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Homeomorphisms of composants of Knaster continua

by

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Abstract. The Knaster continuum K_p is defined as the inverse limit of the pth degree tent map. On every composant of the Knaster continuum we introduce an order and we consider some special points of the composant. These are used to describe the structure of the composants. We then prove that, for any integer $p \geq 2$, all composants of K_p having no endpoints are homeomorphic. This generalizes Bandt's result which concerns the case $p = 2$.

1. Introduction. For an integer $p \geq 2$, let $f_p : [0,1] \to [0,1]$ be the pth degree tent map, shown in Fig. 1.

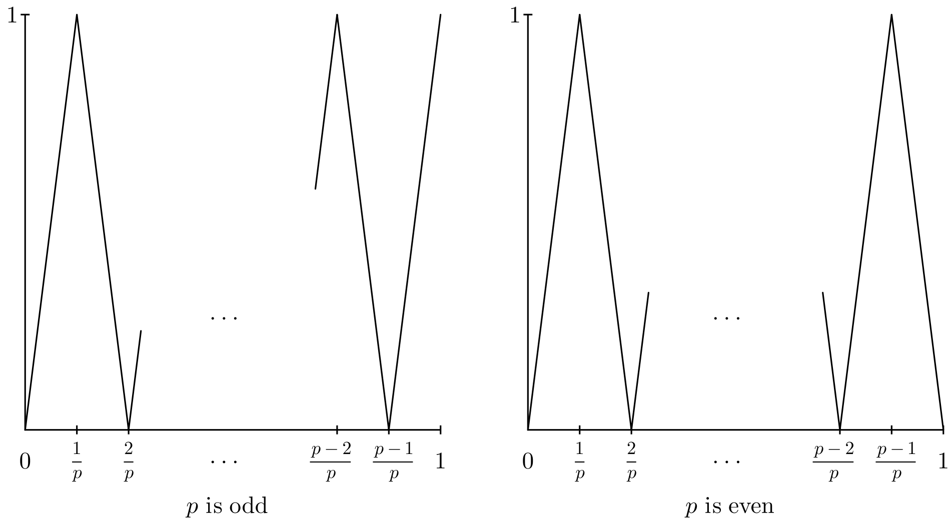

Fig. 1

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The inverse limit K_p of the tent map f_p is given by

$$K_p = \varprojlim\{[0,1], f_p\} = \{(\ldots x_{-3}x_{-2}x_{-1}) \in [0,1]^{\mathbb{N}} : x_{-i} = f_p(x_{-i-1}),\ i \in \mathbb{N}\}.$$

Although this notation may seem somewhat unusual, it will turn out to be useful later on. The spaces K_p are often called Knaster continua since K_2 is, in fact, the Knaster "bucket handle" ([N], [W]). The bucket handle K_2 was constructed in 1922. Kuratowski attributed the idea to Knaster ([K1]). In the same volume of Fundamenta Mathematicae, Knaster gave credit to Kuratowski for the corresponding construction of K_3. In connection with dynamical systems, the space K_2 and related spaces have become known in the sixties as the "horseshoe"—the attractor of a suitably chosen nonlinear map.

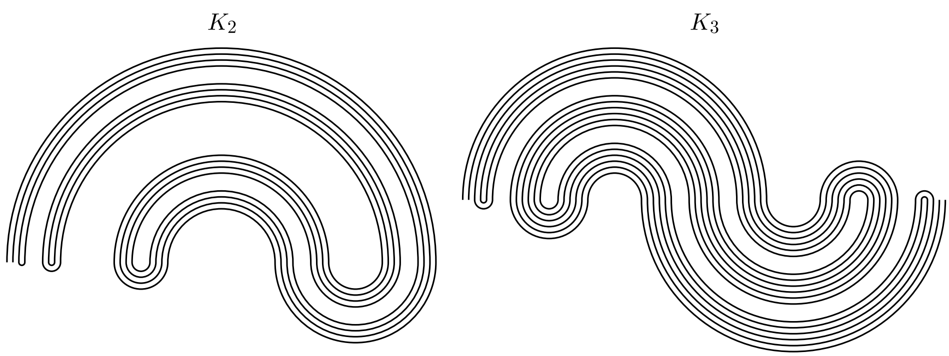

Fig. 2

For every integer $p \geq 2$, the Knaster continuum K_p is an indecomposable continuum. There are uncountably many arcwise connected components of K_p which coincide with its composants, and with the unstable manifolds in the dynamical setting. The Knaster continuum K_p is not arcwise connected. All continua K_p contain one special composant, that of the point zero, i.e., 0^∞. This composant is a one-to-one continuous image of the half-line, and the point zero is its endpoint. If $p \geq 2$, and p is odd, the continuum K_p contains one more special composant, that of the point one, i.e., 1^∞. It is also a one-to-one continuous image of the half-line, and the point one is its endpoint. All other composants of K_p are one-to-one continuous images of the whole straight line and contain only cut points ([K2], [N]).

In the fifties or even earlier, Knaster asked at his seminar whether all non-zero composants of K_2 are homeomorphic. In 1994, C. Bandt answered Knaster's question by proving the following theorem: All non-zero composants of K_2 are homeomorphic ([B]). In this paper we prove the following generalization of Bandt's result:

THEOREM. *Let $p \geq 2$ be an integer. All composants of K_p having no endpoints are homeomorphic.*

The key step in both proofs consists in exhibiting suitable descriptions of the composants of K_p. These descriptions are different. Bandt [B] describes points of K_2 as two-sided sequences, and composants of K_2 as copies of the real line consisting of some special intervals. These intervals are described by one-sided sequences. The structure of a particular composant is given by relations between intervals, i.e., between the corresponding one-sided sequences. Using some special properties of the structure of a composant, the homeomorphism is constructed.

We take advantage of Bandt's idea to represent K_p as a quotient space of the space of two-sided sequences of p symbols by a certain equivalence relation. We define an order on every composant of K_p. This makes possible a description of the structure of composants by relations between some special points, called i-points. When the structure of composants is given, the construction of the homeomorphism is a rather straightforward generalization of Bandt's construction.

In the remaining part of the introduction, we set up our notation and give some preliminaries.

Let $I_m = [m/p, (m+1)/p]$, $m \in \{0, \ldots, p-1\}$. The tent map f_p is a *Markov map*, i.e. f_p is surjective, C^1 and monotone on each of the open intervals $\operatorname{int} I_m$, and has the following additional properties:

(i) There exists $\alpha > 1$ such that $|f_p'(x)| \geq \alpha$ for each $x \in I_m$, $m \in \{0, \ldots, p-1\}$.

(ii) If $f_p(\operatorname{int} I_i) \cap \operatorname{int} I_j \neq \emptyset$ then $f_p(\operatorname{int} I_i) \supset \operatorname{int} I_j$, for $i, j \in \{0, \ldots, p-1\}$ (see [P-Y], p. 39).

The *transition matrix* A of the Markov map f_p is defined by

$$A(i,j) = \begin{cases} 1 & \text{if } f_p(\operatorname{int} I_i) \supset \operatorname{int} I_j, \\ 0 & \text{if } f_p(\operatorname{int} I_i) \cap \operatorname{int} I_j = \emptyset, \end{cases}$$

$0 \leq i, j \leq p-1$. In our case $A(i,j) = 1$ for all $0 \leq i, j \leq p-1$. In general, one defines the space X_A^+ of sequences by

$$X_A^+ = \Big\{ \overline{x} = (x_i)_{i \in \mathbb{Z}^+} \in \prod_{\mathbb{Z}^+} \{0, \ldots, p-1\} : A(x_i, x_{i+1}) = 1 \text{ for } i \geq 0 \Big\}.$$

Hence, denoting in our case X_A^+ by X_p^+, we see that $X_p^+ = \prod_{\mathbb{Z}^+} \{0, \ldots, p-1\}$.

The one-sided shift $\sigma : X_p^+ \to X_p^+$ is given by $\sigma((x_i)_{i \in \mathbb{Z}^+}) = (x_{i+1})_{i \in \mathbb{Z}^+}$.

Applying a known result on Markov maps (see [P-Y], pp. 41–43), we obtain the following lemma:

Lemma 1.1. *There exists a continuous mapping* $\pi : X_p^+ \to [0,1]$ *having the following properties*:

(1) π *is a semi-conjugacy*, *i.e.* π *is surjective and* $\pi \circ \sigma = f_p \circ \pi$,

(2) *points* $y \in [0,1]$ *have exactly one or two pre-images in* X_p^+, *i.e. for every* $y \in [0,1]$ *the set* $E(y) = \{\overline{x} \in X_p^+ : \pi(\overline{x}) = y\}$ *consists of either one or two points*,

(3) *the set of points* $y \in I$ *such that* $E(y)$ *consists of two points is equal to the countable set* $\bigcup_{i\in\mathbb{Z}^+} f_p^{-i}\{1/p, 2/p, \ldots, (p-1)/p\}$.

The mapping π can be defined by $\pi(\overline{x}) = \bigcap_{i=0}^{\infty} f_p^{-i}(I_{x_i})$, where $\overline{x} = (x_i)_{i\in\mathbb{Z}^+} \in X_p^+$, i.e. $\pi(\overline{x})$ corresponds to the only point $y \in [0,1]$ such that $f_p^i(y) \in I_{x_i}$ for $i \geq 0$.

Let $X_p = \prod_{\mathbb{Z}}\{0,\ldots,p-1\}$ denote the space of all two-sided sequences of p symbols. To avoid confusion, we denote left-infinite sequences by $\overleftarrow{x} = (x_{-i})_{i\in\mathbb{N}} = \ldots x_{-3}x_{-2}x_{-1}$, right-infinite sequences by $\overrightarrow{x} = (x_i)_{i\in\mathbb{Z}^+} = x_0x_1x_2\ldots$, and two-sided sequences by $\overline{x} = (x_i)_{\in\mathbb{Z}} = \ldots x_{-2}x_{-1}x_0x_1x_2\ldots$ The metric d on X_p is given as follows: For $\overline{x} = (x_i)_{i\in\mathbb{Z}}$ and $\overline{y} = (y_i)_{i\in\mathbb{Z}}$, let $l(\overline{x},\overline{y}) = \min\{l \in \mathbb{N}_0 : x_l \neq y_l \text{ or } x_{-l} \neq y_{-l}\}$. Then

$$d(\overline{x},\overline{y}) = \begin{cases} 2^{-l(\overline{x},\overline{y})} & \text{if } \overline{x} \neq \overline{y}, \\ 0 & \text{otherwise.} \end{cases}$$

2. Structure of composants. In order to describe the structure of composants we code the Knaster continuum by means of two-sided sequences. We start by defining an equivalence relation $\sim$ on the space X_p^+. Let p be odd. Two sequences $\overrightarrow{x}, \overrightarrow{y} \in X_p^+$, $\overrightarrow{x} = (x_i)_{i\in\mathbb{Z}^+}$, $\overrightarrow{y} = (y_i)_{i\in\mathbb{Z}^+}$ are equivalent if there is an $l \in \mathbb{Z}^+$ such that

(1) $x_i = y_i$ for $0 \leq i < l$,

(2) $|x_l - y_l| = 1$,

(3) for $i > l$,

$$x_i = y_i = \begin{cases} p-1 & \text{if } \min\{x_l, y_l\} \text{ is even,} \\ 0 & \text{if } \min\{x_l, y_l\} \text{ is odd.} \end{cases}$$

When p is even condition (3) should be modified as follows:

$$x_{l+1} = y_{l+1} = \begin{cases} p-1 & \text{if } \min\{x_l, y_l\} \text{ is even,} \\ 0 & \text{if } \min\{x_l, y_l\} \text{ is odd,} \end{cases} \tag{3'}$$

and $x_i = y_i = 0$ for $i > l+1$.

The quotient map $\widetilde{\pi} : X_p^+/{\sim} \to [0,1]$ is defined by $\widetilde{\pi}([\overrightarrow{x}]) = \pi(\overrightarrow{x})$. Note that $\overrightarrow{x}$ and $\overrightarrow{y}$ are equivalent if and only if $\pi(\overrightarrow{x}) = \pi(\overrightarrow{y})$. In particular, $\widetilde{\pi}$ is a homeomorphism. This enables us to use Bandt's [B] Propositions 2 and 3 to obtain the following assertions.

• $K_p = X_p/\approx$ where $(x_i)_{i\in\mathbb{Z}} \approx (y_i)_{i\in\mathbb{Z}}$ if there is m with $x_i = y_i$ for $i \le m$ and $x_{m+1}x_{m+2}\ldots \sim y_{m+1}y_{m+2}\cdots$

• Each left-infinite sequence $\overleftarrow{s} = \ldots s_{-3}s_{-2}s_{-1}$ describes one composant in K_p which is just the set of two-sided sequences having a left tail common to $\overleftarrow{s}$. Two sequences $\overleftarrow{s}$ and $\overleftarrow{t}$ describe the same composant if and only if they have a common left tail.

We now fix a left-infinite sequence $\overleftarrow{s} = \ldots s_{-3}s_{-2}s_{-1}$. Denote the corresponding composant of K_p by I. The composant I consists of unit length arcs I_v^0, $v = v_{-k}\ldots v_{-1}$, $v_{-i} \in \{0,\ldots,p-1\}$, $1 \le i \le k$, given by $I_v^0 = \{(x_i)_{i\in\mathbb{Z}} : x_{-i} = s_{-i} \text{ for } i > k,\ x_{-i} = v_{-i} \text{ for } i = 1,\ldots,k\}$ and $I^0 = \{(x_i)_{i\in\mathbb{Z}} : x_{-i} = s_{-i} \text{ for } i > 0\}$. Longer arcs I_v^n and I^n, of length p^n, are given by $I_v^n = \{(x_i)_{i\in\mathbb{Z}} : x_{-i} = s_{-i} \text{ for } i > k+n,\ x_{-i-n} = v_{-i} \text{ for } i = 1,\ldots,k\}$ and $I^n = \{(x_i)_{i\in\mathbb{Z}} : x_{-i} = s_{-i} \text{ for } i > n\}$. We can require $v_{-k} \ne s_{-k-n}$, but whenever we handle two arcs I_v^n and I_w^n, for simplicity, we will suppose that v and w have the same number of digits. This is possible because, if $v = v_{-k_1}\ldots v_{-1}$, $w = w_{-k_2}\ldots w_{-1}$ and $k_1 \ge k_2$, we require additionally $w_{-i} = s_{-i-n}$ for $k_2 < i \le k_1$. Two arcs I_v^n, $v = v_{-k}\ldots v_{-1}$, and I_w^n, $w = w_{-k}\ldots w_{-1}$, are *neighboring arcs* if they have a common endpoint. Therefore I_v^n and I_w^n are neighboring if and only if there is $m \in \mathbb{N}$, $m \le k$, such that:

(1) $v_{-i} = w_{-i}$ for $m < i \le k$,
(2) $|v_{-m} - w_{-m}| = 1$,
(3) for $1 \le i < m$,

$$v_{-i} = w_{-i} = \begin{cases} p-1 & \text{if } \min\{v_{-m}, w_{-m}\} \text{ is even,} \\ 0 & \text{if } \min\{v_{-m}, w_{-m}\} \text{ is odd,} \end{cases}$$

for p odd. When p is even condition (3) should be modified as follows:

$$(3') \qquad v_{-m+1} = w_{-m+1} = \begin{cases} p-1 & \text{if } \min\{v_{-m}, w_{-m}\} \text{ is even,} \\ 0 & \text{if } \min\{v_{-m}, w_{-m}\} \text{ is odd,} \end{cases}$$

and $v_{-i} = w_{-i} = 0$ for $1 \le i < m-1$.

DEFINITION 2.1. Fix $n \in \mathbb{N}_0$. Let

$$P(n) = \operatorname{card}\{s_{-i} : s_{-i} \text{ is odd, } 1 \le i \le n\}.$$

If $n = 0$ let $P(0) = 0$. We say that an arc I^n is *even* (respectively *odd*) if $P(n)$ is even (respectively odd). An arc I_v^n, $v = v_{-k}\ldots v_{-1}$, $v_{-k} \ne s_{-k-n}$, is *even* (respectively *odd*) if $(-1)^{P(n+k)} = \prod_{i=1}^k (-1)^{v_{-i}}$ (respectively $(-1)^{P(n+k)} \ne \prod_{i=1}^k (-1)^{v_{-i}}$).

We will now introduce an order structure on the composant I. For $\overline{x} = (x_i)_{i\in\mathbb{Z}}, \overline{y} = (y_i)_{i\in\mathbb{Z}} \in I$ let

$$k = k(\overline{x}, \overline{y}) = \max\{i \in \mathbb{N} : x_{-i} \ne s_{-i} \text{ or } y_{-i} \ne s_{-i}\}.$$

If $x_{-i} = s_{-i}$ and $y_{-i} = s_{-i}$ for all $i \in \mathbb{N}$, let $k = 0$.

DEFINITION 2.2. The *generalized lexicographical ordering* $\prec$ on I is defined as follows: We say that $\overline{x} \prec \overline{y}$ if either $(-1)^{P(k)} x_{-k} < (-1)^{P(k)} y_{-k}$ or there exists $l \in \mathbb{Z}$, $l > -k$, such that $x_i = y_i$ for $-k \leq i < l$ and

$$(-1)^{P(k)} \Big(\prod_{i=-k}^{l-1} (-1)^{x_i} \Big) x_l < (-1)^{P(k)} \Big(\prod_{i=-k}^{l-1} (-1)^{x_i} \Big) y_l.$$

We also say $\overline{x} \preceq \overline{y}$ if $\overline{x} \prec \overline{y}$ or $\overline{x} = \overline{y}$.

Note that the order depends on the chosen left-infinite sequence $\overleftarrow{s}$. Choosing another representative of this particular composant would lead either to the same, or to the opposite order.

REMARK 2.3. The ordering $\preceq$ on the composant I is natural because there exists an order-preserving bijection ϑ between the real line endowed with its natural order and I endowed with the ordering $\preceq$. Note that ϑ is continuous but its inverse is not.

Since we are interested only in composants without endpoints let us assume the following additional condition on the left-infinite sequence $\overleftarrow{s} = \ldots s_{-3} s_{-2} s_{-1}$: for p odd, there is no $m \in \mathbb{N}$ such that either $s_{-i} = 0$ for all $i \geq m$ or $s_{-i} = p - 1$ for all $i \geq m$; for p even, there is no $m \in \mathbb{N}$ such that $s_{-i} = 0$ for all $i \geq m$.

In the next definition we have to treat the cases of p odd and p even separately. This is a consequence of the difference in the definition of the equivalence relation between sequences in these two cases.

DEFINITION 2.4. Let p be odd (respectively even). Let $\overline{x}$ be a sequence such that there is $m \in \mathbb{N}$ with $x_{-i} = 0$ for all $i < m$, or $x_{-i} = p - 1$ for all $i < m$ (respectively $x_{-i} = 0$ for all $i < m$, or $x_{-m+1} = p - 1$ and $x_{-i} = 0$ for all $i < m - 1$). Such an $\overline{x}$ will be called an *identification point* or briefly an *i-point*. Fix $n \in \mathbb{N}_0$. Let $\overline{x}$ be an i-point and $m \in \mathbb{N}$ be such that $n + m = \max\{j : x_{-i} = 0 \text{ for all } i < j \text{ or } x_{-i} = p - 1 \text{ for all } i < j\}$ (respectively $n + m = \max\{j : x_{-i} = 0 \text{ for all } i < j, \text{ or } x_{-j+1} = p - 1 \text{ and } x_{-i} = 0 \text{ for all } i < j - 1\}$). Define the *n-level* of $\overline{x}$ by $L_n(\overline{x}) = m$.

The geometrical meaning of the n-level of $\overline{x}$ is visible from the following remark:

REMARK 2.5. Fix $n \in \mathbb{N}_0$. Let I^n_v and I^n_w be two neighboring arcs. Let $m \in \mathbb{N}$ be such that $v_{-m} \neq w_{-m}$. Let $\overline{x}$ be the common endpoint of I^n_v and I^n_w. Then $L_n(\overline{x}) = m$. Note that a similar definition can be found in [KL1] and [KL2], but these papers take a topological approach to this concept. For fixed $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$ there are countably many $\overline{x} \in I$ such that $L_n(\overline{x}) = m$.

From now on until the end of this section, we assume that p is odd. The case of p even can be treated analogously.

LEMMA 2.6. *Fix $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$, $m > 1$. Let $\overline{x}, \overline{y} \in I$ be two i-points such that*:

(1) $L_n(\overline{x}) = m$ *and* $L_n(\overline{y}) \geq m$,

(2) *there is no i-point $\overline{z}$ between $\overline{x}$ and $\overline{y}$ such that $L_n(\overline{z}) \geq m$.*

Then $\overline{x}$ and $\overline{y}$ are the endpoints of an arc I_v^{n+m-1} and there are exactly $p-1$ i-points $\overline{z}_1, \ldots, \overline{z}_{p-1}$ between $\overline{x}$ and $\overline{y}$ such that $L_n(\overline{z}_i) = m-1$ for $1 \leq i \leq p-1$.

Proof. Since there is no i-point $\overline{z}$ between $\overline{x}$ and $\overline{y}$ with $L_n(\overline{z}) \geq m$, we have $x_{-i} = y_{-i}$ for $i \geq m$. Since $\overline{x} \neq \overline{y}$, if $x_i = 0$ for $i > m$, then $y_i = p-1$ for $i > m$, and vice versa. Therefore $\overline{x}$ and $\overline{y}$ are the endpoints of the arc I_v^{n+m-1} of length p^{n+m-1} with $v = x_{-m-k} \ldots x_{-m}$, where k is the largest integer such that $x_{-m-k} \neq s_{-m-k}$. Suppose that the arc I_v^{n+m-1} is even. The i-points

$$\begin{aligned}
\overline{z}_1 &= \ldots x_{-m-1}x_{-m}0(p-1)^\infty \sim \ldots x_{-m-1}x_{-m}1(p-1)^\infty \prec \\
\overline{z}_2 &= \ldots x_{-m-1}x_{-m}10^\infty \sim \ldots x_{-m-1}x_{-m}20^\infty \prec \\
&\vdots \\
\overline{z}_{p-1} &= \ldots x_{-m-1}x_{-m}(p-2)0^\infty \sim \ldots x_{-m-1}x_{-m}(p-1)0^\infty
\end{aligned}$$

are the only i-points with $\overline{z}_i \in \operatorname{int} I_v^{n+m-1}$ and $L_n(\overline{z}_i) = m-1$. If I_v^{n+m-1} is odd, the conclusion is the same with $\overline{z}_{p-1} \prec \overline{z}_{p-2} \prec \ldots \prec \overline{z}_1$. ∎

A direct consequence of the previous lemma is the following remark:

REMARK 2.7. Each arc I^{n+1} of length p^{n+1} consists of p arcs $I_0^n, \ldots, I_{p-1}^n$ of length p^n and one of them is I^n. The arcs $I^0, I^1, I^2, \ldots$ form a nested sequence. If $s_{n+1} \neq 0$ and $s_{n+1} \neq p-1$, then I^n is one of the middle arcs of I^{n+1}. Hence, if there is no $m \in \mathbb{N}$ such that either $s_{-i} = 0$ for all $i \geq m$ or $s_{-i} = p-1$ for all $i \geq m$, the union of all I^n is all of the composant I.

LEMMA 2.8. *Fix $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Let $\overline{x}, \overline{y} \in I$ be two i-points such that*:

(1) $L_n(\overline{x}) = m$ *and* $L_n(\overline{y}) = l > m$,

(2) *there is no i-point $\overline{u}$ between $\overline{x}$ and $\overline{y}$ such that $L_n(\overline{u}) \geq L_n(\overline{y})$.*

Then there is an i-point $\overline{z} \in I$, $\overline{z} \neq \overline{x}$, such that:

(i) $L_n(\overline{z}) = m$,

(ii) *there is no i-point $\overline{u}$ between $\overline{z}$ and $\overline{y}$ with $L_n(\overline{u}) \geq L_n(\overline{y})$,*

(iii) $d(\overline{x}, \overline{y}) = d(\overline{y}, \overline{z})$, *where d is the natural inner metric on I_v^{n+l}, $v = y_{-n-l-k} \ldots y_{-n-l-1}$, and k is the largest integer such that $y_{-n-l-k} \neq s_{-n-l-k}$.*

Proof. The i-points $\overline{x}, \overline{y}$ belong to the arc I_w^{n+l-1}, $w = y_{-n-l-k} \dots y_{-n-l}$, k is the largest integer such that $y_{-n-l-k} \neq s_{-n-l-k}$, and $\overline{y}$ is an endpoint of I_w^{n+l-1}. Let I_v^{n+l-1} be a neighboring arc of I_w^{n+l-1} with the common endpoint $\overline{y}$. Since $L_n(\overline{x}) < L_n(\overline{y})$, the point $\overline{x}$ is not an endpoint of I_w^{n+l-1}. Since f_p is symmetric on the open interval $\langle (m-1)/p, (m+1)/p \rangle$, $m \in \{1, \dots, p-1\}$, neighboring arcs have symmetric interiors with the common endpoint as their center of symmetry. Therefore, there is an i-point $\overline{z} \in I_v^{n+l-1}$ which is symmetric to the point $\overline{x}$, and (i)–(iii) are satisfied. ∎

The next remark is a direct consequence of the previous lemma:

REMARK 2.9. Fix $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Let $w = w_{-m} \dots w_{-1}$, $w_{-i} \in \{0, \dots, p-1\}$, $1 \leq i \leq m$. Among any $2p^m$ consecutive arcs I_v^n there are exactly two of type I_w^n, and they have different parity.

Let $0 = n_1 < n_2 < \dots$ be a sequence of integers. Let $d_k = n_{k+1} - n_k$. By Lemma 2.6, between any two consecutive points $\overline{x}$ and $\overline{y}$ with $L_{n_k}(\overline{x}) \geq d_k$ and $L_{n_k}(\overline{y}) \geq d_k$ (respectively $L_{n_k}(\overline{x}) \geq d_k + d_{k+1}$ and $L_{n_k}(\overline{y}) \geq d_k + d_{k+1}$), there is an arc $I_v^{n_k}$ with $v_{-d_k} \dots v_{-1} = s_{-n_{k+1}} \dots s_{-n_k-1}$ (respectively $v_{-d_k-d_{k+1}} \dots v_{-1} = s_{-n_{k+2}} \dots s_{-n_k-1}$). With respect to the sequence $n_1, n_2, \dots$, the arc $I_v^{n_k}$ is called a *return arc* of order n_k (respectively a *close return arc* of order n_k). By Lemma 2.6, each arc $I_v^{n_{k+2}}$ contains $p^{d_k+d_{k+1}}$ arcs $I_w^{n_k}$; $p^{d_{k+1}}$ of them are return arcs of order n_k, and exactly one of them is a close return arc of order n_k.

Fix $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$. For $v = v_{-m} \dots v_{-1}$, $v_{-i} \in \{0, \dots, p-1\}$, $1 \leq i \leq m$, define a map $\phi_v^n : I^n \to I_v^n$ as follows: For $\overline{x} \in I^n$, $\overline{x} = \dots x_{-2} x_{-1} x_0 x_1 x_2 \dots$, let $\phi_v^n(\overline{x}) = \dots x_{-m-1} v_{-m} \dots v_{-1} x_0 x_1 \dots \in I_v^n$. Let us describe sequences $(\overline{x}_i)_{i \in \mathbb{N}}$ in I which converge to a given $\overline{x}$. The convergence on I is the coordinatewise convergence of sequences up to identification of equivalent i-points. For given $\overline{x}$ there is $n \in \mathbb{N}$ with $\overline{x} \in \operatorname{int} I^n$. A sequence $(\overline{x}_i)_{i \in \mathbb{N}}$ in I converges to $\overline{x}$ if and only if the following two conditions are satisfied:

(1) For each $n_j > n$, the points of the sequence $(\overline{x}_i)_{i \in \mathbb{N}}$ eventually belong to return arcs, i.e., there is an $i_0 \in \mathbb{N}$ such that for each $i \geq i_0$ there is a finite sequence $v(i)$ such that $\overline{x}_i \in I_{v(i)}^{n_j}$, and $I_{v(i)}^{n_j}$ is a return arc.

(2) The position of $\overline{x}_i$ stabilizes, i.e., $\lim_{i \to \infty} (\phi_{v(i)}^{n_j})^{-1}(\overline{x}_i) = \overline{x}$ in I^{n_j}.

3. Construction of the homeomorphism. Having described the structure of a composant, we are ready to prove the following theorem.

THEOREM 3.1. *Let $p \geq 2$ be an integer. All composants of K_p having no endpoint are homeomorphic.*

To prove the theorem, we construct a homeomorphism h from an arbitrary composant I with characteristic left-infinite sequence $\overleftarrow{s} = \dots s_{-3} s_{-2} s_{-1}$

to the particular composant J characterized by the left-infinite sequence $\left(\frac{p-1}{2}\right)^\infty = \ldots \frac{p-1}{2}\frac{p-1}{2}\frac{p-1}{2}$ if p is odd, and $\left(\frac{p}{2}\right)^\infty = \ldots \frac{p}{2}\frac{p}{2}\frac{p}{2}$ if p is even. This construction is a rather straightforward generalization of Bandt's construction. We give some details for reader's convenience, but omit the proofs which can be easily reconstructed from [B]. Again, we restrict ourselves to the case of p odd, since the case of p even is analogous.

We denote arcs of J by J^n_w and the corresponding maps by $\psi^n_w : J^n \to J^n_w$. Since there is no $m \in \mathbb{N}$ with either $s_i = 0$ for all $i \geq m$, or $s_i = p-1$ for all $i \geq m$, it is easy to choose a sequence $0 = n_1 < n_2 < \ldots$ with

(i) $d_j = n_{j+1} - n_j \geq 10 + j$ for $j \geq 1$,
(ii) $s_{-n_j}s_{-n_j+1} \neq 00$ and $s_{-n_j}s_{-n_j+1} \neq (p-1)(p-1)$ for $j \geq 2$.

Let $h : I \to J$ be a one-to-one map with the following properties, with respect to the chosen sequence:

(a) h maps I^{n_j} continuously onto J^{n_j} for all j.
(b) h maps each close return arc $I^{n_j}_v$ onto a return arc $J^{n_j}_w$ in the same way as I^{n_j} is mapped onto J^{n_j}, i.e., $h\phi^{n_j}_v(\overline{z}) = \psi^{n_j}_w h(\overline{z})$ for $\overline{z} \in I^{n_j}$.
(c) h^{-1} maps each close return arc $J^{n_j}_v$ onto a return arc $I^{n_j}_w$ in the same way as it maps J^{n_j} onto I^{n_j}.

Then h is a homeomorphism.

Lemma 3.2. *Let $m \geq 2$. Let $\widetilde{I}$ be a union of c consecutive arcs of order n_m in I, and $\widetilde{J}$ a union of d consecutive arcs of order n_m in J, and let the length ratio of these two arcs be $q = \max\{c/d, d/c\} < p^2$. Assume that between any return arc of order n_m and an endpoint in $\widetilde{I}$ and $\widetilde{J}$ there are at least p^{d_m-2} arcs of order n_m. Then there are partitions of $\widetilde{I}$ and $\widetilde{J}$ into finitely many subarcs and a correspondence between the first, second, ..., kth element* (*counted from the* 0*-endpoint*) *of these partitions such that*:

(1) *Each close return arc of order n_{m-1} in $\widetilde{I}$ or $\widetilde{J}$ is a partition element and corresponds to a return arc of order n_{m-1} in the other partition which has the same parity* (*respectively opposite parity*) *if $I^{n_{m-1}}$ and $J^{n_{m-1}}$ are of the same parity* (*respectively opposite parity*).

(2) *The other partition elements are unions of consecutive arcs of order n_{m-1}. Between an endpoint of a partition arc and the next return arc of order n_{m-1} inside that arc, there are at least $p^{d_{m-1}-2}$ other arcs of order n_{m-1}. The length ratio of two corresponding partition elements is at most $q + p^{-m}$.*

Proof. Consider the smaller endpoint of $\widetilde{I}$ (with respect to the generalized lexicographical ordering) and the smaller endpoint of $\widetilde{J}$ as 0-endpoints. Moreover, consider arcs of order n_m as units of measurement. In this way we define a linear scale on $\widetilde{I}$ and $\widetilde{J}$. There is a unique linear orientationpreserving map χ from $\widetilde{I}$ into $\widetilde{J}$. Assume that $c > d$. Let $\overline{x}_1, \dots, \overline{x}_t$ and $\overline{y}_1, \dots, \overline{y}_r$ be the smaller endpoints of the close return arcs of order n_{m-1} in $\widetilde{I}$ and $\widetilde{J}$, respectively. Let $\overline{y}'_i = \chi(\overline{x}_i)$ for $i = 1, \dots, t$ and $\overline{x}'_i = \chi^{-1}(\overline{y}_i)$ for $i = 1, \dots, r$. Also, denote the endpoints of $\widetilde{I}$ by $\overline{x}_0$ and $\overline{x}_{t+1}$, and the endpoints of $\widetilde{J}$ by $\overline{y}_0$ and $\overline{y}_{r+1}$.

Two close return arcs of order n_{m-1} are contained in different return arcs of order n_m, each of which is one of the middle arcs of two different arcs of order n_{m+1}. So, between $\overline{x}_{i+1}$ and $\overline{x}_i$, $i = 1, \dots, t-1$, there are at least p^{d_m-2} arcs of order n_m. The assumption of the lemma implies that this remains true for $i = 0$ and $i = t$. Similarly, between $\overline{y}'_{i+1}$ and $\overline{y}'_i$, $i = 1, \dots, t-1$, there are at least p^{d_m-2} arcs of order n_m.

Let $\mathcal{P}$ be the partition of $\widetilde{I}$ with vertices $\overline{x}_i$, $i = 0, \dots, t+1$, and $\overline{y}'_i$, $i = 1, \dots, t$. Then at least one of any two neighboring arcs of $\mathcal{P}$ is longer than p^{d_m-3}. For the partition $\mathcal{Q}$ of $\widetilde{J}$, induced by $\overline{y}_i$ and $\overline{x}'_i$, at least one of any two neighboring arcs is longer than p^{d_m-3}/q.

For every point $\overline{y}'_i$ in $\widetilde{I}$ let $\overline{y}''_i$ in $\widetilde{I}$ be the nearest point to $\overline{y}'_i$ with the following properties:

- $\overline{y}''_i$ is the smaller endpoint of a return arc of order n_{m-1} which has the same parity (respectively, opposite parity) as the close return arc given by $\overline{y}_i$, if $I^{n_{m-1}}$ and $J^{n_{m-1}}$ are of the same parity (respectively, opposite parity),
- if $\overline{y}'_i$ is the endpoint of a short arc, i.e., of an arc whose length is less than or equal to p^{d_m-3}, then we choose $\overline{y}''_i$ inside this short arc.

By Remark 2.9, the distance between $\overline{y}'_i$ and $\overline{y}''_i$ is at most two units.

Analogously, we choose points $\overline{x}''_i$ in $\widetilde{J}$, but now we require that if $\overline{x}'_i$ is the endpoint of an arc of length smaller than p^{d_m-3}/q, then $\overline{x}''_i$ is chosen outside the short arc. Consider now $\overline{x}_i$ and $\overline{y}''_i$ as vertices of a new partition $\mathcal{P}'$ of $\widetilde{I}$, and $\overline{y}_i$ and $\overline{x}''_i$ as vertices of a new partition $\mathcal{Q}'$ of $\widetilde{J}$. These partitions have the same number of elements, and the first, second, ..., kth arcs correspond to each other. All pairs of corresponding arcs, except the first one, begin with a close return arc in $\widetilde{I}$ and a return arc in $\widetilde{J}$, or conversely. Adding the larger endpoints of these beginning arcs to the vertices of the partitions $\mathcal{P}'$ and $\mathcal{Q}'$ one obtains the desired partitions of $\widetilde{I}$ and $\widetilde{J}$.

Now we have to show the last statement of (2). For all pairs of short arcs the ratio of their lengths is smaller than q. For other pairs of arcs, the largest relative increase is p^2/p^{d_m-3}, and the largest relative decrease is qp^2/p^{d_m-3}. Since $d_m \geq 10 + m$ we have

$$q' \leq q\frac{1 + p^2/p^{d_m-3}}{1 - qp^2/p^{d_m-3}} \leq q\frac{1 + p^{-5-m}}{1 - p^{-3-m}} = q + q\frac{p^2 + 1}{p^2(p^{3+m} - 1)} \leq q + p^{-m}. \blacksquare$$

Proof of Theorem 3.1. We are going to construct h by induction on the intervals I^{n_j} such that properties (a)–(c) are valid.

First we fix an order-preserving linear map h from I^0 onto J^0. By Remark 2.7, $J^{n_2} \setminus J^{n_1}$ consists of two arcs of length $(p^{n_2} - 1)/2$ each. By the same remark and by (ii) of Section 3, $I^{n_2} \setminus I^{n_1}$ also consists of two arcs of length at least p^{n_2-2} and at most $p^{n_2} - p^{n_2-2}$ each. We extend h to a homeomorphism from I^{n_2} onto J^{n_2} in such a way that the two arcs of $I^{n_2} \setminus I^{n_1}$ are mapped linearly onto the two corresponding arcs of $J^{n_2} \setminus J^{n_1}$. Thus $h : I^{n_2} \to J^{n_2}$ is defined as a piecewise linear bijection.

Suppose $h : I^{n_k} \to J^{n_k}$ has already been defined and satisfies (a)–(c). We now define h on $I^{n_{k+1}} \setminus I^{n_k}$ so that (a)–(c) hold. We first apply Lemma 3.2 with $m = k - 1$, $q = p^2 - 1$ and taking as $\widetilde{I}$ each of the two arcs of $I^{n_{k+1}} \setminus I^{n_k}$ which are unions of arcs of order $n_k = n_{m+1}$. Since every return arc of order n_m is one of the middle arcs of a larger arc of order n_{m+1} (in the same way as I^{n_m} is contained in $I^{n_{m+1}}$), there are at least p^{d_m-2} arcs of order n_m between such a return arc and an endpoint of $\widetilde{I}$. Lemma 3.2 gives a correspondence from close return arcs of order $n_{m-1} = n_{k-2}$ of I to return arcs of J and vice versa. The definition of $h : I^{n_{k-2}} \to J^{n_{k-2}}$ is now transferred to each pair of such arcs. So far, (a)–(c) are satisfied.

By (2), we may again apply Lemma 3.2 to each pair of the remaining arcs, with $m = k - 2$ and $q = p^2 - 1 + p^{-(k-2)}$, then with $m = k - 3$, and by induction to all return arcs of orders down to $n_1 = 0$. Since we have a geometric series, q stays below p^2. Therefore, h is defined on $I^{n_{k+1}}$ and by induction on all of I in such a way that (a)–(c) hold. Thus $h : I \to J$ is a homeomorphism. ■

REMARK 3.3. Besides the tent maps f_p one can consider analogous tent maps g_p, where $g_p(0) = 1$. Note that the continuum defined as $\varprojlim\{[0,1], g_p\}$ is homeomorphic to the Knaster continuum K_p. This is so because the mappings g_p^2 and f_p^2 are conjugate. Indeed, $g_p^2\varphi = \varphi f_p^2$, where $\varphi : [0,1] \to [0,1]$ is the homeomorphism given by $\varphi(x) = 1 - x$.

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