FUNDAMENTA
MATHEMATICAE
171 (2002)

The covering number for category and partition relations on $P_\omega(\lambda)$

by

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Abstract. We show that $\mathrm{cov}(M)$ is the least infinite cardinal λ such that $P_\omega(\lambda)$ (the set of all finite subsets of λ) fails to satisfy a certain natural generalization of Ramsey's Theorem.

0. Introduction. The covering number for category ($\mathrm{cov}(M)$) is known to play an important part in the study of partition properties of ideals on ω. Namely, we have $K^+ \to (K^+)^2$ for every ideal K on ω with less than $\mathrm{cov}(M)$ generators (see [14]). On the other hand, there exists an ideal K on ω generated by $\mathrm{cov}(M)$ sets such that $K^+ \nrightarrow (K^+, \omega)^2$ (see [10]). In the present paper, we investigate the relationship between $\mathrm{cov}(M)$ and partition properties for ideals on $P_\omega(\lambda)$, λ an infinite cardinal. The partition relation we are mostly interested in, $J^+ \xrightarrow{\omega} (J^+)^2$, is of a mixed type, in the sense that its definition involves functions $F : \omega \times P_\omega(\lambda) \to 2$. We show, as above, that $J^+ \xrightarrow{\omega} (J^+)^2$ for every fine ideal J on $P_\omega(\lambda)$ generated by less than $\mathrm{cov}(M)$ sets. In particular, $I^+_{\omega,\lambda} \xrightarrow{\omega} (I^+_{\omega,\lambda})^2$ for every $\lambda < \mathrm{cov}(M)$, where $I_{\omega,\lambda}$ denotes the smallest fine ideal on $P_\omega(\lambda)$. Observe that for $\lambda = \omega$, $I^+_{\omega,\lambda} \xrightarrow{\omega} (I^+_{\omega,\lambda})^2$ is just a reformulation of Ramsey's Theorem [13]. We also show that $I^+_{\omega,\lambda} \overset{\omega}{\nrightarrow} (I^+_{\omega,\lambda})^2$ for all $\lambda \geq \mathrm{cov}(M)$. This result emphasizes the heterogeneity of $I_{\omega,\lambda}$, i.e. the fact that the members of $I^+_{\omega,\lambda}$ are not all alike, since it was shown in [8] that for every λ, there is a fine ideal J on $P_\omega(\lambda)$ such that $J^+ \xrightarrow{\omega} (J^+)^2$.

The paper is organized as follows. The two results mentioned above are to be found in Sections 3 and 4. Section 1 deals with notation and basic

2000 *Mathematics Subject Classification*: 03E02, 03E17.
Key words and phrases: partition relations, $P_\omega(\lambda)$, covering number.

definitions. It is shown in Section 2 that $I_{\omega,\lambda}$ is a weak χ-point if and only if $\lambda < \text{cov}(M)$. We prove in Section 5 that $\text{cov}(M) = 2^\lambda$ if and only if every fine ideal J on $P_\omega(\lambda)$ with less than 2^λ generators can be extended to a prime ideal K on $P_\omega(\lambda)$ such that $K^+ \overset{\omega}{\to} (K^+)^2$. A companion result deals with extension to a prime χ-point. Finally, in Section 6 we derive another remarkable property shared by all fine ideals J on $P_\omega(\lambda)$ which are generated by less than $\text{cov}(M)$ sets.

1. Notation. In this section we review some basic definitions.

Given an infinite set S, an *ideal on* S is a collection J of subsets of S such that (i) $\{s\} \in J$ for every $s \in S$, (ii) $P(A) \subseteq J$ for all $A \in J$, (iii) $A \cup B \in J$ whenever $A, B \in J$, and (iv) $S \notin J$.

Let J be an ideal on S.

- $\text{cof}(J)$ denotes the least cardinality of any $X \subseteq J$ such that $J = \bigcup_{A \in X} P(A)$.
- J is *prime* if $J \cap \{A, S - A\} \neq \emptyset$ for all $A \subseteq S$.
- $J^+ = P(S) - J$ and $J^* = \{A \subseteq S : S - A \in J\}$.
- $J|A = \{B \subseteq S : B \cap A \in J\}$ for all $A \in J^+$.

Let K be an ideal on ω.

- K is *weakly selective* if given $A \in K^+$ and $B_n \in K$ for $n \in \omega$, there is $C \in K^+ \cap P(A)$ such that $m \notin B_n$ for all $n, m \in C$ with $n < m$.
- K is a *P-point* if given $B_n \in K$ for $n \in \omega$, there is $C \in K^*$ such that $C \cap B_n$ is finite for every $n \in \omega$.
- K is a *Q-point* if given $g : \omega \to \omega$, there is $A \in K^*$ such that $g(n) \leq m$ for all $n, m \in A$ with $n < m$.
- $K^+ \to (K^+)^2$ asserts that given $A \in K^+$ and $F : \omega \times \omega \to 2$, there is $B \in K^+ \cap P(A)$ such that F is constant on $\{(n, m) \in B \times B : n < m\}$.
- $K^+ \nrightarrow (K^+)^2$ means that $K^+ \to (K^+)^2$ does not hold.
- For each ordinal α with $2 \leq \alpha \leq \omega$, $K^+ \to (K^+, \alpha)^2$ means that given $A \in K^+$ and $F : \omega \times \omega \to 2$, there is either $B \in K^+ \cap P(A)$ such that F is identically 0 on $\{(n, m) \in B \times B : n < m\}$, or else $n_i \in A$ for $i < \alpha$ such that $n_j < n_i$ for all $j < i$, and F is identically 1 on $\{(n_j, n_i) : j < i < \alpha\}$.
- $K^+ \nrightarrow (K^+, \alpha)^2$ is the negation of $K^+ \to (K^+, \alpha)^2$.
- For any set A, $P_\omega(A)$ denotes the collection of all finite subsets of A.
- λ is a fixed infinite cardinal.
- $\widehat{a} = \{b \in P_\omega(\lambda) : a \subseteq b\}$ for every $a \in P_\omega(\lambda)$.
- $I_{\omega,\lambda}$ denotes the set of all $A \subseteq P_\omega(\lambda)$ such that $A \cap \widehat{a} = \emptyset$ for some $a \in P_\omega(\lambda)$.
- An ideal J on $P_\omega(\lambda)$ is *fine* if $I_{\omega,\lambda} \subseteq J$. It is easy to see that $\text{cof}(J) \geq \lambda$ for every fine ideal J on $P_\omega(\lambda)$. As is readily verified, $I_{\omega,\lambda}$ is a fine ideal on $P_\omega(\lambda)$ and $\text{cof}(I_{\omega,\lambda}) = \lambda$.

• Given two sets X and Y, Y^X denotes the set of all functions from X to Y. We endow the set 2^ω with the product topology, where 2 is given the discrete topology.

• $O_s = \{f \in 2^\omega : s \subset f\}$ for all $s \in \bigcup_{n\in\omega} 2^n$.

• M denotes the collection of all meager subsets of 2^ω.

• $\mathrm{cov}(M)$ is the least cardinality of any $X \subseteq M$ such that $2^\omega = \cup X$.

• $\mathfrak{d}$ is the least cardinality of any $F \subseteq \omega^\omega$ with the property that for every $g \in \omega^\omega$, there is $f \in F$ such that $g(n) \le f(n)$ for all $n \in \omega$. It is well known (see e.g. [15]) that $\omega_1 \le \mathrm{cov}(M) \le \mathfrak{d}$.

• $\mathfrak{d}^\omega_{\omega,\lambda}$ is the least cardinality of any family F of functions from ω to $P_\omega(\lambda)$ with the property that for every $g : \omega \to P_\omega(\lambda)$, there is $f \in F$ such that $g(n) \subseteq f(n)$ for all $n \in \omega$. It is shown in [12] that $\mathfrak{d}^\omega_{\omega,\omega} = \mathfrak{d}$, and that $\mathfrak{d}^\omega_{\omega,\lambda} = \max\{\mathfrak{d}, u(\omega_1, \lambda)\}$ if $\lambda > \omega$, where $u(\omega_1, \lambda)$ is the least cardinality of any family X of countable subsets of λ such that for every countable $a \subseteq \lambda$, there is $b \in X$ with $a \subseteq b$.

2. Weak χ-points. In this section we introduce the property of being a weak χ-point and determine when $I_{\omega,\lambda}$ has this property.

An ideal J on $P_\omega(\lambda)$ is a *weak χ-point* if given $A \in J^+$ and $g : \omega \to P_\omega(\lambda)$, there is $C \in J^+ \cap P(A)$ such that $g(\max(a \cap \omega)) \subseteq b$ for all $a, b \in C$ with $\max(a \cap \omega) < \max(b \cap \omega)$.

LEMMA 2.1. *Let J be a fine ideal on $P_\omega(\lambda)$ such that $\mathrm{cof}(J) < \mathrm{cov}(M)$. Then J is a weak χ-point.*

Proof. Fix $A \in J^+$ and $g : \omega \to P_\omega(\lambda)$. Set

$$A_n = \{a \in A : \max(a \cap \omega) = n\}$$

for each $n \in \omega$. Pick $B_\alpha \in J$ for $\alpha < \mathrm{cof}(J)$ so that $J = \bigcup_{\alpha<\mathrm{cof}(J)} P(B_\alpha)$. For $\alpha < \mathrm{cof}(J)$ and $n \in \omega$, let D^n_α be the set of all $s \in 2^{n+1}$ such that $s(n) = 1$ and there is $a \in A_n - B_\alpha$ with the property that $g(m) \subseteq a$ for all $m < n$ with $s(m) = 1$. Given $\alpha < \mathrm{cof}(J)$, let $D_\alpha = \bigcup_{n\in\omega} D^n_\alpha$ and $U_\alpha = \bigcup_{s\in D_\alpha} O_s$. Let us show that the open set U_α is dense. Thus let $k \in \omega$ and $p \in 2^k$. Put $y = \{m < k : p(m) = 1\}$. Pick $b \in A - B_\alpha$ so that $\bigcup_{m\in y} g(m) \subseteq b$ and $k \le \max(b \cap \omega)$. Now define $q \supset p$ by $\mathrm{dom}(q) = \max(b \cap \omega) + 1$, $q(\max(b \cap \omega)) = 1$ and $q(i) = 0$ for all i with $k \le i < \max(b \cap \omega)$. Then clearly $q \in D^{\max(b\cap\omega)}_\alpha$.

Now select $f \in \bigcap_{\alpha<\mathrm{cof}(J)} U_\alpha$. For each $\alpha < \mathrm{cof}(J)$, pick $s_\alpha \in D_\alpha$ with $s_\alpha \subset f$. Put

$$Y = \{\max(\mathrm{dom}(s_\alpha)) : \alpha < \mathrm{cof}(J)\}$$

and let $m_0, m_1, \ldots$ be the increasing enumeration of Y. Set $E_0 = A_{m_0}$ and for each $l \in \omega$,

$$E_{l+1} = \Big\{ a \in A_{m_{l+1}} : \bigcup_{i \le l} g(m_i) \subseteq a \Big\}.$$

Finally define $C = \bigcup_{l \in \omega} E_l$. Given $\alpha < \operatorname{cof}(J)$, let l be such that $\max(\operatorname{dom}(s_\alpha)) = m_l$. Then $E_l - B_\alpha \neq \emptyset$ since $s_\alpha \in D_\alpha^{m_l}$ and $s_\alpha(m_i) = 1$ for all $i < l$. Thus $C \in J^+$. ∎

We will need the following result from [10].

LEMMA 2.2. $\operatorname{cov}(M)$ *is the least cardinal* μ *with the property that there is an ideal* K *on* ω *such that* $\operatorname{cof}(K) = \mu$ *and* K *is not weakly selective.*

PROPOSITION 2.3. $I_{\omega,\lambda}$ *is a weak* χ*-point if and only if* $\lambda < \operatorname{cov}(M)$.

Proof. The right-to-left direction is immediate from Lemma 2.1. For the other implication, assume $\lambda > \omega$ and $I_{\omega,\lambda}$ is a weak χ-point. We will show that every ideal K on ω such that $\operatorname{cof}(K) \le \lambda$ is weakly selective. This will give $\lambda < \operatorname{cov}(M)$ by Lemma 2.2.

Thus let K be a fixed ideal on ω with $\operatorname{cof}(K) \le \lambda$, and let $A \in K^+$. Pick $x \subseteq \lambda - \omega$ with $|x| = \operatorname{cof}(K)$, and a one-to-one $h : x \to K$ with the property that $K = \bigcup_{\beta \in x} P(h(\beta))$. For each $n \in A$, let X_n be the set of all $a \in P_\omega(\lambda)$ such that $a \cap \omega = n + 1$ and $a \cap x \subseteq \{\alpha \in x : n \notin h(\alpha)\}$. We let $B = \bigcup_{n \in A} X_n$. Given $d \in P_\omega(\lambda)$, we have $\bigcup_{\beta \in d \cap x} h(\beta) \in K$ and therefore there is $k \in A$ such that $k \ge \max(d \cap \omega)$ and $k \notin \bigcup_{\beta \in d \cap x} h(\beta)$. Setting $c = (k+1) \cup (d - \omega)$, we have $d \subseteq c$ and $c \in X_k$. Hence $B \in I^+_{\omega,\lambda}$.

Now let $E_n \in K$ for $n \in \omega$. Define $p : \omega \to x$ so that $E_n \subseteq h(p(n))$ for all $n \in \omega$. Since $I_{\omega,\lambda}$ is a weak χ-point, there is $C \in I^+_{\omega,\lambda} \cap P(B)$ such that $p(\max(b \cap \omega)) \in e$ for all $b, e \in C$ with $\max(b \cap \omega) < \max(e \cap \omega)$. Set $D = \{\max(a \cap \omega) : a \in C\}$. Given $\alpha \in x$, there is $a \in C$ such that $\alpha \in a$. Then $\max(a \cap \omega) \notin h(\alpha)$. Hence $D \in K^+$. Moreover, $D \subseteq A$. Finally let $b, e \in C$ with $\max(b \cap \omega) < \max(e \cap \omega)$. As $p(\max(b \cap \omega)) \in e$, we have $\max(e \cap \omega) \notin h(p(\max(b \cap \omega)))$ and therefore $\max(e \cap \omega) \notin E_{\max(b \cap \omega)}$. ∎

3. $J^+ \xrightarrow{\omega} (J^+)^2$**.** We now introduce the partition property $J^+ \xrightarrow{\omega} (J^+)^2$ and show that it is satisfied whenever J has a small (meaning $< \operatorname{cov}(M)$) number of generators. We start with a few definitions.

Let J be an ideal on $P_\omega(\lambda)$.

- $J^+ \xrightarrow{\omega} (J^+)^2$ asserts that given $A \in J^+$ and $F : \omega \times P_\omega(\lambda) \to 2$, there is $B \in J^+ \cap P(A)$ such that F is constant on

$$\{(\max(a \cap \omega), b) \in \omega \times B : a \in B \text{ and} \max(a \cap \omega) < \max(b \cap \omega)\}.$$

• J is *almost* $(\omega,2)$*-distributive* if given $A \in J^+$ and $B_n \subseteq P_\omega(\lambda)$ for $n \in \omega$, there is $C \in J^+ \cap P(A)$ such that

$$\{C - B_{\max(a\cap\omega)}, C \cap B_{\max(a\cap\omega)}\} \cap J \neq \emptyset$$

for all $a \in C$.

• J is a *weak* π*-point* if given $A \in J^+$ and $B_n \in J$ for $n \in \omega$, there is $C \in J^+ \cap P(A)$ such that $C \cap B_n \in I_{\omega,\lambda}$ for all $n \in \omega$.

LEMMA 3.1. *Let J be a fine ideal on $P_\omega(\lambda)$. Then the following are equivalent*:

(i) $J^+ \overset{\omega}{\to} (J^+)^2$.

(ii) *J is almost $(\omega,2)$-distributive and both a weak χ-point and a weak π-point.*

Proof. (i)⇒(ii). Assume $J^+ \overset{\omega}{\to} (J^+)^2$. Then given $A \in J^+$ and $B_n \subseteq P_\omega(\lambda)$ for $n \in \omega$, there is $C \in J^+ \cap P(A)$ such that either $b \in B_{\max(a\cap\omega)}$ for all $a, b \in C$ with $\max(a\cap\omega) < \max(b\cap\omega)$, or else $b \notin B_{\max(a\cap\omega)}$ for all $a, b \in C$ with $\max(a\cap\omega) < \max(b\cap\omega)$. It easily follows that (ii) holds.

(ii)⇒(i). Assume (ii), and fix $A \in J^+$ and $F : \omega \times P_\omega(\lambda) \to 2$. For $n \in \omega$ and $i < 2$, set $B_n^i = \{b \in A : F(n,b) = i\}$. Since J is almost $(\omega,2)$-distributive, there are $C \in J^+ \cap P(A)$ and $h : \{\max(a\cap\omega) : a \in C\} \to 2$ with $C - B_{\max(a\cap\omega)}^{h(\max(a\cap\omega))} \in J$ for all $a \in C$. Set $C_i = \{a \in C : h(\max(a\cap\omega)) = i\}$ for each $i < 2$. Pick $j < 2$ so that $C_j \in J^+$. Since J is a weak π-point and a weak χ-point, there is $D \in J^+ \cap P(C_j)$ such that $b \in B^j_{\max(a\cap\omega)}$ for all $a, b \in D$ with $\max(a\cap\omega) < \max(b\cap\omega)$. Then F takes the constant value j on

$$\{(\max(a\cap\omega), b) \in \omega \times D : a \in D \text{ and } \max(a\cap\omega) < \max(b\cap\omega)\}. \ \blacksquare$$

The following is proved in [11].

LEMMA 3.2. *Let J be a fine ideal on $P_\omega(\lambda)$ such that* $\mathrm{cof}(J) < \mathfrak{d}^\omega_{\omega,\lambda}$. *Then J is almost $(\omega,2)$-distributive and a weak π-point.*

PROPOSITION 3.3. *Let J be a fine ideal on $P_\omega(\lambda)$ such that* $\mathrm{cof}(J) < \mathrm{cov}(M)$. *Then* $J^+ \overset{\omega}{\to} (J^+)^2$.

Proof. By Lemmas 2.1, 3.1 and 3.2. ■

The following is immediate from Proposition 3.3.

COROLLARY 3.4. *If* $\lambda < \mathrm{cov}(M)$, *then* $I^+_{\omega,\lambda} \overset{\omega}{\to} (I^+_{\omega,\lambda})^2$.

4. $I^+_{\omega,\lambda} \underset{\omega}{\overset{\omega}{\to}} (I^+_{\omega,\lambda}, \alpha)^2$**.** This section deals with negative partition properties. Let J be an ideal on $P_\omega(\lambda)$ and α an ordinal with $2 \leq \alpha \leq \omega$.

• $J^+ \overset{\omega}{\underset{\omega}{\to}} (J^+,\alpha)^2$ means that for all $A \in J^+$ and $f : \omega\times\omega \to 2$, either there is $B \in J^+ \cap P(A)$ such that f is identically 0 on

$$\{(\max(a\cap\omega), \max(b\cap\omega)) : a, b \in B \text{ and } \max(a\cap\omega) < \max(b\cap\omega)\},$$

or there are $a_n \in A$ for $n < \alpha$ such that $\max(a_m \cap \omega) < \max(a_n \cap \omega)$ for all $m < n$, and f is identically 1 on $\{(\max(a_m\cap\omega), \max(a_n\cap\omega)) : m < n < \alpha\}$.

• $J^+ \overset{\omega}{\underset{\omega}{\not\to}} (J^+,\alpha)^2$ means that $J^+ \overset{\omega}{\underset{\omega}{\to}} (J^+,\alpha)^2$ does not hold.

Lemma 4.1. *Let α be such that $2 \le \alpha \le \omega$ and $I^+_{\omega,\lambda} \overset{\omega}{\underset{\omega}{\to}} (I^+_{\omega,\lambda},\alpha)^2$, and let K be an ideal on ω with* $\mathrm{cof}(K) \le \lambda$. *Then $K^+ \to (K^+,\alpha)^2$.*

Proof. Fix $A \in K^+$ and $f : \omega\times\omega \to 2$. Let B be defined as in the proof of Proposition 2.3. If f is identically 0 on

$$\{(\max(a\cap\omega), \max(b\cap\omega)) : a, b \in C \text{ and } \max(a\cap\omega) < \max(b\cap\omega)\}$$

for some $C \in I^+_{\omega,\lambda} \cap P(B)$, then setting $D = \{\max(a\cap\omega) : a \in C\}$, we see that $D \in K^+ \cap P(A)$ and f is identically 0 on $\{(n,m) \in D\times D : n < m\}$. On the other hand, if f is identically 1 on $\{\big(\max(a_p\cap\omega), \max(a_q\cap\omega)\big) : p < q < \alpha\}$, where $\{a_q : q < \alpha\} \subseteq B$ and $\max(a_p\cap\omega) < \max(a_q\cap\omega)$ whenever $p < q < \alpha$, then setting $E = \{\max(a_q \cap \omega) : q < \alpha\}$, we find that $E \subseteq A$, $|E| = \alpha$ and f is identically 1 on $\{(n,m) \in E\times E : n < m\}$. ■

The following result is folklore. As we do not know any explicit reference for it, a proof is provided.

Lemma 4.2. *Given an ideal K on ω, the following are equivalent*:

(i) *K is weakly selective.*
(ii) *$K^+ \to (K^+,\omega)^2$.*

Proof. (i)⇒(ii). Assume (i) and fix $A \in K^+$ and $F : \omega\times\omega \to 2$. Set $E_n = \{m > n : F(n,m) = 1\}$ for all $n \in \omega$. First suppose there is $B \in K^+ \cap P(A)$ such that $B \cap E_n \in K$ for every $n \in B$. Pick $C \in K^+ \cap P(B)$ so that $m \notin B \cap E_n$ for all $n, m \in C$ with $n < m$. Then F takes the constant value 0 on $\{(n,m) \in C\times C : n < m\}$. Now suppose there is $\varphi : K^+ \cap P(A) \to A$ such that $\varphi(D) \in D$ and $D \cap E_{\varphi(D)} \in K^+$ for all $D \in K^+ \cap P(A)$. Define $n_i \in A$ for $i < \omega$ by $n_0 = \varphi(A)$ and $n_{i+1} = \varphi(A \cap \bigcap_{j\le i} E_{n_j})$. Then F is identically 1 on $\{(n_j, n_i) : j < i < \omega\}$.

(ii)⇒(i). Assume (ii) and fix $A \in K^+$ and $B_i \in K$ for $i < \omega$. We must find $H \in K^+ \cap P(A)$ such that $m \notin B_n$ for all $n, m \in H$ with $n < m$. If $A - \bigcup_{i<\omega} B_i \in K^+$, we can set $H = A - \bigcup_{i<\omega} B_i$. Otherwise we put $C = A \cap \bigcup_{i<\omega} B_i$ and define $h : C \to \omega$ by $h(n) =$ the least i such that $n \in B_i$. Now define $F : C\times C \to 2$ by $F(n,m) = 1$ precisely when $h(n) > h(m)$. Then clearly, there is $D \in K^+ \cap P(C)$ such that F is identically 0 on

$\{(n,m) \in D \times D : n < m\}$. Notice that $D \cap B_i$ is finite for each $i < \omega$, since $D \cap B_i \subseteq m$ whenever $m \in D - \bigcup_{j \le i} B_j$. Finally, define $G : D \times D \to 2$ by $G(n,m) = 1$ if and only if $m \in B_n$. Clearly, there is $H \in K^+ \cap P(D)$ such that F takes the constant value 0 on $\{(n,m) \in H \times H : n < m\}$. Then H is as desired. ∎

The following shows that Proposition 3.3 is optimal.

PROPOSITION 4.3. *If* $\lambda \ge \mathrm{cov}(M)$, *then* $I^+_{\omega,\lambda} \overset{\omega}{\underset{\omega}{\nrightarrow}} (I^+_{\omega,\lambda}, \omega)^2$.

Proof. By Lemmas 2.2, 4.1 and 4.2. ∎

For each ordinal α with $3 \le \alpha \le \omega$, let $\mathfrak{par}_\alpha$ be the least cardinal μ with the property that there is an ideal K on ω such that $\mathrm{cof}(K) = \mu$ and $K^+ \nrightarrow (K^+, \alpha)^2$.

It follows from Lemmas 2.2 and 4.2 that $\mathfrak{par}_\omega = \mathrm{cov}(M)$. The exact value of $\mathfrak{par}_3$ is not known, but one has the following upper bound (see [4], p. 63, and [2], p. 7).

PROPOSITION 4.4. $\mathfrak{par}_3 \le \mathfrak{d}$.

Proof. Fix a bijection $j : \omega \times \omega \times \omega \to \omega$, and let K be the set of all $B \subseteq \omega$ such that

$$\{m \in \omega : \{n \in \omega : \{p \in \omega : j(m,n,p) \in B\} \text{ is infinite}\} \text{ is infinite}\} \text{ is finite.}$$

It is easy to check that K is an ideal on ω. To see that $K^+ \nrightarrow (K^+, 3)^2$, consider $F : \omega \times \omega \to 2$ defined by $F(j(m,n,p), j(m',n',p')) = 1$ if and only if $m < n < m' < p < n' < p'$.

It remains to check that $\mathrm{cof}(K) \le \mathfrak{d}$. Select $X \subseteq \omega^\omega$ so that for every $f \in \omega^\omega$, there is $g \in X$ with the property that $f(n) \le g(n)$ for all $n \in \omega$. Fix a bijection $k : \omega \times \omega \to \omega$. For $m \in \omega$ and $f, g \in X$, set

$$\begin{aligned} A_m &= j[m \times \omega \times \omega], \\ B_{m,f} &= \bigcup_{n \ge m} j[\{n\} \times f(n) \times \omega], \\ C_{m,f,g} &= \bigcup_{n \ge m} \bigcup_{p \ge f(n)} j[\{n\} \times \{p\} \times g(k(n,p))], \\ D_{m,f,g} &= A_m \cup B_{m,f} \cup C_{m,f,g}. \end{aligned}$$

It is readily verified that $K = \bigcup\{P(D_{m,f,g}) : m \in \omega \text{ and } f, g \in X\}$. ∎

Jörg Brendle has shown that the inequality in Proposition 4.4 can consistently be strict. His result is included here with his kind permission.

PROPOSITION 4.5. *It is consistent with* ZFC *that* $\mathfrak{par}_3 < \mathfrak{d}$.

Proof. Let $V \models$ ZFC + GCH. By a result of Baumgartner and Taylor (Corollary 4.12 in [3]), there is in V a prime P-point ideal K on ω such that $K^+ \not\to (K^+,3)^2$. Now let Q be an ω_2-stage countable-support iteration of Miller's rational perfect set forcing. In V^Q the following hold (see [5]): (a) $\mathfrak{d} = \aleph_2 = 2^{\aleph_0}$, and (b) $J = \bigcup_{B\in K} P(B)$ is a prime ideal on ω. Clearly, $\mathrm{cof}(J) = \aleph_1$. Moreover, $J^+ \not\to (J^+,3)^2$. ■

PROPOSITION 4.6. *If* $\lambda \geq \mathfrak{par}_3$, *then* $I^+_{\omega,\lambda} \overset{\omega}{\underset{\omega}{\not\to}} (I^+_{\omega,\lambda},3)^2$.

Proof. By Lemma 4.1. ■

5. Extending ideals. Suppose we are given a property P of ideals and a cardinal $\mu > \lambda$. Then one might ask whether it is possible to extend every fine ideal J on $P_\omega(\lambda)$ such that $\mathrm{cof}(J) < \mu$ to an ideal K on $P_\omega(\lambda)$ with the property P. In this section we will consider several questions of this type. We start with a lemma.

LEMMA 5.1. *There is a fine ideal* J *on* $P_\omega(\lambda)$ *such that* $\mathrm{cof}(J) = \max\{\lambda, \mathrm{cov}(M)\}$ *and* $J^+ \overset{\omega}{\underset{\omega}{\not\to}} (J^+,\omega)^2$.

Proof. By Lemmas 2.2 and 4.2, we can find an ideal K on ω with $\mathrm{cof}(K) = \mathrm{cov}(M)$, $E \in K^+$ and $f : \omega\times\omega \to 2$ such that (a) there is no $B \in K^+ \cap P(E)$ such that f is identically 0 on $\{(n,m) \in B\times B : n < m\}$, and (b) there is no infinite subset C of E such that f is identically 1 on $\{(n,m) \in C\times C : n < m\}$. Define $\varphi : E \times P(P_\omega(\lambda)) \to P(P_\omega(\lambda))$ by letting $\varphi(n,A) = \{b \in A : \max(b\cap\omega) = n\}$. For $a \in P_\omega(\lambda)$ and $A \subseteq P_\omega(\lambda)$, set $Y^a_A = \{n \in E : \widehat{a} \cap \varphi(n,A) \neq \emptyset\}$. Now define $J \subseteq P(P_\omega(\lambda))$ by letting $A \in J$ if and only if $Y^a_A \in K$ for some $a \in P_\omega(\lambda)$. It is immediate from the following easy facts that J is a fine ideal on $P_\omega(\lambda)$:

(i) the set $E - Y^a_{P_\omega(\lambda)}$ is finite for all $a \in P_\omega(\lambda)$;
(ii) $Y^a_A \subseteq Y^a_B$ for all $a \in P_\omega(\lambda)$ and $A, B \subseteq P_\omega(\lambda)$ with $A \subseteq B$;
(iii) $Y^{a\cup b}_{A\cup B} \subseteq Y^a_A \cup Y^b_B$ for all $a, b \in P_\omega(\lambda)$ and $A, B \subseteq P_\omega(\lambda)$;
(iv) if $A \in I_{\omega,\lambda}$, then $Y^a_A = \emptyset$ for some $a \in P_\omega(\lambda)$.

Set $A = \{a \in P_\omega(\lambda) : \max(a\cap\omega) \in E\}$. Then $A \in J^*$, as $Y^{\emptyset}_{P_\omega(\lambda)-A} = \emptyset$. Given $B \in J^+ \cap P(A)$, set $C = \{\max(a\cap\omega) : a \in B\}$. Then $C \in K^+$ since $C = Y^{\emptyset}_B$, and therefore f is not constantly 0 on

$$\{(\max(a\cap\omega), \max(b\cap\omega)) : a, b \in B \text{ and } \max(a\cap\omega) < \max(b\cap\omega)\}.$$

It easily follows that $J^+ \overset{\omega}{\underset{\omega}{\not\to}} (J^+,\omega)^2$.

It remains to compute $\mathrm{cof}(J)$. Given $D \subseteq P_\omega(\lambda)$, we know that $D \in J$ if and only if there are $a \in P_\omega(\lambda)$ and $H \in K$ such that

$$D \subseteq (P_\omega(\lambda) - \widehat{a}) \cup \{b \in P_\omega(\lambda) : \max(b\cap\omega) \in H \cup (\omega - E)\}.$$

It clearly follows that $\mathrm{cof}(J) \le \max\{\lambda, \mathrm{cof}(K)\}$. On the other hand, $\mathrm{cof}(J) \ge \mathrm{cov}(M)$ by Proposition 3.3. Hence $\mathrm{cof}(J) = \max\{\lambda, \mathrm{cov}(M)\}$. ∎

Ketonen [7] showed that if $\mathrm{cov}(M) = 2^{\aleph_0}$, then every ideal on ω generated by less than $2^{\aleph_0}$ sets can be extended to a prime ideal K on ω such that $K^+ \to (K^+)^2$. The converse was proved by Canjar [6] and by Bartoszyński and Judah [1]. The equivalence can be generalized as follows.

Proposition 5.2. *The following are equivalent*:

(i) $\mathrm{cov}(M) = 2^\lambda$.

(ii) *If J is a fine ideal on $P_\omega(\lambda)$ with* $\mathrm{cof}(J) < 2^\lambda$, *then there is a prime ideal K on $P_\omega(\lambda)$ such that $J \subseteq K$ and $K^+ \xrightarrow{\omega} (K^+)^2$.*

(iii) *If J is a fine ideal on $P_\omega(\lambda)$ with* $\mathrm{cof}(J) < 2^\lambda$, *then there is an ideal K on $P_\omega(\lambda)$ such that $J \subseteq K$ and $K^+ \xrightarrow[\omega]{\omega} (K^+, \omega)^2$.*

Proof. (i)⇒(ii). Assume $\mathrm{cov}(M) = 2^\lambda$, and let J be a fine ideal on $P_\omega(\lambda)$ with $\mathrm{cof}(J) < 2^\lambda$. Let F_α for $\alpha < 2^\lambda$ be an enumeration of the set of all $F : \omega \times P_\omega(\lambda) \to 2$. Using Proposition 3.3, we define for each $\alpha < 2^\lambda$ a fine ideal J_α on $P_\omega(\lambda)$ with $\mathrm{cof}(J_\alpha) \le \max\{|\alpha|, \mathrm{cof}(J)\}$, and $A_\alpha \in J_\alpha^+$ so that

(0) $J_0 = J$;

(1) F_α is constant on

$$\{(\max(a \cap \omega), b) \in \omega \times A_\alpha : a \in A_\alpha \text{ and } \max(a \cap \omega) < \max(b \cap \omega)\};$$

(2) $J_{\alpha+1} = J_\alpha | A_\alpha$;

(3) $J_\alpha = \bigcup_{\beta<\alpha} J_\beta$ if α is a limit ordinal > 0.

Finally set $K = \bigcup_{\alpha<2^\lambda} J_\alpha$. Then clearly K is an ideal on $P_\omega(\lambda)$ such that $J \subseteq K$. Moreover, as $A_\alpha \in K^*$ for every $\alpha < 2^\lambda$, we conclude that $K^+ \xrightarrow{\omega} (K^+)^2$ and K is prime.

(ii)⇒(iii). Trivial.

(iii)⇒(i). Assume $\mathrm{cov}(M) \ne 2^\lambda$. Then $\mathrm{cov}(M) < 2^\lambda$ because $\mathrm{cov}(M) \le 2^{\aleph_0}$. By Lemma 5.1 one can find a fine ideal J on $P_\omega(\lambda)$ with $\mathrm{cof}(J) = \max\{\lambda, \mathrm{cov}(M)\}$, $A \in J^+$ and $F : \omega \times \omega \to 2$ with the property that (a) there is no $C \in J^+ \cap P(A)$ such that F is constantly 0 on

$$\{(\max(a \cap \omega), \max(b \cap \omega)) : a, b \in C \text{ and } \max(a \cap \omega) < \max(b \cap \omega)\},$$

and (b) there is no $f \in A^\omega$ such that $\max(f(n) \cap \omega) < \max(f(m) \cap \omega)$ whenever $n < m < \omega$, and F is identically 1 on

$$\{(\max(f(n) \cap \omega), \max(f(m) \cap \omega)) : n < m < \omega\}.$$

We have $\mathrm{cof}(J|A) < 2^\lambda$. Moreover, $K^+ \underset{\omega}{\overset{\omega}{\not\to}} (K^+, \omega)^2$ for every ideal K on $P_\omega(\lambda)$ with $J|A \subseteq K$. ∎

Canjar [6] showed that $\mathrm{cov}(M) = \mathfrak{d}$ if and only if every ideal on ω generated by less than $\mathfrak{d}$ sets can be extended to a Q-point. We will now generalize his result.

An ideal J on $P_\omega(\lambda)$ is a *χ-point* if for every $g : \omega \to P_\omega(\lambda)$, there is $A \in J^*$ such that $g(\max(a \cap \omega)) \subseteq b$ for all $a, b \in A$ with $\max(a \cap \omega) < \max(b \cap \omega)$.

Notice that if an ideal J on $P_\omega(\lambda)$ is a χ-point, then so is every ideal K on $P_\omega(\lambda)$ with $J \subseteq K$.

PROPOSITION 5.3. *Let J be a fine ideal on $P_\omega(\lambda)$ that is a χ-point. Then* $\mathrm{cof}(J) \geq \mathfrak{d}^\omega_{\omega,\lambda}$.

Proof. Let $X \subseteq J^*$ be such that $J = \bigcup_{A \in X} P(P_\omega(\lambda) - A)$. Given $A \in X$, we define $f_A : \omega \to P_\omega(\lambda)$ as follows. Let $m^A_0, m^A_1, \ldots$ be the increasing enumeration of the elements of the set $\{\max(a \cap \omega) : a \in A\}$. For each $n \in \omega$, pick $a^A_n \in A$ with $\max(a^A_n \cap \omega) = m^A_n$. We put $f_A(n) = a^A_{n+1}$ for all $n \in \omega$.

Now fix $g : \omega \to P_\omega(\lambda)$. Define $h : \omega \to P_\omega(\lambda)$ by $h(m) = \bigcup_{n \leq m} g(n)$. Select $A \in X$ so that $h(\max(a \cap \omega)) \subseteq b$ for all $a, b \in A$ satisfying $\max(a \cap \omega) < \max(b \cap \omega)$. For each $n \in \omega$, we have $g(n) \subseteq h(m^A_n) \subseteq f_A(n)$. Thus $|X| \geq \mathfrak{d}^\omega_{\omega,\lambda}$. ∎

The proof of the following shows that assuming $\lambda < \mathrm{cov}(M) = \mathfrak{d}^\omega_{\omega,\lambda}$, there is a fine ideal J on $P_\omega(\lambda)$ such that J is a χ-point and $\mathrm{cof}(J) = \mathfrak{d}^\omega_{\omega,\lambda}$.

PROPOSITION 5.4. *Assuming $\lambda < \mathfrak{d}^\omega_{\omega,\lambda}$ the following are equivalent*:

(i) $\mathrm{cov}(M) = \mathfrak{d}^\omega_{\omega,\lambda}$.

(ii) *If J is a fine ideal on $P_\omega(\lambda)$ with* $\mathrm{cof}(J) < \mathfrak{d}^\omega_{\omega,\lambda}$, *then there is a χ-point ideal K on $P_\omega(\lambda)$ such that $J \subseteq K$.*

Proof. (i)⇒(ii). Assume (i), and let J be a fine ideal on $P_\omega(\lambda)$ with $\mathrm{cof}(J) < \mathfrak{d}^\omega_{\omega,\lambda}$. Pick $f_\alpha : \omega \to P_\omega(\lambda)$ for $\alpha < \mathrm{cov}(M)$ so that for every $g : \omega \to P_\omega(\lambda)$, there is $\alpha < \mathrm{cov}(M)$ such that $g(n) \subseteq f(n)$ for all $n \in \omega$. Using Lemma 2.1, define for each $\alpha < \mathrm{cov}(M)$ an ideal J_α on $P_\omega(\lambda)$ with $\mathrm{cof}(J_\alpha) \leq \max\{|\alpha|, \mathrm{cof}(J)\}$, and $A_\alpha \in J^+_\alpha$ so that

(0) $J_0 = J$;

(1) $f_\alpha(\max(a \cap \omega)) \subseteq b$ for all $a, b \in A_\alpha$ with $\max(a \cap \omega) < \max(b \cap \omega)$;

(2) $J_{\alpha+1} = J_\alpha | A_\alpha$;

(3) $J_\alpha = \bigcup_{\beta < \alpha} J_\beta$ if α is a limit ordinal > 0.

Setting $K = \bigcup_{\alpha < \mathrm{cov}(M)} J_\alpha$, we clearly find that K is a χ-point ideal on $P_\omega(\lambda)$ extending J.

(ii)⇒(i). Assume (i) does not hold. Then $\mathrm{cov}(M) < \mathfrak{d}^\omega_{\omega,\lambda}$ since $\mathrm{cov}(M) \leq \mathfrak{d} \leq \mathfrak{d}^\omega_{\omega,\lambda}$. Hence by Lemma 5.1, there is a fine ideal J on $P_\omega(\lambda)$ such that

$\mathrm{cof}(J) < \mathfrak{d}^\omega_{\omega,\lambda}$ and $J^+ \overset{\omega}{\not\to} (J^+)^2$. J is not a weak χ-point by Lemmas 3.1 and 3.2, and so we can find $A \in J^+$ and $g : \omega \to P_\omega(\lambda)$ with the property that there is no $B \in J^+ \cap P(A)$ such that $g(\max(a \cap \omega)) \subseteq b$ for all $a, b \in B$ with $\max(a \cap \omega) < \max(b \cap \omega)$. Then there is no weak χ-point ideal K on $P_\omega(\lambda)$ extending $J|A$. We have $\mathrm{cof}(J|A) < \mathfrak{d}^\omega_{\omega,\lambda}$ since $\mathrm{cof}(J|A) \le \mathrm{cof}(J)$. ∎

6. $J^+ \underset{\prec}{\overset{\omega}{\to}} (J^+)^\omega$**.** For $a, b \in P_\omega(\lambda)$, we let $a \prec b$ whenever $a \subseteq b$ and $\max(a \cap \omega) < \max(b \cap \omega)$.

Given an ideal J on $P_\omega(\lambda)$, $J^+ \underset{\prec}{\overset{\omega}{\to}} (J^+)^2$ means that for all $A \in J^+$ and $F : \omega \times P_\omega(\lambda) \to 2$, there is $B \in J^+$ such that F is constant on

$$\{(\max(a \cap \omega), b) \in \omega \times B : a \in B \text{ and } a \prec b\}.$$

This partition property, which is studied in [11] and [9], is clearly weaker than the property $J^+ \overset{\omega}{\to} (J^+)^2$ considered above. The following shows that if $\mathrm{cof}(J) < \mathrm{cov}(M)$, then $J^+ \overset{\omega}{\to} (J^+)^2$ and $J^+ \underset{\prec}{\overset{\omega}{\to}} (J^+)^2$ are equivalent in a strong sense.

PROPOSITION 6.1. *Let J be a fine ideal on $P_\omega(\lambda)$ with* $\mathrm{cof}(J) < \mathrm{cov}(M)$, *and let $A \in J^+$. Then there is $C \in J^+ \cap P(A)$ with the property that for all $a, b \in C$ with* $\max(a \cap \omega) < \max(b \cap \omega)$, *one can find $c \in C$ with $c \prec b$ and* $\max(c \cap \omega) = \max(a \cap \omega)$.

Proof. Set $A_n = \{a \in A : \max(a \cap \omega) = n\}$ for each $n \in \omega$. Define $c_n \in P_\omega(\lambda)$ for $n \in \omega$ as follows. If $A_0 \ne \emptyset$, let c_0 be an arbitrary member of A_0. Otherwise let $c_0 = \{0\}$. Suppose $c_0, \dots, c_m$ have already been constructed. If $A_{m+1} = \emptyset$, put $c_{m+1} = \{0, \dots, m+1\}$. If $A_{m+1} \ne \emptyset$ and $A_{m+1} \cap \widehat{c_0} = \emptyset$, let a_{m+1} be an arbitrary member of A_{m+1}. Finally if $A_{m+1} \cap \widehat{c_0} \ne \emptyset$, let c_{m+1} be an arbitrary member of $A_{m+1} \cap \widehat{\bigcup_{i \le r} c_i}$, where r is the greatest $j \le m$ such that $A_{m+1} \cap \widehat{\bigcup_{i \le j} c_i} \ne \emptyset$.

Select $B_\alpha \in J$ for $\alpha < \mathrm{cof}(J)$ so that $J = \bigcup_{\alpha < \mathrm{cof}(J)} P(B_\alpha)$. For $\alpha < \mathrm{cof}(J)$ and $n \in \omega$, let D^n_α be the set of all $s \in 2^{n+1}$ such that $s(n) = 1$ and there is $b \in A_n - B_\alpha$ with the property that $c_m \subseteq b$ whenever m is less than or equal to some $i < n$ with $s(i) = 1$. Given $\alpha < \mathrm{cof}(J)$, let $D_\alpha = \bigcup_{n \in \omega} D^n_\alpha$ and $U_\alpha = \bigcup_{s \in D_\alpha} O_s$.

Let us prove that the open set U_α is dense. Thus let $k \in \omega$ and $p \in 2^k$. Pick $b \in A - B_\alpha$ so that $\bigcup_{m<k} c_m \subseteq b$ and $\max(b \cap \omega) \ge k$. Define $q \supset p$ by $\mathrm{dom}(q) = \max(b \cap \omega) + 1$, $q(\max(b \cap \omega)) = 1$ and $q(i) = 0$ for all i with $k \le i < \max(b \cap \omega)$. Then $q \in D^{\max(b \cap \omega)}_\alpha$.

Select $f \in \bigcap_{\alpha<\mathrm{cof}(J)} U_\alpha$. For each $\alpha < \mathrm{cof}(J)$, pick $s_\alpha \in D_\alpha$ with $s_\alpha \subset f$. Put

$$Y = \{\max(\mathrm{dom}(s_\alpha)) : \alpha < \mathrm{cof}(J)\}$$

and let $m_0, m_1, \ldots$ be the increasing enumeration of the elements of Y. Set $E_0 = A_{m_0}$ and for each $l \in \omega$,

$$E_{l+1} = \Big\{a \in A_{m_{l+1}} : \bigcup_{i\le l} c_{m_i} \subseteq a\Big\}.$$

Finally set $C = \bigcup_{l\in\omega} E_l$. If $\alpha < \mathrm{cof}(J)$ and $l \in \omega$ are such that $\max(\mathrm{dom}(s_\alpha)) = m_l$, then $s_\alpha \in D_\alpha^{m_l}$ and $s_\alpha(m_i) = 1$ for all $i < l$, and therefore there is $b \in A_{m_l} - B_\alpha$ such that $\bigcup_{j\le m_i} c_j \subseteq b$ for all $i < l$. It clearly follows that $C \in J^+$ and $c_{m_l} \in E_l$ for all $l \in \omega$. Finally given $d, e \in C$ with $\max(d\cap\omega) < \max(e \cap \omega)$, we have $c_{\max(d\cap\omega)} \subseteq e$. ■

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Received 20 July 2000;
in revised form 14 June 2001