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Torsion in one-term distributive homology

by

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Abstract. The one-term distributive homology was introduced in [Prz] as an atomic replacement of rack and quandle homology, which was first introduced and developed by Fenn–Rourke–Sanderson [FRS] and Carter–Kamada–Saito [CKS]. This homology was initially suspected to be torsion-free [Prz], but we show in this paper that the one-term homology of a finite spindle may have torsion. We carefully analyze spindles of block decomposition of type $(n,1)$ and introduce various techniques to compute their homology precisely. In addition, we show that any finite group can appear as the torsion subgroup of the first homology of some finite spindle. Finally, we show that if a shelf satisfies a certain, rather general, condition then the one-term homology is trivial—this answers a conjecture from [Prz] affirmatively.

1. Introduction. For any set X, we can consider colorings of arcs of a link diagram by elements of X. Motivated by a Wirtinger presentation of the fundamental group of a link complement, we may assume that overcrossings preserve colors while undercrossings change them in a way described by some binary operation $\star\colon X \times X \to X$, as shown in Fig. 1.

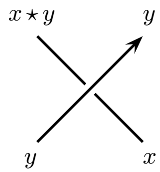

Fig. 1. Propagation of colors at a crossing

The requirement that the Reidemeister moves change the coloring only locally results in several conditions on $(X,\star)$, making it a *quandle* [Joy]

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or a *rack* [FR] (1). However, the most important is the third Reidemeister move, visualized in Fig. 2, because of its close connection to the Yang–Baxter equation [CES, Eis, Prz]. This requires $\star$ to be distributive, i.e. $(x \star y) \star z = (x \star z) \star (y \star z)$, and pairs $(X, \star)$ satisfying this condition are called *shelves.* If $\star$ is also idempotent, i.e. $x \star x = x$, then $(X, \star)$ is a *spindle* [Cr].

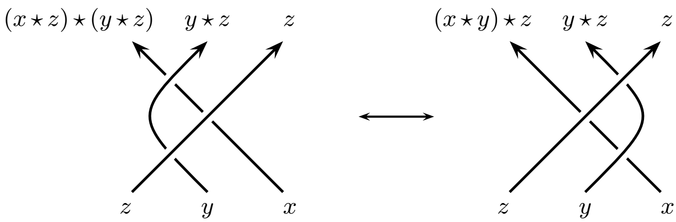

Fig. 2. Third Reidemeister move forces $\star$ to be distributive.

Link invariants come not only from counting colorings by racks or by quandles, but also from their homologies (see [CJKLS, CJKS]). They have a rich algebraic structure [NP-2] and they were computed completely for some dihedral and Alexander quandles [Cla, EG, NP-1, Nos] and partially for other families [Gr, NP-3]. We noticed in [Prz] that homology groups can be defined similarly for any shelf or spindle. Even more, there is a chain complex with a simpler differential, called a *one-term distributive chain complex* $C^\star(X)$ (see Section 2 for a definition). We showed in [Prz, PS] that if $(X, \star)$ is a rack, then $C^\star(X)$ is acyclic.

More generally, to force $C^\star(X)$ to be acyclic it is enough to have just one element $y \in X$ such that $x \mapsto x \star y$ is a bijection. This is perhaps the reason why this homology has never been examined before. At first, one would be tempted to suspect that $H^\star(X)$ is always trivial, but we quickly computed the homology for a right trivial shelf $(X, \dashv)$, where $a \dashv y = y$, and found it to be a large free group [PS]. For a while all one-term homology we computed was free; only in February of 2012 did we find two four-element spindles with torsion in homology. More precisely, our examples are given by the following tables:

$\star_1$	1	2	3	4
1	1	2	3	4
2	1	2	3	4
3	1	2	3	4
4	2	1	1	4

$\star_2$	1	2	3	4
1	1	2	4	3
2	1	2	4	3
3	2	1	3	4
4	2	1	3	4

Using `Mathematica`, we found that the first homology for both spindles has

(1) Involutive quandles, i.e. with $(x \star y) \star y = x$, were considered for the first time in [Tak] under the name *Kei* (圭). Both [Car] and [Fe] provide a nice introduction to the subject.

$\mathbb{Z}_2$-torsion. Namely, we obtained the following groups:

$$
\begin{aligned}
H_0^{\star_1}(X) &= \mathbb{Z}^2, & H_0^{\star_2}(X) &= \mathbb{Z}^2,\\
H_1^{\star_1}(X) &= \mathbb{Z}^2 \oplus \mathbb{Z}_2, & H_1^{\star_2}(X) &= \mathbb{Z}^2 \oplus \mathbb{Z}_2^4,\\
H_2^{\star_1}(X) &= \mathbb{Z}^8 \oplus \mathbb{Z}_2^4, & H_2^{\star_2}(X) &= \mathbb{Z}^8 \oplus \mathbb{Z}_2^{12}.
\end{aligned}
$$

In this paper, we compute the homology of the first spindle and, more generally, of other *f-spindles*, which are spindles given by a function $f\colon X_0 \to X_0$ where $X = X_0 \sqcup \{b\}$ and $x \star y = y$, unless $x = b$, in which case $b \star y = f(y)$ (see Definition 3.1). This family of spindles was introduced in [PS]. If X is finite, we prove in Section 4 the following formulas for normalized homology (see Section 2 for a definition of a normalized complex):

THEOREM 4.3. *Assume X is a finite f-spindle. Then its homology is given by the formulas*

$$
\begin{cases}
\widetilde{H}_0^N(X) = \mathbb{Z}^{\mathrm{orb}(f)},\\
H_1^N(X) = \mathbb{Z}^{(\mathrm{orb}(f)-1)|X_0|+2\mathrm{orb}(f)} \oplus \mathbb{Z}_\ell^{\mathrm{init}(f)},\\
H_n^N(X) = (\mathbb{Z}^{(\mathrm{orb}(f)-1)|X|^2+|X|} \oplus \mathbb{Z}_\ell^{\mathrm{init}(f)|X|})^{\oplus(|X|-1)^{n-2}} & \text{for } n \geq 2.
\end{cases}
$$

In particular, $H_{n+1}^N(X) = H_n^N(X)^{\oplus(|X|-1)}$ for $n \geq 2$.

Here, $\mathrm{orb}(f)$ and $\mathrm{init}(f)$ stand, respectively, for the number of orbits of f and the number of elements that are not in the image of f. This shows that any power of a cyclic group can appear as the torsion subgroup of $H_1(X)$ for some spindle. The other finite abelian groups are realized by *block spindles*, defined in Section 5. The idea is that we take several blocks X_i and a function $f_i\colon X_i \to X_i$ for each of them, and we take as X their disjoint sum together with a one-element block $\{b\}$. Then each $X_i^+ := X_i \sqcup \{b\}$ is a subspindle, which contributes some torsion to $H_1(X)$. We show that, in fact, there is no more torsion.

THEOREM 5.4. *Assume a block spindle X has a one-element block $\{b\}$. Then*

$$
H_1(X) \cong F \oplus \bigoplus_{i\in I} H_1(X_i^+),
$$

where F is a free abelian group of rank $\sum_{i\neq j} \mathrm{orb}(f_i)|X_j|$. In particular, every finite abelian group can be realized as the torsion subgroup of $H_1(X)$ for some spindle X.

This paper is arranged as follows. We provide basic definitions in Section 2, including the construction of a distributive chain complex and its variants: augmented, reduced, and related chain complexes. We also include a discussion about degenerate and normalized complexes and how they are related to each other.

The next two sections are devoted to the calculation of homology groups for f-spindles. In Section 3 we define an f-spindle, provide a few examples, and then compute the first homology group. Then in Section 4 we generalize these calculations for any homology groups. We conclude that section with a presentation of homology groups in terms of generators and relations for any f-spindle, not necessarily finite.

The final section is split into four parts. In the first, we give a presentation of the relative homology groups with respect to the subspindle $X_0 \subset X$. The second part contains a proof of Theorem 5.4 and the third discusses the Growth Conjecture from [PS]. The last part contains a result about the acyclicity of a distributive chain complex under a small condition—all that was known previously was that homology was annihilated by some number, leaving it with a possibility to have torsion [Prz].

2. Distributive homology. A *spindle* $(X, \star)$ consists of a set X equipped with a binary operation $\star\colon X \times X \to X$ that is

(i) idempotent, $x \star x = x$, and
(ii) self-distributive, $(x \star y) \star z = (x \star z) \star (y \star z)$.

A (*one-term*) *distributive chain complex* $C^\star(X)$ of X is defined as follows (see also [Prz, PS]):

$$C^\star_n(X) := \mathbb{Z}X^{n+1} = \mathbb{Z}\langle (x_0, \dots, x_n) \mid x_i \in X \rangle, \tag{1}$$

$$\partial_n := \sum_{i=0}^{n} (-1)^i d^i, \tag{2}$$

where the maps d^i are given by the formulas

$$d^0(x_0, \dots, x_n) = (x_1, \dots, x_n), \tag{3}$$

$$d^i(x_0, \dots, x_n) = (x_0 \star x_i, \dots, x_{i-1} \star x_i, x_{i+1}, \dots, x_n). \tag{4}$$

We check that $d^i d^j = d^{j-1} d^i$ whenever $i < j$, which implies $\partial^2 = 0$. The homology of this chain complex is called the (*one-term*) *distributive homology* of $(X, \star)$ and will be denoted by $H^\star(X)$. There is also an *augmented* version, $\widetilde{C}(X)$, with $\widetilde{C}^\star_n(X) = C^\star_n(X)$ for $n \geq 0$, but $\widetilde{C}^\star_{-1}(X) = \mathbb{Z}$ and $\partial_0(x) = 1$. Its homology, called the *augmented distributive homology* $\widetilde{H}^\star(X)$, satisfies the following, as in the classical case:

$$H^\star_n(X) = \begin{cases} \mathbb{Z} \oplus \widetilde{H}^\star_n(X), & n = 0, \\ \widetilde{H}^\star_n(X), & n > 0. \end{cases} \tag{5}$$

For simplicity, we will omit $\star$ and write $C(X)$ and $H(X)$ for the distributive chain complex and its homology, and similarly for the augmented versions. Furthermore, we will use the shorthand notation $\underline{x} := (x_0, \dots, x_n)$

for a sequence or a subsequence of elements, omitting brackets in the latter case (2), and occasionally a multilinear notation (3) $(\ldots, x_i + x_i', \ldots) := (\ldots, x_i, \ldots) + (\ldots, x_i', \ldots)$. In particular, $(0, \underline{x}) = 0$.

Assume $Y \subset X$ is a subspindle of X, i.e. $x \star y \in Y$ whenever $x, y \in Y$. It follows directly from the definition above that the chain complex $C(Y)$ is a subcomplex of $C(X)$. The quotient $C(X,Y) := C(X)/C(Y)$ is called the *relative chain complex* of X modulo Y. It is spanned by sequences $\underline{x}$ where not all entries are from Y. Clearly, there is a long exact sequence of homology

$$\cdots \to H_n(Y) \to H_n(X) \to H_n(X,Y) \to H_{n-1}(Y) \to \cdots \tag{6}$$

and an analogous sequence when we replace the homologies of Y and X with their augmented versions.

Let $f\colon X \to Y$ be a homomorphism of spindles, i.e. $f(x \star x') = f(x) \star f(x')$. Then there is an induced chain map $f_\sharp\colon C(X) \to C(Y)$ sending a sequence $(x_0, \ldots, x_n)$ to $(f(x_0), \ldots, f(x_n))$. In the case where $r\colon X \to X$ is a retraction on a subspindle Y (i.e. $r(X) = Y$ and $r|_Y = \mathrm{id}$), one has a decomposition $C(X) \cong C(Y) \oplus C(X,Y)$. In particular, for any element $b \in X$ one has $C(X) \cong C(b) \oplus C(X,b)$, so that $C(X,b)$ is independent of the choice of b. It is called the *reduced chain complex* (see [PP-1]). As a subcomplex of $C(X)$, it is generated by differences $\underline{x} - \underline{b}$.

Idempotency of the spindle operation in X implies that its distributive chain complex $C(X)$ is in fact a *weak simplicial module* (see [Prz, PP-1]). In particular, there are notions of degenerate and normalized complexes. Indeed, if $\underline{x}$ has a repetition, say $x_i = x_{i+1}$, so does each entry in $\partial\underline{x}$, as $d^i\underline{x} = d^{i+1}\underline{x}$ cancel each other and other faces preserve the repetition. Hence, sequences with repetition span a subcomplex $C^D(X) \subset C(X)$, called the *degenerate complex* of X. Explicitly,

$$C_n^D(X) := \mathbb{Z}\langle \underline{x} \mid x_i = x_{i+1} \text{ for some } 0 \le i < n\rangle. \tag{7}$$

The quotient $C^N(X) := C(X)/C^D(X)$ is called the *normalized complex* and is generated by sequences with no repetitions. Degenerate and normalized homology are written, respectively, as $H^D(X)$ and $H^N(X)$. In classical homology theories (simplicial homology, group homology, etc.) the degenerate complex is acyclic, so that $H^N \cong H$. However, this does not hold for a weak simplicial module and we can have nontrivial degenerate homology in the distributive case, so that normalized homology $H^N(X)$ is usually different from $H(X)$. However, we can split the degenerate complex apart. This was first shown in [LN] for quandles (for the two-term variant of distributive homology) and an explicit formula for the splitting map appeared for

(2) For instance, $(a, \underline{x})$ stands for $(a, x_0, \ldots, x_n)$, not for $(a, (x_0, \ldots, x_n))$.

(3) Think of $(x_0, \ldots, x_n)$ as an element $x_0 \otimes \cdots \otimes x_n$ in $\mathbb{Z}X^{\otimes(n+1)}$.

the first time in [NP-1]. It was observed in [Prz, PP-1] that the same map works for the one-term variant as well.

THEOREM 2.1 (cf. [Prz, PP-1]). *Let $(X,\star)$ be a spindle. Then the exact sequence of complexes*

$$0 \to C^D(X) \to C(X) \to C^N(X) \to 0 \tag{8}$$

splits. In particular, $H(X) \cong H^N(X) \oplus H^D(X)$.

EXAMPLE 2.2. A normalized complex for a one-element spindle $\{b\}$ has a unique generator in degree 0. Since a retraction splits a normalized complex as well, we obtain an isomorphism $\widetilde{H}^N(X) \cong H^N(X,b)$ for any $b \in X$, so that the normalized versions of reduced and augmented homologies coincide. In fact, the inclusion $C^N(X,b) \subset \widetilde{C}^N(X)$ is a homotopy equivalence.

In [PP-2] we canonically decomposed the degenerate complex into a bunch of copies of the normalized complex. Therefore, normalized homology carries all information and there is no need to bother with the degenerate part.

THEOREM 2.3 (cf. [PP-2]). *Let $(X,\star)$ be a spindle. Then the degenerate complex decomposes as*

$$C^D_n(X) = \bigoplus_{p+q=n-2} \widetilde{C}_p(X) \otimes C^N_q(X) \tag{9}$$

with the differential acting only on the first factor: $\partial(\underline{x} \otimes \underline{y}) = \partial\underline{x} \otimes \underline{y}$.

In particular, $H^D_0(X) = H^D_1(X) = 0$ and $H^D_2(X) = \widetilde{H}_0(X) \otimes \mathbb{Z}X$.

3. A family of spindles with torsion. In this section we construct a family of spindles that have torsion in their homology groups. Namely, we can realize every power of a cyclic group as a torsion subgroup of H_1.

DEFINITION 3.1. Choose a set together with a basepoint, (X,b), and set $X_0 = X - \{b\}$. Any function $f\colon X_0 \to X_0$ induces a spindle on X by defining

$$x \star y = \begin{cases} f(y) & \text{if } x = p, \\ y & \text{if } x \neq p. \end{cases} \tag{10}$$

We call $(X,\star)$ an *f-spindle* and denote it by X_f.

The function f induces a discrete semidynamical system on X_0. We can visualize it as a graph Γ_f whose vertices are elements of X_0 and with directed edges $x \to f(x)$. Every vertex in this graph has exactly one outcoming edge. If a vertex v has no incoming edges, it is called an *initial* vertex or a *source*. The initial vertices are precisely the elements of X_0 that are not in the image of f. The number of such elements will be denoted by $\operatorname{init}(f)$. Finally, connected components of Γ_f correspond to orbits of the semidynamical system induced by f. Their number will be denoted by $\operatorname{orb}(f)$. The orbit of an element x will be written as $\bar{x}$.

Consider a connected component Γ_f^0 of Γ_f. Then either Γ_f^0 is an infinite directed tree with no loops (so that $f^i(x) \neq x$ for any $i > 0$) or there exists a number $k > 0$ such that for any vertex $v \in \Gamma_f^0$ we have $f^{i+k}(v) = f^i(v)$ for i large enough. If we choose the smallest such k, then the set $\{f^i(v), \ldots, f^{i+k-1}(v)\}$ is a unique cycle in Γ_f^0, which we call a *soma* of Γ_f^0. Clearly, the component Γ_f^0 consists of this cycle and *dendrites*, possibly infinite, as can be seen in Fig. 3.

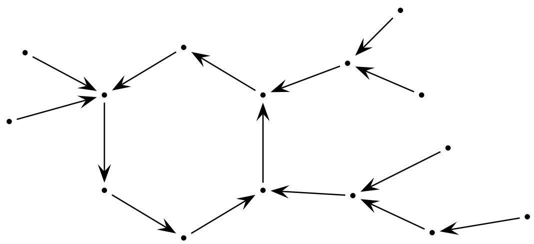

Fig. 3. A typical connected component of Γ_f. It has four dendrites and six initial vertices.

Finally, we choose a single vertex v^i from any component of Γ_f and define ℓ to be the greatest common divisor of the lengths of all cycles in Γ_f. If Γ_f has no cycles at all, set $\ell = 0$.

EXAMPLE 3.2. Let $X = \{0, \ldots, k+1\}$ for some $k \geq 1$ and set $b = 0$ so that $X_0 = \{1, \ldots, k+1\}$. Define $\sigma_k \colon X_0 \to X_0$ as follows:

$$\sigma_k(n) := \begin{cases} n+1 & \text{if } n < k, \\ 1 & \text{if } n = k, k+1. \end{cases} \tag{11}$$

The graph for σ_5 is shown in Fig. 4. It has one component with a cycle of length $k = 5$ and a unique initial vertex.

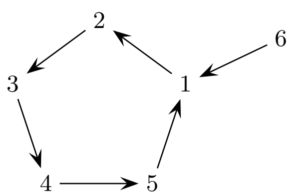

Fig. 4. The graph of the function σ_5 from Example 3.2

It appears that the first homology group of the spindle obtained from σ_k has $\mathbb{Z}_k$ as a direct summand. Indeed, we have the following formula:

Proposition 3.3. *Let* $X = \{x_0, \ldots, x_{k+1}\}$ *and* $\sigma_l \colon X_0 \to X_0$ *be as in Example* 3.2. *Then*

$$H_1(X_{\sigma_k}) = \mathbb{Z}^2 \oplus \mathbb{Z}_k. \tag{12}$$

In particular, every finite cyclic group appears as the torsion of the first homology of some spindle.

This proposition follows from a more general result, concerning any f-spindle.

Theorem 3.4. *The first homology group* $H_1(X_f)$ *of an* f*-spindle* X_f *is generated by*

(1) *pairs* $(f(y), y)$, *one per each initial element* $y \in X_0$,
(2) *pairs* (v^i, b) *and* (v^i, y), *where* $y \in X_0$ *is not in the same orbit as* v^i, *and*
(3) *sums* $(b, c_1) + \cdots + (b, c_k)$, *one for each cycle* $(c_1, \ldots, c_k)$ *in* Γ_f,

subject to the relation $\ell \cdot (f(y), y) \equiv 0$. *In particular,*

$$H_1(X_f) = \mathbb{Z}^{|X_0|(\mathrm{orb}(f)-1)+2\mathrm{orb}(f)} \oplus \mathbb{Z}_\ell^{\mathrm{init}(f)} \tag{13}$$

if X *is a finite set.*

Corollary 3.5. *Every power of a finite cyclic group can be realized as torsion of the first homology for some spindle. Namely, let* $X_0 = \{1, \ldots, k+r\}$ *and define* $\sigma_{k,r} \colon X_0 \to X_0$ *by the formula*

$$\sigma_{k,r}(n) := \begin{cases} n+1 & \text{if } n < k, \\ 1 & \text{if } n \geq k. \end{cases} \tag{14}$$

Then the torsion subgroup $H_1(X_{\sigma_{k,r}})$ *is isomorphic to* $\mathbb{Z}_k^r$.

We need one technical, but useful, fact before we prove Theorem 3.4. It will be an important tool for the calculation of higher homology groups in the next section.

Lemma 3.6. *Choose* $\underline{y} \in C_n^N(X)$ *with* $y_0 \neq b$ *and an orbit* $\bar{a}$ *of* $a \in X_0$. *Let* $V \subset C_{n+2}^N(X)$ *and* $W \subset C_{n+1}^N(X)$ *be the subgroups spanned by all sequences* $(b, x, \underline{y})$ *and* $(x, \underline{y})$ *respectively, with* $x \in \bar{a}$. *If* $\bar{a} \neq \bar{y}_0$ *we also add* $(f(y_0), \underline{y})$ *to the list of generators of* W. *The restricted differential* $\partial \colon V \to W$ *is injective and* coker ∂ *is generated by* $(a, \underline{y})$, *if* $\bar{a} \neq \bar{y}_0$, *and* $(f(y_0), \underline{y})$ *subject to the relation* $k \cdot (f(y_0), \underline{y}) \equiv 0$, *if* $\bar{y}_0$ *has a cycle of length* k.

Proof. We will prove this lemma by computing $Q := \operatorname{coker} \partial / (f(y_0), \underline{y})$. Each element $\partial(b, x, \underline{y})$ gives a relation in Q:

$$(x, \underline{y}) \equiv (f(x), \underline{y}). \tag{15}$$

Hence, we can replace x with any other element from its orbit. In particular $Q = 0$ if y_0 and a are in the same orbit. Otherwise, it is freely generated by

$(a, \underline{y})$. On the other hand, the kernel of the composition

$$V \xrightarrow{\partial} \operatorname{coker} \partial \to Q \tag{16}$$

is trivial if the orbit of a is a directed tree, and one-dimensional otherwise, generated by a sum $(b, c_1, \underline{y}) + \cdots + (b, c_k, \underline{y})$, where $(c_1, \ldots, c_k)$ is a cycle in $\bar{a}$. The latter is mapped by ∂ to $k(f(y_0), \underline{y})$. Hence, $\ker \partial = 0$ and the cokernel is as expected. ■

Proof of Theorem 3.4. Because for a spindle we have $H_1(X) = H_1^N(X)$, we will consider only sequences without repetitions. The first differential $\partial \colon C_1^N(X_f) \to C_0^N(X_f)$ is given by the formula

$$\partial(x, y) = y - x \star y = \begin{cases} 0 & \text{if } x \neq b, \\ y - f(y) & \text{if } x = b. \end{cases} \tag{17}$$

Hence, the kernel of ∂ is freely generated by

- pairs (x, y) with $x \neq b$, and
- sums $(b, c_1) + \cdots + (b, c_k)$, where $(c_1, \ldots, c_k)$ is a cycle in Γ_f.

Now consider relations introduced by $\partial(x, y, z)$. If $x, y \neq b$, then $\partial(x, y, z) = (z, z) = 0$. When only $y \neq b$, the relations are

$$(f(y), z) \equiv (y, z) + (f(z), z) \quad \text{if } z \neq b, \tag{18}$$

$$(f(y), b) \equiv (y, b). \tag{19}$$

According to Lemma 3.6, this restricts pairs (x, y) to (v^i, y), where v^i and y are from different orbits, and to $(f(y), y)$ (with $y \neq b$). The latter is annihilated by the length of any cycle in the graph Γ_f.

If y is initial, there are no more relations among the generators (x, y). Otherwise, for $y = f(z)$ we have $\partial(x, b, z) = (z, f(z)) = (z, y)$, which forces $(f(y), y)$ to be zero:

$$(f(y), y) \equiv (z, y) + (f(y), y) \equiv (f(z), y) = (y, y) \equiv 0. \tag{20}$$

This fulfils all relations. In particular, each cycle $\underline{c}$ in Γ_f contributes a free generator to $H_1^N(X_f)$, and sequences $(f(y), y)$ have order ℓ. ■

COROLLARY 3.7. *The first homology of an f-spindle X_f has torsion if and only if the following three conditions hold:*

1. *f has an initial element,*
2. *f has a cycle,*
3. *the lengths of the cycles of f are not co-prime, i.e. they have a common divisor $d > 1$.*

The second condtition is automatic if X is finite, but not the others.

4. Higher homology groups for f-spindles. We will now compute higher homology groups for an f-spindle, and for simplicity we will restrict to the normalized part. Doing so already determines the whole homology, as explained in Theorem 2.3 (see Corollary 5.7).

In this section, X will always stand for an f-spindle induced by a fixed function $f\colon X_0 \to X_0$, where $X = X_0 \cup \{b\}$. Recall from the previous section that each connected component Γ_f^0 of the graph Γ_f is represented by some vertex v^i and either it is an infinite directed tree, or it contains a unique cycle $\underline{c} = (c_1, \dots, c_k)$ of length k. In particular, the set of distinguished vertices $\{v^i\}$ parametrizes the set of orbits in X different from $\{b\}$. Finally, ℓ denotes the greatest common divisor of the lengths of all cycles in Γ_f (we set $\ell = 0$ if Γ_f has no cycles).

According to Theorem 3.4, the generators of $H_1(X)$ split into two groups: sequences with two entries from the same orbit or from two different orbits. The first generate the torsion subgroup and the latter are free. A similar phenomenon occurred in Lemma 3.6, where we compare the orbits of the first two entries in a sequence. This observation motivates the following splitting of $C^N(X)$.

Let $C^{ND}(X)$ be spanned by sequences $\underline{x}$ of length at least two, with x_0 and x_1 from the same orbit. Clearly, for such a sequence $d^j\underline{x} = 0$ if $j \geq 2$ and $d^0\underline{x} = d^1\underline{x}$. Hence, $C^{ND}(X)$ is a subcomplex of $C^N(X)$ and has a trivial differential. The quotient complex $C^{NN}(X) := C^N(X)/C^{ND}(X)$ is freely spanned by sequences $\underline{x}$ of length 1 or with x_0 and x_1 lying in two different orbits (in particular, we can take b as one of them). Since $d^j\underline{x} \in C^{ND}(X)$ for any sequence $\underline{x}$ as long as $j \geq 2$, the differential in $C^{NN}(X)$ has only two terms: $\partial = d^0 - d^1$.

Lemma 4.1. *The homology $H^{NN}(X)$ is freely generated by three types of chains:*

- *type I: $(v^i, x_1, \dots, x_n)$, where x_1 and v^i are in different orbits,*
- *type II: $(b, x_1, x_2, \dots, x_n)$, where x_1 and x_2 are in the same orbit,*
- *type III: $\sum_{i=1}^k (b, c_i, x_2, \dots, x_n)$, where $(c_1, \dots, c_k)$ is a cycle from beyond the orbit of x_2.*

In all cases, neighboring entries are never equal.

Proof. The only case for which $\partial\underline{x} \neq 0$ is when $x_0 = b$ and the orbits of x_1 and x_2 are not the same (or simply $\underline{x} = (b, x_1)$). In that case

$$\partial(b, x_1, \underline{y}) = (x_1, \underline{y}) - (f(x_1), \underline{y}). \tag{21}$$

This has two consequences:

(i) cycles are the chains listed in the lemma, except that in the first case all sequences $\underline{x}$ with $x_0 \neq b$ are allowed,

(ii) boundaries (21) only restrict type I generators: we can replace x_0 in $\underline{x}$ by any other element from the same orbit; in particular by v^i.

This gives the desired presentation of $H^{NN}(X)$. ■

The chain complexes described above induce a long exact sequence of homology

$$\cdots \to C_n^{ND}(X) \to H_n^N(X) \to H_n^{NN}(X) \xrightarrow{\delta_n} C_{n-1}^{ND}(X) \to \cdots \tag{22}$$

where $\delta_n([a]) = \sum_{i=2}^n (-1)^i d^n a = \partial a$ is induced by the full differential in $C^N(X)$. Due to Lemma 4.1 the groups $H_n^{NN}(X)$ are free, and so are $\ker \delta_n$, which results in a splitting formula

$$H_n^N(X) \cong \ker \delta_n \oplus \operatorname{coker} \delta_{n+1}. \tag{23}$$

It remains to compute both summands.

LEMMA 4.2. *The cokernel of δ_n is a free $\mathbb{Z}_\ell$-module with basis consisting of all sequences $(f(x), x, \dots)$ and $(f^2(x), f(x), x, \dots)$, where x is initial in both cases.*

Proof. Since $C_n^{ND}(X) = 0$ for $n \le 1$, $\operatorname{coker} \delta_n = 0$ as well. This agrees with the statement above, as there are no such sequences of length smaller than 2. Hence, we will assume $n \ge 2$.

According to Lemma 3.6, the generators of $H_n^{NN}(X)$ of the second type are crucial: they are orthogonal to $\ker \delta_n$ and their images restrict generators of $\operatorname{coker} \delta_n$ to sequences $(f(y), y, \dots)$. Type III generators, in turn, show that the length of any cycle in Γ_f annihilates $\operatorname{coker} \delta_n$:

$$0 \equiv \partial\Big(\sum_{i=1}^{k} (b, c_i, x_2, \underline{z})\Big) = k(f(x_2), x_2, \underline{z}), \tag{24}$$

so that $\operatorname{coker} \delta_n$ is a $\mathbb{Z}_\ell$-module. To restrict the set of generators even further, take a type I generator with $x_1 = b$ and $x_2, x_3 \in X_0$ (or just $x_2 \in X_0$ if $n = 2$). Then

$$0 \equiv \partial(v^i, b, x_2, x_3, \underline{z}) = (x_2, f(x_2), x_3, \underline{z}) - (x_3, f(x_3), x_3, \underline{z}) \tag{25}$$

makes it possible to replace $(f^2(x), f(x), y, \dots)$ with $(f^2(y), f(y), y, \dots)$, or to kill $(f^2(x), f(x))$ in case $n = 2$, as we did in Theorem 3.4. Also, y must be initial—otherwise (25) forces $(f^2(y), f(y), y, \dots) \equiv 0$, if we pick $x_3 = y$ and x_2 such that $f(x_2) = y$. All the remaining relations are induced by sequences of the form

$$\underline{x} = (v^i, b, z_0, b, z_1, b, \dots, b, z_k, z_{k+1}, \dots), \tag{26}$$

perhaps ending at the b before z_k or at z_k. Because $\partial\underline{x}$ is independent of v^i, we can choose one particular element. Then $\partial\underline{x}$ determines $\underline{x}$ completely, so that all these boundaries are linearly independent. Each of them allows us to eliminate one more sequence from the list of generators: $(z_0, f(z_0), b, \ldots)$ can be expressed as a linear sum of sequences of type $(y, f(y), y, \ldots) \equiv (f^2(y), f(y), y, \ldots)$. This results in the desired presentation of $\operatorname{coker}\delta_n$. ∎

If X is finite, every component of Γ_f must have a cycle. Therefore, Lemma 3.6 implies that δ_n, when restricted to type II generators, is an isomorphism over $\mathbb{Q}$. Hence, it is enough to count the other generators to find the rank of the distributive homology of X.

THEOREM 4.3. *Assume X is a finite f-spindle. Then its homology is given by the formulas*

$$(27)\quad \begin{cases} \widetilde{H}_0^N(X) = \mathbb{Z}^{\mathrm{orb}(f)}, \\ H_1^N(X) = \mathbb{Z}^{(\mathrm{orb}(f)-1)|X_0|+2\mathrm{orb}(f)} \oplus \mathbb{Z}_\ell^{\mathrm{init}(f)}, \\ H_n^N(X) = (\mathbb{Z}^{(\mathrm{orb}(f)-1)|X|^2+|X|} \oplus \mathbb{Z}_\ell^{\mathrm{init}(f)|X|})^{\oplus(|X|-1)^{n-2}} & \text{for } n \geq 2. \end{cases}$$

In particular, $H_{n+1}^N(X) = H_n^N(X)^{\oplus(|X|-1)}$ for $n \geq 2$.

Proof. Clearly, $\operatorname{rk}\widetilde{H}_0^{NN}(X) = \mathrm{orb}(f)$, since the only possible generators are (v^i). For higher n, the generators are counted in Table 1. The last two

Table 1. Numbers of generators in $H_n^N(X)$

Type of generators	$n = 1$	$n \geq 2$
$(v^i, x, \ldots)$, $x \in X_0$	$(\mathrm{orb}(f)-1)\lvert X_0\rvert$	$(\mathrm{orb}(f)-1)(\lvert X\rvert - 1)^n$
$(v^i, b, \ldots)$	$\mathrm{orb}(f)$	$\mathrm{orb}(f)(\lvert X\rvert - 1)^{n-1}$
$\sum_{i=1}^{k}(b, c_i, x, \ldots)$, $\bar{x} \neq \bar{c}_i$, $x \in X_0$	$\mathrm{orb}(f)$	$(\mathrm{orb}(f)-1)(\lvert X\rvert - 1)^{n-1}$
$\sum_{i=1}^{k}(b, c_i, b, \ldots)$	0	$\mathrm{orb}(f)(\lvert X\rvert - 1)^{n-2}$
$(f(y), y, \ldots)$, y initial	$\mathrm{init}(f)$	$\mathrm{init}(f)(\lvert X\rvert - 1)^{n-1}$
$(f^2(y), f(y), y, \ldots)$, y initial	0	$\mathrm{init}(f)(\lvert X\rvert - 1)^{n-2}$

rows correspond to the torsion part. Summing up these numbers results in the formula (27). ∎

We can enhance the theorem above by giving an actual presentation of homology, including the case of infinite f-spindles. Indeed, since $\operatorname{im}\delta_n$ is a free group, there is a decomposition $H_n^{NN}(X) = \ker\delta_n \oplus V_n$ with $V_n \cong \operatorname{im}\delta_n$ and we can naturally identify $\ker\delta_n$ with $H_n^{NN}(X)/V_n$. To construct such a V_n, we first assume v^i belongs to a cycle, if its orbit has one, and we choose a section $g\colon f(X_0) \to X_0$ of f. Furthermore, if $\ell \neq 0$, we choose cycles $\underline{c}^1, \ldots, \underline{c}^r$ and nonzero numbers $\alpha_1, \ldots, \alpha_r$ such that $\sum_{i=1}^{r} \alpha_i k^i = \ell$, where k^i is the length of the cycle $\underline{c}^i$. We then use the chosen cycles to construct a *base cycle*

$$\underline{\mathfrak{c}} := \sum_{i=1}^{r} \alpha^i (c_1^i + \cdots + c_{k^i}^i). \tag{28}$$

Notice that $\partial(b, \underline{\mathfrak{c}}, x_2, \ldots, x_n) = \ell \cdot (f(x_2), x_2, \ldots x_n)$.

Lemma 4.4. *Fix an element v^0 from among v^i's and let $V_n \subset H_n^{NN}(X)$ be generated by the sequences*

1. $(b, x_1, \ldots, x_n)$ *with* $\bar{x}_1 = \bar{x}_2$, *unless* $x_1 = v^i$ *or* $x_1 = f(v^i)$, *if already* $x_2 = v^i$,
2. $(v^0, b, g(y), x_3, \ldots, x_n)$ *with* $x_3 = b$ *or* $y \neq f(x_3)$, *and*
3. *if* Γ_f *has cycles, chains* $(b, \underline{\mathfrak{c}}, x_2, \ldots, x_n)$ *with either an initial* x_2 *or* $x_2 = f(x_3)$ *and an initial* x_3.

Then $\delta_n|_{V_n}$ is injective and $\delta_n(V_n) = \operatorname{im} \delta_n$.

Proof. Injectivity follows from Lemma 3.6 and carefully choosing the other generators. Indeed, since we removed one sequence $(b, x_1, x_2, \ldots)$ for every cycle in the orbit of x_2, the quotient by the first group of generators is freely generated by sequences $(f(y), y, \ldots)$. Then, as seen in the proof of Lemma 4.2, every sequence $(v^0, b, g(y), x_3, \ldots, x_n)$ lowers the rank of the cokernel by one and each chain from the last group turns one of the remaining generators into torsion of order ℓ. This also shows $\delta_n(V_n) = \operatorname{im} \delta_n$. ∎

Corollary 4.5. *Let X be an f-spindle, not necessarily finite. Construct V_n as above and choose a cycle $\underline{c}^0$, if Γ_f has one. Then the generators of the free part of $H_n^N(X)$ are given modulo V_n by the following chains:*

1. *sequences* $(b, f(v^i), v^i, x_3, \ldots, x_n)$ *and* $(b, v^i, x_2, \ldots, x_n)$ *with* $\bar{v}^i = \bar{x}_2$,
2. *sequences* $(v^i, x_1, \ldots, x_n)$ *with* $\bar{v}^i \neq \bar{x}_1$ *and* $x_1 \neq b$,
3. *sequences* $(v^i, b, x_2, \ldots, x_n)$ *with* $v^i \neq v^0$ *or* $x_2 \notin g(X_0')$, *and*
4. *sums* $\sum_{i=1}^{k} (b, c_i, x_2, \ldots, x_n)$, *one per cycle* $(c_1, \ldots, c_k)$ *from a different orbit than* x_2, *except* $\underline{c}^0$, *when* x_2 *is initial or* $x_2 = f(x_3)$ *and* x_3 *is initial.*

The torsion subgroup [4] *of $H_n^N(X)$ is a $\mathbb{Z}_\ell$-module generated by sequences $(f(y), y, x_2, \ldots, x_n)$ and $(f^2(y), f(y), y, x_3, \ldots, x_n)$, where y is initial.*

If X is finite, this presentation is coherent with Theorem 4.3: although we restrict free generators in the last two groups, we include the same number of generators in the first group that have not been counted before.

[4] If $\ell = 0$, these generators also contribute to the free part and there is no torsion. In the other extreme case $\ell = 1$ the torsion subgroup is trivial.

5. Odds and ends

Relative homology. If X is an f-spindle, then $X_0 = X - \{b\}$ is a trivial spindle (i.e. $x \star y = y$), so that $H^N(X_0) = C^N(X_0)$. This makes it easy to compute the relative homology $H^N(X, X_0)$. Indeed, the long exact sequence

$$\cdots \to C_n^N(X_0) \xrightarrow{i_n} H_n^N(X) \to H_n^N(X, X_0) \to C_{n-1}^N(X_0) \xrightarrow{i_{n-1}} \cdots \tag{29}$$

implies $H_n^N(X, X_0) \cong \ker i_{n-1} \oplus H_n^N(X)/\operatorname{im} i_n$, since $\ker i_{n-1}$ is free. Hence, we can obtain a presentation for $H_n^N(X, X_0)$ as follows:

1. (1) Take a presentation for $H_n^N(X)$.
2. (2) Remove the generators $(v^i, \underline{x})$ with $\underline{x} \in C_{n-1}^N(X_0)$. Notice that this kills both free and torsion generators.
3. (3) Add the free generators coming from $\ker i_{n-1}$.

Although this procedure results in a presentation of relative homology, it misses a very nice structure of these groups. Every sequence from $C^N(X, X_0)$ can be written uniquely as $(\underline{x}, b, \underline{y})$, where each y_i is different from b (both $\underline{x}$ and $\underline{y}$ might be empty). Because $b \star y_i \neq b$, higher faces vanish so that in the quotient complex we have

$$\partial(\underline{x}, b, \underline{y}) = \begin{cases} 0 & \text{if } \underline{x} = \emptyset \text{ or } \underline{x} = (x_0), \\ (\partial\underline{x}, b, \underline{y}) & \text{otherwise.} \end{cases} \tag{30}$$

In particular, the sequence $\underline{y}$ is preserved. This proves the decomposition

$$C_{n+1}^N(X, X_0) = \bigoplus_{p+q=n} C_p^{N,b}(X) \otimes \widetilde{C}_q^N(X_0), \tag{31}$$

where $C^{N,b}(X)$ is spanned by sequences ending with b. Notice that the differential in $\widetilde{C}^N(X_0)$ is trivial, so the formula above shows $C^N(X, X_0)$ is a shifted total complex of the bicomplex $C^{N,b}(X) \otimes \widetilde{C}^N(X_0)$.

To compute $H_p^{N,b}(X)$ we note first that the normalized complex $C^N(X)$ splits into two copies of $C^{N,b}(X)$. Indeed, consider the homomorphism $h : C^N(X) \to C^N(X)[1]$ given by $h(\underline{x}) = (\underline{x}, b)$. It commutes with differentials ([5]) and $C^{N,b}(X) = \ker h$. Moreover, the image of h is the shifted reduced complex $\widetilde{C}^{N,b}(X, b)[1]$, because we can use h to obtain all sequences except (b). Finally, the short exact sequence

$$0 \to C^{N,b}(X) \to C^N(X) \xrightarrow{h} C^{N,b}(X, b)[1] \to 0 \tag{32}$$

splits via a homomorphism $u\colon C^{N,b}(X, b)[1] \to C^N(X)$ that forgets the b standing at the end. Hence, $H_n^N(X) \cong H_n^{N,b}(X) \oplus \widetilde{H}_{n+1}^{N,b}(X)$ and the group $H_n^{N,b}(X) = \ker h_*$ is generated by classes represented by sequences with

([5]) Recall that in the shifted complex $C[1]_n = C_{n+1}$ and $\partial[1]_n = -\partial_{n+1}$ changes sign.

b at the end. This, together with (31), results in another presentation for $H^N(X, X_0)$.

We finish this part by computing $H^{N,b}(X)$ for a finite X. This can be easily done using the split exact sequence (32) and Theorem 4.3.

PROPOSITION 5.1. *Assume X is a finite f-spindle. Then*

$$\begin{cases} H_0^{N,b}(X) = \mathbb{Z}, \\ H_1^{N,b}(X) = \mathbb{Z}^{\mathrm{orb}(f)}, \\ H_n^{N,b}(X) = (\mathbb{Z}^{\mathrm{orb}(f)|X|-|X|+1} \oplus \mathbb{Z}_\ell^{\mathrm{init}(f)})^{\oplus(|X|-1)^{n-2}} \quad \text{for } n \geq 2. \end{cases} \tag{33}$$

Proof. Clearly, $H_0^{N,b}(X) = \mathbb{Z}$, generated by (b). Directly from (32) we compute

$$\operatorname{rk} H_1^{N,b}(X) = \operatorname{rk} H_0^N(X) - \operatorname{rk} H_0^{N,b}(X) = \mathrm{orb}(f), \tag{34}$$

$$\operatorname{rk} H_2^{N,b}(X) = \operatorname{rk} H_1^N(X) - \operatorname{rk} H_1^{N,b}(X) = \mathrm{orb}(f)|X| - (|X| - 1). \tag{35}$$

We observe that $H_0^N(X)$ is free, so is $H_1^{N,b}(X)$, and the torsion subgroup of $H_2^{N,b}(X)$ is equal to the one of $H_1^N(X)$. For higher n we use induction:

$$\begin{aligned} \operatorname{rk} H_{n+3}^{N,b}(X) &= \operatorname{rk} H_{n+2}^N(X) - \operatorname{rk} H_{n+2}^N(X) \\ &= (|X| - 1)^n \big((\mathrm{orb}(f) - 1)|X|^2 + |X| - \mathrm{orb}(f)|X| + |X| - 1\big) \\ &= (|X| - 1)^n (\mathrm{orb}(f)|X|(|X| - 1) - (|X|^2 - 2|X| + 1)) \\ &= (|X| - 1)^{n+1} (\mathrm{orb}(f)|X| - |X| + 1), \qquad n \geq 0. \end{aligned} \tag{36}$$

Torsion is even simpler to check. ■

Realization of any finite abelian group. We prove that every finite abelian group can be realized as the torsion subgroup of $H_1(X)$ for some spindle X. For this, we will first generalize Definition 3.1 to several functions (see [PS]).

DEFINITION 5.2. Choose a family $\{X_i\}_{i \in I}$ of sets, not necessarily finite, and functions $f_i : X_i \to X_i$. Define the spindle product on $X := \coprod_{i \in I} X_i$ for $x \in X_i$ and $y \in X_j$ by the formula

$$x \star y := \begin{cases} y & \text{if } i = j, \\ f_j(y) & \text{if } i \neq j. \end{cases} \tag{37}$$

The subsets $X_i \subset X$ are called the *blocks* of the spindle X and f_i's are called the *block functions*. We will write $f : X \to X$ for the function induced by all block functions.

EXAMPLE 5.3. Consider an f-spindle which has two blocks, X_0 and $\{b\}$. The block functions are given by $f : X_0 \to X_0$ and a constant function on $\{b\}$.

From now on we assume X has a one-element block $\{b\}$. Then for every other block X_i, the sum $X_i^+ := X_i \sqcup \{b\}$ is an f_i-spindle that is a rectract of X, where the retraction $r\colon X \to X_i^+$ is the identity on X_i and maps everything else onto b. Hence, $C^N(X_i^+)$ is a direct summand of $C^N(X)$.

THEOREM 5.4. *Assume a block spindle X has a one-element block $\{b\}$. Then*

$$H_1(X) \cong F \oplus \bigoplus_{i\in I} H_1(X_i^+), \tag{38}$$

where F is a free abelian group of rank $\sum_{i\neq j} orb(f_i)|X_j|$. In particular, every finite abelian group can be realized as the torsion subgroup of $H_1(X)$ for some spindle X.

Proof. We will assume there are at least two blocks different than $\{b\}$—otherwise the statement is trivial. Since $H_1(X) = H_1(X, b)$, we will compute reduced homology. Each of $C(X_i^+, b)$ is still a direct summand of $C(X, b)$, but now they have trivial intersections: no two of them have a generator in common. This implies

$$C(X, b) \cong Q \oplus \bigoplus_{i\in I} C(X_i^+, b), \tag{39}$$

where Q is a chain complex isomorphic to the quotient of $C(X, b)$ by the big direct sum. To compute $H_1(Q)$, we first notice that $Q_0 = 0$. Therefore, all 1-chains are cycles and $H_1(Q) = \operatorname{coker} \partial$. Pick any sequence $(x, y, z) \in Q_2$. Its boundary is equal to

$$\partial(x, y, z) = \begin{cases} 0 & \text{if } x \text{ and } y \text{ are from the same block,} \\ (y, z) - (f(y), z) & \text{otherwise.} \end{cases} \tag{40}$$

The induced relation only identifies some generators and does not introduce torsion. Namely, we can replace (y, z) by any other pair (y', z) with y' from the same orbit as y. A simple counting results in the desired rank of $H_1(Q)$. ∎

REMARK 5.5. The homology groups $H_n(Q)$ are usually not free when $n > 1$, and the same holds for their normalized versions $H_n^N(Q)$.

REMARK 5.6. The method of this paper can be applied to the more general case of block spindles, even with no one-element block $\{b\}$. The proof is, however, much more involved and is postponed for future work.

The degenerate part and growth conjectures. We can easily compute the distributive homology for an f-spindle X using formula (9) from Theorem 2.3. Indeed, (9) implies the relation

$$C_{n+1}^D(X) \cong C_n^D(X)^{\oplus(|X|-1)} \oplus \widetilde{C}_{n-1}(X)^{\oplus|X|} \tag{41}$$

and assuming $n \geq 2$, we can combine this with the formula for the normalized part from Theorem 4.3 to obtain an isomorphism of homology

$$H_{n+1}(X) \cong H_n(X)^{\oplus(|X|-1)} \oplus H_{n-1}(X)^{\oplus|X|} \tag{42}$$

for $n \geq 2$. In the case where X is an f-spindle, one has

$$\begin{aligned} \operatorname{rk} H_2(X) &= \operatorname{rk} H_2^N(X) + \operatorname{rk} H_2^D(X) \\ &= \big((\operatorname{orb}(f)-1)|X| + 1 + \operatorname{orb}(f)\big)|X| \\ &= \big((\operatorname{orb}(f)-1)|X_0| + 2\operatorname{orb}(f)\big)|X| = |X| \operatorname{rk} H_1(X), \end{aligned} \tag{43}$$

which implies $H_1(X)^{\oplus|X|} \cong H_2(X)$, resulting in $H_{n+1}(X) \cong H_n(X)^{\oplus|X|}$ for $n \geq 1$.

Corollary 5.7. *The whole distributive homology for an f-spindle X is given by the formulas*

$$\begin{cases} \widetilde{H}_0(X) = \mathbb{Z}^{\operatorname{orb}(f)}, \\ H_n(X) = (\mathbb{Z}^{\operatorname{orb}(f)(|X|+1)-(|X|-1)} \oplus \mathbb{Z}_\ell^{\operatorname{init}(f)})^{\oplus|X|^{(n-1)}} & \text{for } n \geq 1. \end{cases} \tag{44}$$

In particular, $H_{n+1}(X) \cong H_n(X)^{\oplus|X|}$ for $n \geq 1$.

In [PS] the following conjecture was stated:

Conjecture 5.8 (Rank Growth Conjecture). *Let $(X, \star)$ be a shelf. Then for $n \geq |X| - 2$ one has* $\operatorname{rk} H_{n+1}(X) = |X| \operatorname{rk} H_n(X)$.

Using formula (9) for the degenerate subcomplex one can show that the rank of the normalized homology grows by a factor of $|X|-1$ (see [PP-2]).

Conjecture 5.9 (Normalized Rank Growth Conjecture). *Let $(X, \star)$ be a spindle. Then* $\operatorname{rk} H_{n+1}^N(X) = (|X|-1) \operatorname{rk} H_n^N(X)$ *for $n \geq |X| - 1$.*

Conjecture 5.8 implies Conjecture 5.9, but not the other way round. Indeed, we cannot expect more than formula (42). Although the authors do not know of any example of a spindle that does not satisfy Conjecture 5.8, there are spindles where $H_{n+1}(X) \not\cong H_n(X)^{\oplus|X|}$ because of torsion.

Example 5.10. Let $X = \{1, 2, 3, 4\}$ and define $\star \colon X \times X \to X$ by the following table:

$\star$	1	2	3	4
1	1	2	4	3
2	1	2	4	3
3	2	1	3	4
4	2	1	3	4

Computer calculations resulted in the following groups:

$$\begin{aligned} H_0(X) &= \mathbb{Z}^2, & H_3(X) &= \mathbb{Z}^{32} \oplus \mathbb{Z}_2^{52}, \\ H_1(X) &= \mathbb{Z}^2 \oplus \mathbb{Z}_2^4, & H_4(X) &= \mathbb{Z}^{128} \oplus \mathbb{Z}_2^{204}, \\ H_2(X) &= \mathbb{Z}^8 \oplus \mathbb{Z}_2^{12}, & H_5(X) &= \mathbb{Z}^{512} \oplus \mathbb{Z}_2^{820}. \end{aligned}$$

One can easily check that $H_n(X) \cong H_{n-1}(X)^{\oplus 3} \oplus H_{n-2}(X)^{\oplus 4}$ for $3 \leq n \leq 5$ and that the Rank Growth Conjecture holds. However, the torsion subgroup does not grow by the factor of 4.

This suggests the following Growth Conjecture for distributive homology, including torsion.

Conjecture 5.11 (Growth Conjecture). *Let $(X, \star)$ be a shelf. Then for $n \geq |X| - 2$,*

$$H_{n+1}(X) \cong H_n(X)^{\oplus(|X|-1)} \oplus H_{n-1}(X)^{\oplus|X|}. \tag{45}$$

Furthermore, if X is a spindle, then also $H^N_{n+1}(X) \cong H^N_n(X)^{\oplus(|X|-1)}$.

Theorem 4.3 shows f-spindles satisfy all of these conjectures. Also, the authors tested plenty of other block spindles in attempts to find a counterexample to these conjectures, but they did not succeed.

Acyclicity results. Let X be a shelf and $A \subset X$ a subset such that X acts on A from the right by permutations, i.e. $a \star x \in A$ whenever $a \in A$ and the map $a \mapsto a \star x$ is a permutation of A for every $x \in X$. If such an A exists and is finite, it was proved in [Prz] that $H(X)$ is annihilated by $|A|$. It was expected to be trivial as one-term distributive homology was supposed to be torsion-free. However, we have already seen the latter is not true and it is no longer obvious why homology groups of such a spindle should vanish. We prove this below. To simplify notation, we will omit $\star$ and use the left-first convention for bracketing:

$$x_1 \cdots x_n := ((x_1 \star x_2) \star \cdots) \star x_n. \tag{46}$$

One can easily check that distributivity of the operation $\star$ implies the generalized distributivity: $(x_1 \cdots x_n) \star y = (x_1 \star y) \cdots (x_n \star y)$.

Proposition 5.12. *Let $(X, \star)$ be a shelf with a subset $A \subset X$ on which X acts from the right by permutations. Then the complex $\widetilde{C}(X)$ is acyclic.*

Proof. We will construct a contracting homotopy $h\colon \widetilde{C}_n(X) \to \widetilde{C}_{n+1}(X)$. First, notice that for every element $a \in A$ and $x \in X$ we can find a unique $a' \in A$ such that $a = a' \star x$. More generally, for a fixed $a \in A$ there is a unique solution $a_{\underline{x}}$ to the equation $a = a_{\underline{x}} x_0 \cdots x_n$ for any sequence $\underline{x} = (x_0, \ldots, x_n)$. Using the distributivity of $\star$ we can transform the right hand side by moving

x_i to the left, which results in the equality

$$a = (a_{\underline{x}} \star x_i) \cdots (x_{i-1} \star x_i) x_{i+1} \cdots x_n. \tag{47}$$

This means that $a_{\underline{x}} \star x_i = a_{d^i \underline{x}}$ and the map $h(\underline{x}) := (a_{\underline{x}}, \underline{x})$ satisfies

$$d^{i+1} h(\underline{x}) = (a_{\underline{x}} \star x_i, d^i \underline{x}) = h(d^i \underline{x}) \tag{48}$$

for every $0 \leq i \leq n$. Hence, $\partial h(\underline{x}) + h(\partial \underline{x}) = d^0 h(\underline{x}) = \underline{x}$ and the identity homomorphism on $\widetilde{C}(X)$ is nullhomotopic. ∎

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