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Link invariants from finite racks

by

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Abstract. We define ambient isotopy invariants of oriented knots and links using the counting invariants of framed links defined by finite racks. These invariants reduce to the usual quandle counting invariant when the rack in question is a quandle. We are able to further enhance these counting invariants with 2-cocycles from the coloring rack's second rack cohomology satisfying a new degeneracy condition which reduces to the usual case for quandles.

1. Introduction. A *rack* is a generally non-associative algebraic structure whose axioms correspond to blackboard-framed isotopy moves on link diagrams. Racks generalize *quandles*, an algebraic structure whose axioms correspond to the three Reidemeister moves which combinatorially encode ambient isotopy of knot diagrams.

Given a finite quandle T, the set of quandle homomorphisms from a knot quandle $Q(K)$ to T gives us an easily computed knot invariant, namely its cardinality $|\mathrm{Hom}(Q(K), T)|$. This is the *quandle counting invariant*, also sometimes called the *quandle coloring invariant* since each homomorphism $f : Q(K) \to T$ can be pictured as a "coloring" of the knot diagram assigning to each arc x_i in a knot diagram the element $f(x_i) \in T$ such that a quandle coloring condition is satisfied at every crossing. Indeed, Fox 3-coloring is the simplest non-trivial example of a quandle coloring invariant for knots.

If T is a non-quandle rack, the set of colorings of arcs of a link diagram by elements of T satisfying the coloring condition at every crossing is invariant only under blackboard-framed isotopy—type I Reidemeister moves which change the framing of the knot also change the number of colorings. In this paper we will exploit a property of finite coloring racks to define computable invariants of ambient isotopy of knots and links incorporating these framed isotopy coloring invariants. The usual quandle coloring invariants then form a special case of these more general rack coloring invariants.

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The paper is organized as follows. In Section 2 we review the basics of racks, framed links and virtual links. In Section 3 we define finite rack based counting invariants and give some examples. In particular, we show that the writhe-enhanced invariant specializes to the integral invariant and contains more information. In Section 4 we enhance the rack counting invariants with 2-cocycles in the style of [3]. We provide an example showing that the cocycle-enhanced invariant contains more information than the writhe-enhanced and integral rack counting invariants. In Section 5 we collect questions for future research.

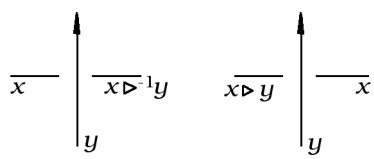
2. Basic definitions. In this section we review the basic definitions we will need for the remainder of the paper.

2.1. Racks. We begin with a definition from [6].

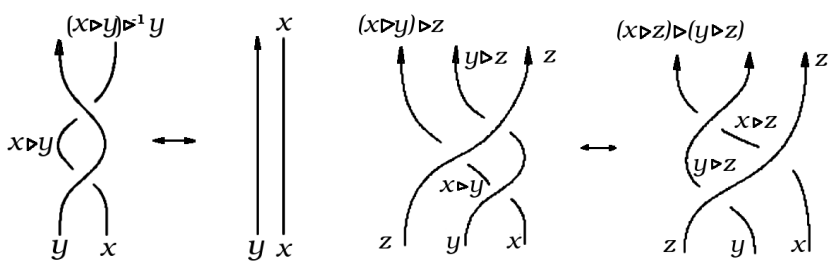
DEFINITION 1. A *rack* is a set R with a binary operation $\triangleright : R \times R \to R$ satisfying

(i) for all $x \in R$, the map $f_x : R \to R$ defined by $f_x(y) = y \triangleright x$ is invertible, with inverse $f_x^{-1}(y)$ denoted $y \triangleright^{-1} x$, and
(ii) for all $x, y, z \in R$, we have $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

A rack in which $x \triangleright x = x$ for all $x \in R$ is a *quandle.* The operation $\triangleright^{-1}$ is the *dual* rack operation—it is also self-distributive, and the two operations are mutually distributive. Note that in [6], $x \triangleright y$ is denoted x^y and $x \triangleright^{-1} y$ is denoted $x^{\overline{y}}$.



The rack axioms correspond to Reidemeister moves II and III, where we think of rack elements as arcs in an oriented link diagram and $\triangleright$ means crossing under from right to left when looking in the positive direction of the overcrossing strand. The dual operation $\triangleright^{-1}$ can then be interpreted as crossing under from left to right.



EXAMPLE 1. Perhaps the simplest example of a rack structure on a finite set $R = \{x_1, \ldots, x_n\}$ is the *constant action rack* or *permutation rack* on R associated to a permutation $\sigma \in S_n$. Specifically, set

$$x_i \triangleright x_j = x_{\sigma(i)}$$

for all $i = 1, \ldots, n$; then the action of $y \in R$ on R remains constant as y varies. It is easy to verify that this definition gives us a rack structure, since $x_i \triangleright^{-1} x_j = x_{\sigma^{-1}(i)}$ and we have

$$(x_i \triangleright x_j) \triangleright x_k = x_{\sigma^2(i)} = (x_i \triangleright x_k) \triangleright (x_j \triangleright x_k).$$

If a constant action rack is a quandle, then $x \triangleright x = x$ and consequently $x \triangleright y = x$ for all $x, y \in R$; such a quandle is called *trivial*. There is one trivial quandle for each cardinality n, denoted T_n. We will denote the constant action rack associated to $\sigma \in S_n$ by T_σ.

EXAMPLE 2. A simple example of a non-trivial rack structure from [6] is the (t, s)-*rack* structure: let $\ddot{A}$ be the ring $\mathbb{Z}[t, t^{-1}, s]$ modulo the ideal generated by $s^2 - (1 - t)s$. Then any $\ddot{A}$-module M is a rack under the operation

$$x \triangleright y = tx + sy.$$

For instance, we can take $M = \mathbb{Z}_n$ and choose $t, s \in M$ such that $\gcd(n, t) = 1$ and $s^2 = (1 - t)s$, e.g. $M = \mathbb{Z}_8$ with $t = 3$ and $s = 2$. If $s = 1 - t$ then M is a quandle, known as an *Alexander quandle*.

One useful way to describe a rack operation $\triangleright$ on a finite set $\{x_1, \ldots, x_n\}$ is to encode its operation table as a matrix M whose entry in row i column j is k where $x_k = x_i \triangleright x_j$. Thus, the constant action rack on $R = \{x_1, x_2, x_3\}$ defined by $\sigma = (123)$ has matrix

$$M_{(123)} = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

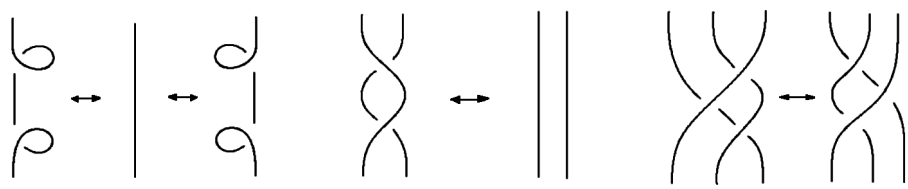
and the (t, s)-rack $M = \mathbb{Z}_8$ with $t = 3$ and $s = 2$ has matrix

$$M_{(\mathbb{Z}_8,3,2)} = \begin{bmatrix} 5 & 7 & 1 & 3 & 5 & 7 & 1 & 3 \\ 8 & 2 & 4 & 6 & 8 & 2 & 4 & 6 \\ 3 & 5 & 7 & 1 & 3 & 5 & 7 & 1 \\ 6 & 8 & 2 & 4 & 6 & 8 & 2 & 4 \\ 1 & 3 & 5 & 7 & 1 & 3 & 5 & 7 \\ 4 & 6 & 8 & 2 & 4 & 6 & 8 & 2 \\ 7 & 1 & 3 & 5 & 7 & 1 & 3 & 5 \\ 2 & 4 & 6 & 8 & 2 & 4 & 6 & 8 \end{bmatrix}.$$

Rack axiom (i) requires the columns of a rack matrix to be permutations. See [8] for more.

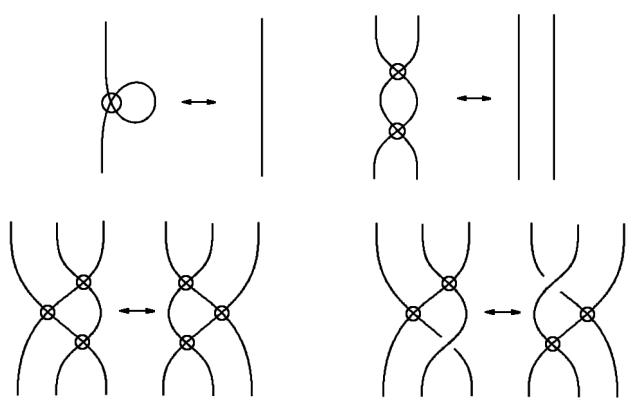
2.2. Framed links. Recall that a *framed link* is a link L with a choice of *framing curve* F_i for every component C_i of L, i.e. F_i is a longitude of a regular neighborhood of C_i. Framing curves are determined up to isotopy by their linking numbers with C_i. In terms of diagrams, we can bestow a canonical framing on every component of a link via the *blackboard framing*, i.e. drawing a framing curve for each C_i parallel to C_i. This gives a framing with linking number given by the *writhe* $w(C_i) = \sum_{x \in S_i} \operatorname{sign}(x)$, where S_i is the set of crossings where C_i crosses itself, $\operatorname{sign}(\diagup\!\!\diagdown) = 1$ and $\operatorname{sign}(\diagdown\!\!\diagup) = -1$.

Combinatorially, blackboard-framed links can be regarded as equivalence classes of link diagrams under the equivalence relation generated by Reidemeister moves II and III together with a doubled type I move which preserves the framing of each component (see [11, 6]).

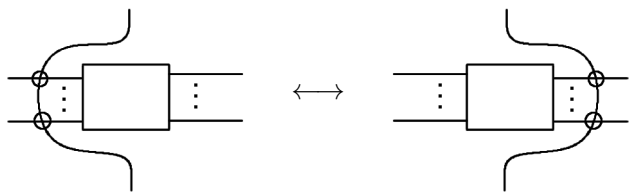


2.3. Virtual links. Virtual knot theory is a combinatorial generalization of ordinary *classical* knot theory; geometrically, a virtual link is an ordinary link in which the ambient space is not S^3 or $\mathbb{R}^3$ but $\Sigma \times [0,1]$ for some compact orientable surface Σ, considered up to stabilization (see [9, 4]). More formally, we have:

Definition 2. A *virtual link* is an equivalence class of link diagrams with an extra crossing type known as a *virtual crossing*, ⊠, under the equivalence relation determined by the usual Reidemeister moves together with the four *virtual moves*



We can summarize the rules for virtual moves with the *detour move*, which says that any strand with only virtual crossings can be replaced by any other strand with the same endpoints with only virtual crossings. That is, a strand with only virtual crossings can move past any virtual tangle:



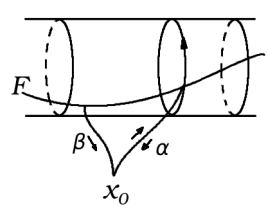
Virtual crossings have no intrinsic over- or under-sense, as they are artifacts of drawing non-planar link diagrams on planar paper. Classical links are then virtual links whose underlying surface Σ is S^2. Replacing the classical Reidemeister I move with the doubled version yields *framed virtual links*. For the remainder of this paper, we will use "link diagram" to mean "oriented blackboard-framed virtual or classical link diagram". See [9] for more.

2.4. The fundamental rack(s) of a link. Associated in [6] to a framed link L is a rack known as the *fundamental rack* of L, which we will denote by $\mathrm{FR}(L)$ ([1]).

Geometrically, elements of $\mathrm{FR}(L)$ are homotopy classes of paths in the link complement $X = S^3 \setminus (L \times \mathrm{Int}(B^2))$ from the framing curves $\bigcup F_i \subset \partial(X)$ to a fixed base point $x_0 \in X$ where the terminal point is fixed but the initial point is permitted to wander along the framing curve F_i during the homotopy. Any such path $\alpha : [0,1] \to X$ has an associated element $\pi(\alpha)$ of the fundamental group $\pi_1(X, x_0)$ defined by traveling backwards along α, then going around the canonical meridian in $\partial(X)$ intersecting $\alpha(0)$, then going forwards along α. The rack operation is then

$$[\beta] \triangleright [\alpha] = [\beta * \pi(\alpha)],$$

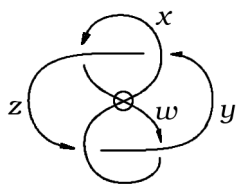
where $*$ is concatenation of paths.



([1]) Does the notation "FR" stand for "fundamental rack" or "Fenn and Rourke"? Perhaps both!

Combinatorially, given a diagram of L, the fundamental rack of L consists of equivalence classes of rack words in generators corresponding to arcs in the diagram of L under the equivalence relation generated by the rack axioms together with the relations imposed at each crossing. If L is a virtual link, we simply ignore the virtual crossings.

Example 3. The pictured blackboard-framed virtual link has fundamental rack with generators x, y and relation $x \triangleright y = x \triangleright (y \triangleright x)$:



$$\begin{aligned}\mathrm{FR}(L) &= \langle x, y, z, w \mid y \triangleright x = z,\ x \triangleright z = w,\ x \triangleright y = w \rangle \\ &= \langle x, y, w \mid x \triangleright y = w,\ x \triangleright (y \triangleright x) = w \rangle \\ &= \langle x, y \mid x \triangleright y = x \triangleright (y \triangleright x) \rangle.\end{aligned}$$

For each framing of a given link, we have a fundamental rack, generally distinct from the racks of the other framings. All of these racks have a common quotient quandle obtained by setting $a \triangleright a = a$ for all elements $a \in \mathrm{FR}(L)$, which is the *knot quandle* $Q(L)$ of the unframed link L. Elements of the knot quandle may be interpreted geometrically as homotopy classes of paths where the initial point is permitted to wander not just along the framing curve but along all of $\partial(X)$. See [6] for more.

3. Racks and counting invariants. Let L be an unframed link with an ordering on the components. If L has n components, then the framings of L may be indexed by n-tuples $\mathbf{w} \in \mathbb{Z}^n$, each with its own *a priori* distinct fundamental rack. At the most basic level, then, there are infinitely many rack counting invariants for a given link with respect to any choice of finite target rack T. However, we can make a useful observation which enables us to get computable ambient isotopy invariants from the $\mathbb{Z}^n$-set of racks of L.

Definition 3. Let T be a rack. For any $x \in T$, let $x^{\triangleright n}$ for $n \in \mathbb{Z}_+$ be defined recursively by

$$x^{\triangleright 1} = x \triangleright x \quad \text{and} \quad x^{\triangleright(k+1)} = x^{\triangleright k} \triangleright x^{\triangleright k}.$$

For each element $x \in T$, the *rack rank* of $x \in T$, denoted $N(x)$, is the minimal natural number $N \in \mathbb{Z}_+$ such that $x^{\triangleright N} = x$, or $N(x) = \infty$ if $x^{\triangleright N} \neq x$ for all $N \in \mathbb{N}$. The *rack rank* of T, denoted $N(T)$ or just N if T is understood, is the least common multiple of the rack ranks of the elements of T,

$$N(T) = \mathrm{lcm}\{N(x) \mid x \in T\}.$$

To see that $N(x)$ is well defined for all $x \in T$, we first need a lemma.

LEMMA 1. *Let T be any rack and $x, y \in T$. Then $y \triangleright (x \triangleright x) = y \triangleright x$.*

Proof. We compute

$$y \triangleright (x \triangleright x) = [(y \triangleright^{-1} x) \triangleright x] \triangleright (x \triangleright x) = [(y \triangleright^{-1} x) \triangleright x] \triangleright x = y \triangleright x. \blacksquare$$

EXAMPLE 4. Two elements $x, y \in T$ are *operator equivalent* if $z \triangleright x = z \triangleright y$ for all $z \in T$. If T is a finite rack, then two elements are operator equivalent iff their columns in the matrix of T are identical. Lemma 1 says that the $\triangleright$-powers of $x \in T$ are all operator equivalent. Indeed, the set of operator equivalence classes of a rack forms a quandle under the natural operation $[x] \triangleright [y] = [x \triangleright y]$.

COROLLARY 2. *Let T be a rack. If $x \triangleright x = y \triangleright y$, then $x = y$.*

Proof. Suppose $x \triangleright x = y \triangleright y = z$. We have $x \triangleright x = x \triangleright (x \triangleright x) = x \triangleright z$ and $y \triangleright y = y \triangleright (y \triangleright y) = y \triangleright z$. Then $x \triangleright x = y \triangleright y$ implies $x \triangleright z = y \triangleright z$ and rack axiom (i) implies $x = y$. $\blacksquare$

In terms of rack matrices, Corollary 2 says that like the columns of a rack matrix, the diagonal of a rack matrix must be a permutation. Indeed, if we define $\pi : T \to T$ by $\pi(x) = x \triangleright x$ then the diagonal of a rack matrix tells us the permutation π. It then easily follows that $N(x) < |T|$ for any $x \in T$ where T is a finite rack—indeed, $N(T)$ is just the exponent of $\pi \in S_{|T|}$. This fact also follows from [10, Proposition 7.3].

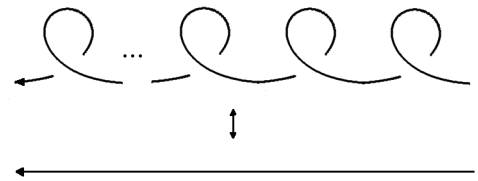
We will also need the following standard result (see e.g. [6] or [11]):

THEOREM 3. *If D and D' are ambient isotopic link diagrams, we can modify D' to obtain a diagram D'' which is framed isotopic to D by selecting an arc on each component of D' and adding positive or negative kinks until the framings match.*

The proof of Theorem 3 involves taking any Reidemeister move sequence starting with D and ending with D', and replacing every type I move with a double I move to adjust the framed isotopy class; at the end, we can then slide the extra crossings along the component until they arrive at the chosen arc. Note that this argument applies to virtual links as well as classical links, since we can slide a classical kink past a virtual crossing using a detour move. Note also that without loss of generality we can assume that all kinks added have positive winding number since we need not preserve the regular isotopy class, only the blackboard-framed isotopy class.

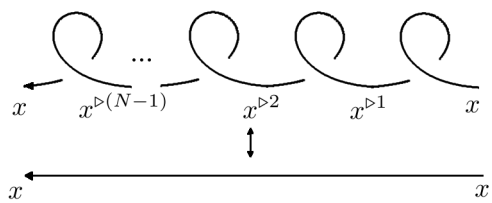
DEFINITION 4. Let $N \in \mathbb{N}$. We say two blackboard-framed oriented link diagrams are *N-phone cord equivalent* if one may be obtained from the other by a finite sequence of Reidemeister II and III moves and the following

N-phone cord move, where N is the number of loops:



PROPOSITION 4. *Let T be a finite rack with rack rank N. If two link diagrams D and D' are N-phone cord isotopic then* $|\mathrm{Hom}(\mathrm{FR}(D), T)| = |\mathrm{Hom}(\mathrm{FR}(D'), T)|$.

Proof. From the definition of rack rank, it is easy to see that N-phone cord moves induce a bijection on the set of colorings as illustrated:



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For two n-tuples $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$, let us write $\mathbf{v} \equiv \mathbf{w} \bmod N$ if for all components $i = 1, \ldots, n$ we have $v_i \equiv w_i \bmod N$.

COROLLARY 5. *Let T be a finite rack with rack rank N. If two link diagrams D and D' are ambient isotopic and have writhe vectors congruent modulo N, then* $|\mathrm{Hom}(\mathrm{FR}(D), T)| = |\mathrm{Hom}(\mathrm{FR}(D'), T)|$.

Note that if T is a finite rack with rack rank N, and L is a link, the set of writhes of each component of L modulo N can be indexed by $\mathbf{w} \in (\mathbb{Z}_N)^c$, where c is the number of components of L. For ease of notation, when N and c are understood let us write $(\mathbb{Z}_N)^c = W$ and denote by $(D, \mathbf{w})$ a blackboard-framed diagram of D with writhe vector $\mathbf{w} \in W$.

We can now define computable unframed knot and link invariants using these cardinalities.

DEFINITION 5. Let T be a finite rack and L a link with c components. The *integral rack counting invariant* of L with respect to T is

$$\Phi_T^{\mathbb{Z}}(L) = \sum_{\mathbf{w} \in W} |\mathrm{Hom}(\mathrm{FR}(D, \mathbf{w}), T)|.$$

Note that if T is a quandle, then we have $N(T) = 1$ and $\Phi_T^{\mathbb{Z}}(L)$ is the ordinary quandle counting invariant $|\mathrm{Hom}(Q(L), T)|$. Hence the integral rack counting invariant is the natural generalization of the quandle counting invariant to the finite rack case.

EXAMPLE 5. If $T = \{x_1, \ldots, x_n\}$ is a constant action rack defined by an n-cycle, an undercrossing color τ becomes $\sigma(\tau)$ if going right-to-left and $\sigma^{-1}(\tau)$ if going left-to-right, so pushing a color around the knot yields an ending color of $\sigma^{w(K)}(\tau)$, where $w(K)$ is the writhe of K and our starting color was τ. Thus, there is a rack coloring of a framed knot K by T if and only if the writhe of K is zero modulo n. Indeed, there are n such colorings for the 0-framing modulo n and none for the others, and we have $\mathrm{SR}(K,T) = n + (n-1)0 = n$ for any knot K. This generalizes the fact that $|\mathrm{Hom}(K,T)| = n$ for T a trivial quandle of cardinality n and K a knot.

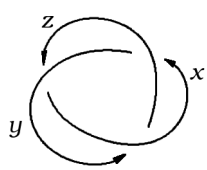
EXAMPLE 6. Let T_* be the rack with matrix

$$M_{T_*} = \begin{bmatrix} 1 & 3 & 2 & 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 & 2 & 2 & 2 \\ 2 & 1 & 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 6 & 4 & 6 & 4 \\ 5 & 5 & 5 & 5 & 7 & 5 & 7 \\ 6 & 6 & 6 & 4 & 6 & 4 & 6 \\ 7 & 7 & 7 & 7 & 5 & 7 & 5 \end{bmatrix}.$$

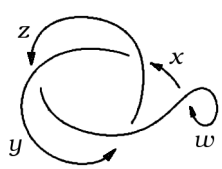
The integer rack counting invariant with respect to T_* distinguishes the trefoil 3_1 from the unknot U_1 with $\Phi^{\mathbb{Z}}_{T_*}(U_1) = 10$ and $\Phi^{\mathbb{Z}}_{T_*}(3_1) = 22$, as the reader can verify from the tables of colorings listed in Table 1. Here $N(T_*) = 2$, so we need only consider one diagram of each of U_1 and 3_1 with odd writhe and one of each with even writhe.

Table 1. Rack colorings of 3_1 and U_1 by T_*

D	Colorings by T_*

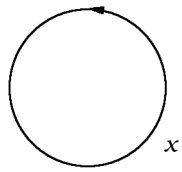


x	1	1	1	2	2	2	3	3	3
y	1	2	3	1	2	3	1	2	3
z	1	3	2	3	2	1	2	1	3



x	1	1	1	2	2	2	3	3	3	6	4	5	7
y	1	2	3	1	2	3	1	2	3	4	6	7	5
z	1	3	2	3	2	1	2	1	3	6	4	5	7
w	1	1	1	2	2	2	3	3	3	4	6	7	5

Table 1 (cont.)

D	Colorings by T_*
	x | 1 2 3 4 5 6 7
	x | 1 2 3

We can enhance the integer rack counting invariant by keeping track of which framings contribute which colorings. For a writhe vector $\mathbf{w} = (w_1, \ldots, w_c) \in W = (\mathbb{Z}_N)^c$ let us denote the product $q_1^{w_1} \ldots q_c^{w_c}$ by $q^{\mathbf{w}}$. Then we have:

Definition 6. Let T be a finite rack and L a link with c components and writhe vector $\mathbf{w} = (w_1, \ldots, w_c) \in W$. The *writhe-enhanced rack counting invariant* of L with respect to T is given by

$$\Phi_T^W(L) = \sum_{\mathbf{w} \in W} |\mathrm{Hom}(\mathrm{FR}(L, \mathbf{w}), T)| q^{\mathbf{w}}.$$

The writhe-enhanced rack counting invariant holds more information than the simple version, as the next example shows.

Example 7. The constant action rack T with rack matrix $\left[\begin{smallmatrix} 2 & 2 \\ 1 & 1 \end{smallmatrix}\right]$ has rack rank $N(T) = 2$. The Hopf link H and the two-component unlink U_2 both have integral rack counting invariant value $\Phi_T^{\mathbb{Z}}(L) = 4$ with respect to T, but the writhe-enhanced rack counting invariants are distinct, with $\Phi_T^W(H) = 4q_1q_2$ and $\Phi_T^W(U_2) = 4$ as the reader can easily verify from Table 2.

Indeed, generalizing the preceding example we have

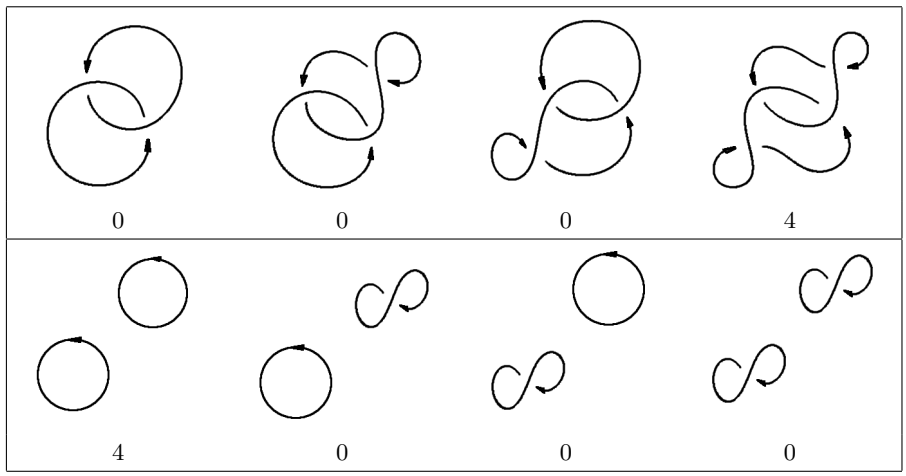
Proposition 6. *Let L be a two-component classical link and $T = T_\sigma$ a constant action rack with $\sigma \in S_N$ an N-cycle. Then the writhe-enhanced rack counting invariant has the form*

$$\Phi_{T_\sigma}^W(L) = N^2 q_1^l q_2^l$$

where l is the negative of the linking number $\mathrm{lk}(L_1, L_2)$ of L modulo N.

Proof. Traveling around a component, to get a valid coloring, the end color must match the initial color, so we must go through N crossings (counted algebraically). Since $\mathrm{lk}(L_1, L_2)$ of these are multi-component

Table 2. Numbers of colorings of H and U_2 by $T_{(12)}$



crossings which do not contribute to the component's writhe, we must have

$$l + \mathrm{lk}(L_1, L_2) = N.$$

The same holds for both components if L is classical. There are N choices of starting color for each component and every pair produces exactly one coloring, so there are N^2 total colorings. ∎

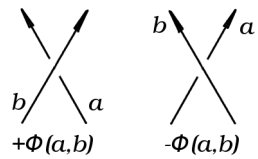
COROLLARY 7. *If L is a two-component link and $T = T_\sigma$ a constant action rack with $\sigma \in S_N$ an N-cycle such that the exponents of q_1 and q_2 differ in any term of $\Phi^W_{T_\sigma}(L)$, then L is non-classical.*

4. Rack cocycle invariants. In this section we generalize the quandle 2-cocycle invariants defined in [3] to the finite rack case.

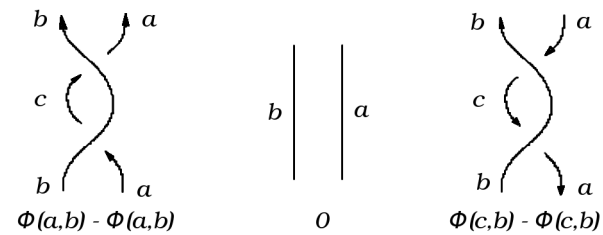
The rack counting invariants described in the last section are cardinalities of sets of homomorphisms which are unchanged by Reidemeister moves. However, a set is more than a cardinality, and we would like to recover as much information from these sets of homomorphisms as possible.

In [3], the idea is to associate a sum in an abelian group A called a *Boltzmann weight* to a quandle-colored knot diagram in such a way that the sum does not change under Reidemeister moves. Then, instead of counting "1" for each homomorphism, we count its Boltzmann weight, transforming the set of colorings into a multiset of these weights. Such multisets are commonly encoded as polynomials by taking a generating function, i.e. by converting the multiset elements to exponents and multiplicities to coefficients of a dummy variable, e.g. $\{1, 1, 1, 4, 4\}$ becomes $3t + 2t^4$.

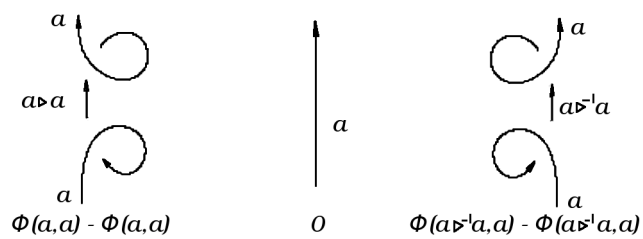
The Boltzmann weights are defined as follows: at every crossing in a rack-colored link diagram, we want to count $\phi(a, b)$ at a positive crossing or $-\phi(a, b)$ at a negative crossing, where b is the color on the overcrossing strand and a is the color on the inbound understrand for positive crossings and the outbound understrand for negative crossings:



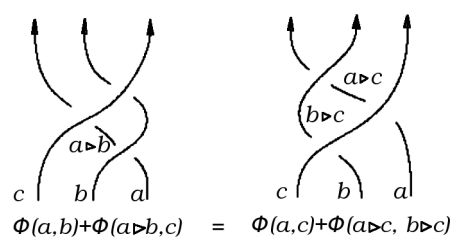
This weighting rule has the advantage that the contributions from the two crossings in a Reidemeister type II move cancel, so the sum is automatically invariant under II moves:



We also note that the weighting rule gives invariance under the doubled type I moves required for blackboard-framed isotopy:



The condition for the sum to be unchanged by Reidemeister III moves is pictured below:



This turns out to be the condition that ϕ is a cocycle in the second rack cohomology $H^2_R(T;A)$ of the rack T with coefficients in A. Specifically, the A-module spanned by n-tuples of elements of T is the space of *rack n-chains* $C^R_n(T;A) = A[T^n]$; its dual is the space of *rack n-cochains* $C^n_R(T;A) = \mathrm{Hom}(C^R_n(T;A), A)$. Note that $C^n_R(T;A)$ has A-generating set $\{\chi_{\mathbf{x}} \mid \mathbf{x} \in T^n\}$ where

$$\chi_{\mathbf{x}}(\mathbf{y}) = \begin{cases} 1 & \text{for } \mathbf{x} = \mathbf{y}, \\ 0 & \text{otherwise,} \end{cases}$$

for $\mathbf{y} \in T^n$. Next, we define a coboundary map $\delta^n : C^n_R(T;A) \to C^{n+1}_R(T;A)$ by

$$(\delta^n\phi)(x_1,\dots,x_{n+1}) = \sum_{i=1}^{n+1}(-1)^{i-1}\big(\phi(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1}) - \phi(x_1 \triangleright x_i,\dots,x_{i-1}\triangleright x_i,x_{i+1},\dots,x_{n+1})\big).$$

Then for ϕ to yield a Boltzmann weight, we need $\phi \in \mathrm{Ker}(\delta^2)$.

To get invariance under the Reidemeister I move in the quandle case, we require that $\phi(x,x) = 0$ for all $x \in T$. This condition also turns out to have a homological interpretation—the cocycles we want to kill live in a subcomplex called the *degenerate* cochains. In the non-quandle rack case, however, a weaker condition suffices.

Definition 7. Let T be a finite rack with rack rank N, A an abelian group and $\phi \in C^2_R(T;F)$. Say ϕ is *N-reduced* if

$$\sum_{k=1}^{N} \phi(a^{\triangleright k}, a^{\triangleright k}) = 0$$

for all $a \in T$.

Now we can define an enhanced version of the polynomial rack counting invariant:

Definition 8. Let L be an oriented blackboard-framed link, T a finite rack and ϕ an N-reduced rack 2-cocycle. For a rack-colored framed diagram $(D,\mathbf{w})$ of L, the *Boltzmann weight* $\mathrm{BW}(f)$ of the coloring $f \in \mathrm{Hom}(\mathrm{FR}(D,\mathbf{w}),T)$ is the sum over the crossings in D of the crossing weights,

$$\mathrm{BW}(f) = \sum_{c \text{ a crossing}} \mathrm{sign}(c)\phi(a,b).$$

Then the *rack cocycle invariant* of L with respect to T is

$$\Phi^{\phi}_T(L) = \sum_{\mathbf{w}\in W}\Big(\sum_{f\in\mathrm{Hom}(\mathrm{FR}(D,\mathbf{w}),T)} z^{\mathrm{BW}(f)}\Big)$$

and the *writhe-enhanced rack cocycle invariant* is

$$\Phi_T^{\phi,W}(L) = \sum_{\mathbf{w}\in W} \Big(\sum_{f\in \mathrm{Hom}(\mathrm{FR}(D,\mathbf{w}),T)} z^{\mathrm{BW}(f)} \Big) q^{\mathbf{w}}.$$

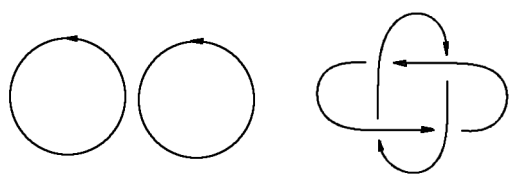
Note that if T is a quandle, then $N(T) = 1$ and ϕ is 1-reduced iff $\phi(x,x) = 0$ for all degenerate cycles $(x,x) \in C_2^R(T;A)$; in this case we also have $W = \{(0,\dots,0)\}$. Indeed, in the quandle case this rack cocycle invariant becomes the usual CJKLS quandle 2-cocycle invariant from [3].

Specializing $z = 1$ in $\Phi_T^{\phi,W}(L)$ recovers the writhe-enhanced rack counting invariant, and likewise in the integral case. To see that the rack cocycle invariant is stronger than $\Phi_T^{\mathbb{Z}}(L)$, consider the following simple example.

Example 8. The rack T with rack matrix

$$M_T = \begin{bmatrix} 3 & 1 & 3 & 1 \\ 2 & 4 & 2 & 4 \\ 1 & 3 & 1 & 3 \\ 4 & 2 & 4 & 2 \end{bmatrix}$$

has a reduced cocycle $\phi = \chi_{(12)} + \chi_{(14)} + \chi_{(32)} + \chi_{(34)}$ with $\mathbb{Z}_{13}$ coefficients. Then the $(4,2)$-torus link is distinguished from the two-component unlink by Φ_ϕ:



$\Phi_\phi = 16$ $\qquad$ $\Phi_\phi = 8 + 8z^{12}$

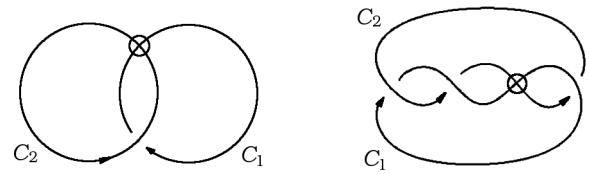
Our final example illustrates a pair of virtual links with equal writhe-enhanced rack counting invariant values which are distinguished when we include the rack cocycle information.

Example 9. Again let T be the rack with rack matrix

$$M_T = \begin{bmatrix} 3 & 1 & 3 & 1 \\ 2 & 4 & 2 & 4 \\ 1 & 3 & 1 & 3 \\ 4 & 2 & 4 & 2 \end{bmatrix}$$

and $\phi = \chi_{(1,2)} + \chi_{(1,4)} + \chi_{(3,2)} + \chi_{(3,4)} \in C_R^2(T;\mathbb{Z}_{13})$. Then Φ_ϕ distinguishes the two virtual links pictured below, both of which have $\Phi_T^W(L) = 8 + 8q_1$.

Note that the subscripts on q correspond to the component ordering.



$\Phi_T^{\phi,W} = 4 + 4z + 8q_1 \qquad \Phi_T^{\phi,W} = 4z + 4z^2 + 8q_1$

5. Questions. In this section we collect a few questions for future research.

Rack and quandle (co)homology has been generalized in various ways including *twisted quandle (co)homology* in [2], quandle (co)homology with coefficients in quandle modules in [1], and more. How does the rack cocycle invariant change in these cases?

Quandle 3-cocycles have been used to enhance quandle counting invariants of surface knots, i.e. embeddings of compact orientable 2-manifolds in S^4. How do the rack counting and cocycle invariants extend to the surface knot case?

Other ways of enhancing the quandle counting invariants include using quandle polynomials and exploiting any extra structure the quandle may have (symplectic vector space, R-module, etc.); generalizing these ideas to the rack case will be the subject of future papers.

Replacing the arcs in the combinatorial motivation for the rack axioms with semiarcs yields *biracks*, also known as invertible switches or *Yang–Baxter sets* (see [7, 5]). The birack analogues of the simple and polynomial rack counting invariants will be the subject of another future paper.

A `Python` code for computing rack counting invariants, reduced rack 2-cocycles with $\mathbb{Z}_n$ coefficients, and rack cocycle invariants is available for download from `www.esotericka.org`.

References

[1] J. S. Carter, M. Elhamdadi, M. Graña and M. Saito, *Cocycle knot invariants from quandle modules and generalized quandle homology*, Osaka J. Math. 42 (2005), 499–541.

[2] J. S. Carter, M. Elhamdadi and M. Saito, *Twisted quandle homology theory and cocycle knot invariants*, Algebr. Geom. Topol. 2 (2002), 95–135.

[3] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford and M. Saito, *Quandle cohomology and state-sum invariants of knotted curves and surfaces*, Trans. Amer. Math. Soc. 355 (2003), 3947–3989.

[4] J. S. Carter, S. Kamada and M. Saito, *Stable equivalence of knots on surfaces and virtual knot cobordisms*, J. Knot Theory Ramif. 11 (2002), 311–322.

[5] R. Fenn, M. Jordan-Santana and L. Kauffman, *Biquandles and virtual links*, Topology Appl. 145 (2004), 157–175.
[6] R. Fenn and C. Rourke, *Racks and links in codimension two*, J. Knot Theory Ramif. 1 (1992), 343–406.
[7] R. Fenn, C. Rourke and B. Sanderson, *Trunks and classifying spaces*, Appl. Categ. Structures 3 (1995), 321–356.
[8] B. Ho and S. Nelson, *Matrices and finite quandles*, Homology Homotopy Appl. 7 (2005), 197–208.
[9] L. H. Kauffman, *Virtual knot theory*, Eur. J. Combin. 20 (1999), 663–690.
[10] P. Lopes and D. Roseman, *On finite racks and quandles*, Comm. Algebra 34 (2006), 371–406.
[11] B. Trace, *On the Reidemeister moves of a classical knot*, Proc. Amer. Math. Soc. 89 (1983), 722–724.

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