FUNDAMENTA
MATHEMATICAE
225 (2014)

Heisenberg algebra and a graphical calculus

by

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Abstract. A new calculus of planar diagrams involving diagrammatics for biadjoint functors and degenerate affine Hecke algebras is introduced. The calculus leads to an additive monoidal category whose Grothendieck ring contains an integral form of the Heisenberg algebra in infinitely many variables. We construct bases of the vector spaces of morphisms between products of generating objects in this category.

Contents

1. Introduction. In this paper we propose a graphical calculus for a categorification of the Heisenberg algebra. The one-variable Heisenberg algebra has generators p, q, one defining relation $pq - qp = 1$, and appears as the algebra of operators in the quantization of the harmonic oscillator. A fundamental role in quantum field theory is played by its infinitely generated analogue, the algebra with generators p_i, q_i for i in some infinite set I, and relations

$$p_i q_j = q_j p_i + \delta_{i,j} 1, \qquad p_i p_j = p_j p_i, \qquad q_i q_j = q_j q_i. \tag{1}$$

2010 *Mathematics Subject Classification*: 18D10, 19A22.
Key words and phrases: categorification of Heisenberg algebra, biadjoint functors, symmetric groups, induction and restriction functors, graphical calculus.

DOI: 10.4064/fm225-1-8

In Section 2 we define a strict monoidal category $\mathcal{H}'$ with two generating objects Q_+ and Q_-, and morphisms between tensor products of these objects given by linear combinations of certain planar diagrams modulo local relations. The category is $\Bbbk$-linear over a ground commutative ring $\Bbbk$, and we specialize $\Bbbk$ to a field of characteristic 0. The endomorphism rings of the tensor powers $Q_+^{\otimes n}$ and $Q_-^{\otimes n}$ contain the group algebra $\Bbbk[S_n]$ of the symmetric group. The symmetrization and antisymmetrization idempotents in $\Bbbk[S_n]$ produce objects in the Karoubi envelope $\mathcal{H}$ of $\mathcal{H}'$. These objects can be viewed as symmetric and exterior powers of the generating objects Q_+ and Q_-. Consequently, we denote them by

(2) $S_+^n := S^n(Q_+), \quad \Lambda_+^n := \Lambda^n(Q_+), \quad S_-^n := S^n(Q_-), \quad \Lambda_-^n := \Lambda^n(Q_-),$

and call them the *symmetric* and *exterior powers* of Q_+ and Q_-. When $n = 0$,

$$S_+^0 \cong S_-^0 \cong \Lambda_+^0 \cong \Lambda_-^0 \cong \mathbf{1},$$

where $\mathbf{1}$ is the identity object of the monoidal category $\mathcal{H}$, with $\mathbf{1} \otimes M = M$ for any M. We also set

$$S_+^n = S_-^n = \Lambda_+^n = \Lambda_-^n = 0 \quad \text{if } n < 0.$$

Proposition 1. *There are canonical isomorphisms in $\mathcal{H}$:*

$$\begin{aligned} S_-^n \otimes \Lambda_+^m &\cong (\Lambda_+^m \otimes S_-^n) \oplus (\Lambda_+^{m-1} \otimes S_-^{n-1}), \\ S_-^n \otimes S_-^m &\cong S_-^m \otimes S_-^n, \\ \Lambda_+^n \otimes \Lambda_+^m &\cong \Lambda_+^m \otimes \Lambda_+^n. \end{aligned}$$

These isomorphisms are constructed in Section 2.2. Since $\mathcal{H}$ is monoidal, its Grothendieck group $K_0(\mathcal{H})$ is a ring. It has generators $[M]$ for all objects M of $\mathcal{H}$ and relations $[M_1] = [M_2] + [M_3]$ whenever $M_1 \cong M_2 \oplus M_3$. Multiplication is defined by $[M_1][M_2] := [M_1 \otimes M_2]$.

Corollary 1. *The following equalities hold in $K_0(\mathcal{H})$:*

$$\begin{aligned} [S_-^n][\Lambda_+^m] &= [\Lambda_+^m][S_-^n] + [\Lambda_+^{m-1}][S_-^{n-1}], \\ [S_-^n][S_-^m] &= [S_-^m][S_-^n], \\ [\Lambda_+^n][\Lambda_+^m] &= [\Lambda_+^m][\Lambda_+^n]. \end{aligned}$$

Let $H_{\mathbb{Z}}$ be the unital ring with generators a_n, b_n, $n \geq 1$ and defining relations

$$a_n b_m = b_m a_n + b_{m-1} a_{n-1}, \tag{3}$$

$$a_n a_m = a_m a_n, \tag{4}$$

$$b_n b_m = b_m b_n. \tag{5}$$

Here we have simply rewritten the relations in Corollary 1 using a_n in place of $[S_-^n]$ and b_m instead of $[\Lambda_+^m]$. Also set $a_0 = b_0 = 1$, $a_n = b_n = 0$ for $n < 0$, and require that the above relations hold for any $n, m \in \mathbb{Z}$. Any product of a's and b's can be converted into a linear combination with nonnegative integer coefficients of monomials in b's times monomials in a's,

$$b_{m_1} \dots b_{m_k} a_{n_1} \dots a_{n_r} \tag{6}$$

with $1 \le m_1 \le \dots \le m_k$, $1 \le n_1 \le \dots \le n_r$. The Bergman diamond lemma [4] tells us that this set of elements is a basis of $H_{\mathbb{Z}}$ viewed as a free abelian group. Let $H = H_{\mathbb{Z}} \otimes \mathbb{C}$ be the $\mathbb{C}$-algebra with the same generators and relations as $H_{\mathbb{Z}}$.

Forming the generating functions

$$A(t) = 1 + a_1 t + a_2 t^2 + \cdots, \qquad B(u) = 1 + b_1 u + b_2 u^2 + \cdots,$$

we can rewrite relation (3) as

$$A(t)B(u) = B(u)A(t)(1 + tu).$$

Let

$$\widetilde{A}(t) = 1 + tA'(-t)A(-t)^{-1}, \qquad \widetilde{A}(t) = 1 + \tilde{a}_1 t + \tilde{a}_2 t^2 + \cdots.$$

It is easy to check that $\tilde{a}_1, \tilde{a}_2, \dots$ generate the same subalgebra of H as $a_1, a_2, \dots$, and that

$$\widetilde{A}(t)B(u) = B(u)\widetilde{A}(t) + \frac{tu}{1 - tu}B(u).$$

Defining

$$\widetilde{B}(u) = 1 + uB'(-u)B(-u)^{-1},$$

we see that

$$[\tilde{a}_n, \tilde{b}_m] = (-1)^{n-1} n \delta_{n,m}, \tag{7}$$

so that H is isomorphic to the usual Heisenberg algebra.

Corollary 1 gives a ring homomorphism

$$\gamma : H_{\mathbb{Z}} \to K_0(\mathcal{H}) \tag{8}$$

that takes a_n to $[S_-^n]$ and b_n to $[\Lambda_+^n]$.

Theorem 1. *The map γ is injective.*

This theorem is proved in Section 3.3.

Conjecture 1. *The map γ is an isomorphism.*

If true, this conjecture would allow us to view the additive monoidal category $\mathcal{H}$ as a categorification of the integral form $H_{\mathbb{Z}}$ of the Heisenberg algebra.

The degenerate affine Hecke algebra, which we call degenerate AHA following a suggestion of Etingof, was introduced by Drinfeld [14] in the GL(n) case and by Lusztig [27] in the general case. Cherednik [9] classified finite-dimensional irreducible representations of degenerate AHA; its centralizing properties were studied by him and Olshanski in [10, 36]. We denote by DH_n the degenerate AHA for $\mathrm{GL}(n)$, over the base field $\Bbbk$. Under the canonical homomorphism [9, 14] from DH_n to the group algebra $\Bbbk[S_n]$ of the symmetric group, the polynomial generators of DH_n go to the Jucys–Murphy elements. Okounkov and Vershik [34, 35] presented a detailed derivation of the basic representation theory of the symmetric group via these elements (see also [25, Chapter 2] and [8, 13]). For some other uses of Jucys–Murphy's elements and degenerate AHA we refer the reader to [20, 29, 33, 40].

We will prove in Section 4 that the ring of endomorphisms of the object $Q_+^{\otimes n}$ in our category is isomorphic to the tensor product of DH_n and the polynomial algebra in infinitely many variables. Thus, the degenerate AHA for $\mathrm{GL}(n)$ emerges naturally in our approach as part of a larger structure. The polynomial generators of DH_n acquire graphical interpretation in our calculus as right-twisted curls on strands. We also describe a basis, given diagrammatically, of the vector spaces of morphisms between arbitrary tensor products of the generators Q_+ and Q_- of $\mathcal{H}'$.

To prove our results, we construct a family of functors from $\mathcal{H}'$ to the category $\mathcal{S}'$ whose objects are compositions of the induction and restriction functors between the group algebras $\Bbbk[S_n]$ of the symmetric group, and morphisms are natural transformations between these functors. The image under these functors of the endomorphism of Q_+ given by the right curl diagram is the Jucys–Murphy element. The image of the counterclockwise circle diagram with k right curls is the kth moment of the Jucys–Murphy element. Products of these moments were investigated in [20, 33, 38, 40] in relation to the asymptotic representation theory of the symmetric group and free probability. It also seems that our construction should be related to the circle of ideas considered by Guionnet, Jones and Shlyakhtenko [19] that intertwine planar algebras and free probability. In addition, one would hope for a relation between our calculus and the geometrization of the Heisenberg algebra via Hilbert schemes by Nakajima [31, 32] and Grojnowski [18], and for a possible link with Frenkel, Jing and Wang [16].

We discovered the monoidal category $\mathcal{H}'$ by considering compositions of the induction and restriction functors for the standard inclusions of the symmetric group algebras $\Bbbk[S_n] \subset \Bbbk[S_{n+1}]$. The induction and restriction functors for inclusions of finite groups are biadjoint, and biadjointness natural transformations can be depicted via cap and cup planar diagrams. Furthermore, the composition of two induction functors admits a natural endotransformation given by right multiplication by the transposition $(n+1, n+2)$, an endomorphism of $\Bbbk[S_{n+2}]$ viewed as a left $\Bbbk[S_{n+2}]$-module and a right $\Bbbk[S_n]$-module. We denote this natural transformation by the crossing of two upward-oriented strands. Relations for compositions of the crossing, cup, and cap transformations that hold for all n (universal relations) are given by equations (9)–(11). These relations together with isotopies of diagrams are exactly the defining relations for the additive monoidal category $\mathcal{H}'$.

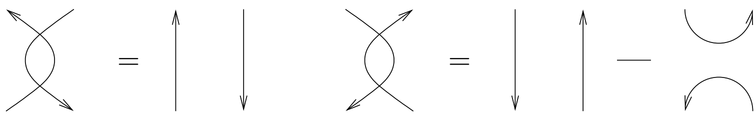
Cautis and Licata [7] introduced graded relatives of $\mathcal{H}'$ and $\mathcal{H}$ associated to finite subgroups of SU(2), identified their Grothendieck rings with certain quantized Heisenberg algebras, and constructed an action of their categories on derived categories of coherent sheaves on Hilbert schemes of points on the ALE spaces. Hom spaces in Cautis–Licata monoidal categories carry a natural grading (absent in our case) with finite-dimensional homogeneous terms and vanishing negative degree homs on certain objects, leading to a proof that their analogue of the map γ is an isomorphism.

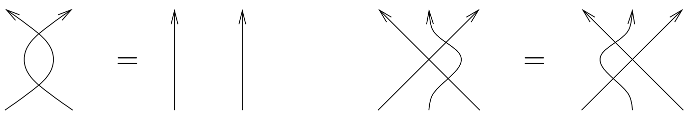
By themselves, Heisenberg algebras are rather simple constructs. Their value is in the structures that quantum field theory builds on top of them, for instance, the structures of vertex operator algebras. The problem posed by Igor Frenkel [15] to categorify just the simplest vertex operator algebra remains wide open—perhaps our paper will serve as a small step towards this goal.

2. A new graphical calculus

2.1. Local moves and their consequences. Fix a commutative ring $\Bbbk$ and consider the following additive $\Bbbk$-linear monoidal category $\mathcal{H}'$ generated by two objects Q_+ and Q_-. An object of $\mathcal{H}'$ is a finite direct sum of tensor products $Q_{\epsilon_1}\otimes\cdots\otimes Q_{\epsilon_m}$, denoted Q_ϵ, where $\epsilon = \epsilon_1 \dots \epsilon_m$ are finite sequences of signs. Thus, $Q_{\epsilon\epsilon'} \cong Q_\epsilon \otimes Q_{\epsilon'}$ for sequences ϵ, ϵ' and their concatenation $\epsilon\epsilon'$. The unit object corresponds to the empty sequence: $\mathbf{1} = Q_\emptyset$.

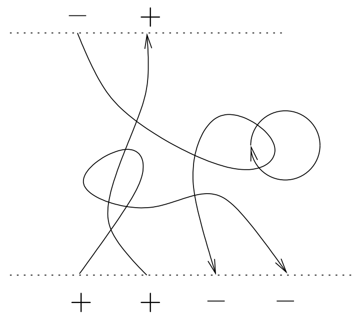
The space of homomorphisms $\mathrm{Hom}_{\mathcal{H}'}(Q_\epsilon, Q_{\epsilon'})$ for sequences ϵ, ϵ' is the $\Bbbk$-module generated by suitable planar diagrams, modulo local relations. The diagrams consist of oriented compact one-manifolds immersed into the plane strip $\mathbb{R} \times [0, 1]$, modulo rel boundary isotopies. The relations are

(9) 

(10) 

(11) 

We require that the endpoints of the one-manifold are located at $\{1, \dots, m\} \times \{0\}$ and $\{1, \dots, k\} \times \{1\}$, and call these the lower and upper endpoints, respectively, where m and k are the lengths of the sequences ϵ and ϵ'. Moreover, orientation of the one-manifold at the endpoints must match the signs in the sequences ϵ and ϵ'. For instance, the diagram



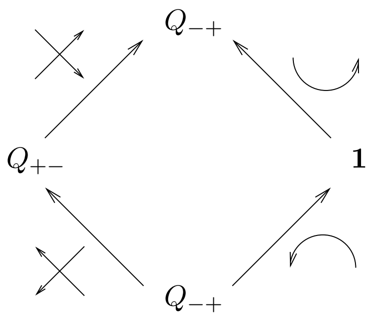
is a morphism from Q_{++--} to Q_{-+}. A diagram without endpoints is an endomorphism of $\mathbf{1}$. Composition of morphisms is given by concatenating the diagrams. The sequence of n pluses is denoted $+^n$, the sequence of n minuses $-^n$.

We have the Heisenberg relation

$$Q_{-+} \cong Q_{+-} \oplus \mathbf{1}.$$

This isomorphism is canonical and comes from the maps between these objects encoded in the following diagram:

(12)



The four arrows are given by four morphisms, two of which are crossings and two are U-turns. The condition that these maps describe a decomposition of Q_{-+} as the direct sum of Q_{-+} and $\mathbf{1}$ is equivalent to (9) and (11), modulo the condition that an isotopy of a diagram does not change the morphism. The latter condition is equivalent to the biadjointness of the functors of tensoring with Q_+ and Q_-, with the biadjointness transformations given by the four U-turns

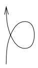
(see Section 3.1 for details).

Moving the lower endpoints of a diagram up via a multiple cups diagram leads to canonical isomorphisms

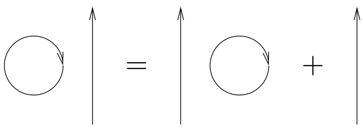
$$\mathrm{Hom}_{\mathcal{H}'}(Q_\epsilon, Q_{\epsilon'}) \cong \mathrm{Hom}_{\mathcal{H}'}(\mathbf{1}, Q_{\overline{\epsilon}\epsilon'}) \cong \mathrm{Hom}_{\mathcal{H}'}(\mathbf{1}, Q_{\epsilon'\overline{\epsilon}}), \tag{13}$$

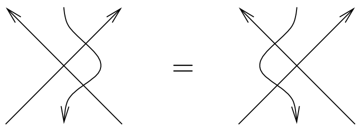
related to the biadjointness of tensoring with Q_ϵ and $Q_{\overline{\epsilon}}$. Here $\overline{\epsilon}$ is the sequence ϵ with the order and all signs reversed. Biadjointness natural transformations satisfy the cyclicity condition [1, 2, 11, 26], which follows at once from the definition of $\mathcal{H}'$.

The two relations in (9) allow simplification of a double crossing for oppositely oriented intervals. The first relation in (11) says that a counterclockwise-oriented circle equals one. Thus, an innermost counterclockwise circle can be erased from the diagram without changing the value of the diagram viewed as an element of the hom space between the functors.

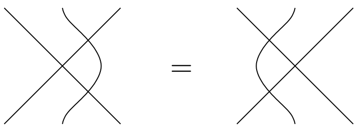
There are two possible types of curls on strands: a left curl  and a right curl . The second relation in (11) says that a diagram that contains a left curl subdiagram is zero.

The defining local relations in $\mathcal{H}'$ imply the following relations:

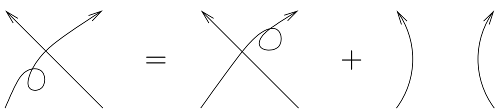


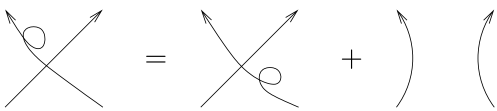


The second relation, jointly with the original ones, implies that the triple intersection move holds for any orientation of the three strands:

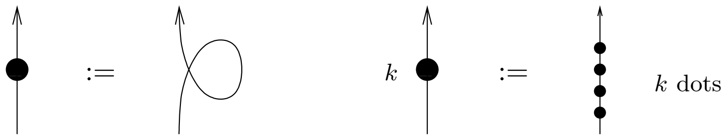


Furthermore, right curls can be moved across intersection points, modulo simpler diagrams:

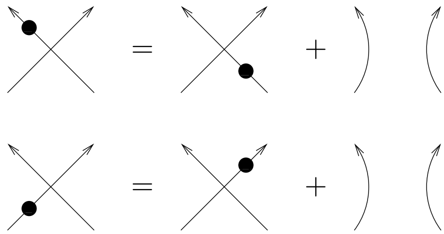




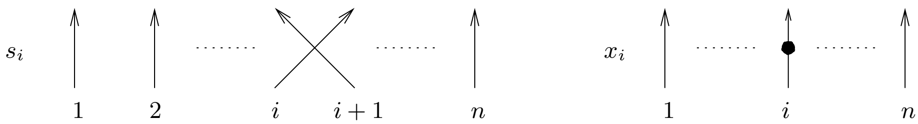
It will be convenient to denote a right curl by a dot on a strand, and kth power of a right curl by a dot with k next to it:

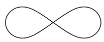


The above relations can be rewritten as

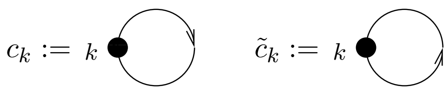


Together with the earlier ones, these relations show that there is a homomorphism from the degenerate affine Hecke algebra DH_n with coefficients in $\Bbbk$ to the $\Bbbk$-algebra of endomorphisms of the object Q_{+^n}. The permutation generator s_i of DH_n goes to the permutation diagram of the ith and $(i+1)$st strands, and the polynomial generator x_i goes to the dot on the ith strand:



Note that  $= 0$. Indeed, the figure eight diagram, for any orientation, contains both left and right curls, and therefore equals zero in our calculus.

A strand with k dots can be closed into either a clockwise-oriented or a counterclockwise-oriented circle with k dots. Denote these circles by c_k and $\tilde{c}_k$, respectively:

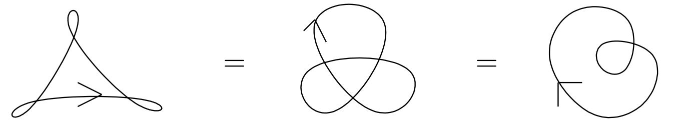


Counterclockwise circles can be expressed as linear combinations of products of clockwise circles. For the first few values of k, these are

$$\begin{aligned} \tilde{c}_0 &= 1, & \tilde{c}_3 &= c_1, \\ \tilde{c}_1 &= 0, & \tilde{c}_4 &= c_2 + c_0^2, \\ \tilde{c}_2 &= c_0, & \tilde{c}_5 &= c_3 + 2c_1c_0. \end{aligned}$$

These equations are obtained by expanding each dot into a left curl and then operating on the resulting diagram via the rules of the graphical calculus.

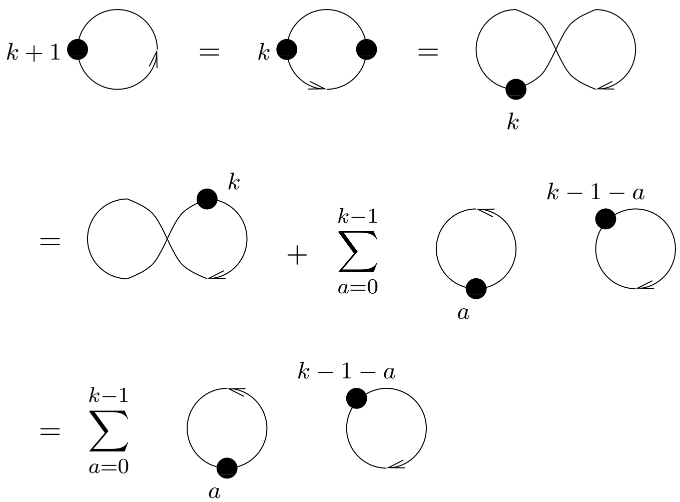
A counterclockwise circle with one dot expands into the figure eight diagram, which is 0. For another example,

$\tilde{c}_3 =$  $= c_1$

PROPOSITION 2. *For $k > 0$ we have*

$$\tilde{c}_{k+1} = \sum_{a=0}^{k-1} \tilde{c}_a c_{k-1-a}. \tag{14}$$

Proof. We compute



In the second equality, we converted a dot into a right curl, and in the third equality, we moved k dots through a crossing. The first term on the second line equals 0, since it contains a left curl. ■

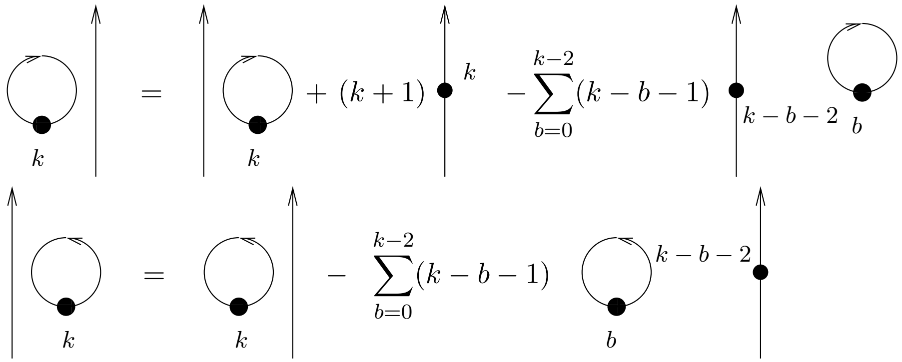
Iterating this formula, one obtains an expression for $\tilde{c}_k$ as a polynomial function of c_m, $m \leq k-2$. Vice versa, each c_m can be written as a polynomial in $\tilde{c}_k$, $k \leq m+2$. Let t be a formal variable and write

$$c(t) = \sum_{i=0}^{\infty} c_i t^i, \qquad \tilde{c}(t) = \sum_{i=0}^{\infty} \tilde{c}_i t^i.$$

Formula (14) turns into $t^2 c(t)\tilde{c}(t) = \tilde{c}(t) - 1$, so that

$$\tilde{c}(t) = \frac{1}{1 - t^2 c(t)}.$$

The following identities, called *bubble moves* by analogy with [26], hold:



A closed diagram D defines an endomorphism of the object $\mathbf{1}$ of $\mathcal{H}'$. Using the local moves, such a diagram D can be converted into a linear combinations of crossingless diagrams that consist of nested dotted circles. Furthermore, bubble moves can be used to split apart nested circles. Lastly, convert counterclockwise circles into linear combinations of products of clockwise circles. Therefore, a closed diagram can be written as a linear combination of products of dotted clockwise circles. We see that the endomorphism algebra $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ is a quotient of the polynomial algebra $\Pi := \Bbbk[c_0, c_1, c_2, \dots]$ in countably many variables via the map

$$\psi_0 : \Pi = \Bbbk[c_0, c_1, c_2, \dots] \to \mathrm{End}_{\mathcal{H}'}(\mathbf{1}) \tag{15}$$

that takes c_k to the clockwise circle with k dots (we took the liberty of using c_k to denote both a formal variable and its image in the endomorphism algebra).

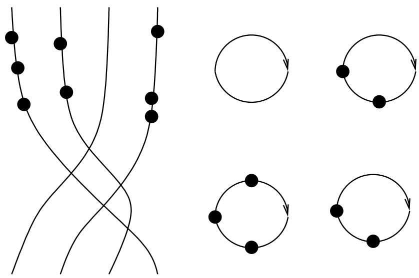
Proposition 3. *The map ψ_0 is an isomorphism.*

This proposition will be proved in Section 4.

The endomorphism algebra of Q_{+^m} is spanned by all diagrams that have m upper and m lower endpoints and are such that at each endpoint the strand is oriented upward. A homomorphism from the degenerate affine Hecke algebra DH_m to $\mathrm{End}_{\mathcal{H}'}(Q_{+^m})$ was described earlier. Placing a closed diagram to the right of a diagram representing an element of DH_m gives a homomorphism

$$\psi_m : \mathrm{DH}_m \otimes \Pi \to \mathrm{End}_{\mathcal{H}'}(Q_{+^m}). \tag{16}$$

It is easy to see that ψ_m is surjective, by taking a diagram representing an element on the right hand side, and inductively simplifying it to a linear combination of diagrams that come from a standard basis of the left hand side. Namely, any diagram representing an endomorphism of Q_{+^m} is a combination of diagrams that consist of a permutation $\sigma \in S_m$, some number (possibly zero) of dots on each strand above the permutation diagram, and a monomial in dotted clockwise circles to the right:



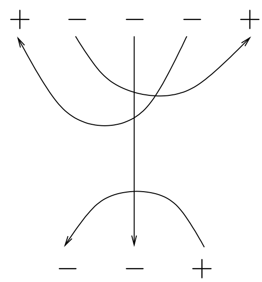
We write these basis elements as $x_1^{a_1} \dots x_m^{a_m} \cdot \sigma \cdot c_0^{b_0} c_1^{b_1} \dots c_k^{b_k}$. In the above example, the element is $x_1^3 x_2^2 x_4^3 \cdot (1324) \cdot c_0 c_2^2 c_3$.

Surjectivity of ψ_m can be strengthened to the following result.

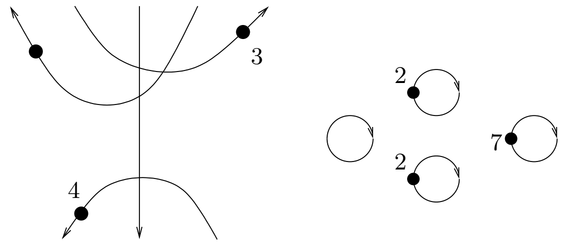
PROPOSITION 4. *The map ψ_m is an isomorphism.*

Injectivity of ψ_m is proved in Section 4.

We now describe a spanning set in the $\Bbbk$-module $\mathrm{Hom}_{\mathcal{H}'}(Q_\epsilon, Q_{\epsilon'})$ for any sequences ϵ, ϵ'. Denote by $\langle \epsilon \rangle$ the difference between the number of pluses and the number of minuses in ϵ (the *weight* of ϵ). The hom space is nontrivial iff $\langle \epsilon \rangle = \langle \epsilon' \rangle$, which we assume to be the case. Let k be half the sum of the lengths of ϵ and ϵ'. The spanning set, denoted $B(\epsilon, \epsilon')$, is obtained by forming all possible oriented matchings of the sequences ϵ, ϵ' of signed points via k oriented segments in the plane strip $\mathbb{R} \times [0,1]$. We assume that the sequences ϵ and ϵ' are written at the bottom and top of the strip, the segments are embedded in the strip, and their orientations at the endpoints match the corresponding elements of ϵ and ϵ'. Any two segments intersect at most once, and no triple intersections are allowed; see the example below for $\epsilon = --+$ and $\epsilon' = +---+$.



Select an interval disjoint from intersections near the out endpoint of each interval and put any number (perhaps zero) of dots on it. In the rightmost region of the diagram, draw some number of clockwise-oriented disjoint nonnested circles with no dots, some number of such circles with one dot, two dots, etc., with finitely many circles in total. The resulting set of diagrams $B(\epsilon, \epsilon')$ is parametrized by $k!$ possible matchings of the $2k$ oriented endpoints, by a sequence of k nonnegative integers describing the number of dots on each interval, and by a finite sequence of nonnegative integers listing the number of clockwise-oriented circles with no dots, one dot, and so on. An example of a diagram in $B(--+, +----+)$ is depicted below:



In this picture, we put no dots, one dot, three dots, and four dots on the four arcs of the matching, and added one bubble with no dots, two with two dots and one with five dots.

It is rather straightforward to check that $B(\epsilon, \epsilon')$ is a spanning set of the $\Bbbk$-vector space $\mathrm{Hom}_{\mathcal{H}'}(Q_\epsilon, Q_{\epsilon'})$.

Proposition 5. *For any sign sequences ϵ, ϵ' the set $B(\epsilon, \epsilon')$ constitutes a basis of the $\Bbbk$-vector space $\mathrm{Hom}_{\mathcal{H}'}(Q_\epsilon, Q_{\epsilon'})$.*

Proof. We claim that the set $B(\epsilon, \epsilon')$ is linearly independent. The proposition holds as well for $\Bbbk$ being any commutative ring rather than a field, with $B(\epsilon, \epsilon')$ being a basis of the free $\Bbbk$-module $\mathrm{Hom}_{\mathcal{H}'}(Q_\epsilon, Q_{\epsilon'})$.

Notice that Proposition 4 is a special case of this proposition, for $\epsilon = \epsilon' = +^m$. Proposition 5 follows from Proposition 4, the functor isomorphisms $Q_{-+} \cong Q_{+-} \oplus \mathrm{Id}$ and arguments similar to the ones in [24, Section 2.2]. First, canonical isomorphisms (13) take the set $B(\epsilon, \epsilon')$ to $B(\emptyset, \overline{\epsilon}\epsilon')$ and $B(\emptyset, \epsilon'\overline{\epsilon})$, respectively, and it is then enough to show that $B(\emptyset, \epsilon)$ is linearly independent for any sequence ϵ with k pluses and k minuses.

Proposition 4 implies linear independence for $k = 0, 1$, and for the sequence $+^k-^k$ for any k. Assume that $\epsilon = \epsilon_1 - + \epsilon_2$ for some sequences ϵ_1, ϵ_2. Assume by induction on k and by induction on the lexicographic order among length $2k$ sequences that the sets $B(\emptyset, \epsilon_1\epsilon_2)$ and $B(\emptyset, \epsilon_1 + - \epsilon_2)$ are linearly independent in their respective hom spaces.

The two upper arrows in the diagram (12) lead to a canonical decomposition

$$Q_{\epsilon_1 - + \epsilon_2} \cong Q_{\epsilon_1 + - \epsilon_2} \oplus Q_{\epsilon_1 \epsilon_2}.$$

Under this isomorphism the sets $B(\emptyset, \epsilon_1\epsilon_2)$ and $B(\emptyset, \epsilon_1 + - \epsilon_2)$ get mapped to two subsets of $\mathrm{Hom}_{\mathcal{H}'}(\mathbf{1}, Q_{\epsilon_1 - + \epsilon_2})$. Denote by B the union of these two subsets. It is easy to see that linear independence of $B(\emptyset, \epsilon_1 - + \epsilon_2)$ is equivalent to linear independence of B, which we know by induction. Proposition 5 follows. ∎

By the *thickness* of a diagram in $B(\epsilon, \epsilon')$ we understand the number of arcs connecting lower and upper endpoints. The diagram depicted earlier has thickness one. For any k and ϵ, the subset of diagrams of thickness at most k is a 2-sided ideal in the endomorphism ring of Q_ϵ, since thickness cannot increase upon composition. For $\epsilon = +^n -^m$ and $k = n + m - 1$ we denote the corresponding ideal by $J_{n,m}$. It is spanned by diagrams with at least one arc connecting a pair of upper endpoints (and, necessarily, at least one arc connecting a pair of lower endpoints). It is easy to see that the quotient of the endomorphism ring of $Q_{+^n -^m}$ by this ideal is naturally isomorphic to the tensor product $\mathrm{DH}_n \otimes \mathrm{DH}_m^{\mathrm{op}} \otimes \Pi$, and the short exact sequence

$$0 \to J_{n,m} \to \mathrm{End}_{\mathcal{H}'}(Q_{+^n -^m}) \to \mathrm{DH}_n \otimes \mathrm{DH}_m^{\mathrm{op}} \otimes \Pi \to 0 \tag{17}$$

admits a canonical splitting. Notice also that $\mathrm{DH}_m^{\mathrm{op}} \cong \mathrm{DH}_m$.

We now list some obvious symmetries of $\mathcal{H}'$. The map that assigns $(-1)^{w(D)} D$ to a diagram D, where $w(D)$ is the number of crossings plus the number of dots of D, extends to an involutive autoequivalence ξ_1 of $\mathcal{H}'$. We have $\xi_1^2 = \mathrm{Id}$ (equality and not just isomorphism). The autoequivalence ξ_1 exchanges S_+^n with Λ_+^n and S_-^n with Λ_-^n.

Denote by ξ_2 the symmetry of the category $\mathcal{H}'$ given on diagrams by reflecting about the x-axis and reversing orientation. This symmetry is an involutive monoidal contravariant autoequivalence of $\mathcal{H}'$.

Denote by ξ_3 the symmetry of the category $\mathcal{H}'$ given on diagrams by reflecting about the y-axis and reversing orientation. This symmetry is an involutive antimonoidal autoequivalence of $\mathcal{H}'$. Being antimonoidal means reversing the order of elements in the tensor product: $\xi_3(M \otimes N) = \xi_3(N) \otimes \xi_3(M)$.

The symmetries ξ_1, ξ_2, ξ_3 pairwise commute and generate an action of $(\mathbb{Z}/2)^3$.

2.2. Karoubi envelope and projectors. The two relations in (10) tell us that upward-oriented crossings satisfy the symmetric group relations and give us a canonical homomorphism

$$\Bbbk[S_n] \to \mathrm{End}_{\mathcal{H}'}(Q_{+^n})$$

from the group algebra of the symmetric group to the endomorphism ring of the nth tensor power of Q_+. Turning the diagrams by 180 degrees, we obtain a canonical homomorphism

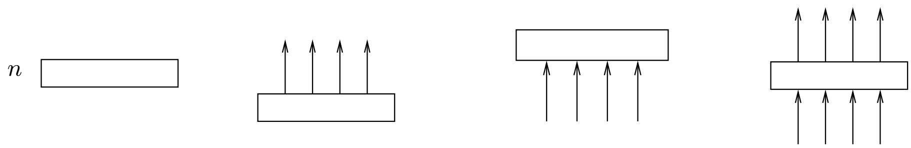
$$\Bbbk[S_n] \to \mathrm{End}_{\mathcal{H}'}(Q_{-^n}).$$

Assume that $\Bbbk$ is a field of characteristic 0. Then we can use symmetrizers and antisymmetrizers, and more generally, Young symmetrizers, to produce idempotents in $\mathrm{End}_{\mathcal{H}'}(Q_{+^n})$. At this point it is convenient to introduce the *Karoubi envelope* of $\mathcal{H}'$, the category $\mathcal{H}$ whose objects are pairs (P, e), where P is an object of $\mathcal{H}'$ and $e : P \to P$ is an idempotent endomorphism, $e^2 = e$. Morphisms from (P, e) to (P', e') are maps $f : P \to P'$ in $\mathcal{H}'$ such that $e'fe = f$. It is immediate that $\mathcal{H}$ is a $\Bbbk$-linear additive monoidal category.

To the complete symmetrizer

$$e(n) \in \Bbbk[S_n], \qquad e(n) = \frac{1}{n!}\sum_{\sigma\in S_n} \sigma$$

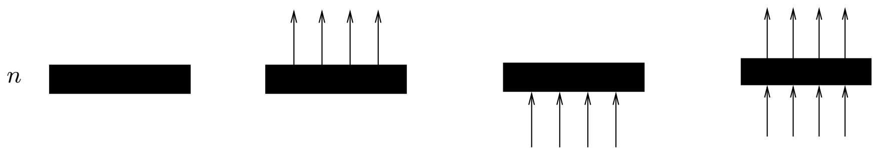
we assign the object $S_+^n := (Q_{+^n}, e(n))$ in $\mathcal{H}$. Following Cvitanović [12], which contains diagrammatics for Young symmetrizers and antisymmetrizers, we depict S_+^n as a white box labelled n. The inclusion morphism $S_+^n \to Q_{+^n}$ is depicted by a white box with n upward-oriented lines emanating from the top. The projection $Q_{+^n} \to S_+^n$ is depicted by a white box with n upward-oriented lines at the bottom. The composition $Q_{+^n} \to S_+^n \to Q_{+^n}$ is depicted likewise.



To the complete antisymmetrizer

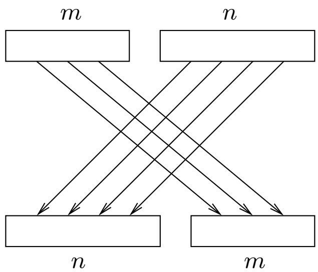
$$e'(n) \in \Bbbk[S_n], \qquad e'(n) = \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sign}(\sigma)\sigma$$

we assign the object $\Lambda_+^n := (Q_{+^n}, e'(n))$ in $\mathcal{H}$ and depict it and related inclusions and projections to and from Q_{+^n} by black boxes with up arrows

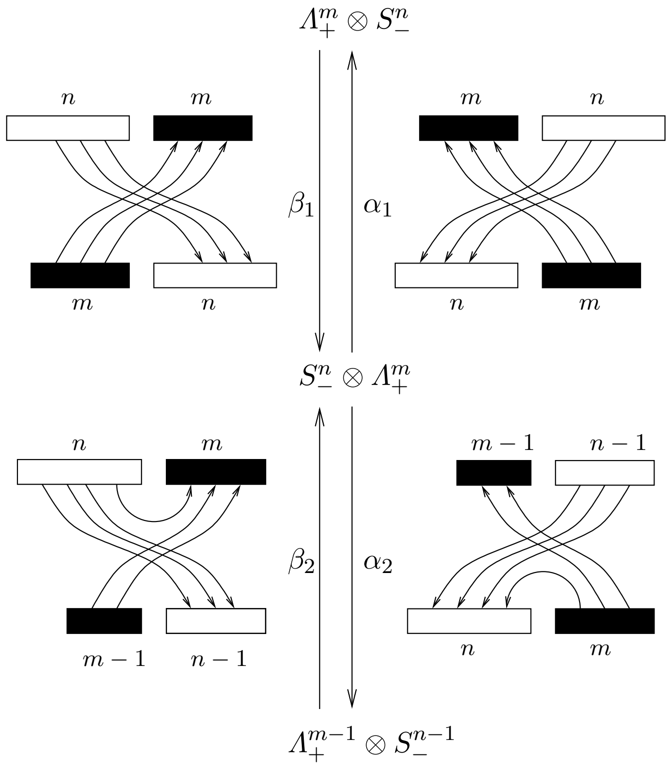


Define the objects $S_n^- := (Q_{-^n}, e(n))$ and $\Lambda_n^- := (Q_{-^n}, e'(n))$ as the subobjects of Q_{-^n} associated to the symmetrizer $e(n)$ and the antisymmetrizer $e'(n)$ idempotents, respectively, under the canonical homomorphism $\Bbbk[S_n] \to \mathrm{End}_{\mathcal{H}'}(Q_{-^n})$. We draw S_n^- and Λ_n^- as white, respectively black, boxes, but with the lines at the boxes oriented downward.

We plan to develop the graphical calculus of these diagrams elsewhere. Part of the calculus that deals with the lines oriented only upwards (or only downwards) is the graphical calculus of symmetrizers and antisymmetrizers in the symmetric group, and can be found in [12]. The latter calculus implies the second and third isomorphisms from Proposition 1 in the introduction. For instance, the second isomorphism is realized by the diagram



The first family of isomorphisms in Proposition 1 is realized by the maps



A straightforward manipulation of diagrams shows that

$$\alpha_1\beta_1 = \mathrm{Id}, \quad \alpha_1\beta_2 = 0, \quad \alpha_2\beta_2 = \frac{1}{mn}\mathrm{Id}, \quad \alpha_2\beta_1 = 0.$$

Let $\beta_2' = mn\beta_2$. Then the maps

$$\Lambda_+^m \otimes S_-^n \underset{\beta_1}{\overset{\alpha_1}{\leftrightarrows}} S_-^n \otimes \Lambda_+^m \underset{\beta_2}{\overset{\alpha_2}{\leftrightarrows}} \Lambda_+^{m-1} \otimes S_-^{n-1}$$

satisfy

$$\alpha_1\beta_1 = \mathrm{Id}, \quad \alpha_1\beta_2' = 0, \quad \alpha_2\beta_2' = \mathrm{Id}, \quad \alpha_2\beta_1 = 0, \quad \beta_1\alpha_1 + \beta_2'\alpha_2 = \mathrm{Id}.$$

The last equality follows from a direct diagrammatic manipulation as well. Thus, there is an isomorphism

$$S_-^n \otimes \Lambda_+^m \cong (\Lambda_+^m \otimes S_-^n) \oplus (\Lambda_+^{m-1} \otimes S_-^{n-1}),$$

concluding the proof of Proposition 1 and Corollary 1.

There is a natural bijection between partitions λ of n and (isomorphism classes of) irreducible representations of $\Bbbk[S_n]$. To each partition $\lambda = (\lambda_1, \dots, \lambda_k)$ with $|\lambda| = \lambda_1 + \dots + \lambda_k = n$ there corresponds the unique common irreducible summand L_λ of the representation induced from the trivial representation of the parabolic subgroup $S_\lambda = S_{\lambda_1} \times \dots \times S_{\lambda_k} \subset S_n$ and the representation induced from the sign representation of the parabolic subgroup $S_{\lambda^*} = S_{\lambda_1^*} \times \dots \times S_{\lambda_m^*}$, where λ^* is the dual partition. Let $e_\lambda \in \Bbbk[S_n]$ be the Young idempotent, so that $e_\lambda^2 = e_\lambda$ and $L_\lambda \cong \Bbbk[S_n]e_\lambda$.

We denote by $Q_{+,\lambda} := (Q_{+^n}, e_\lambda)$ the object of $\mathcal{H}$ which is the direct summand of Q_{+^n} corresponding to the idempotent e_λ, where we view the latter as an idempotent in the endomorphism ring via the standard homomorphism $\Bbbk[S_n] \to \mathrm{End}_{\mathcal{H}}(Q_{+^n})$. Likewise, let $Q_{-,\lambda} := (Q_{-^n}, e_\lambda)$ be the corresponding direct summand of Q_{-^n}, where we view e_λ as an endomorphism of the latter object. In particular,

$$S_+^n = Q_{+,(n)}, \quad \Lambda_+^n = Q_{+,(1^n)}, \quad S_-^n = Q_{-,(n)}, \quad \Lambda_-^n = Q_{-,(1^n)}.$$

The *Grothendieck ring* $K_0(\mathcal{H})$ is the abelian group with generators the symbols $[M]$ for all objects M of $\mathcal{H}$, and defining relations $[M_1] = [M_2] + [M_3]$ whenever $M_1 \cong M_2 \oplus M_3$. The monoidal structure on $\mathcal{H}$ descends to an associative multiplication on $K_0(\mathcal{H})$, with $[\mathbf{1}]$ being the identity for multiplication. Hence, $K_0(\mathcal{H})$ is an associative unital ring.

Recall the ring $H_{\mathbb{Z}}$ from the introduction. We can now define the homomorphism $\gamma : H_{\mathbb{Z}} \to K_0(\mathcal{H})$ discussed there:

$$\gamma(a_n) = [Q_{-,(n)}] = [S_-^n], \quad \gamma(b_m) = [Q_{+,(1^m)}] = [\Lambda_+^m].$$

If we identify the subring of $H_{\mathbb{Z}}$ generated by the a_n's with the ring Sym of symmetric functions so that a_n corresponds to the nth complete symmetric function h_n, then γ will take the Schur function associated to the partition λ to $[Q_{+,\lambda}]$. This function is often denoted s_λ; for us it is convenient to call it a_λ, so that $a_{(n)} = a_n$.

Similarly, we identify the subring generated by the b_m's with Sym by taking b_m to the mth elementary symmetric function e_m. Denote by b_λ the polynomial in b_m's that corresponds to the Schur function s_λ under this identification. In particular, $b_{(1^m)} = b_m$. We have

$$\gamma(a_\lambda) = [Q_{-,\lambda}], \qquad \gamma(b_\lambda) = [Q_{+,\lambda^*}].$$

The Littlewood–Richardson coefficients $r^\nu_{\lambda,\mu}$ that appear in decompositions of the product of the Schur functions

$$a_\lambda a_\mu = \sum_\nu r^\nu_{\lambda,\mu} a_\nu, \qquad b_\lambda b_\mu = \sum_\nu r^\nu_{\lambda,\mu} b_\nu$$

also appear in the following isomorphism formulas in $\mathcal{H}$:

$$Q_{+,\lambda} \otimes Q_{+,\mu} \cong \bigoplus_\nu (Q_{+,\nu})^{r^\nu_{\lambda,\mu}}, \qquad Q_{-,\lambda} \otimes Q_{-,\mu} \cong \bigoplus_\nu (Q_{-,\nu})^{r^\nu_{\lambda,\mu}}.$$

Descending to the Grothendieck ring, we have

$$[Q_{+,\lambda}][Q_{+,\mu}] = \sum_\nu r^\nu_{\lambda,\mu}[Q_{+,\nu}], \qquad [Q_{-,\lambda}][Q_{-,\mu}] = \sum_\nu r^\nu_{\lambda,\mu}[Q_{-,\nu}].$$

The ring $H_\mathbb{Z}$ has a basis $\{b_\mu a_\lambda\}_{\lambda,\mu}$ over all partitions λ, μ. Consequently, the elements $[Q_{+,\mu}][Q_{-,\lambda}]$ over all λ, μ span the subring $\gamma(H_\mathbb{Z})$ of $K_0(\mathcal{H})$.

Remark. The symmetries ξ_1, ξ_2, ξ_3 of $\mathcal{H}'$ extend to self-equivalences of the category $\mathcal{H}$, also denoted ξ_1, ξ_2, ξ_3. On objects $Q_{+,\lambda} \otimes Q_{-,\mu}$ they act as follows:

$$\xi_1(Q_{+,\mu} \otimes Q_{-,\lambda}) = Q_{+,\mu^*} \otimes Q_{-,\lambda^*},$$
$$\xi_2(Q_{+,\mu} \otimes Q_{-,\lambda}) = Q_{+,\mu} \otimes Q_{-,\lambda},$$
$$\xi_3(Q_{+,\mu} \otimes Q_{-,\lambda}) = Q_{+,\lambda} \otimes Q_{-,\mu}.$$

These self-equivalences induce involutions $[\xi_1], [\xi_2]$ and an antiinvolution $[\xi_3]$ on $K_0(\mathcal{H})$. The involution of $H_\mathbb{Z}$ corresponding to $[\xi_2]$ is the identity. We do not know whether $[\xi_2]$ is the identity involution on the entire $K_0(\mathcal{H})$; this would follow from Conjecture 1.

3. Diagrammatics for induction and restriction functors

3.1. Biadjoint functors. Recall [28] that a functor $L : \mathcal{A} \to \mathcal{B}$ between categories $\mathcal{A}$ and $\mathcal{B}$ is *left adjoint* to a functor $R : \mathcal{B} \to \mathcal{A}$ whenever there are natural transformations

$$\alpha : LR \Rightarrow \mathrm{Id}_\mathcal{B}, \qquad \beta : \mathrm{Id}_\mathcal{A} \Rightarrow RL \tag{18}$$

that satisfy the relations

$$(\alpha \circ \mathrm{Id}_L)(\mathrm{Id}_L \circ \beta) = \mathrm{Id}_L, \qquad (\mathrm{Id}_R \circ \alpha)(\beta \circ \mathrm{Id}_R) = \mathrm{Id}_R. \tag{19}$$

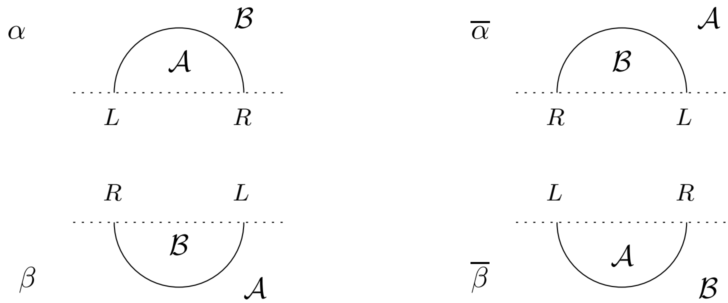
Assume that L is both left and right adjoint to R, and the second adjunction maps

$$\overline{\alpha} : RL \Rightarrow \mathrm{Id}_\mathcal{A}, \qquad \overline{\beta} : \mathrm{Id}_\mathcal{B} \Rightarrow LR \tag{20}$$

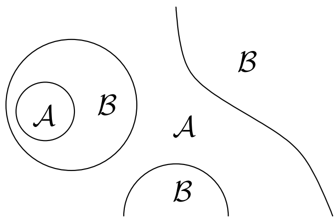
are fixed as well. They satisfy

$$(\overline{\alpha} \circ \mathrm{Id}_R)(\mathrm{Id}_R \circ \overline{\beta}) = \mathrm{Id}_R, \qquad (\mathrm{Id}_L \circ \overline{\alpha})(\overline{\beta} \circ \mathrm{Id}_L) = \mathrm{Id}_L. \tag{21}$$

Out of $\alpha, \overline{\alpha}, \beta, \overline{\beta}$ one can construct more general natural transformations between compositions of functors L and R by placing the basic four transformations in various locations in the composition of functors, and then composing several such transformations. It is convenient to draw these compositions via planar diagrams, with transformations (18), (20) depicted as U-turns:



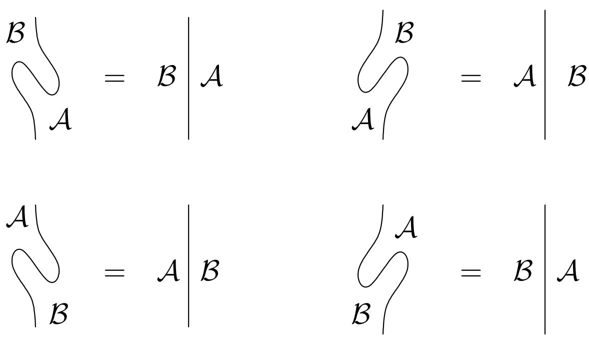
General diagrams are built of U-turns and vertical lines, the latter denoting identity natural transformations of R and L. For instance,



is the following natural transformation from RLR to R:

$$(\overline{\alpha} \circ \mathrm{Id})(\mathrm{Id} \circ \alpha \circ \mathrm{Id}^{\otimes 2})(\mathrm{Id} \circ \overline{\beta} \circ \mathrm{Id}^{\otimes 2})(\beta \circ \mathrm{Id})\overline{\alpha}$$

The four biadjointness equations (19), (21), which can be drawn as

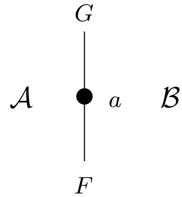


are equivalent to the condition that the lines and circles can be isotoped without changing the natural transformation associated to the diagram. This was observed in [11, 30].

The graphical calculus of biadjoints can be further enhanced. Assume given a collection of categories and a collection of functors between them such that each functor has a biadjoint, which is also in the collection, and the biadjointness transformations are fixed. Natural transformations generated by the biadjointness ones can be drawn via diagrams on the plane strip $\mathbb{R} \times [0,1]$, with lines and circles labelled by functors and regions labelled by categories, with arbitrary rel boundary isotopies allowed.

Furthermore, any element z in the centre of a category $\mathcal{A}$ (i.e. z is the endomorphism of the identity functor $\mathrm{Id}_{\mathcal{A}}$) can be shown as freely floating in a region labelled $\mathcal{A}$. Two such central elements can freely move past each other.

Given two functors $F, G : \mathcal{A} \to \mathcal{B}$ in the collection, a natural transformation $a : F \to G$ can be depicted as a labelled dot on a line separating a segment labelled F from a segment labelled G:



When a is dragged through a U-turn, it can change in two possible ways, depending on the type of the U-turn, into natural transformations $a^*, {}^*a : G' \to F'$, where F', G' are the biadjoints of F and G. If ${}^*a = a^*$, we say that the biadjointness data is *cyclic*. For more on the cyclicity condition see [1, 2, 11, 26].

Biadjoint functors appear throughout categorification and TQFTs, see discussions in [21, Section 5], [22] and references therein, also [6].

3.2. Induction and restriction between finite groups in pictures. We fix a commutative ring $\Bbbk$ and denote by $\Bbbk G$ the group algebra of finite group G with coefficients in $\Bbbk$. We denote by (G) the group algebra $\Bbbk G$ viewed as a $(\Bbbk G, \Bbbk G)$-bimodule. If H is a subgroup of G and we view $\Bbbk G$ as a $(\Bbbk G, \Bbbk H)$-bimodule via the left action of $\Bbbk G$ and the right action of $\Bbbk H$, we denote it by (G_H). When viewing $\Bbbk G$ as a $\Bbbk H$-bimodule, denote it by $({}_H G_H)$, etc. Similar shortcut notation is adopted for tensor products of bimodules. For instance, $(G_H G)$ denotes the $\Bbbk G$-bimodule $\Bbbk G \otimes_{\Bbbk H} \Bbbk G$, while $({}_H G_H G)$ denotes the same space, but viewed as a $(\Bbbk H, \Bbbk G)$-bimodule.

Start with the 2-category BFin whose objects are finite groups G, morphisms from G to H are $(\Bbbk H, \Bbbk G)$-bimodules, and 2-morphisms are bimodule homomorphisms. Consider the 2-subcategory IRFin$'$ of BFin with the same objects as BFin, while morphisms are finite direct sums of tensor products of bimodules (G_H) and $({}_HG)$ corresponding to the induction and restriction functors between categories of H- and G-modules. Thus, a 1-morphism from G to G' is a finite direct sum of bimodules isomorphic to

$$(G_n\ {}_{H_{n-1}}G_{n-1}\ {}_{H_{n-2}} \cdots {}_{H_2}G_2\ {}_{H_1}G_1)$$

where $G' = G_n \supset H_{n-1} \subset G_{n-1} \supset \cdots \subset G_2 \supset H_1 \subset G_1 = G$ is a zigzag of inclusions between finite groups. The 2-morphisms in IRFin$'$ are bimodule homomorphisms. Alternatively, we can think of 1-morphisms in IRFin$'$ as given by compositions of induction and restriction functors between categories of (left) G-modules, over finite groups G.

In this section, we develop basics of a graphical calculus for studying 2-morphisms in IRFin$'$. In general, given a unital inclusion of rings $B \subset A$, the induction functor Ind : B-mod $\to$ A-mod that takes M to $A \otimes_B M$ is left adjoint to the restriction functor. An inclusion $\iota : H \subset G$ of finite groups produces an inclusion $\Bbbk H \subset \Bbbk G$ of group algebras, with the induction functor

$$\mathrm{Ind}_H^G : \Bbbk H\text{-mod} \to \Bbbk G\text{-mod}$$

being both left and right adjoint (i.e. biadjoint) to the restriction functor

$$\mathrm{Res}_G^H : \Bbbk G\text{-mod} \to \Bbbk H\text{-mod}.$$

The biadjointness endomorphisms are given by the following four bimodule maps:

1) $(G_HG) \to (G)$, $x \otimes y \mapsto xy$, $x, y \in (G)$,
2) $(H) \to ({}_HG_GG_H)$, $x \mapsto x \otimes 1 = 1 \otimes x$, $x \in (H)$,
3) $({}_HG_GG_H) \cong ({}_HG_H) \to (H)$, $g \mapsto g$ if $g \in H$, $g \mapsto 0$ if $g \in G \setminus H$.

We denote this projection map by $p_H : ({}_HG_H) \to (H)$, $p_H(g) = g$ if $g \in H$ and $p_H(g) = 0$ if $g \in G \setminus H$, extended by $\Bbbk$-linearity. Clearly, p_H is a map of $\Bbbk H$-bimodules.

4) Let $G = \bigsqcup_{i=1}^m Hg_i$ be a decomposition of G into left H-cosets, so that $m = [G : H]$ is the index of H in G. Notice that the element

$$\sum_{i=1}^m g_i^{-1} \otimes g_i \ \in (G_HG)$$

does not depend on the choice of coset representatives $\{g_i\}_{i=1}^m$ of H in G: if $g_i' = h_i g_i$ then

$$\sum_{i=1}^m g_i'^{-1} \otimes g_i' = \sum_{i=1}^m g_i^{-1} h_i^{-1} \otimes h_i g_i = \sum_{i=1}^m g_i^{-1} \otimes g_i,$$

since the tensor product is over $\Bbbk[H]$, and h_i can be moved through the tensor product sign. Define a bimodule map

$$(G) \to (G_H G) \tag{22}$$

by the condition that

$$1 \mapsto \sum_{i=1}^{m} g_i^{-1} \otimes g_i,$$

so that

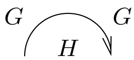
$$g \mapsto \sum_{i=1}^{m} g_i^{-1} \otimes g_i g = \sum_{i=1}^{m} g g_i^{-1} \otimes g_i.$$

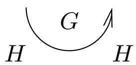
The second equality, needed to ensure that one does get a bimodule map, follows from the following computation: $g_i g = h_i g_{i'}$ for some $h_i \in H$ and $i' \in \{1, \dots, m\}$. The assignment $i \mapsto i'$ is a bijection of $\{1, \dots, m\}$. We have

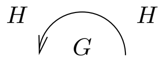
$$\begin{aligned}\sum_{i=1}^{m} g_i^{-1} \otimes g_i g = \sum_{i=1}^{m} g_i^{-1} \otimes h_i g_{i'} = \sum_{i=1}^{m} g_i^{-1} h_i \otimes g_{i'} &= \sum_{i'=1}^{m} g g_{i'}^{-1} \otimes g_{i'} \\ &= \sum_{i=1}^{m} g g_i^{-1} \otimes g_i.\end{aligned}$$

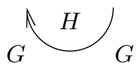
Combining this with the earlier remark, we see that (22) is a bimodule map which does not depend on the choices of H-coset representatives g_i.

We associate to these four bimodule maps the following four pictures:

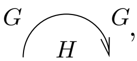






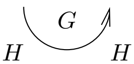


Thus:



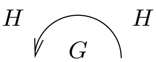
, denoted α_H^G, is the map

$$(G_H G) \to (G), \quad x \otimes y \mapsto xy, \qquad x, y \in (G).$$



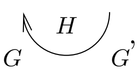
, denoted β_H^G, is the map

$$(H) \to ({}_H G_G G_H) \cong ({}_H G_H), \quad x \mapsto x \otimes 1 = 1 \otimes x, \qquad x \in (H).$$



, denoted $\overline{\alpha}_G^H$, is the map p_H described earlier,

$$({}_H G_G G_H) \cong ({}_H G_H) \to (H), \quad x \mapsto p_H(x), \qquad x \in (G).$$

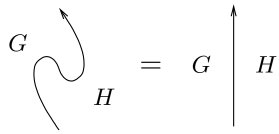


, denoted $\overline{\beta}_G^H$, is the $\Bbbk$-linear map (22),

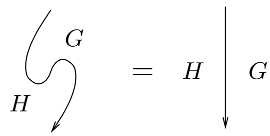
$$(G) \to (G_H G), \quad g \mapsto \sum_{i=1}^{m} g_i^{-1} \otimes g_i g, \qquad g \in G.$$

THEOREM 2. *These four bimodule maps turn the induction and restriction functors* Ind_H^G *and* Res_G^H *into a cyclic biadjoint pair.*

Proof. First, we check that the adjointness equations (19) and (21) hold for these maps. The bimodule map $(G_H) \to (G_H G_H) \to (G_H)$ corresponding to the left hand side of the first equation in (19) is given by $g \mapsto g \otimes 1 \mapsto g1 = g$, hence the map is the identity:



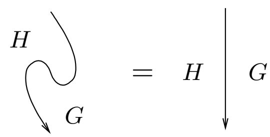
The bimodule map $({}_H G) \to ({}_H G {}_H G) \to ({}_H G)$ for the left hand side of the second equation in (19) is given by $g \mapsto 1 \otimes g \mapsto 1g = g$, and the map is the identity:



The bimodule map $({}_H G) \to ({}_H G {}_H G) \to ({}_H G)$ for the left hand side of the first equation in (21) is given by

$$g \mapsto \sum_{i=1}^{m} g g_i^{-1} \otimes g_i \mapsto \sum_{i=1}^{m} p_H(g g_i^{-1}) g_i.$$

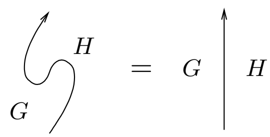
Notice that $p_H(g g_i^{-1}) = 0$ iff $g \notin H g_i$, and $p_H(h g_i g_i^{-1}) = h$. Therefore, $g \mapsto g$ under the map, and the first equation in (21) holds:



The bimodule map $(G_H) \to (G_H G_H) \to (G_H)$ for the left hand side of the second equation in (21) is given by

$$g \mapsto \sum_{i=1}^{m} g_i^{-1} \otimes g_i g \mapsto \sum_{i=1}^{m} g_i^{-1} p(g_i g) = g$$

by a similar computation, so that



and the four bimodule maps above determine biadjointness morphisms for induction and restriction functors between H and G.

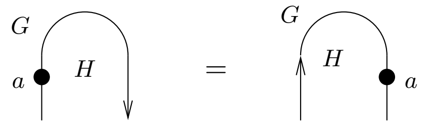
Consider the $\Bbbk$-algebra

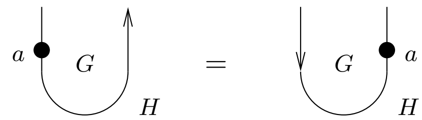
$$\Bbbk G^H := \{a \in \Bbbk G \mid ha = ah \ \forall h \in H\}$$

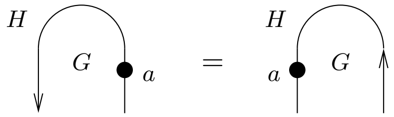
of H-invariants in $\Bbbk G$ with respect to the conjugation action. This algebra is canonically isomorphic to the endomorphism ring of the bimodule $({}_H G)$, and therefore to the endomorphism ring of the functor Res^H_G, via the map that assigns to $a \in \Bbbk G^H$ the endomorphism $'a(x) := ax$, where $x \in ({}_H G)$. Likewise, the opposite algebra of $\Bbbk G^H$ is canonically isomorphic to the endomorphism ring of the bimodule (G_H), and therefore to that of the functor Ind^G_H, via the map that assigns to $a \in \Bbbk G^H$ the endomorphism $a'(x) := xa$, where $x \in (G_H)$.

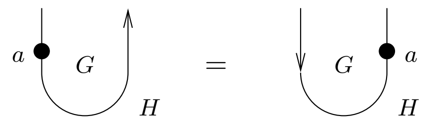
Thus, to $a \in \Bbbk G^H$ we assign endomorphisms a' and $'a$ of Ind^G_H and Res^H_G and depict them by

Lemma 1. *For any $a \in \Bbbk G^H$ there are equalities of bimodule homomorphisms*

(23)  $\alpha^G_H(a' \circ \mathrm{Id}) = \alpha^G_H(\mathrm{Id} \circ {'a})$,

(24)  $({'a} \circ \mathrm{Id})\beta^G_H = (\mathrm{Id} \circ a')\beta^G_H$,

(25)  $\overline{\alpha}^H_G({'a} \circ \mathrm{Id}) = \overline{\alpha}^H_G(\mathrm{Id} \circ a')$,

(26)  $(a' \circ \mathrm{Id})\overline{\beta}^H_G = (\mathrm{Id} \circ {'a})\overline{\beta}^H_G$.

The left hand side of the first equality is a map $(G_H G) \to (G)$ given by $g_1 \otimes g_2 \mapsto g_1 a \otimes g_2 \mapsto g_1 a g_2$. The right hand side is $g_1 \otimes g_2 \mapsto g_1 \otimes a g_2 \mapsto g_1 a g_2$, and the equality is obvious.

The second equality follows from an equally trivial computation.

The third equality is the equation $p_H(ga) = p_H(ag)$ for $g \in G$ and $a \in \Bbbk G^H$. It suffices to check it when $\Bbbk = \mathbb{Z}$ and $a = \sum_{h \in H} hkh^{-1}$ for some

$k \in G$. The equation becomes

$$\sum_{h\in H} p_H(ghkh^{-1}) = \sum_{h\in H} p_H(hkh^{-1}g). \tag{27}$$

The left hand side equals

$$\sum_{h\in H} p_H(ghk)h^{-1} = \sum_{h^{-1}g^{-1}u=k} p_H(u)p_H(h^{-1}) = \sum_{hg^{-1}u=k} p_H(u)p_H(h),$$

where in the first equality we set $u = ghk$, the sum being over all $u, h \in G$ with $h^{-1}g^{-1}u = k$. For the second equality, we converted h to h^{-1}.

The right hand side of (27) equals

$$\sum_{h\in H} hp_H(kh^{-1}g) = \sum_{ug^{-1}h=k} p_H(h)p_H(u),$$

where we set $u = kh^{-1}g$ and the sum is over all $u, h \in G$. Interchanging h and u, we see that (27) holds.

For the last of the four equations, it suffices to check that the image of $1 \in G$ is the same under these two bimodule homomorphisms. For the one on the left,

$$1 \mapsto \sum_{i=1}^{m} g_i^{-1} \otimes g_i \mapsto \sum_{i=1}^{m} g_i^{-1} \otimes ag_i.$$

For the one on the right,

$$1 \mapsto \sum_{i=1}^{m} g_i^{-1} \otimes g_i \mapsto \sum_{i=1}^{m} g_i^{-1}a \otimes g_i.$$

Again, we can assume $\Bbbk = \mathbb{Z}$ and $a = \sum_{h\in H} hkh^{-1}$ for some $k \in G$. The equation becomes

$$\sum_{i,h} g_i^{-1} \otimes hkh^{-1}g_i = \sum_{i,h} g_i^{-1}hkh^{-1} \otimes g_i, \tag{28}$$

with sum over $1 \le i \le m$ and $h \in H$. We have

$$\sum_{i,h} g_i^{-1} \otimes hkh^{-1}g_i = \sum_{i,h} g_i^{-1}h \otimes kh^{-1}g_i = \sum_{u\in G} u \otimes ku^{-1},$$

where, in the first equality, h is moved to the left (the tensor product is over $\Bbbk H$), and in the second equality, $u = g_i^{-1}h$ runs over all elements of G as i changes from 1 to m and h runs over all elements of H. Likewise,

$$\sum_{i,h} g_i^{-1}hkh^{-1} \otimes g_i = \sum_{i,h} g_i^{-1}hk \otimes h^{-1}g_i = \sum_{u\in G} uk \otimes u^{-1}.$$

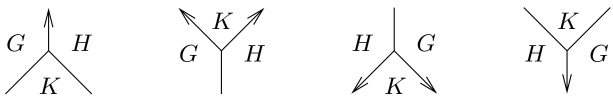
Equation (28) and Lemma 1 follow. ∎

Since the box labelled by $a \in \Bbbk G^H$ can be dragged through any U-turn, we see that the biadjointness maps have the cyclic property—dragging the box labelled a all the way along a circle brings us back to the original diagram. This concludes the proof of Theorem 2. ∎

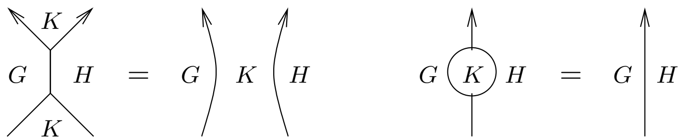
There are obvious simplification relations

The first relation says that a counterclockwise bubble with G inside and H outside can be erased. The second relation allows us to remove a clockwise bubble at the cost of multiplying the diagram by the index of H in G.

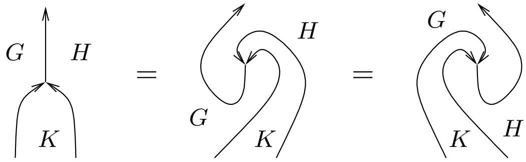
For each inclusion of finite groups $H \subset K \subset G$ there is a canonical isomorphism between the induction functors $\mathrm{Ind}_H^G \cong \mathrm{Ind}_K^G \circ \mathrm{Ind}_H^K$ which corresponds to the canonical isomorphism of bimodules $(G)_H \cong (G)_K(K)_H$. Likewise, the canonical isomorphism between restrictions $\mathrm{Res}_G^H \cong \mathrm{Res}_K^H \circ \mathrm{Res}_G^K$ is given by the natural isomorphism of bimodules ${}_H(G) \cong {}_H(K)_K(G)$. We draw these isomorphisms via trivalent diagrams



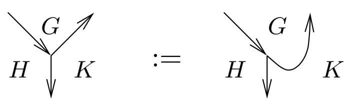
Since the isomorphisms are mutually inverse, we have, for the first two isomorphisms,



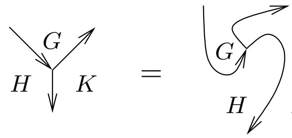
Mutual inversion of the other two isomorphisms can be similarly depicted. These definitions are compatible with isotopies—we have the identities



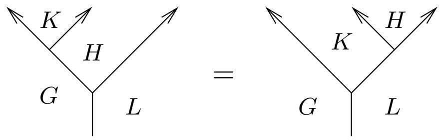
(likewise for the other pair of isomorphisms). These identities imply that various definitions of trivalent vertices in other positions relative to the y-coordinate are all the same. For instance, if we define



then



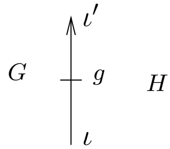
The associativity relation for the inclusions of four groups $L \subset H \subset K \subset G$ has the form



The induction and restriction functors for H and G depend on the inclusion $\iota : H \hookrightarrow G$. Conjugating the inclusion by an element $g \in G$, so that $\iota'(h) = ghg^{-1}$, $\iota' : H \hookrightarrow G$, leads to induction and restriction functors isomorphic to the original ones, via bimodule maps

$$(G)_{\iota(H)} \to (G)_{\iota'(H)}, \quad f \mapsto fg^{-1}, \quad f \in G,$$
$$_{\iota(H)}(G) \to {}_{\iota'(H)}(G), \quad f \mapsto gf, \quad f \in G.$$

We depict these conjugation isomorphisms via a mark on a line with g next to it:



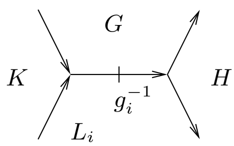
The Mackey induction-restriction theorem says that, given subgroups H, K of a finite group G, there is an isomorphism

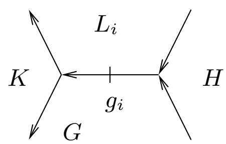
$$\mathrm{Res}^K_G \circ \mathrm{Ind}^G_H \cong \bigoplus_{i \in I} \mathrm{Ind}^K_{K \cap g_i H g_i^{-1}} \circ \mathrm{Res}^{K \cap g_i H g_i^{-1}}_H, \tag{29}$$

where the sum is over representatives g_i of (K, H)-cosets of G,

$$G = \bigsqcup_{i \in I} K g_i H.$$

Let $L_i = K \cap g_i H g_i^{-1}$. The diagrams





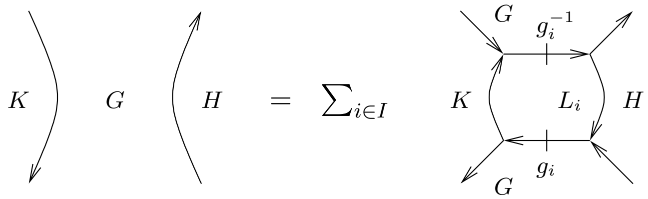
define (K, H)-bimodule maps

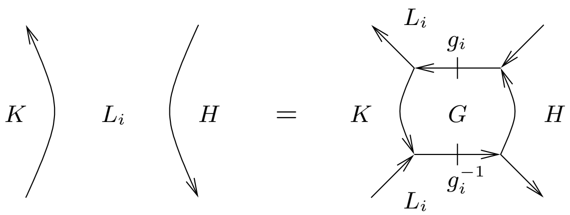
$$\alpha_i : (K)_{L_i}{}'(H) \to {}_K(G)_H, \qquad \beta_i : {}_K(G)_H \to (K)_{L_i}{}'(H).$$

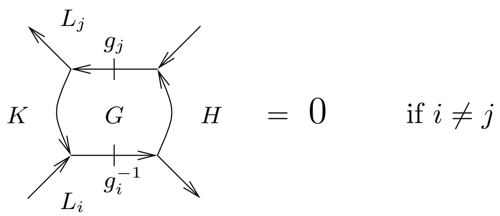
Here $(K)_{L_i}{}'(H)$ is $\Bbbk[K] \otimes_{\Bbbk[L_i]} \Bbbk[H]$, with $x \in L_i$ acting on H by right multiplication by $g_i^{-1} x g_i$.

Proposition 6. *The maps* $\sum_{i \in I} \alpha_i$ *and* $\sum_{i \in I} \beta_i$ *are mutually inverse isomorphisms of the bimodules* $\bigoplus_{i \in I} (K)_{L_i}{}'(H)$ *and* ${}_K(G)_H$.

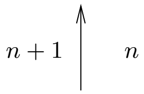
This proposition is a pictorial restatement of the Mackey theorem. The proof is left to the reader and amounts to checking the following relations:







3.3. Induction and restriction between symmetric groups. We now specialize the earlier construction to the case of the symmetric group S_n, viewed as the permutation group of $\{1, \ldots, n\}$, and induction/restriction functors for inclusions $S_n \subset S_{n+1}$, where S_n is identified with the stabilizer of $n+1$ in S_{n+1}. Notations for bimodules will be further simplified, so that, for instance, ${}_n(n+1)_{n-1}$ stands for $\Bbbk[S_{n+1}]$, viewed as a $(\Bbbk[S_n], \Bbbk[S_{n-1}])$-bimodule for the standard inclusions $S_n \subset S_{n+1} \supset S_{n-1}$, and ${}_n(n{+}1)_n(n{+}2)$ stands for $\Bbbk[S_{n+1}] \otimes_{\Bbbk[S_n]} \Bbbk[S_{n+2}]$, viewed as a $(\Bbbk[S_n], \Bbbk[S_{n+2}])$-bimodule. The regions of the strip $\mathbb{R} \times [0, 1]$ are now labelled by nonnegative integers n. An upward-oriented line separating regions labelled n and $n+1$:

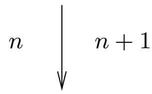


denotes the identity endomorphism of the induction functor

$$\mathrm{Ind}_n^{n+1} \ : \ \Bbbk[S_n]\text{-mod} \to \Bbbk[S_{n+1}]\text{-mod}.$$

This is the functor of tensoring with the bimodule $(n+1)_n$.

A downward-oriented line separating regions $n+1$ and n:

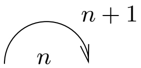


denotes the identity endomorphism of the restriction functor

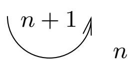
$$\mathrm{Res}_{n+1}^{n} : \Bbbk[S_{n+1}]\text{-mod} \to \Bbbk[S_n]\text{-mod}.$$

The bimodule for this functor is ${}_n(n+1)$.

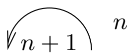
The four U-turns are given by the following bimodule maps:

(30) 

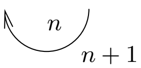
$(n+1)_n(n+1) \to (n+1), \quad g \otimes h \mapsto gh, \ g, h \in S_{n+1},$

(31) 

$(n) \to {}_n(n+1)_n, \quad g \mapsto g, \ g \in S_n$

(32) 

$p_n : {}_n(n+1)_n \to (n), \quad p_n(g) = \begin{cases} g & \text{if } g \in S_n, \\ 0 & \text{otherwise,} \end{cases}$

(33) 

$q_n : (n+1) \to (n+1)_n(n+1),$

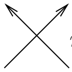
where the bimodule map q_n is determined by the condition

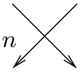
$$q_n(1) = \sum_{i=1}^{n+1} s_i s_{i+1} \dots s_n \otimes s_n \dots s_{i+1} s_i, \qquad s_i = (i, i+1).$$

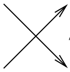
Notice that $\{s_n \dots s_2 s_1, s_n \dots s_3 s_2, \dots, s_n s_{n-1}, s_n, 1\}$ are $n+1$ coset representatives of $S_n \subset S_{n+1}$, and

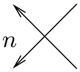
$$\begin{aligned} q_n(g) &= \sum_{i=1}^{n+1} g s_i s_{i+1} \dots s_n \otimes s_n \dots s_{i+1} s_i \\ &= \sum_{i=1}^{n+1} s_i s_{i+1} \dots s_n \otimes s_n \dots s_{i+1} s_i g, \qquad g \in S_{n+1}. \end{aligned}$$

The bimodule maps p_n and q_n are the second adjointness maps for the group inclusion $S_n \subset S_{n+1}$. For an arbitrary inclusion of finite groups $H \subset G$ these maps were described in the previous subsection.

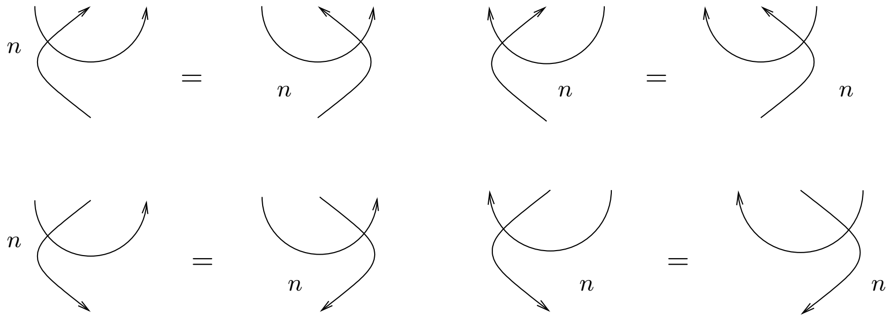
Denote by an upward-pointing crossing  n the endomorphism of $(n+2)_n$ given by right multiplication by s_{n+1}, so that $g \mapsto gs_{n+1}$, $g \in S_{n+2}$.

Denote by a downward-pointing crossing n  the endomorphism of ${}_n(n+2)$ given by left multiplication by s_{n+1}, so that $g \mapsto s_{n+1}g$, $g \in S_{n+2}$.

Denote by a right-pointing crossing  n the bimodule endomorphism $(n)_{n-1}(n) \ \rightarrow \ {}_n(n+1)_n$ that takes $g \otimes h$ for $g, h \in S_n$ to $gs_nh \in S_{n+1}$.

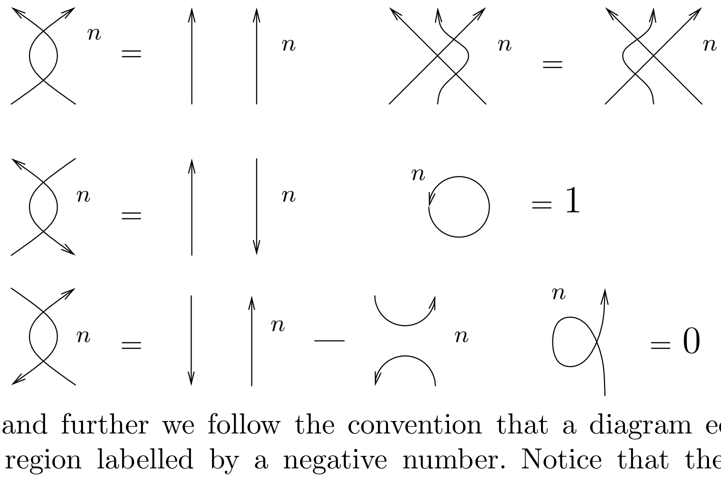
Denote by a left-pointing crossing n  the bimodule endomorphism ${}_n(n+1)_n \ \rightarrow \ (n)_{n-1}(n)$ that takes $g \in S_n \subset S_{n+1}$ to 0 and gs_nh for $g, h \in S_n$ to $g \otimes h \in (n)_{n-1}(n)$.

These four definitions-notations are compatible with the isotopies of diagrams in the plane strip—there are equalities of bimodule endomorphisms



They can be checked by direct computations.

Proposition 7. *The following relations hold for any $n \in \mathbb{Z}$:*

(34) 

Here and further we follow the convention that a diagram equals 0 if it has a region labelled by a negative number. Notice that the top two relations in (34) come from the relations in the symmetric groups: $s_{n+1}^2 = 1$

and $s_{n+1}s_{n+2}s_{n+1} = s_{n+2}s_{n+1}s_{n+2}$, and follow at once from the definition of the bimodule homomorphism associated to an upward-pointing crossing. The four remaining relations encode the bimodule decomposition

$$_n(n+1)_n \cong (n)_{n-1}(n) \oplus (n)$$

giving an isomorphism

$$\mathrm{Res}^n_{n+1} \circ \mathrm{Ind}^{n+1}_n \cong \mathrm{Ind}^n_{n-1} \circ \mathrm{Res}^{n-1}_n \oplus \mathrm{Id}$$

of endofunctors in the category $\Bbbk[S_n]$-mod. This is a special case of the Mackey decomposition theorem which was given a diagrammatic interpretation in an earlier section. The relations in Proposition 7 are identical to the ones in the definition of the category $\mathcal{H}'$ (see Section 2.1).

Let $\mathcal{S}'$ be the category whose objects are compositions of induction and restriction functors between symmetric groups for standard embeddings $S_k \subset S_{k+1}$. The morphisms are natural transformations of functors (again, we work over a field $\Bbbk$ of characteristic 0). The category $\mathcal{S}'$ is the sum of categories $\mathcal{S}'_k$ over $k \geq 0$; in the latter the first induction or restriction starts from S_k. For instance, $\mathrm{Ind}^{k+2}_{k+1} \circ \mathrm{Ind}^{k+1}_k \circ \mathrm{Res}^k_{k+1} \circ \mathrm{Ind}^{k+1}_k$ is an object of $\mathcal{S}'_k$. Morphisms in $\mathcal{S}'_k$ are natural transformations of functors and can be identified with homomorphisms of associated bimodules.

Thus, for each $k \geq 0$, there is a functor $\mathcal{F}'_k : \mathcal{H}' \to \mathcal{S}'_k$ that takes Q_ϵ to the corresponding composition of induction and restriction functors. For instance,

$$\mathcal{F}'_k(Q_{++-+}) = \mathrm{Ind}^{k+2}_{k+1} \circ \mathrm{Ind}^{k+1}_k \circ \mathrm{Res}^k_{k+1} \circ \mathrm{Ind}^{k+1}_k.$$

If, for some m, the last m terms of ϵ contain at least $k+1$ more minuses than pluses, then $\mathcal{F}'_k(Q_\epsilon) = 0$. On morphisms, $\mathcal{F}'_k$ is defined as follows. It takes a diagram representing a morphism in $\mathcal{H}'$, labels the rightmost region of the diagram by k, and views the diagram as a natural transformation between compositions of induction and restriction functors. The functor $\mathcal{F}'_k$ is not monoidal, since $\mathcal{S}'_k$ does not have a monoidal structure matching that of $\mathcal{H}'$.

Let $\mathcal{S}$, respectively $\mathcal{S}_k$, be the Karoubi envelope of $\mathcal{S}'$, respectively $\mathcal{S}'_k$. The functor $\mathcal{F}'_k$ induces a functor on Karoubi envelopes $\mathcal{F}_k : \mathcal{H} \to \mathcal{S}_k$. We summarize relevant categories and functors below:

$$\mathcal{S} = \mathrm{Kar}(\mathcal{S}'), \quad \mathcal{S}_k = \mathrm{Kar}(\mathcal{S}'_k), \quad \mathcal{F}'_k : \mathcal{H}' \to \mathcal{S}'_k, \quad \mathcal{F}_k : \mathcal{H} \to \mathcal{S}_k.$$

$$\begin{array}{ccc} \mathcal{S}' & = & \bigoplus_{k\geq 0}\mathcal{S}'_k \\ {\scriptstyle \mathrm{Kar}}\downarrow & & \downarrow{\scriptstyle \mathrm{Kar}} \\ \mathcal{S} & = & \bigoplus_{k\geq 0}\mathcal{S}_k \end{array} \qquad \begin{array}{ccc} \mathcal{H}' & \xrightarrow{\mathcal{F}'_k} & \mathcal{S}'_k \\ {\scriptstyle \mathrm{Kar}}\downarrow & & \downarrow{\scriptstyle \mathrm{Kar}} \\ \mathcal{H} & \xrightarrow{\mathcal{F}_k} & \mathcal{S}_k \end{array}$$

The functor $\mathcal{F}_k$ induces a homomorphism of abelian groups

$$[\mathcal{F}_k] : K_0(\mathcal{H}) \to K_0(\mathcal{S}_k). \tag{35}$$

Notice that $K_0(\mathcal{H})$ is a ring, while $K_0(\mathcal{S}_k)$ is only an abelian group.

An object of $\mathcal{S}_k$ is a direct summand of a finite sum of compositions of induction and restriction functors that start with the category of $\Bbbk[S_k]$-modules, thus it takes any finite-dimensional $\Bbbk[S_k]$-module to a module over $\bigoplus_{m\geq 0} \Bbbk[S_m]$. Descending to Grothendieck groups, we obtain a homomorphism

$$\theta_k : K_0(\mathcal{S}_k) \to \mathrm{Hom}_{\mathbb{Z}}\Big(K_0(\Bbbk[S_k]), \bigoplus_{m\geq 0} K_0(\Bbbk[S_m])\Big).$$

From now on until the end of this paper we assume that $\mathrm{char}(\Bbbk) = 0$. Consider the composite homomorphism

$$\theta_k[\mathcal{F}_k] : K_0(\mathcal{H}) \to \mathrm{Hom}_{\mathbb{Z}}\Big(K_0(\Bbbk[S_k]), \bigoplus_{m\geq 0} K_0(\Bbbk[S_m])\Big).$$

This homomorphism takes $[Q_{+,\mu}]$ to a map that assigns to $[M] \in K_0(\Bbbk[S_k])$, for a $\Bbbk[S_k]$-module M, the symbol $[\mathrm{Ind}_{S_{|\mu|}\times S_k}^{S_{|\mu|+k}}(L_\mu \otimes M)]$ of the induced module over $\Bbbk[S_{|\mu|+k}]$. In other words, tensor M with L_μ, producing a module over $\Bbbk[S_{|\mu|}] \times S_k$, induce to $\Bbbk[S_{|\mu|+k}]$, then pass to the Grothendieck group.

Likewise, $\theta_k[\mathcal{F}_k]$ takes $[Q_{-,\lambda}]$ to the zero map if $|\lambda| > k$, and if $k \geq |\lambda|$, to the map that assigns to $[M]$ as above the symbol of the module

$$\mathrm{Hom}_{\Bbbk[S_{|\lambda|}]}(L_\lambda, M) \in \Bbbk[S_{k-|\lambda|}]\text{-mod}.$$

In other words, restrict M to being a module over the group algebra of $S_{|\lambda|} \times S_{k-|\lambda|} \subset S_k$, and form homs from the simple module L_λ over $S_{|\lambda|}$. The result is a representation of the symmetric group $S_{k-|\lambda|}$.

Now consider the composition

$$\theta_k[\mathcal{F}_k]\gamma : H_{\mathbb{Z}} \to \mathrm{Hom}_{\mathbb{Z}}\Big(K_0(\Bbbk[S_k]), \bigoplus_{m\geq 0} K_0(\Bbbk[S_m])\Big).$$

We claim that the sum of these maps, over all $k \geq 0$, is injective. Let

$$y = \sum_{\lambda,\mu} y_{\lambda,\mu} b_\mu a_\lambda, \qquad y_{\lambda,\mu} \in \mathbb{Z},$$

be an arbitrary nonzero element of $H_{\mathbb{Z}}$. We have

$$\gamma(y) = \sum_{\lambda,\mu} y_{\lambda,\mu} [Q_{+,\mu^*}][Q_{-,\lambda}].$$

When $|\lambda| = k$, the map

$$\theta_k[\mathcal{F}_k]\gamma(a_\lambda) = \theta_k[\mathcal{F}_k]([Q_{-,\lambda}])$$

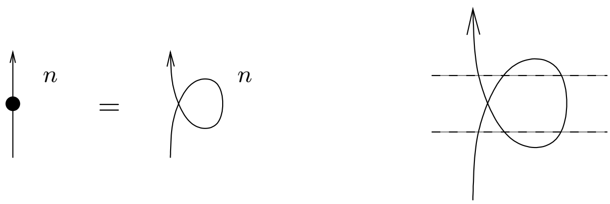
takes $[L_\nu]$ to 0 if $|\nu| = k$ and $\nu \neq \lambda$. The same map takes $[L_\lambda]$ to $[L_\emptyset]$, the symbol of the simple module over $\Bbbk[S_0] = \Bbbk$.

Choose k such that $y_{\lambda,\mu} \neq 0$ for some λ with $|\lambda| = k$ and some μ, while $y_{\lambda,\mu} = 0$ for all μ whenever $|\lambda| < k$. Also choose ν with $|\nu| = k$ and $y_{\nu,\mu} \neq 0$ for some μ. The map $\theta_k[\mathcal{F}_k]\gamma(y)$ takes $[L_\nu]$ to

$$\sum_{\mu} y_{\nu,\mu}[L_{\mu^*}] \neq 0.$$

Therefore, $\theta_k[\mathcal{F}_k]\gamma(y)$ is a nonzero map, and $\gamma(y) \neq 0$. This concludes the proof that γ is injective (Theorem 1).

4. The size of morphism spaces in $\mathcal{H}'$. In this section we will prove Propositions 3 and 4. Consider the right curl with the rightmost region labelled n (also recall the shorthand of denoting this curl by a dot). This curl can be realized as the composition of a cup with a crossing with a cap:



The corresponding endomorphism of the bimodule $(n+1)_n$ takes 1 to

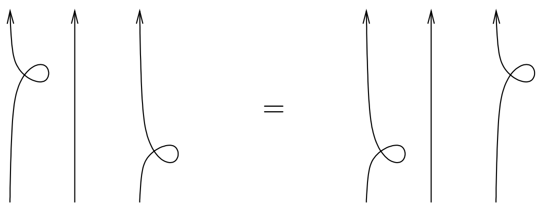
$$J_n := \sum_{i=1}^{n} s_i \dots s_{n-1} s_n s_{n-1} \dots s_i = (1, n+1) + (2, n+1) + \dots + (n, n+1).$$

This endomorphism of $(n+1)_n$ is right multiplication by J_n:

$$g \mapsto g J_n, \qquad g \in S_{n+1}.$$

Notice that J_n is the Jucys–Murphy element, ubiquitous in the representation theory of the symmetric group. Our diagrammatics realizes it via the right curl and interprets the endomorphism of multiplication by J_n as the composition of three natural transformations, two of which (cup and cap) come from biadjointness of the induction and restriction functors.

Commutativity of Jucys–Murphy elements now acquires a graphical interpretation as isotopies of curls (or dots) on upward-oriented disjoint strands past each other:



For each $n \geq 0$ the functor $\mathcal{F}'_n$, applied to $\mathbf{1}$ and its endomorphisms, produces a homomorphism from $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ to $Z(\Bbbk[S_n])$, the centre of the group ring of the symmetric group. Composing with the homomorphism ψ_0, we obtain the homomorphism

$$\psi_{0,n} : \Pi \cong \Bbbk[c_0, c_1, \dots] \to Z(\Bbbk[S_n])$$

that takes c_k to

$$\begin{aligned} c_{k,n} &= \sum_{i=1}^{n} s_i \dots s_{n-1} J_{n-1}^k s_{n-1} \dots s_i \\ &= \sum_{i=1}^{n} (i, i+1, \dots, n)\big((1,n) + \dots + (n-1,n)\big)^k (n, n-1, \dots, i). \end{aligned}$$

We would like to show that the union of $\psi_{0,n}$ over all n,

$$\Pi \to \bigoplus_{n \geq 0} Z(\Bbbk[S_n]),$$

is injective. For the first few values of k we have

$$c_{0,n} = n, \qquad c_{1,n} = 2 \sum_{1 \leq i < j \leq n} (i,j), \qquad c_{2,n} = 3 \sum (i_1, i_2, i_3) + n(n-1),$$

where the last sum is over all 3-cycles. In general,

$$c_{k,n} = (k+1) \sum (i_1, \dots, i_{k+1}) + \text{l.o.t.}, \tag{36}$$

where the sum is over all $k+1$-cycles. For $k > 0$, the lower order terms is a sum over permutations of disturbance less than $k+1$, where we define the *disturbance* of a permutation $\sigma \in S_n$ as the number of elements moved by σ:

$$\mathrm{dist}(\sigma) = |\{i \mid 1 \leq i \leq n,\, \sigma(i) \neq i\}|.$$

Notice that $c_{k,n}$ is the sum of conjugates of J_{n-1}^k and contains terms of disturbance at most $k+1$, for $k > 0$.

Since $\mathrm{char}(\Bbbk) = 0$, the coefficient of the sum of $k+1$-cycles in (36) is nonzero. Consider an increasing filtration

$$Z_0 = \Bbbk \subset Z_2 \subset Z_3 \subset \dots \subset Z_n = \mathbb{Z}(\Bbbk[S_n])$$

where Z_i is the span of conjugacy classes that consist of permutations of disturbance at most i. We make $\Bbbk[c_0, c_1, \dots]$ graded by $\deg(c_0) = 0$ and $\deg(c_k) = k+1$ for $k > 0$, and then consider the associated increasing filtration on $\Bbbk[c_0, c_1, \dots]$, where the ith filtered terms is the sum of graded terms of degree at most i. The homomorphism $\psi_{0,n}$ respects the two filtrations.

To show asymptotic faithfulness of $\psi_{0,n}$ as $n \to \infty$ we examine the induced homomorphism of adjoint graded rings. Assume that there is a universal relation

$$\sum_I a_I c_I = 0$$

in $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ which holds for all sufficiently large n. Here $I = (a_0, a_1, \dots, a_i)$ is a finite sequence of nonnegative integers, $a_I \in \Bbbk$, and $c_I = c_0^{a_0} c_1^{a_1} \dots c_i^{a_i}$ is the corresponding monomial in $\Bbbk[c_0, c_1, \dots]$. The sum is over finitely many sequences I. This implies the relations

$$\sum_I a_I c_{I,n} = 0$$

for all n, where $c_{I,n} = c_{0,n}^{a_0} c_{1,n}^{a_1} \dots c_{i,n}^{a_i}$. Choose any term from the sum corresponding to $I = (a_0, a_1, \dots, a_i)$ with the highest possible degree monomial c_I. The only terms in the sum $\sum_I a_I c_{I,n}$ that contribute to the conjugacy class of type $(a_1 + 1, \dots, a_i + 1)$ can come from sequences $I' = (a, a_1, \dots, a_i)$ that differ from I only in the first term. Since, for such I',

$$c_{I',n} = n^a c_{1,n}^{a_1} \dots c_{i,n}^{a_i}$$

the relation $\sum_I a_I c_{I,n} = 0$ implies $\sum_a a_{I'} n^a = 0$, which, in turn, leads to $a_{I'} = 0$ for all I' as above, since the only polynomial with infinitely many positive integers n as roots is the zero polynomial. This contradiction implies asymptotic faithfulness of $\psi_{0,n}$. Therefore, ψ_0 is injective, which concludes the proof of Proposition 3 when $\mathrm{char}(\Bbbk) = 0$. Take $\Bbbk = \mathbb{Q}$, then pass to the subring $\mathbb{Z}$ of $\mathbb{Q}$. Our arguments imply that ψ_0 is an isomorphism over $\mathbb{Z}$, and hence, over any commutative ring $\Bbbk$, including any field.

We next prove injectivity of ψ_m (Proposition 4) (see equation (16)), again first in the characteristic zero case. The algebra $\mathrm{DH}_m \otimes \Pi$ has a basis of elements $x_1^{a_1} \dots x_m^{a_m} \cdot \sigma \cdot c_0^{b_0} c_1^{b_1} \dots c_k^{b_k}$ over permutations $\sigma \in S_m$ and $a_i, b_j \in \mathbb{Z}_+$ (see discussion before Proposition 4).

Applying the functor $\mathcal{F}'_n$ to Q_{+^m} and its endomorphism ring gives us a homomorphism

$$\mathrm{End}_{\mathcal{H}'}(Q_{+^m}) \to \mathrm{End}((n+m)_n)$$

to the endomorphism ring of the $(\Bbbk[S_{n+m}], \Bbbk[S_n])$-bimodule $\Bbbk[S_{n+m}]$, which we also denote $(n+m)_n$. The composite homomorphism

$$\psi_{m,n} : \mathrm{DH}_m \otimes \Pi \to \mathrm{End}((n+m)_n)$$

takes elements of $\mathrm{DH}_m \otimes \Pi$ to endomorphisms given by right multiplication by suitable elements of $\Bbbk[S_{n+m}]$. Namely, $\psi_{m,n}(\sigma)$, for a permutation $\sigma \in S_m \subset \mathrm{DH}_m$, is right multiplication by $\underline{\sigma}$, where we define the latter as the translate of σ by n:

$$\underline{\sigma}(i+n) = \sigma(i) + n, \quad 1 \le i \le m, \qquad \underline{\sigma}(i) = i, \quad 1 \le i \le n.$$

The map $\psi_{m,n}(x_i)$, with x_i the diagram of m vertical lines with the dot (right curl) on the ith strand from the left, is right multiplication by J_{n+m-i}, and $\psi_{m,n}(c_k)$ is right multiplication by $c_{k,n}$. The map $\psi_{m,n}$ is described by the corresponding homomorphism $\psi'_{m,n}$ from $\mathrm{DH}_m \otimes \Pi$ to the opposite of the group algebra, $\Bbbk[S_{n+m}]^{\mathrm{op}} \supset \mathrm{End}((n+m)_n)$, with $\psi'_{m,n}(x_i) = J_{n+m-i}$, $\psi'_{m,n}(c_k) = c_{k,n}$, etc. We need to take the opposite algebra since the ring of endomorphisms of a ring A viewed as a left A-module is naturally isomorphic to the opposite of A: $\mathrm{End}_A({}_AA, {}_AA) \cong A^{\mathrm{op}}$.

Define the *m-disturbance* of a permutation $\sigma \in S_{n+m}$ as the number of integers between 1 and n that are moved by σ:

$$\mathrm{dist}_m(\sigma) = |\{i \mid 1 \le i \le n,\ \sigma(i) \ne i\}|.$$

Notice that $\mathrm{dist}_m(\sigma\tau) \le \mathrm{dist}_m(\sigma) + \mathrm{dist}_m(\tau)$. On the group algebra $\Bbbk[S_{n+m}]$ we can introduce an increasing filtration

$$\Bbbk[S_m] = Z_0^m \subset Z_1^m \subset \cdots \subset Z_n^m = \Bbbk[S_{n+m}]$$

where Z_k^m is spanned by all permutations of disturbance at most k.

We turn $\mathrm{DH}_m \otimes \Pi$ into a filtered algebra by setting $\deg(c_0) = 0$, $\deg(c_k) = k+1$ if $k > 0$, $\deg(x_i) = 1$ and $\deg(\sigma) = 0$, and then spanning the kth term in the increasing filtration by the basis elements of total degree at most k.

The homomorphism $\psi'_{m,n}$ is that of filtered algebras, and we can pass to a homomorphism of the associated graded algebras. To show asymptotic faithfulness of $\psi'_{m,n}$ we fix m and will be taking n large compared to m. Assume that there exists a relation

$$\sum d_{\sigma,\mathbf{a},\mathbf{b}} x_1^{a_1} \dots x_m^{a_m} \cdot \sigma \cdot c_0^{b_0} c_1^{b_1} \dots c_r^{b_r} = 0 \tag{37}$$

in $\mathrm{End}_{\mathcal{H}'}(Q_{+^m})$ for some $d_{\sigma,\mathbf{a},\mathbf{b}} \in \Bbbk \setminus \{0\}$, with $\mathbf{a} = (a_1, \dots, a_m)$, $\mathbf{b} = (b_1, \dots, b_r)$, the sum being over finitely many triples $(\sigma, \mathbf{a}, \mathbf{b})$. Let

$$x(\sigma, \mathbf{a}, \mathbf{b}) = x_1^{a_1} \dots x_m^{a_m} \cdot \sigma \cdot c_0^{b_0} c_1^{b_1} \dots c_r^{b_r}$$

denote the elements of our basis of $\mathrm{DH}_m \otimes \Pi$. The element $\psi'_{m,n}(x(\sigma, \mathbf{a}, \mathbf{b})) \in \Bbbk[S_{n+m}]$ belongs to the kth term of the filtration of the latter, where

$$k = a_1 + \dots + a_m + 2b_1 + 3b_2 + \dots + (r+1)b_r,$$

but not to the $(k-1)$st term. Among the terms $x(\sigma, \mathbf{a}, \mathbf{b})$ in the sum select only those with the maximal possible k (denote such k by k_0). It is enough to show that, as we sum over only those terms, the image of $\sum \psi'_{m,n}(x(\sigma, \mathbf{a}, \mathbf{b}))$ in the associated graded ring of $\Bbbk[S_{n+m}]$ relative to the above filtration is nonzero. In other words, we need to show that the coefficients of permutations of disturbance k_0 are not all zero for some sufficiently large n in the expansion of $\psi_{m,n}$ applied to the LHS of (37).

This is obtained by looking at the structure of these permutations. They are disjoint unions of cycles, with some of the cycles containing one or more elements of the set $P = \{n+1, \dots, n+m\}$. The relative positions of elements of P in the cycles, the lengths of the portions of the cycles between elements of P, together with the number of cycles of each length without elements of P uniquely determine σ, $a_1, \dots, a_m$ and $c_1, \dots, c_r$ that can contribute to the coefficient of the permutation. The coefficients at different powers of c_0 are taken care of in the same way as in the $m = 0$ case. Linear independence of our spanning set of $\mathrm{End}_{\mathcal{H}'}(Q_{+^m})$ and Proposition 4 follow when $\mathrm{char}(\Bbbk) = 0$. The same argument as in the $m = 0$ case then implies that ψ_m is an isomorphism over any commutative ring $\Bbbk$.

The formula (36) implies that the natural homomorphism $\mathrm{End}_{\mathcal{H}'}(\mathbf{1}) \to Z(\Bbbk[S_n])$ from the endomorphisms of the identity object of $\mathcal{H}'$ to the center of the group algebra is surjective when the field $\Bbbk$ has characteristic 0. Combining this with the result of Cherednik [10] and Olshanski [36], [8, Theorem 3.2.6] that the centralizer algebra of $\Bbbk[S_n]$ in $\Bbbk[S_{n+m}]$ is generated by DH_m and the center of $\Bbbk[S_n]$, we deduce that the homomorphism $\mathrm{DH}_m \otimes \Pi \to \mathrm{End}((n+m)_n)$ introduced above is surjective when $\mathrm{char}(\Bbbk) = 0$.

5. Remarks on the Grothendieck ring of $\mathcal{H}$

5.1. Idempotented rings from $\mathcal{H}'$. For a sequence ϵ of pluses and minuses denote by $\langle \epsilon \rangle$ the difference between the number of pluses and minuses in ϵ (the weight of ϵ). Then $\mathrm{Hom}_{\mathcal{H}'}(Q_\epsilon, Q_{\epsilon'}) = 0$ if and only if $\langle \epsilon \rangle \neq \langle \epsilon' \rangle$. The "if" part of this observation implies that $\mathcal{H}'$ and $\mathcal{H}$, viewed as additive categories, decompose into the direct sums of subcategories

$$\mathcal{H}' = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{H}'_\ell, \qquad \mathcal{H} = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{H}_\ell,$$

where $\mathcal{H}'_\ell$ is a full subcategory of $\mathcal{H}'$ which contains objects Q_ϵ over all sequences of weight ℓ, and $\mathcal{H}_\ell$ is the Karoubi envelope of $\mathcal{H}'_\ell$.

This direct sum decomposition induces a grading on the Grothendieck ring,

$$K_0(\mathcal{H}) = \bigoplus_{\ell \in \mathbb{Z}} K_0(\mathcal{H}_\ell).$$

The Heisenberg algebra H and its integral form $H_{\mathbb{Z}}$ are graded by $\deg(a_n) = n = -\deg(b_n)$, and the homomorphism $\gamma : H_{\mathbb{Z}} \to K_0(\mathcal{H})$ is that of graded rings.

We can redefine the Grothendieck groups $K_0(\mathcal{H})$ and $K_0(\mathcal{H}_\ell)$ via idempotented rings. For a sequence ϵ let

$$\mathrm{End}(\epsilon) := \mathrm{End}_{\mathcal{H}}(Q_\epsilon)$$

denote the endomorphism algebra of Q_ϵ. Likewise, denote

$$\mathrm{Hom}(\epsilon, \epsilon') := \mathrm{Hom}_{\mathcal{H}}(Q_\epsilon, Q_{\epsilon'}).$$

To the category $\mathcal{H}$ we can assign the idempotented ring of all homomorphisms between various tensor products of generating objects Q_+ and Q_-:

$$\underline{R} := \bigoplus_{\epsilon,\epsilon'} \mathrm{Hom}(\epsilon,\epsilon'),$$

the sum being over all sequences ϵ, ϵ'. The ring $\underline{R}$ is nonunital, but has a family of idempotents $1_\epsilon = 1 \in \mathrm{End}(\epsilon)$.

The right projective $\underline{R}$-modules $1_\epsilon \underline{R}$ correspond to the objects Q_ϵ, in the sense that

$$\mathrm{Hom}_{\underline{R}}(1_\epsilon \underline{R}, 1_{\epsilon'}\underline{R}) = \mathrm{Hom}(\epsilon,\epsilon') = \mathrm{Hom}_{\mathcal{H}'}(Q_\epsilon, Q_{\epsilon'}), \tag{38}$$

and the Grothendieck group of finitely generated projective right $\underline{R}$-modules is canonically isomorphic to the Grothendieck group of $\mathcal{H}$:

$$K_0(\underline{R}) \cong K_0(\mathcal{H}).$$

This isomorphism takes $[1_\epsilon \underline{R}]$ to $[Q_\epsilon]$. Usually we use $K_0(A)$ to denote the Grothendieck group of finitely generated projective left, not right, A-modules. Here, because of (38), we use right $\underline{R}$-modules in the definition of $K_0(\underline{R})$. Alternatively, we could use left $\underline{R}^{\mathrm{op}}$-modules, or even left $\underline{R}$-modules after fixing an isomorphism $\underline{R} \cong \underline{R}^{\mathrm{op}}$ (the involution ξ_2 induces such an isomorphism). We have

$$\underline{R} = \bigoplus_{\ell\in\mathbb{Z}} \underline{R}_\ell, \qquad \underline{R}_\ell := \bigoplus_{\langle\epsilon\rangle=\langle\epsilon'\rangle=\ell} \mathrm{Hom}(\epsilon,\epsilon').$$

Assume from now on that $\ell \geq 0$ (the other case can be treated similarly, or by applying the symmetry ξ_3 to reverse ℓ). Given a sequence ϵ with $n+\ell$ pluses and n minuses, the object Q_ϵ of $\mathcal{H}_\ell$ decomposes into the direct sum of objects $Q_{+^{k+\ell}-^k}$ with $0 \leq k \leq n$ with some multiplicities. Hence, $\underline{R}_\ell$ is Morita equivalent to the idempotented ring

$$R_\ell = \bigoplus_{k,k'=0}^{\infty} \mathrm{Hom}(+^{k+\ell}-^k, +^{k'+\ell}-^{k'}),$$

and the inclusion $R_\ell \subset \underline{R}_\ell$ induces an isomorphism of Grothendieck groups $K_0(R_\ell) \cong K_0(\underline{R}_\ell)$. Let

$$R_{\ell,m} = \bigoplus_{k,k'=0}^{m} \mathrm{Hom}(+^{k+\ell}-^k, +^{k'+\ell}-^{k'}).$$

The ring R_ℓ is the union of rings in the increasing chain $R_{\ell,0} \subset R_{\ell,1} \subset \cdots$. Formation of Grothendieck group commutes with direct limits, implying that $K_0(R_\ell)$ is the direct limit of $K_0(R_{\ell,m})$ as m goes to infinity. Thus, there is an isomorphism

$$K_0(\mathcal{H}_\ell) \cong \lim_{m\to\infty} K_0(R_{\ell,m}).$$

For each k between 0 and m, the natural inclusion of rings $\mathrm{End}(+^{k+\ell}-^{k}) \subset R_{\ell,m}$ induces a homomorphism of groups $K_0(\mathrm{End}(+^{k+\ell}-^{k})) \to K_0(R_{\ell,m})$. Conjecture 1 would follow from the following two conjectures:

Conjecture 1.1. *The standard inclusion* $\Bbbk[S_n \times S_m] \subset \mathrm{End}(+^{n}-^{m})$ *induces an isomorphism of Grothendieck groups of these two rings.*

Conjecture 1.2. *The ring inclusion*

$$\bigoplus_{k=0}^{m} \mathrm{End}(+^{k+\ell}-^{k}) \subset R_{\ell,m}$$

induces an isomorphism on Grothendieck groups.

We do not know how to prove either statement, but will now present some weak evidence in favor of Conjecture 1.1.

5.2. Grothendieck group of a degenerate affine Hecke algebra. Here we prove Conjecture 5.1 in the case $m = 0$ (the $n = 0$ case follows by symmetry). By Proposition 4, the endomorphism ring of the object Q_{+^n} of $\mathcal{H}$ is isomorphic to the tensor product of the degenerate affine Hecke algebra DH_n and the polynomial algebra Π:

$$\mathrm{End}(+^{n}) \cong \mathrm{DH}_n \otimes \Pi.$$

The inclusion $\Bbbk[S_n] \hookrightarrow \mathrm{DH}_n$ is split, via the homomorphism $\tau_n : \mathrm{DH}_n \to \Bbbk[S_n]$ which takes the generators s_i of DH_i to the transpositions $(i, i+1)$ and the generators x_i to the Jucys–Murphy elements. The split inclusion induces a split short exact sequence of two rings and an ideal

$$0 \to \ker(\tau_n) \to \mathrm{DH}_n \to \Bbbk[S_n] \to 0,$$

which, in turn, induces a split short exact sequence of K_0-groups

$$0 \to K_0(\ker(\tau_n)) \to K_0(\mathrm{DH}_n) \to K_0(\Bbbk[S_n]) \to 0,$$

(see [39, 41]). Introduce an increasing filtration

$$0 = Z_{-1}\,\mathrm{DH}_n \subset Z_0\,\mathrm{DH}_n \subset Z_1\,\mathrm{DH}_n \subset \cdots$$

on DH_n, where $Z_k\,\mathrm{DH}_n$ is spanned by elements of the form $x_1^{a_1} \dots x_n^{a_n}\sigma$ over all $\sigma \in S_n$ and $a_1 + \cdots + a_n \le k$. Then $Z_k\,\mathrm{DH}_n \times Z_m\,\mathrm{DH}_n \subset Z_{k+m}\,\mathrm{DH}_n$ and $Z_0\,\mathrm{DH}_n = \Bbbk[S_n]$. Let $B = \mathrm{gr}\,\mathrm{DH}_n$ with respect to this filtration. B is a graded algebra isomorphic to the cross-product of the polynomial algebra on n generators with the group algebra of the symmetric group, $B \cong \Bbbk[x_1, \dots, x_n] * \Bbbk[S_n]$.

The algebra B is Koszul, with the Koszul dual algebra isomorphic to the cross-product of the exterior algebra on n generators with the group algebra of the symmetric group (recall that $\mathrm{char}(\Bbbk) = 0$). Hence, B has finite Tor dimension and, in particular, $Z_0\,\mathrm{DH}_n = \Bbbk[S_n]$ has Tor dimension n as a right B-module. Furthermore, B has Tor dimension 0 as a right

$Z_0\,\mathrm{DH}_n$-module, since $\Bbbk[S_n]$ is semisimple. We are in a position to invoke Quillen's theorem [37, Theorem 7, p. 112], [5].

THEOREM. *Let A be a ring equipped with an increasing filtration Z_kA, and such that Z_0A is regular. Suppose that $B = \operatorname{gr} A$ has finite Tor dimension as a right Z_0A-module and that Z_0A has finite Tor dimension as a right B-module. Then the inclusion $Z_0A \subset A$ induces an isomorphism $K_i(Z_0A) \cong K_i(A)$.*

In our case $A = \mathrm{DH}_n$. The regularity of $Z_0\,\mathrm{DH}_n$ is obvious due to it being semisimple (a *regular* ring is a noetherian ring such that every left module has finite projective dimension). Applying the theorem in the $i = 0$ case we obtain

PROPOSITION 8. *The inclusion $\Bbbk[S_n] \subset \mathrm{DH}_n$ and the surjection $\mathrm{DH}_n \to \Bbbk[S_n]$ induce mutually inverse isomorphisms $K_0(\Bbbk[S_n]) \cong K_0(\mathrm{DH}_n)$.*

The same argument shows that the inclusion

$$\Bbbk[S_n] \subset \mathrm{DH}_n \otimes \Bbbk[c_0, \dots, c_r]$$

induces an isomorphism of Grothendieck groups

$$K_0(\Bbbk[S_n]) \cong K_0(\mathrm{DH}_n \otimes \Bbbk[c_0, \dots, c_r]).$$

Formation of Grothendieck groups commutes with taking direct limits of rings [39, Section 1.2], and Π is the limit of $\Bbbk[c_0, \dots, c_r]$ as $r \to \infty$. We obtain an isomorphism

$$K_0(\Bbbk[S_n]) \cong K_0(\mathrm{End}(+^n))$$

proving Conjecture 1.1 when $m = 0$. ∎

The induction and restriction functors for the inclusions

$$\Bbbk[S_n] \otimes \Bbbk[S_m] \subset \Bbbk[S_{n+m}], \qquad \mathrm{DH}_n \otimes \mathrm{DH}_m \subset \mathrm{DH}_{n+m}$$

induce "multiplication" and "comultiplication" maps on the Grothendieck groups that turn

$$\bigoplus_{n\geq 0} K_0(\Bbbk[S_n]) \quad \text{and} \quad \bigoplus_{n\geq 0} K_0(\mathrm{DH}_n)$$

into graded birings (see Geissinger [17] for the symmetric group, Zelevinsky [42] for semisimple generalizations, and Bergeron and Li [3], Khovanov and Lauda [23] for nonsemisimple ones). We write *birings* rather than *bialgebras*, since these K_0 groups are abelian groups rather than vector spaces over some field. Isomorphisms in the above proposition are compatible with multiplication and comultiplication, and induce a biring isomorphism

$$\bigoplus_{n\geq 0} K_0(\mathrm{DH}_n) \cong \bigoplus_{n\geq 0} K_0(\Bbbk[S_n]).$$

The second biring can be canonically identified [17, 42] with an integral form Sym of the ring of symmetric functions in infinitely many variables.

What can we say about K_0 of $\mathrm{End}(+^n-^m)$ when $n, m > 0$? Recall (end of Section 2.1) that $\mathrm{End}(+^n-^m)$ contains the 2-sided ideal $J_{n,m}$ spanned by the basis elements of thickness less than $n+m$, which fits into the split short exact sequence

$$0 \to J_{n,m} \to \mathrm{End}(+^n-^m) \to \mathrm{DH}_{n,m} \to 0, \tag{39}$$

where

$$\mathrm{DH}_{n,m} := \mathrm{DH}_n \otimes \mathrm{DH}_m \otimes \Pi$$

is the tensor product of two degenerate AHA with the polynomial algebra in infinitely many generators. The earlier argument via Quillen's theorem shows that the inclusion

$$\Bbbk[S_n] \otimes \Bbbk[S_m] \to \mathrm{DH}_n \otimes \mathrm{DH}_m \otimes \Bbbk[c_0, c_1, \dots, c_r]$$

induces an isomorphism of K_0-groups

$$K_0(\Bbbk[S_n] \otimes \Bbbk[S_m]) \cong K_0(\mathrm{DH}_n \otimes \mathrm{DH}_m \otimes \Bbbk[c_0, c_1, \dots, c_r]).$$

Passing to the limit as $r \to \infty$, the inclusion

$$\Bbbk[S_n] \otimes \Bbbk[S_m] \to \mathrm{DH}_{n,m}$$

induces an isomorphism of Grothendieck groups

$$K_0(\Bbbk[S_n] \otimes \Bbbk[S_m]) \cong K_0(\mathrm{DH}_{n,m}). \tag{40}$$

The split short exact sequence (39) gives rise to a split short exact sequence

$$0 \to K_0(J_{n,m}) \to K_0(\mathrm{End}(+^n-^m)) \to K_0(\mathrm{DH}_{n,m}) \to 0$$

(for the definition of the K-group of a 2-sided ideal see [39, 41]) and canonical decomposition

$$K_0(\mathrm{End}(+^n-^m)) \cong K_0(\mathrm{DH}_{n,m}) \oplus K_0(J_{n,m}).$$

Conjecture 1.1 is equivalent to the vanishing of $K_0(J_{n,m})$ for all n, m.

Acknowledgements. On a number of occasions Igor Frenkel emphasized to the author the importance of categorifying various structures related to the symmetric functions and Heisenberg algebras. More recently, Tony Licata asked directly if there exists a categorification of the Heisenberg algebra, and this paper was inspired by his question. I am grateful to Marco Mackaay and Tony Licata for pointing out an error in the computations preceding formula (7) in the original version. The author would like to acknowledge partial support by the National Science Foundation via grants DMS-0706924 and DMS-0739392.

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Received 9 June 2013;
in revised form 29 July 2013