idea is to split our set A into different "fractals", on each of which the local dimension is constant. Hence, for $\alpha \in \mathbb{R}$ we define $L(\alpha) := \{x \in A : \mathrm{ld}(x) = \alpha\}$. Then the map $\alpha \mapsto \mathrm{HD}(L(\alpha))$ is called the *multifractal spectrum* of A.

In our first example the only nonzero value of the multifractal spectrum is $\log 2/\log 3$, which is attained at the point $\log 2/\log 3$. However, it is much more difficult to evaluate the multifractal spectrum in the second example.

For $s \in \mathbb{R}$ "define" the multifractal Hausdorff dimension $d_s(A)$ by

$$d_s(A) := \sup\Big\{t : \lim_{n\to\infty} \sum_{Z\in\mathcal{Z}_n} m(Z)^s |Z|^t = \infty\Big\},$$

where $|Z|$ denotes the length of Z (a more exact definition will be given later). Then for each $s \in \mathbb{R}$ there is a unique $\tau(s) \in \mathbb{R}$ with $p(A, T, -s\log 2 - \tau(s)\log|T'|) = 0$, where $p(\cdot,\cdot,\cdot)$ denotes the topological pressure. According to [10] (cf. also [6]), $d_s(A) = \tau(s)$ for all $s \in \mathbb{R}$ and the multifractal spectrum of A equals the Legendre transform $\widehat{\tau}$ of τ, that is, $\widehat{\tau}(\alpha) := \inf\{\tau(s) + \alpha s : s \in \mathbb{R}\}$. This means that in the case of cookie-cutters the multifractal spectrum can be determined if one knows the multifractal Hausdorff dimension.

The multifractal Hausdorff dimension in our first example is the map $s \mapsto (1-s)\log 2/\log 3$, hence its graph is a straight line. In our second example the graph of the multifractal Hausdorff dimension is shown in Figure 1 below.

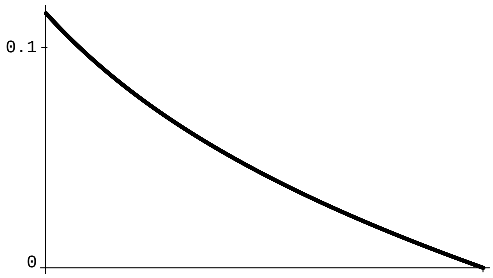

Fig. 1. The graph of the function $s \mapsto d_s(A_2)$ on $[0,1]$

Results similar to those described above have been obtained by several authors even in higher dimensional systems (see e.g. [6] and [7]). However, these results work only under assumptions on the dynamical system implying the existence of a Markov partition. Unfortunately the maps considered in this paper need not have a Markov partition. In the general situation considered here we are not able to prove that the multifractal spectrum equals the Legendre transform of τ. However, under some additional assumptions (namely that T is expanding) it has been proved in [3] that the multifractal spectrum equals the Legendre transform of τ.

In the situation of cookie-cutters we have chosen a very special measure m. The question arises which measure should play the rôle of m in the general situation. It turns out that we can choose any conformal measure m (also in the case of cookie-cutters we could choose another measure). On the other hand, the restriction to conformal measures is necessary, because we need a relation between the measure m and the map T.

It is preferable to give a definition of $d_s(A)$ which does not involve n-cylinders. This is done by replacing n-cylinders by balls of radius smaller than ε. However, it turns out that then there are two approaches to multifractal dimensions. The first one uses covers of A by balls and leads to the multifractal Hausdorff dimension $d_s(A)$, while the second approach uses packings of A by balls and leads to the multifractal packing dimension $D_s(A)$. For cookie-cutters $d_s(A) = D_s(A)$ (see e.g. [6]), but this need not be true in general.

We will prove that a formula analogous to $d_s(A) = D_s(A) = \tau(s)$ holds for ergodic probability measures μ, where the multifractal dimensions $d_s(\mu)$ and $D_s(\mu)$ of μ are defined as the infima of the multifractal dimensions of sets of full measure. Unfortunately we cannot prove that $d_s(A) = D_s(A) = \tau(s)$ in general. Therefore we define the essential multifractal dimensions $e_s(A)$ and $E_s(A)$ as the suprema of the multifractal dimensions of ergodic measures with positive entropy. It is not known when $e_s(A) = d_s(A)$ and $E_s(A) = D_s(A)$. Using a modified version of the topological pressure we will prove that $e_s(A) = E_s(A) = \tau(s)$. Finally we will prove that for expanding maps T the formula $d_s(A) = e_s(A) = E_s(A) = \tau(s)$ holds even if one uses the usual definition of the topological pressure. Applying this result and using Theorem 6.1 of [6] we deduce that in the situation of cookie-cutters $d_s(A) = D_s(A) = e_s(A) = E_s(A) = \tau(s)$ (essential multifractal dimensions have been considered neither in [6] nor in [10]). Hence in this case the essential multifractal dimensions coincide with the usual multifractal dimensions.

1. Multifractal dimensions. The aim of this paper is to investigate multifractal generalizations of Hausdorff dimension and packing dimension of an invariant subset of a piecewise monotonic interval map. We relate them to zeros of certain pressure functions, as is done in [8] for the usual Hausdorff dimension. Similar results have been obtained by several authors even in higher dimensional systems (see e.g. [6] and [7]). However, these results work only under assumptions on the dynamical system implying the existence of a Markov partition. In contrast to this situation the maps considered in this paper need not have a Markov partition. Therefore in the proofs different techniques than those described in [6] or [7] have to be used.

We call $\mathcal{Z}$ a *finite partition* of $[0,1]$ if $\mathcal{Z}$ consists of finitely many pairwise disjoint open intervals with $\bigcup_{Z\in\mathcal{Z}} \overline{Z} = [0,1]$. A map $T:[0,1]\to[0,1]$ is called *piecewise monotonic* if there exists a finite partition $\mathcal{Z}$ of $[0,1]$ such that for every $Z\in\mathcal{Z}$ the function $T|_Z$ is continuous and strictly monotonic. If for every $Z\in\mathcal{Z}$ the function $T|_Z$ is differentiable and $T'|_Z$ can be extended to a continuous function on the closure of Z, then T is called *piecewise continuously differentiable*. A finite partition $\mathcal{Y}$ of $[0,1]$ is called a *generator* if for every sequence $Y_0, Y_1, Y_2, \ldots$ in $\mathcal{Y}$ the set $\bigcap_{j=0}^{\infty} \overline{T^{-j}Y_j}$ contains at most one element.

Let $T:[0,1]\to[0,1]$ be a piecewise continuously differentiable piecewise monotonic map. Set $\varphi := -\log|T'|$. Then φ is continuous on $[0,1]\setminus P$, where $P := \{\inf Z, \sup Z : Z\in\mathcal{Z}\}$, φ is bounded from below, but φ need not be bounded from above. It describes how T deforms the geometry on $[0,1]$. For an interval I define $|I| := \sup_{x,y\in I}|x-y|$. If Y is an interval contained in an element of $\mathcal{Z}$, then

$$|TY| = \int_Y e^{-\varphi(x)}\,dx. \tag{1.1}$$

In order to define multifractal dimensions we introduce a second geometry using a conformal measure m. Let $\psi:[0,1]\to\mathbb{R}$ be a function such that for every $Z\in\mathcal{Z}$ the function $\psi|_Z$ can be extended to a continuous function on the closure of Z. A Borel probability measure m on $[0,1]$ is called $e^{-\psi}$*-conformal* if

$$m(TY) = \int_Y e^{-\psi(x)}\,dm(x) \tag{1.2}$$

for every interval Y contained in an element of $\mathcal{Z}$. Then $e^{-\psi}$ equals locally the Radon–Nikodým derivative $d(m\circ T)/dm$. In Theorem 2 of [5] conditions on ψ are described which imply the existence of an $e^{-\psi}$-conformal measure m (called almost $e^{-\psi}$-conformal measure there).

We give a brief explanation why we cannot take an arbitrary Borel probability measure m instead of the $e^{-\psi}$-conformal measure. In the proofs we will have to estimate $m(Y)$ for sets Y contained in an interval of monotonicity of T^n. Roughly speaking, we deduce from (1.2) that $m(Y)\approx m(T^nY)\exp(\sum_{j=0}^{n-1}\psi(T^jx))$. This “equality” turns out to be crucial in the proofs.

The definition of multifractal Hausdorff dimension and multifractal packing dimension and information about their history can be found in [6] (cf. also [7]). These notions can be defined on metric spaces and with respect to a (general) Borel measure m. For the convenience of the reader we recall these definitions for the setting considered in this paper. Note that we have fixed the $e^{-\psi}$-conformal measure m above.

Let $E, F \subseteq [0,1]$, and let $s \in \mathbb{R}$. For $\varepsilon > 0$ we call $\mathcal{C}$ a *centered ε-cover* of F if $F \subseteq \bigcup_{C\in\mathcal{C}} C$ and for every $C \in \mathcal{C}$ there exist $x \in F$ and $\alpha \in (0,\varepsilon]$ with $C = (x-\alpha, x+\alpha)$. Denote by $\mathcal{U}_\varepsilon(F)$ the collection of all centered ε-covers of F. Now define for $t \in \mathbb{R}$,

$$\nu_{s,t}(E) := \sup_{F\subseteq E} \lim_{\varepsilon\to 0} \inf_{\mathcal{C}\in\mathcal{U}_\varepsilon(F)} \sum_{C\in\mathcal{C}} m(C)^s |C|^t. \tag{1.3}$$

Then $\nu_{s,t}$ is a Borel measure on $[0,1]$ (cf. [6]). Note that we would not get a measure if we omitted the "$\sup_{F\subseteq E}$" in (1.3) (see [6] for a discussion of this fact). For $s = 0$, $\nu_{0,t}$ is the t-dimensional Hausdorff measure. Now set

$$d_s(E) := \sup\{t \in \mathbb{R} : \nu_{s,t}(E) = \infty\}, \tag{1.4}$$

where the values $-\infty$ and ∞ are allowed for $d_s(E)$. By Proposition 1.1 of [6] we have

$$\begin{aligned} \nu_{s,t}(E) &= \infty \quad \text{for } t < d_s(E),\\ \nu_{s,t}(E) &= 0 \quad \text{for } t > d_s(E). \end{aligned} \tag{1.5}$$

Note that $d_0(E)$ is the usual Hausdorff dimension of E. Hence in [6], $d_s(E)$ is called a multifractal analogue of the Hausdorff dimension of E. For simplicity we call it the *multifractal Hausdorff dimension* of E.

For $\varepsilon > 0$ we call $\mathcal{C}$ a *centered ε-packing* of F if $\mathcal{C}$ consists of pairwise disjoint elements and for every $C \in \mathcal{C}$ there exist $x \in F$ and $\alpha \in (0,\varepsilon]$ with $C = (x-\alpha, x+\alpha)$. Denote by $\mathcal{V}_\varepsilon(F)$ the collection of all centered ε-packings of F. We call $\mathcal{F}$ a *cover* of E if $\mathcal{F}$ consists of at most countably many subsets of $[0,1]$ with $E \subseteq \bigcup_{F\in\mathcal{F}} F$. Let $\mathcal{F}(E)$ be the collection of all covers of E. Now define for $t \in \mathbb{R}$,

$$\pi_{s,t}(E) := \inf_{\mathcal{F}\in\mathcal{F}(E)} \sum_{F\in\mathcal{F}} \Big(\lim_{\varepsilon\to 0} \sup_{\mathcal{C}\in\mathcal{V}_\varepsilon(F)} \sum_{C\in\mathcal{C}} m(C)^s |C|^t \Big). \tag{1.6}$$

Then $\pi_{s,t}$ is a Borel measure on $[0,1]$ (cf. [6]). Omitting "$\inf_{\mathcal{F}\in\mathcal{F}(E)} \sum_{F\in\mathcal{F}}$" in (1.6) we would not get a measure (see [6] for a discussion of this fact). For $s = 0$, $\pi_{0,t}$ is the t-dimensional packing measure. Now set

$$D_s(E) := \sup\{t \in \mathbb{R} : \pi_{s,t}(E) = \infty\}, \tag{1.7}$$

where the values $-\infty$ and ∞ are allowed for $D_s(E)$. By Proposition 1.1 of [6] we have

$$\begin{aligned} \pi_{s,t}(E) &= \infty \quad \text{for } t < D_s(E),\\ \pi_{s,t}(E) &= 0 \quad \text{for } t > D_s(E). \end{aligned} \tag{1.8}$$

Note that $D_0(E)$ is the usual packing dimension of E. Hence in [6], $D_s(E)$ is called a multifractal analogue of the packing dimension of E. For simplicity we call it the *multifractal packing dimension* of E. We have

$$d_s(E) \le D_s(E) \tag{1.9}$$

for all $E \subseteq [0,1]$ (cf. Proposition 2.4 of [6]).

Now let μ be a Borel probability measure on $[0,1]$. Denote the collection of all Borel subsets of $[0,1]$ by $\mathcal{B}$. Define

$$d_s(\mu) := \inf_{\substack{E\in\mathcal{B}\\ \mu(E)=1}} d_s(E), \qquad D_s(\mu) := \inf_{\substack{E\in\mathcal{B}\\ \mu(E)=1}} D_s(E). \tag{1.10}$$

We call $d_s(\mu)$ the *multifractal Hausdorff dimension* of μ, and $D_s(\mu)$ the *multifractal packing dimension* of μ.

In Theorem 1 it will be shown that for every ergodic T-invariant Borel probability measure μ with $h_\mu(T) > 0$ and $\mu(\operatorname{supp} m) = 1$ we have

$$d_s(\mu) = D_s(\mu) = \frac{h_\mu(T) + s\int\psi\,d\mu}{-\int\varphi\,d\mu}$$

if $T \in C^2([0,1])$ and $\mathcal{Z}$ is a generator. For $s = 0$ this formula is well known (cf. [4]). Similar results can also be found in [7] (even for higher dimensional systems), but only in situations where the transformation admits a Markov partition.

Next we investigate completely invariant sets A. A set $A \subseteq [0,1]$ is called *completely invariant* if for every $x \in [0,1]\backslash P$ the property $x \in A$ is equivalent to $Tx \in A$. We assume that $\mathcal{Z}$ is a generator and that $A \subseteq \operatorname{supp} m$. These assumptions will be discussed below. In Section 2 we introduce a pressure $q(A,T,f)$ for functions $f : [0,1] \to \mathbb{R}$ which are continuous on $[0,1] \setminus P$, but not necessarily bounded. If T and f are continuous, then $q(A,T,f)$ equals the usual topological pressure $p(A,T,f)$ (see Proposition 1 of [3]). For a fixed $s \in \mathbb{R}$ define

$$z_s(A) := \sup\{t \in \mathbb{R} : q(A,T,t\varphi + s\psi) > 0\}.$$

Moreover set

$$s_A := \inf\{s \in \mathbb{R} : q(A,T,s\psi) = 0\}.$$

Denote by $\mathcal{E}^+(A,T)$ the set of all ergodic Borel probability measures on A with $h_\mu(T) > 0$. Now define

$$e_s(A) := \sup_{\mu\in\mathcal{E}^+(A,T)} d_s(\mu), \qquad E_s(A) := \sup_{\mu\in\mathcal{E}^+(A,T)} D_s(\mu). \tag{1.11}$$

The number $e_s(A)$ is called the *essential multifractal Hausdorff dimension* of A, and $E_s(A)$ is called the *essential multifractal packing dimension* of A. These "essential multifractal dimensions" measure the "size" of the part of A which is "dynamically important". In general it is an open problem when $e_s(A) = d_s(A)$ and $E_s(A) = D_s(A)$ (obviously $e_s(A) \le d_s(A)$ and $E_s(A) \le D_s(A)$).

We will show in Theorem 2 that

$$e_s(A) = E_s(A) = z_s(A)$$

if $s \in [0, s_A)$. This generalizes the results of Section 5 of [5].

If $\sup_{x\in A}\varphi(x)<0$ and $\sup_{x\in A}\psi(x)<0$, then we will show in Theorem 3 that

$$d_s(A)=e_s(A)=E_s(A)=z_s(A)=\widetilde{z}_s(A)$$

for every $s\in[0,s_A)$, where $\widetilde{z}_s(A)$ is the unique zero of $t\mapsto p(A,T,t\varphi+s\psi)$ ($p(A,T,t\varphi+s\psi)$ denotes the usual topological pressure of $t\varphi+s\psi$). This is a generalization of Theorem 2 of [8].

For certain dynamical systems, namely graph directed self-similar sets in $\mathbb{R}^d$ and cookie-cutters in $[0,1]$, these results are a part of Theorems 5.1 and 6.1 of [6]. In contrast to our situation the maps considered in [6] admit a Markov partition. A very general construction, called C-structures, is introduced in [7]. The multifractal dimensions considered in the present paper are a special case of that general construction. Results similar to our results described above are obtained in [7] for C-structures generated by certain dynamical systems. Although these transformations may be higher dimensional, they always admit a Markov partition.

In order to overcome the problems arising in our situation (e.g. in the Markovian case the "bounded distortion principle" is frequently used) we use an approximation by Markov maps and the methods developed in [4].

Finally we describe how our results can be applied to a certain class of sets which are not necessarily completely invariant. Let G be a finite union of open intervals, and define

$$A:=[0,1]\setminus\bigcup_{j=0}^{\infty}T^{-j}G.$$

Assume that $A\neq\emptyset$. We show that our results hold for A. To this end we change T on G so that T remains a piecewise monotonic map with respect to a generator $\mathcal{Y}$ and $TG\subseteq G$. Then also the functions φ and ψ change on G, but T, φ and ψ remain unchanged on $[0,1]\setminus G$ (and hence on A). Therefore $z_s(A)$ remains unchanged. By Lemma 4 of [3] the set A is completely invariant for the changed map T.

2. Topological pressure. We introduce a version of the definition of the topological pressure (cf. [3] and [5]). Let $T:[0,1]\to[0,1]$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of $[0,1]$. Let $C([0,1])$ be the set of all functions $f:[0,1]\to\mathbb{R}$ such that for each $Z\in\mathcal{Z}$ the map $f|_Z$ can be extended to a continuous real-valued function on the closure of Z. Furthermore let $D^+([0,1])$ be the set of all functions $f:[0,1]\to(-\infty,\infty]$ such that for each $Z\in\mathcal{Z}$ the map $f|_Z$ is a continuous real-valued function and can be extended to a continuous function $f_Z:\overline{Z}\to(-\infty,\infty]$, and let $D([0,1])$ be the set of all functions $f:[0,1]\to[-\infty,\infty]$ such that for each $Z\in\mathcal{Z}$ the map $f|_Z$ is a continuous real-valued function and can be extended to a continuous function $f_Z : \overline{Z} \to [-\infty, \infty]$. Note that $C([0,1]) \subseteq D^+([0,1]) \subseteq D([0,1])$.

Next let $A \subseteq [0,1]$ be nonempty. A nonempty closed $B \subseteq A$ is called a *Markov subset* of A if there exists a finite partition $\mathcal{Y}$ of $[0,1]$ refining $\mathcal{Z}$ such that $T(B \cap Y) \subseteq B$ for all $Y \in \mathcal{Y}$ and for every $Y_1, Y_2 \in \mathcal{Y}$ we have either $T(B \cap Y_1) \cap Y_2 = \emptyset$ or $B \cap Y_2 \subseteq T(B \cap Y_1)$. Denote by $\mathcal{M}(A)$ the collection of all Markov subsets $B \subseteq A$.

If X is a compact metric space and $T : X \to X$ is a continuous function, then (X,T) is called a *topological dynamical system.* For $\varepsilon > 0$ and $n \in \mathbb{N}$ a set $E \subseteq X$ is said to be (n,ε)*-separated* if for every $x \neq y \in E$ there exists a $j \in \{0,1,\ldots,n-1\}$ with $d(T^j x, T^j y) > \varepsilon$. Let $f : X \to \mathbb{R}$ be a continuous function. Then the *topological pressure* $p(X,T,f)$ is defined by

$$(2.1)\qquad p(X,T,f) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log\Big(\sup_E \sum_{x\in E} \exp\Big(\sum_{j=0}^{n-1} f(T^j x)\Big)\Big),$$

where the supremum is taken over all (n,ε)-separated subsets E of X. Define the *topological entropy* by

$$(2.2)\qquad h_{\mathrm{top}}(X,T) := p(X,T,0).$$

For $x \in X$ we define the ω*-limit set* $\omega(x)$ of x as the set of all limit points of the sequence $(T^n x)_{n\in\mathbb{N}_0}$. A subset $R \subseteq X$ is called *topologically transitive* if there exists an $x \in R$ with $\omega(x) = R$. Note that every topologically transitive $R \subseteq X$ is closed. If μ is a Borel probability measure on X and $f : X \to \mathbb{R}$ is a Borel measurable function which is integrable with respect to μ, then define

$$(2.3)\qquad \mu(f) := \int_X f\,d\mu.$$

We denote the measure-theoretic entropy of T with respect to μ by $h_\mu(T)$ (see e.g. §4.4 of [11] for the definition).

In general a piecewise monotonic map need not be continuous. One can use a standard doubling points construction in order to get a topological dynamical system (see e.g. [9] for the details). Hence for every nonempty closed T-invariant $A \subseteq [0,1]$ and for every $f \in C([0,1])$ we can define the topological pressure $p(A,T,f)$ (and therefore also the topological entropy $h_{\mathrm{top}}(A,T)$). Observe that by Lemma 2 of [8] this definition does not depend on the partition $\mathcal{Z}$.

Let $T : [0,1] \to [0,1]$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of $[0,1]$, and let $A \subseteq [0,1]$ be a nonempty closed T-invariant set. For $f \in D^+([0,1])$ we define

$$q(A,T,f) = \sup_{B\in\mathcal{M}(A)} \sup_{\substack{g\in C([0,1])\\ g\le f}} p(B,T,g). \tag{2.4}$$

Observe that

$$q(A,T,f) = \sup_{B\in\mathcal{M}(A)} p(B,T,f) \quad \text{for } f\in C([0,1]). \tag{2.5}$$

From now on let $T:[0,1]\to[0,1]$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ such that $T'\in C([0,1])$. Then the function

$$\varphi := -\log|T'| \tag{2.6}$$

satisfies $\varphi\in D^+([0,1])$. Furthermore let $\psi\in C([0,1])$ be so that $q([0,1],T,\psi) = 0$. Then by Theorem 2 of [5] there exists an $e^{-\psi}$-conformal measure m on $[0,1]$. Now assume that A is completely invariant, topologically transitive, $A\subseteq \operatorname{supp} m$, and $h_{\mathrm{top}}(A,T)>0$.

Denote by $\mathcal{E}(A,T)$ the set of all ergodic T-invariant Borel probability measures μ on $[0,1]$ with $\mu(A)=1$, and by $\mathcal{E}_{\mathcal{M}}(A,T)$ the set of all ergodic T-invariant Borel probability measures μ on $[0,1]$ with $\mu(B)=1$ for some $B\in\mathcal{M}(A)$. By Lemma 1 of [5] we have

$$q(A,T,f) = \sup_{\mu\in\mathcal{E}_{\mathcal{M}}(A,T)} h_\mu(T)+\mu(f) \tag{2.7}$$

for every $f\in D^+([0,1])$.

Lemma 1. (1) *If $\mu\in\mathcal{E}_{\mathcal{M}}(A,T)$, then $\mu(\varphi)\le 0$ and $\mu(\psi)\le 0$.*

(2) *For every $\mu\in\mathcal{E}_{\mathcal{M}}(A,T)$ with $h_\mu(T)>0$ we have $\mu(\varphi)<0$ and $\mu(\psi)<0$.*

Proof. Since $q([0,1],T,\psi)=0$ we see by (2.7) that $h_\mu(T)+\mu(\psi)\le 0$. Hence $\mu(\psi)\le -h_\mu(T)$.

In order to prove $\mu(\varphi)\le 0$ assume that $\mu(\varphi)>0$. Let $B\in\mathcal{M}(A)$ with $\mu(B)=1$. As $B\in\mathcal{M}(A)$ the map $T|_B$ is a Markov map. By Propositions 21.2 and 21.8 of [1] there exist $p\in B$ and $k\in\mathbb{N}$ with $T^kp=p$ and $\mu_p(\varphi)>0$, where $\mu_p(C) := k^{-1}\sum_{j=0}^{k-1} 1_C(T^jp)$ for every Borel set $C\subseteq[0,1]$. Hence $|(T^k)'p| = e^{-k\mu_p(\varphi)}<1$, and therefore p is an attracting periodic point. Since there is an $x\in A$ whose ω-limit set equals A, we obtain $A=\{T^jp : j\in\{0,1,\dots,k-1\}\}$, which contradicts $h_{\mathrm{top}}(A,T)>0$.

Finally suppose that $h_\mu(T)>0$. Then Theorem 2 of [2] gives $h_\mu(T)\le -\mu(\varphi)$, completing the proof. ∎

Next we prove that $(t,s)\mapsto q(A,T,t\varphi+s\psi)$ is continuous and decreasing. Set $G(T):=\mathbb{R}^2$ and $G_0(T):=\mathbb{R}$ if $\varphi\in C([0,1])$; otherwise set $G(T):=\{(t,s)\in\mathbb{R}^2 : t\ge 0\}$ and $G_0(T):=\{t\in\mathbb{R} : t\ge 0\}$. In the first case $t\varphi+s\psi\in C([0,1])$ for all $(t,s)\in G(T)$, and in the second case $t\varphi+s\psi\in D^+([0,1])$ for all $(t,s)\in G(T)$.

LEMMA 2. *The function* $(t,s) \mapsto q(A,T,t\varphi+s\psi)$ *defined on* $G(T)$ *is continuous. Furthermore*, *for* $(t_1,s_1),(t_2,s_2) \in G(T)$ *with* $t_1 \le t_2$ *and* $s_1 \le s_2$ *we have*

$$q(A,T,t_1\varphi+s_1\psi) \ge q(A,T,t_2\varphi+s_2\psi).$$

Proof. Set $R := -\inf_{x\in[0,1]} \varphi(x)$ and $S := -\inf_{x\in[0,1]} \psi(x)$. First we prove that for every $(t_1,s_1),(t_2,s_2) \in G(T)$ we have

$$|q(A,T,t_1\varphi+s_1\psi) - q(A,T,t_2\varphi+s_2\psi)| \le R|t_1-t_2| + S|s_1-s_2|. \tag{2.8}$$

Fix $(t_1,s_1),(t_2,s_2) \in G(T)$ and let $\varepsilon > 0$. Then by (2.7) there exists a $\mu \in \mathcal{E}_{\mathcal{M}}(A,T)$ with

$$q(A,T,t_1\varphi+s_1\psi) < h_\mu(T) + t_1\mu(\varphi) + s_1\mu(\psi) + \varepsilon.$$

On the other hand, (2.7) also gives $h_\mu(T) + t_2\mu(\varphi) + s_2\mu(\psi) \le q(A,T,t_2\varphi+s_2\psi)$. Hence using also Lemma 1 we obtain (2.8).

Obviously (2.8) implies the continuity of $(t,s) \mapsto q(A,T,t\varphi+s\psi)$. Assume that $t_1 \le t_2$, $s_1 \le s_2$ and $q(A,T,t_1\varphi+s_1\psi) < q(A,T,t_2\varphi+s_2\psi)$. Then (2.7) gives the existence of a $\mu \in \mathcal{E}_{\mathcal{M}}(A,T)$ such that $(t_2-t_1)\mu(\varphi) + (s_2-s_1)\mu(\psi) > 0$. As $\mu(\varphi) \le 0$ and $\mu(\psi) \le 0$ by Lemma 1, we arrive at a contradiction. ■

For $s \in \mathbb{R}$ and $t \in G_0(T)$ we define

$$\varrho_s(t) := q(A,T,t\varphi+s\psi). \tag{2.9}$$

Our next result follows immediately from Lemma 2.

LEMMA 3. *For each* $s \in \mathbb{R}$ *the function* $t \mapsto \varrho_s(t)$ *is continuous and decreasing.*

Now we investigate further properties of the function $t \mapsto \varrho_s(t)$.

LEMMA 4. (1) $\lim_{t\to-\infty} \varrho_s(t) = \infty$ *if* $\varphi \in C([0,1])$.

(2) *If* $s \ge 0$, *then* $\varrho_s(1) \le 0$.

(3) *If* $\sup_{x\in A} \varphi(x) < 0$ *and* $s < 0$, *then there exists a* $t_0 \in \mathbb{R}$ *with* $\varrho_s(t) \le 0$ *for all* $t \ge t_0$.

Proof. (1) Since $h_{\mathrm{top}}(A,T) > 0$, Lemma 6 of [8] shows that there exists a $B \in \mathcal{M}(A)$ with $h_{\mathrm{top}}(B,T) > 0$. Therefore by the variational principle there exists a $\mu \in \mathcal{E}(B,T) \subseteq \mathcal{E}_{\mathcal{M}}(A,T)$ with $h_\mu(T) > 0$. Now Lemma 1(2) implies $\mu(\varphi) < 0$. By (2.7) we obtain $\varrho_s(t) \ge h_\mu(T) + t\mu(\varphi) + s\mu(\psi)$, and hence $\lim_{t\to-\infty} \varrho_s(t) = \infty$.

(2) Assume that $\varrho_s(1) > 0$. Then (2.7) gives the existence of a $\mu \in \mathcal{E}_{\mathcal{M}}(A,T)$ with $h_\mu(T) + \mu(\varphi) + s\mu(\psi) > 0$. As $\mu(\psi) \le 0$ by Lemma 1(1) and $s \ge 0$ this implies $h_\mu(T) > -\mu(\varphi)$. Again by Lemma 1(1) we get $\mu(\varphi) \le 0$, and hence $h_\mu(T) > 0$. Now Theorem 2 of [2] gives $h_\mu(T) \le -\mu(\varphi)$, contradicting $h_\mu(T) > -\mu(\varphi)$.

(3) In this case we have $t\varphi + s\psi \in C([0,1])$ for all $t \in \mathbb{R}$. Suppose that $t \geq 0$ and $\varrho_s(t) > 0$. By (2.7) there exists a $\mu \in \mathcal{E}_{\mathcal{M}}(A,T)$ with

$$0 < h_\mu(T) + t\mu(\varphi) + s\mu(\psi) \leq h_{\mathrm{top}}(A,T) + t \sup_{x\in A} \varphi(x) + |s| \sup_{x\in A} |\psi(x)|.$$

Therefore

$$t < t_0 := \frac{h_{\mathrm{top}}(A,T) + |s| \sup_{x\in A} |\psi(x)|}{-\sup_{x\in A} \varphi(x)}. \blacksquare$$

For $s \in \mathbb{R}$ we define

$$z_s(A) := \sup\{t \in G_0(T) : q(A,T,t\varphi + s\psi) > 0\}, \tag{2.10}$$

where the values $-\infty$ and ∞ are allowed for $z_s(A)$. By Lemma 3,

$$\begin{aligned} &\varrho_s(t) > 0 \quad \text{for } t \in G_0(T) \text{ with } t < z_s(A),\\ &\varrho_s(t) \leq 0 \quad \text{for } t \in G_0(T) \text{ with } t \geq z_s(A). \end{aligned} \tag{2.11}$$

If $\varphi \in C([0,1])$, then Lemma 4(1) implies $z_s(A) \in (-\infty,\infty]$. Furthermore, by Lemma 4(2) we find that in this case $z_s(A) \in \mathbb{R}$ for all $s \geq 0$. Finally, using also Lemma 4(3) we conclude that $z_s(A) \in \mathbb{R}$ for all $s \in \mathbb{R}$ if $\sup_{x\in A} \varphi(x) < 0$.

3. Multifractal dimensions of a measure. Recall that $T : [0,1] \to [0,1]$ is a piecewise monotonic map with respect to a finite partition $\mathcal{Z}$ such that $T' \in C([0,1])$, and that $\varphi := -\log |T'|$. Furthermore recall that $\psi \in C([0,1])$ with $q([0,1],T,\psi) = 0$, and m is an $e^{-\psi}$-conformal measure on $[0,1]$ (its existence follows from Theorem 2 of [5]).

Assume that $\mathcal{Y}$ is a finite or countable collection of pairwise disjoint open intervals which refines $\mathcal{Z}$. Set $E(\mathcal{Y}) := \bigcap_{j=0}^{\infty} T^{-j}(\bigcup_{Y\in\mathcal{Y}} Y)$. Then for every $x \in E(\mathcal{Y})$ and for every $n \in \mathbb{N}$ there exist unique $Y_0, Y_1, \ldots, Y_{n-1} \in \mathcal{Y}$ with $T^j x \in Y_j$ for all $j \in \{0,1,\ldots,n-1\}$. For $x \in E(\mathcal{Y})$ and $n \in \mathbb{N}$ we define

$$Y_n(x) := \bigcap_{j=0}^{n-1} T^{-j} Y_j, \tag{3.1}$$

where $Y_0, Y_1, \ldots, Y_{n-1}$ are as above. If there exists a T-invariant Borel probability measure μ with $\mu(\bigcup_{Y\in\mathcal{Y}} Y) = 1$, then $\mu(E(\mathcal{Y})) = 1$, and hence $Y_n(x)$ is defined for μ-almost all x in this case.

Suppose that $\widetilde{m}$ is a Borel probability measure on $[0,1]$. Let $U \subseteq [0,1]$ be an interval, and assume that x is contained in the interior of U. Then there exist two disjoint intervals U_1 and U_2 with $x \notin U_1$, $x \notin U_2$ and $U = U_1 \cup U_2 \cup \{x\}$. We define

$$d_{\widetilde{m}}(x,U) := \min\{\widetilde{m}(U_1), \widetilde{m}(U_2)\}. \tag{3.2}$$

Fix $p > 0$. For a function $g : [0,1] \to \mathbb{R}$ we define

$$\operatorname{var}^p g := \sup \Big\{ \sum_{j=1}^{n} |g(x_j) - g(x_{j-1})|^p \Big\}, \tag{3.3}$$

where the supremum is taken over all $n \in \mathbb{N}$ and all $x_0, x_1, \ldots, x_n \in [0,1]$ with $x_0 < x_1 < \ldots < x_n$. We say that g is of *bounded p-variation* if $\operatorname{var}^p g < \infty$.

LEMMA 5. *Assume that μ is an ergodic T-invariant Borel probability measure with $h_\mu(T) > 0$. Suppose that $g_1, \ldots, g_q : [0,1] \to \mathbb{R}$ are integrable with respect to μ, and that for each $j \in \{1, \ldots, q\}$, $g_j \in C([0,1])$ or e^{g_j} is of bounded p-variation for some $p > 0$. Then for each $\varepsilon > 0$ there exists a finite or countable collection $\mathcal{Y}$ of pairwise disjoint open intervals refining $\mathcal{Z}$ such that*

(1) $\mu(\bigcup_{Y \in \mathcal{Y}} Y) = 1$,

(2) $\sup_{x \in Y} g_j(x) - \inf_{x \in Y} g_j(x) < \varepsilon$ *for every* $Y \in \mathcal{Y}$ *and every* $j \in \{1, \ldots, q\}$,

(3) $\lim_{n\to\infty} -n^{-1} \log \mu(Y_n(x)) = h_\mu(T)$ *for μ-almost all x, and*

(4) *whenever* $t_1, \ldots, t_q \in \mathbb{R}$ *and* $\widetilde{m}$ *is an* $e^{t_1 g_1 + \ldots + t_q g_q}$*-conformal measure with* $\mu(\operatorname{supp} \widetilde{m}) = 1$, *then* $\lim_{n\to\infty} n^{-1} \log d_{\widetilde{m}}(T^n x, T^n Y_{n+1}(x)) = 0$ *for μ-almost all x.*

Proof. We claim that for each $j \in \{1, \ldots, q\}$ there exists a finite or countable collection $\mathcal{Y}_j$ of pairwise disjoint open intervals refining $\mathcal{Z}$ such that $\mu(\bigcup_{Y \in \mathcal{Y}_j} Y) = 1$, $\sup_{x \in Y} g_j(x) - \inf_{x \in Y} g_j(x) < \varepsilon$ for every $Y \in \mathcal{Y}_j$, and

$$H_\mu(\mathcal{Y}_j) := -\sum_{Y \in \mathcal{Y}_j} \mu(Y) \log \mu(Y) < \infty. \tag{3.4}$$

If e^{g_j} is of bounded p-variation for some $p > 0$, then the claim follows from Lemma 1 of [4]. Otherwise $g_j \in C([0,1])$ and obviously there exists a finite collection $\mathcal{Y}_j$ of pairwise disjoint open intervals refining $\mathcal{Z}$ with $\mu(\bigcup_{Y \in \mathcal{Y}_j} Y) = 1$ and $\sup_{x \in Y} g_j(x) - \inf_{x \in Y} g_j(x) < \varepsilon$ for every $Y \in \mathcal{Y}_j$. As $\mathcal{Y}_j$ is finite, (3.4) holds trivially in this case.

Now let $\mathcal{Y}$ be the collection of all nonempty sets Y of the form $Y = \bigcap_{j=1}^{q} Y_j$ with $Y_j \in \mathcal{Y}_j$ for $j \in \{1, \ldots, q\}$. It is obvious that $\mathcal{Y}$ satisfies (1) and (2). Since $H_\mu(\mathcal{Y}) \le \sum_{j=1}^{q} H_\mu(\mathcal{Y}_j)$ we obtain $H_\mu(\mathcal{Y}) < \infty$.

By (3.4) and the Shannon–McMillan–Breiman Theorem we obtain

$$\lim_{n\to\infty} -\frac{1}{n} \log \mu(Y_n(x)) = h_\mu(T)$$

for μ-almost all x.

Set $\mathcal{D} := \{T^{n-1}Y_n(x) : x \in E(\mathcal{Y}),\, n \in \mathbb{N}\}$. Then $\mathcal{D}$ is a finite or countable set of open intervals. Choose an arbitrary $E \in \mathcal{D}$ with $\mu(E) > 0$. As μ has no atoms, E can be written as a union $C_1 \cup C \cup C_2$ of pairwise disjoint intervals with $C_1 < C < C_2$, $\mu(\operatorname{int} C) > 0$, $\mu(\operatorname{int} C_1) > 0$ and $\mu(\operatorname{int} C_2) > 0$. Since $\mu(\operatorname{supp} \widetilde{m}) = 1$ we get $\widetilde{m}(C_1) > 0$ and $\widetilde{m}(C_2) > 0$. Hence $c := \min\{\widetilde{m}(C_1), \widetilde{m}(C_2)\} > 0$. By (3.2) we obtain

$$d_{\widetilde{m}}(x, E) \geq c > 0 \tag{3.5}$$

for every $x \in C$. A proof analogous to the first part of the proof of Proposition 2 of [4] shows the existence of a set $L \subseteq [0,1]$ with $\mu(L) = 1$ such that for every $x \in L$ there exists a strictly increasing sequence $(n_k(x))_{k\in\mathbb{N}}$ in $\mathbb{N}$ with

(i) $T^{n_k(x)}x \in C$ and $T^{n_k(x)}Y_{n_k(x)+1}(x) = E$ for every $k \in \mathbb{N}$,
(ii) $\lim_{k\to\infty} n_{k+1}(x)/n_k(x) = 1$, and
(iii) $\lim_{n\to\infty} n^{-1}\sum_{j=0}^{n-1} f(T^j x) = \mu(f)$, where $f := t_1 g_1 + \ldots + t_q g_q$.

Next we fix an $x \in L$. For $n \in \mathbb{N}$ set $D_n := T^n Y_{n+1}(x)$ and $r_n(x) := d_{\widetilde{m}}(T^n x, T^n Y_{n+1}(x))$. Since $\widetilde{m}$ is an e^f-conformal measure we get

$$r_{n+1}(x) \leq d_{\widetilde{m}}(T^{n+1}x, TD_n) \leq r_n(x) \sup_{y\in D_n} e^{f(y)},$$

because $T^n x \in D_n$ and $D_{n+1} \subseteq TD_n$. If we set $\alpha := (|t_1| + \ldots + |t_q|)\varepsilon$ this implies $r_{n+1}(x) \leq r_n(x)\exp(f(T^n x) + \alpha)$.

If $l \in \mathbb{N}$ is large enough we can find a $k \in \mathbb{N}$ with $n_{k-1}(x) < l \leq n_k(x)$. Since $T^{n_k(x)}x \in C$ and $D_{n_k(x)} = E$ we deduce by (3.5) that $r_{n_k(x)}(x) \geq c$. Hence

$$c \leq r_{n_k(x)}(x) \leq r_l(x) \exp\Big(\sum_{j=l}^{n_k(x)-1} f(T^j x) + (n_k(x) - l)\alpha\Big).$$

Observe that (ii) implies $\lim_{l\to\infty} n_k(x)/l = 1$. Using this and (iii) we get

$$\begin{aligned}\lim_{l\to\infty} \frac{1}{l} \sum_{j=l}^{n_k(x)-1} & f(T^j x) \\ &= \lim_{l\to\infty} \frac{n_k(x)}{l} \frac{1}{n_k(x)} \sum_{j=0}^{n_k(x)-1} f(T^j x) - \lim_{l\to\infty} \frac{1}{l} \sum_{j=0}^{l-1} f(T^j x) \\ &= \mu(f) - \mu(f) = 0.\end{aligned}$$

As $r_l(x) \leq 1$ the estimates above imply $\lim_{l\to\infty} l^{-1} \log r_l(x) = 0$, which completes the proof. ∎

For $r > 0$ and $x \in [0,1]$ set $B_r(x) := \{y \in [0,1] : |y-x| < r\}$. Throughout this section we assume that $\varphi \in C([0,1])$ or T' is of bounded p-variation for some $p > 0$.

LEMMA 6. *Suppose that $\mathcal{Z}$ is a generator. Let μ be an ergodic T-invariant Borel probability measure with $h_\mu(T) > 0$ and $\mu(\operatorname{supp} m) = 1$. Set*

$$M := \left\{x \in [0,1] : \lim_{r\to 0^+} \frac{\log \mu(B_r(x))}{\log r} = \frac{h_\mu(T)}{\chi_\mu} \text{ and } \lim_{r\to 0^+} \frac{\log m(B_r(x))}{\log r} = \frac{-\mu(\psi)}{\chi_\mu}\right\},$$

where $\chi_\mu := -\mu(\varphi)$. Then $\mu(M) = 1$.

Proof. Define

$$M_1 := \left\{x \in [0,1] : \lim_{r\to 0^+} \frac{\log \mu(B_r(x))}{\log r} = \frac{h_\mu(T)}{\chi_\mu}\right\},$$
$$M_2 := \left\{x \in [0,1] : \lim_{r\to 0^+} \frac{\log m(B_r(x))}{\log r} = \frac{-\mu(\psi)}{\chi_\mu}\right\}$$

and $M := M_1 \cap M_2$. It remains to show that $\mu(M_1) = 1$ and $\mu(M_2) = 1$.

By Theorem 1 of [3] we get $\mu(M_1) = 1$, as $\chi_\mu > 0$ by Theorem 2 of [2].

Choose a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ in $\{x \in \mathbb{R} : x > 0\}$ with $\varepsilon_k < \chi_\mu/2$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \varepsilon_k = 0$. Fix a $k \in \mathbb{N}$. Denote the Lebesgue measure on $[0,1]$ by λ. Observe that for every interval $U \subseteq [0,1]$ we have $\lambda(U) = |U|$. By Lemma 5 there exists an $L_k \subseteq [0,1]$ with $\mu(L_k) = 1$, and there exists a finite or countable collection $\mathcal{Y}$ of pairwise disjoint open intervals refining $\mathcal{Z}$ such that

(1) $\mu(\bigcup_{Y\in\mathcal{Y}} Y) = 1$,
(2) $\sup_{x\in Y} \varphi(x) - \inf_{x\in Y} \varphi(x) < \varepsilon_k$ for every $Y \in \mathcal{Y}$,
(3) $\sup_{x\in Y} \psi(x) - \inf_{x\in Y} \psi(x) < \varepsilon_k$ for every $Y \in \mathcal{Y}$,
(4) $\lim_{n\to\infty} n^{-1} \sum_{j=0}^{n-1} \varphi(T^j x) = \mu(\varphi)$ for every $x \in L_k$,
(5) $\lim_{n\to\infty} n^{-1} \sum_{j=0}^{n-1} \psi(T^j x) = \mu(\psi)$ for every $x \in L_k$,
(6) $\lim_{n\to\infty} n^{-1} \log d_\lambda(T^n x, T^n Y_{n+1}(x)) = 0$ for every $x \in L_k$, and
(7) $\lim_{n\to\infty} n^{-1} \log d_m(T^n x, T^n Y_{n+1}(x)) = 0$ for every $x \in L_k$.

We may assume that $|Y| < 1$ for all $Y \in \mathcal{Y}$.

Let $x \in L_k$. For $n \in \mathbb{N}$ define $r_n(x) := d_\lambda(x, Y_n(x))$. Obviously $r_n(x) \le |Y_n(x)|$ and $r_n(x)$ is decreasing in n. Since $\mathcal{Z}$ is a generator and $\mathcal{Y}$ refines $\mathcal{Z}$, we get $\lim_{n\to\infty} r_n(x) = \lim_{n\to\infty} |Y_n(x)| = 0$. By (1.1) and (3.2),

$$r_n(x) \ge \exp\Big(\sum_{j=0}^{n-1} \varphi(T^j x) - n\varepsilon_k\Big) d_\lambda(T^n x, T^n Y_{n+1}(x)) \tag{3.6}$$

for every $n \in \mathbb{N}$. Therefore (6) above implies $r_n(x) > 0$ for every $n \in \mathbb{N}$. Now (4)–(7) imply that there exists an $n_0 \in \mathbb{N}$ with

$$\begin{aligned}
&n(\mu(\varphi) - \varepsilon_k) \le \sum_{j=0}^{n-1} \varphi(T^j x) \le n(\mu(\varphi) + \varepsilon_k),\\
&n(\mu(\psi) - \varepsilon_k) \le \sum_{j=0}^{n-1} \psi(T^j x) \le n(\mu(\psi) + \varepsilon_k),\\
&-n\varepsilon_k \le \log d_\lambda(T^n x, T^n Y_{n+1}(x)),\\
&-n\varepsilon_k \le \log d_m(T^n x, T^n Y_{n+1}(x)),\\
&\frac{n}{n-1}\,\frac{\mu(\psi) - 3\varepsilon_k}{\mu(\varphi) + 2\varepsilon_k} \le \frac{\mu(\psi) - 4\varepsilon_k}{\mu(\varphi) + 2\varepsilon_k},\\
&\frac{n-1}{n}\,\frac{\mu(\psi) + 2\varepsilon_k}{\mu(\varphi) - 3\varepsilon_k} \ge \frac{\mu(\psi) + 3\varepsilon_k}{\mu(\varphi) - 3\varepsilon_k},
\end{aligned} \tag{3.7}$$

for all $n \ge n_0$. Using (2) and (3) we get, by (1.1), (1.2), (3.2), (3.6), and (3.7),

$$\begin{aligned}
&n(\mu(\psi) - 3\varepsilon_k) \le \log m(Y_n(x)) \le n(\mu(\psi) + 2\varepsilon_k),\\
&\log |Y_n(x)| \le n(\mu(\varphi) + 2\varepsilon_k),\\
&\log r_n(x) \ge n(\mu(\varphi) - 3\varepsilon_k),
\end{aligned} \tag{3.8}$$

for every $n \ge n_0$. Now fix an arbitrary $r > 0$ with $r < r_{n_0}(x)$. We estimate $\log m(B_r(x))/\log r$ from above. By the choice of r there is an $n > n_0$ with $|Y_n(x)| \le r < |Y_{n-1}(x)|$. Hence $Y_n(x) \subseteq B_r(x)$, and therefore

$$\frac{\log m(B_r(x))}{\log r} \le \frac{\log m(Y_n(x))}{\log |Y_{n-1}(x)|}.$$

As by (3.8),

$$\frac{\log m(Y_n(x))}{\log |Y_{n-1}(x)|} \le \frac{n}{n-1}\,\frac{\mu(\psi) - 3\varepsilon_k}{\mu(\varphi) + 2\varepsilon_k},$$

(3.7) gives

$$\frac{\log m(B_r(x))}{\log r} \le \frac{\mu(\psi) - 4\varepsilon_k}{\mu(\varphi) + 2\varepsilon_k}.$$

Therefore

$$\limsup_{r \to 0^+} \frac{\log m(B_r(x))}{\log r} \le \frac{\mu(\psi) - 4\varepsilon_k}{\mu(\varphi) + 2\varepsilon_k} \tag{3.9}$$

for every $x \in L_k$.

Next we have to give a lower bound for $\log m(B_r(x))/\log r$. Since $r < r_{n_0}(x)$ there exists an $n > n_0$ with $r_n(x) \le r < r_{n-1}(x)$. As $r < r_{n-1}(x)$ we

have $B_r(x) \subseteq Y_{n-1}(x)$. This implies
$$\frac{\log m(B_r(x))}{\log r} \geq \frac{\log m(Y_{n-1}(x))}{\log r_n(x)}.$$
By (3.8) we get
$$\frac{\log m(Y_{n-1}(x))}{\log r_n(x)} \geq \frac{n-1}{n}\,\frac{\mu(\psi)+2\varepsilon_k}{\mu(\varphi)-3\varepsilon_k}.$$
Hence (3.7) implies
$$\frac{\log m(B_r(x))}{\log r} \geq \frac{\mu(\psi)+3\varepsilon_k}{\mu(\varphi)-3\varepsilon_k}.$$
Therefore
$$\liminf_{r\to 0^+} \frac{\log m(B_r(x))}{\log r} \geq \frac{\mu(\psi)+3\varepsilon_k}{\mu(\varphi)-3\varepsilon_k} \tag{3.10}$$
for every $x \in L_k$.

Finally set $L := \bigcap_{k=1}^{\infty} L_k$. Since $\mu(L_k) = 1$ for all $k \in \mathbb{N}$ we have $\mu(L) = 1$. Let $x \in L$. Then (3.9) and (3.10) imply
$$\lim_{r\to 0^+} \frac{\log m(B_r(x))}{\log r} = \frac{\mu(\psi)}{\mu(\varphi)},$$
because $\lim_{k\to\infty} \varepsilon_k = 0$. Hence $L \subseteq M_2$. ∎

Now we are able to prove a formula for the multifractal dimension of μ. Recall that we assume that $\varphi \in C([0,1])$ or T' is of bounded p-variation for some $p > 0$, that $\psi \in C([0,1])$, and that $\mathcal{Z}$ is a generator.

Theorem 1. *Let μ be an ergodic T-invariant Borel probability measure with $h_\mu(T) > 0$ and $\mu(\operatorname{supp} m) = 1$. Define $\chi_\mu := -\int \varphi\, d\mu$. Then for every $s \in \mathbb{R}$ we have*
$$d_s(\mu) = D_s(\mu) = \frac{h_\mu(T) + s\int \psi\, d\mu}{\chi_\mu}.$$

Proof. Set $\alpha := h_\mu(T)/\chi_\mu$ and $\beta := -\mu(\psi)/\chi_\mu$. By Theorem 2 of [2] we have $\alpha > 0$. Furthermore Lemma 6 gives $\beta \geq 0$. We will prove $D_s(\mu) \leq \alpha - s\beta \leq d_s(\mu)$.

In order to prove $D_s(\mu) \leq \alpha - s\beta$ choose an arbitrary $t > \alpha - s\beta$ and set $\varepsilon := \frac{1}{2}(t - \alpha + s\beta)$. For $k \in \mathbb{N}$ define
$$\begin{aligned} M_k := \{x \in [0,1] : \mu(B_r(x))^{(t-\varepsilon)/\alpha} \geq k^{-1} r^t \text{ and}& \\ \mu(B_r(x))^{(s\beta-\varepsilon)/\alpha} \geq k^{-1} m(B_r(x))^s \text{ for all } r \in (0,1)\}&. \end{aligned} \tag{3.11}$$
Choose an $x \in [0,1]$ with
$$\lim_{r\to 0^+} \frac{\log \mu(B_r(x))}{\log r} = \alpha \quad \text{and} \quad \lim_{r\to 0^+} \frac{\log m(B_r(x))}{\log r} = \beta.$$

Then there exists an $r_0(x) > 0$ with

$$\frac{t-\varepsilon}{\alpha}\,\frac{\log\mu(B_r(x))}{\log r} \le t,$$
$$\frac{s\beta-\varepsilon}{\alpha}\,\frac{\log\mu(B_r(x))}{\log m(B_r(x))} = \frac{s\beta-\varepsilon}{\alpha}\,\frac{\log\mu(B_r(x))}{\log r}\,\frac{\log r}{\log m(B_r(x))} \le s$$

for all $r \in (0, r_0(x))$ (note that this is also true in the case $\beta = 0$). Hence there is a $k\in\mathbb{N}$ with $x\in M_k$, and therefore Lemma 6 implies $\mu(\bigcup_{k=1}^{\infty} M_k)=1$. As $D_s(\bigcup_{k=1}^{\infty} M_k) = \sup_{k\in\mathbb{N}} D_s(M_k)$ (see e.g. p. 90 of [6]), using (1.8) and (1.10) it remains to show $\pi_{s,t}(M_k) < \infty$ for all $k \in \mathbb{N}$.

Fix a $k \in \mathbb{N}$ and a $\delta \in (0, 1)$, and let $\mathcal{C}$ be a centered δ-packing of M_k. Then (3.11) gives

$$\sum_{C\in\mathcal{C}} m(C)^s|C|^t \le 2^t k^2 \sum_{C\in\mathcal{C}} \mu(C)^{(s\beta-\varepsilon)/\alpha}\mu(C)^{(t-\varepsilon)/\alpha} = 2^t k^2 \sum_{C\in\mathcal{C}} \mu(C) \le 2^t k^2.$$

Now (1.6) shows $\pi_{s,t}(M_k) < \infty$, completing the proof of $D_s(\mu) \le \alpha - s\beta$.

Next we prove that $\alpha - s\beta \le d_s(\mu)$. By (1.10) it suffices to show $\mu(L) = 0$ for every Borel set $L \subseteq [0, 1]$ with $d_s(L) < \alpha - s\beta$. Let L be such a set. Choose a t with $d_s(L) < t < \alpha - s\beta$ and set $\varepsilon := \frac{1}{2}(\alpha - s\beta - t)$. For $k \in \mathbb{N}$ define

$$L_k := \{x \in L : \mu(B_r(x))^{(t+\varepsilon)/\alpha} \le kr^t \text{ and } \mu(B_r(x))^{(s\beta+\varepsilon)/\alpha} \le km(B_r(x))^s \text{ for all } r \in (0, 1)\}. \tag{3.12}$$

If $x \in L$ with

$$\lim_{r\to 0^+} \frac{\log\mu(B_r(x))}{\log r} = \alpha \quad\text{and}\quad \lim_{r\to 0^+} \frac{\log m(B_r(x))}{\log r} = \beta,$$

then there exists an $r_0(x) > 0$ with

$$\frac{t+\varepsilon}{\alpha}\,\frac{\log\mu(B_r(x))}{\log r} \ge t,$$
$$\frac{s\beta+\varepsilon}{\alpha}\,\frac{\log\mu(B_r(x))}{\log m(B_r(x))} = \frac{s\beta+\varepsilon}{\alpha}\,\frac{\log\mu(B_r(x))}{\log r}\,\frac{\log r}{\log m(B_r(x))} \ge s,$$

for all $r \in (0, r_0(x))$ (also in the case $\beta = 0$). Therefore there exists a $k \in \mathbb{N}$ with $x \in L_k$, and we get $\mu(L) = \lim_{k\to\infty} \mu(L_k)$ by Lemma 6. Hence it remains to show $\mu(L_k) = 0$ for all $k \in \mathbb{N}$.

To this end fix a $k \in \mathbb{N}$ and an $\eta > 0$. By (1.3) and (1.5) there exists a $\delta \in (0, 1)$ and a centered δ-cover $\mathcal{C}$ of L_k with

$$\sum_{C\in\mathcal{C}} m(C)^s|C|^t < \frac{2^t}{k^2}\,\eta$$

since $d_s(L) < t$. Now (3.12) implies

$$\mu(L_k) \le \sum_{C\in\mathcal{C}} \mu(C) = \sum_{C\in\mathcal{C}} \mu(C)^{(s\beta+\varepsilon)/\alpha}\mu(C)^{(t+\varepsilon)/\alpha} \le \frac{k^2}{2^t}\sum_{C\in\mathcal{C}} m(C)^s|C|^t < \eta.$$

As $\eta > 0$ was arbitrary we obtain $\mu(L_k) = 0$, which completes the proof. ∎

4. Essential multifractal dimensions of invariant sets. Throughout this section let $T : [0,1] \to [0,1]$ be a piecewise monotonic map with respect to a finite partition $\mathcal{Z}$ such that $T' \in C([0,1])$ and $\mathcal{Z}$ is a generator. Set $\varphi := -\log|T'|$. We assume that $\varphi \in C([0,1])$ or T' is of bounded p-variation for some $p > 0$. Furthermore let $\psi \in C([0,1])$ with $q([0,1],T,\psi) = 0$, and let m be an $e^{-\psi}$-conformal measure on $[0,1]$ (its existence follows from Theorem 2 of [5]). Finally let $A \subseteq [0,1]$ be completely invariant and topologically transitive, and suppose that $A \subseteq \operatorname{supp} m$ and $h_{\mathrm{top}}(A,T) > 0$.

Set

$$s_A := \inf\{s \in \mathbb{R} : q(A,T,s\psi) = 0\}. \tag{4.1}$$

Note that $q(A,T,\psi) \le q([0,1],T,\psi) = 0$ by (2.7). As $h_{\mathrm{top}}(A,T) > 0$, Lemma 6 of [8] gives $q(A,T,0) > 0$. By Lemma 2 the function $s \mapsto q(A,T,s\psi)$ is continuous and decreasing. Hence $s_A \in (0,1]$. Furthermore $s < s_A$ implies $q(A,T,s\psi) > 0$. Recall the definition of $z_s(A)$ given in (2.10). Then Lemmas 2 and 4(2) give

$$z_s(A) \in (0,1] \tag{4.2}$$

for every $s \in [0,s_A)$.

If $s \in \mathbb{R}$, denote by $c_s(A)$ the infimum of all $t \in \mathbb{R}$ such that there exists an $e^{-t\varphi-s\psi}$-conformal measure $\widetilde{m}$ on $[0,1]$ with $\operatorname{supp}\widetilde{m} = A$, where the values $-\infty$ and ∞ are allowed for $c_s(A)$.

LEMMA 7. *Let $t \in \mathbb{R}$ and $s \in \mathbb{R}$. Suppose that there exists an $e^{-t\varphi-s\psi}$-conformal measure $\widetilde{m}$ on $[0,1]$ with $\operatorname{supp}\widetilde{m} = A$. If μ is an ergodic T-invariant Borel probability measure with $h_\mu(T) > 0$ and $\mu(A) = 1$, then*

$$h_\mu(T) + t\mu(\varphi) + s\mu(\psi) \le 0$$

and $\chi_\mu > 0$, where $\chi_\mu := -\mu(\varphi)$.

Proof. First note that $\chi_\mu > 0$ by Theorem 2 of [2].

Assume that $h_\mu(T) + t\mu(\varphi) + s\mu(\psi) > 0$. Choose an $\eta > 0$ with

$$h_\mu(T) + t\mu(\varphi) + s\mu(\psi) \ge 5\eta. \tag{4.3}$$

By Lemma 5 and the Ergodic Theorem there exists a finite or countable collection $\mathcal{Y}$ of pairwise disjoint open intervals refining $\mathcal{Z}$ and a set $M \subseteq A$ with $\mu(M) = 1$ such that

(1) $\sup_{x\in Y}(t\varphi+s\psi)(x)-\inf_{x\in Y}(t\varphi+s\psi)(x)<\eta$ for every $Y\in\mathcal{Y}$,
(2) $\lim_{n\to\infty} n^{-1}\sum_{j=0}^{n-1}(t\varphi+s\psi)(T^j x)=t\mu(\varphi)+s\mu(\psi)$ for all $x\in M$,
(3) $\lim_{n\to\infty} -n^{-1}\log\mu(Y_n(x))=h_\mu(T)$ for all $x\in M$, and
(4) $\lim_{n\to\infty} n^{-1}\log d_{\widetilde{m}}(T^n x, T^n Y_{n+1}(x))=0$ for all $x\in M$.

For $k\in\mathbb{N}$ define

$$\begin{aligned}M_k := \Big\{x\in M : {}& \frac{1}{n}\sum_{j=0}^{n-1}(t\varphi+s\psi)(T^j x)\ge t\mu(\varphi)+s\mu(\psi)-\eta,\\ & -\frac{1}{n}\log\mu(Y_n(x))\ge h_\mu(T)-\eta \text{ and}\\ & \frac{1}{n}\log d_{\widetilde{m}}(T^n x, T^n Y_{n+1}(x))\ge -\eta \text{ for all } n\ge k\Big\}.\end{aligned}$$

Obviously $\bigcup_{k=1}^{\infty} M_k = M$. Therefore we can fix a k with $\mu(M_k)>0$. Now fix an $n\ge k$ with $e^{-n\eta}<\mu(M_k)$.

Choose an $x\in M_k$. As $\widetilde{m}$ is an $e^{-t\varphi-s\psi}$-conformal measure, we get (cf. (1.2))

$$\begin{aligned}d_{\widetilde{m}}(T^n x, T^n Y_{n+1}(x)) &\le \widetilde{m}(T^n Y_n(x))\\ &\le \exp\Big(\sum_{j=0}^{n-1}(-t\varphi-s\psi)(T^j x)\Big)\, e^{n\eta}\,\widetilde{m}(Y_n(x)).\end{aligned}$$

Hence the definition of M_k gives $n^{-1}\log\widetilde{m}(Y_n(x))\ge t\mu(\varphi)+s\mu(\psi)-3\eta$. Using again the definition of M_k we infer by (4.3) that

$$\frac{1}{n}\log\frac{\widetilde{m}(Y_n(x))}{\mu(Y_n(x))}\ge h_\mu(T)+t\mu(\varphi)+s\mu(\psi)-4\eta\ge\eta.$$

This implies $\mu(Y_n(x))\le e^{-n\eta}\widetilde{m}(Y_n(x))$.

Define $\widehat{\mathcal{Y}} := \{Y_n(x) : x\in M_k\}$. Obviously $\mu(M_k)\le\sum_{Y\in\widehat{\mathcal{Y}}}\mu(Y)$ and $\sum_{Y\in\widehat{\mathcal{Y}}}\widetilde{m}(Y)\le 1$. Therefore

$$\mu(M_k)\le\sum_{Y\in\widehat{\mathcal{Y}}}\mu(Y)\le e^{-n\eta}\sum_{Y\in\widehat{\mathcal{Y}}}\widetilde{m}(Y)\le e^{-n\eta},$$

which contradicts $e^{-n\eta}<\mu(M_k)$. ∎

Recall that $G_0(T):=\mathbb{R}$ if $\varphi\in C([0,1])$, and $G_0(T):=\{t\in\mathbb{R} : t\ge 0\}$ otherwise. In the first case $\operatorname{int} G_0(T)=\mathbb{R}$, and in the second case $\operatorname{int} G_0(T)=\{t\in\mathbb{R} : t>0\}$. Observe that by (2.10), $z_s(A)\in\mathbb{R}$ implies $z_s(A)\in\operatorname{int} G_0(T)$.

Proposition 1. *Let* $s\in\mathbb{R}$.

(1) $e_s(A)=E_s(A)\le c_s(A)$.
(2) *If* $z_s(A)\in\mathbb{R}$, *then* $e_s(A)=E_s(A)\le c_s(A)\le z_s(A)$.

Proof. The definitions (1.11) of $e_s(A)$ and $E_s(A)$ together with Theorem 1 give $e_s(A) = E_s(A)$. In order to show $E_s(A) \leq c_s(A)$ choose an arbitrary $t \in \mathbb{R}$ such that there exists an $e^{-t\varphi - s\psi}$-conformal measure $\widetilde{m}$ on $[0,1]$ with $\operatorname{supp} \widetilde{m} = A$. Let $\mu \in \mathcal{E}^+(A,T)$. By Theorem 1 and Lemma 7 we obtain

$$D_s(\mu) = \frac{h_\mu(T) + s\mu(\psi)}{\chi_\mu} \leq t.$$

Hence the definitions of $E_s(A)$ and $c_s(A)$ imply $E_s(A) \leq c_s(A)$, which completes the proof of (1).

Suppose that $z_s(A) \in \mathbb{R}$. As $z_s(A) \in \operatorname{int} G_0(T)$, (2.10) and Lemma 2 imply $q(A,T,t\varphi + s\psi) = 0$ for $t = z_s(A)$. Now Theorem 2 of [5] gives the existence of an $e^{-t\varphi - s\psi}$-conformal measure $\widetilde{m}$ on $[0,1]$ with $\operatorname{supp} \widetilde{m} = A$. Therefore $c_s(A) \leq z_s(A)$, completing the proof. ∎

Remark. Observe that by Lemmas 2 and 4 we have $z_s(A) \in \mathbb{R}$ (and therefore we can apply Proposition 1(2)) if one of the following assumptions is satisfied:

(1) $s \in [0, s_A)$,
(2) $\varphi \in C([0,1])$ and $s \geq 0$, or
(3) $\sup_{x \in A} \varphi(x) < 0$ and $s \in \mathbb{R}$ is arbitrary.

Next we prove the main result of this section.

Theorem 2. *Let $s \in [0, s_A)$. Then*

$$e_s(A) = E_s(A) = c_s(A) = z_s(A).$$

Proof. We observed in (4.2) that $z_s(A) \in (0,1]$. Hence Proposition 1 gives $e_s(A) = E_s(A) \leq c_s(A) \leq z_s(A)$, and it remains to show $z_s(A) \leq e_s(A)$.

To this end choose an arbitrary $t \in (0, z_s(A))$. Then $q(A,T,t\varphi + s\psi) > 0$ by Lemma 2 and (2.10). By (2.7) there exists a $\mu \in \mathcal{E}_{\mathcal{M}}(A,T)$ with

$$h_\mu(T) + t\mu(\varphi) + s\mu(\psi) > 0. \tag{4.4}$$

As $\mu(\varphi) \leq 0$ and $\mu(\psi) \leq 0$ by Lemma 1, (4.4) implies $h_\mu(T) > 0$. Therefore Theorem 1, (1.11) and (4.4) give $t < (h_\mu(T) + s\mu(\psi))/\chi_\mu = d_s(\mu) \leq e_s(A)$, which completes the proof. ∎

5. Multifractal Hausdorff dimensions of invariant sets. In the previous section the essential multifractal dimension was treated. Now we investigate the multifractal Hausdorff dimension of A.

Lemma 8. *Assume that* $\sup_{x \in A} \varphi(x) < 0$. *Let* $s \in \mathbb{R}$.

(1) *The function $t \mapsto p(A,T,t\varphi + s\psi)$ is continuous and strictly decreasing, and there exists a unique $\widetilde{z}_s(A) \in \mathbb{R}$ with $p(A,T,\widetilde{z}_s(A)\varphi + s\psi) = 0$.*

(2) *If* $s \le s_A$, *then* $\widetilde{z}_s(A) \ge 0$.
(3) *If* $s < s_A$, *then* $\widetilde{z}_s(A) > 0$.

Proof. The proof of (1) is completely analogous to the proof of Lemma 3 of [8]. Then (2) and (3) follow from (4.1) and the fact that $p(A,T,s\psi) \ge q(A,T,s\psi)$. ∎

If $\sup_{x\in A}\varphi(x) < 0$ and $s \in \mathbb{R}$, then let $\widetilde{z}_s(A)$ be the number described in Lemma 8(1).

LEMMA 9. *Assume that* $\sup_{x\in A}\varphi(x) < 0$ *and* $\sup_{x\in A}\psi(x) < 0$.

(1) *For every* $t \in \mathbb{R}$ *the function* $s \mapsto p(A,T,t\varphi + s\psi)$ *is continuous and strictly decreasing.*

(2) *If* $s \ge 0$, *then* $\widetilde{z}_s(A) \le 1$.

Proof. A proof analogous to the proof of Lemma 3 of [8] shows (1). By Lemma 3 and Theorem 2 of [8] we get $p(A,T,\varphi) \le 0$. If $s \ge 0$, then $p(A,T,\varphi + s\psi) \le 0$ by (1), and Lemma 8(1) implies $\widetilde{z}_s(A) \le 1$. ∎

Our next result states that $\widetilde{z}_s(A)$ is an upper bound for $d_s(A)$ if $s \in [0, s_A]$. If $f : [0,1] \to \mathbb{R}$ is a function, then we define for $n \in \mathbb{N}$ and $x \in [0,1]$,

$$S_n f(x) := \sum_{j=0}^{n-1} f(T^j x). \tag{5.1}$$

LEMMA 10. *Assume that* $\sup_{x\in A}\varphi(x) < 0$, *and let* $s \in [0, s_A]$. *Then* $d_s(A) \le \widetilde{z}_s(A)$.

Proof. Let $t > \widetilde{z}_s(A)$. By Lemma 8(1) we get $p(A,T,t\varphi + s\psi) < 0$. Let $\eta > 0$ with

$$\eta < -\frac{p(A,T,t\varphi + s\psi)}{t+s+1}.$$

Choose a finite partition $\mathcal{Y}$ of $[0,1]$ refining $\mathcal{Z}$ with

$$\max_{Y\in\mathcal{Y}} \sup_{x,y\in Y} |\varphi(x) - \varphi(y)| < \eta, \qquad \max_{Y\in\mathcal{Y}} \sup_{x,y\in Y} |\psi(x) - \psi(y)| < \eta. \tag{5.2}$$

For $n \in \mathbb{N}$ define $\mathcal{Y}_n(A) := \{Y_n(x) : x \in A \cap E(\mathcal{Y})\}$. Since $\sup_{x\in A}\varphi(x) < 0$ we find that $\mathcal{Y}_n(A)$ is a generator for $T|_A$. Therefore

$$p(A,T,t\varphi + s\psi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{Y\in\mathcal{Y}_n(A)} \sup_{x\in Y} \exp(tS_n\varphi(x) + sS_n\psi(x)) \tag{5.3}$$

by Theorem 9.6 of [11]. By (5.3) and the choice of η there exists an $n_1 \in \mathbb{N}$ such that

$$\sum_{Y\in\mathcal{Y}_n(A)} \sup_{x\in Y} \exp(tS_n\varphi(x) + sS_n\psi(x)) < e^{-n(t+s+1)\eta} \tag{5.4}$$

for all $n \ge n_1$.

Let $F \subseteq A$ and $\varepsilon > 0$. Then there exists an $n_2 \in \mathbb{N}$ with $n_2 \geq n_1$ such that $|Y| < \varepsilon$ for any $Y \in \mathcal{Y}_n(A)$ with $n \geq n_2$. Fix an $n \geq n_2$. If $Y \in \mathcal{Y}_n(A)$ satisfies $Y \cap F \neq \emptyset$, then there exist $x_1, x_2 \in F$ and $\alpha_1, \alpha_2 \in (0, \varepsilon)$ with

$$F \cap Y = F \cap ((x_1 - \alpha_1, x_1 + \alpha_1) \cup (x_2 - \alpha_2, x_2 + \alpha_2)).$$

Hence

$$\inf_{\mathcal{C} \in \mathcal{U}_\varepsilon(F)} \sum_{C \in \mathcal{C}} m(C)^s |C|^t \leq 2 \sum_{Y \in \mathcal{Y}_n(A)} m(Y)^s |Y|^t. \tag{5.5}$$

Using (1.1) we get by induction

$$|T^n Y| = \int_Y e^{-S_n \varphi(x)}\, dx \geq |Y| \exp(-\sup_{x \in Y} S_n \varphi(x))$$

if $Y \in \mathcal{Y}_n(A)$. This implies

$$|Y|^t \leq \exp(t \sup_{x \in Y} S_n \varphi(x)) \tag{5.6}$$

for all $Y \in \mathcal{Y}_n(A)$. Analogously using (1.2) we obtain

$$m(Y)^s \leq \exp(s \sup_{x \in Y} S_n \psi(x)) \tag{5.7}$$

for all $Y \in \mathcal{Y}_n(A)$. By (5.2) we have

$$\begin{aligned} \exp(t \sup_{x \in Y} S_n \varphi(x)) \exp(s \sup_{x \in Y} S_n \psi(x)) \\ \leq e^{n(t+s)\eta} \sup_{x \in Y} \exp(t S_n \varphi(x) + s S_n \psi(x)) \end{aligned} \tag{5.8}$$

whenever $Y \in \mathcal{Y}_n(A)$. Now (5.4)–(5.8) imply

$$\inf_{\mathcal{C} \in \mathcal{U}_\varepsilon(F)} \sum_{C \in \mathcal{C}} m(C)^s |C|^t \leq 2e^{-n\eta}.$$

Therefore (1.3) gives $\nu_{s,t}(A) = 0$, and by (1.5) we obtain $d_s(A) \leq t$. As $t > \widetilde{z}_s(A)$ was arbitrary, this completes the proof. ∎

Now we are able to prove the main result of this section.

THEOREM 3. *Let A be a completely invariant subset of* $[0,1]$. *Suppose that* $\sup_{x \in A} \varphi(x) < 0$ *and* $\sup_{x \in A} \psi(x) < 0$. *Then for every $s \in [0, s_A)$ we have*

$$d_s(A) = e_s(A) = E_s(A) = c_s(A) = z_s(A) = \widetilde{z}_s(A).$$

Proof. Fix an $s \in [0, s_A)$. By (4.2) we get $z_s(A) \in (0,1]$. Next we prove that $\widetilde{z}_s(A) = z_s(A)$. Assume that $t \geq 0$ and $p(A, T, t\varphi + s\psi) > 0$. As $\sup_{x \in A} \varphi(x) < 0$ and $\sup_{x \in A} \psi(x) < 0$, we get

$$\lim_{n \to \infty} \frac{1}{n} \sup_{x \in A} S_n(t\varphi + s\psi)(x) \leq 0 < p(A, T, t\varphi + s\psi).$$

Therefore Lemma 6 of [8] gives $q(A,T,t\varphi+s\psi)=p(A,T,t\varphi+s\psi)$. Hence (2.10) and Lemma 8 imply $\widetilde{z}_s(A)=z_s(A)$.

By Theorem 2 we obtain $e_s(A)=E_s(A)=c_s(A)=z_s(A)=\widetilde{z}_s(A)$. Furthermore $e_s(A)\le d_s(A)$ by (1.10) and (1.11). Using Lemma 10 we obtain $d_s(A)\le\widetilde{z}_s(A)$, which completes the proof. ■

Remark. The result of Theorem 3 remains valid if the conditions $\sup_{x\in A}\varphi(x)<0$ and $\sup_{x\in A}\psi(x)<0$ are replaced by

$$\lim_{n\to\infty}\frac{1}{n}\sup_{x\in A}S_n\varphi(x)<0 \quad\text{and}\quad \lim_{n\to\infty}\frac{1}{n}\sup_{x\in A}S_n\psi(x)<0.$$

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