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An infinite torus braid yields a categorified Jones–Wenzl projector

by

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Abstract. A sequence of Temperley–Lieb algebra elements corresponding to torus braids with growing twisting numbers converges to the Jones–Wenzl projector. We show that a sequence of categorification complexes of these braids also has a limit which may serve as a categorification of the Jones–Wenzl projector.

1. Introduction. The Jones–Wenzl projector P_n is a special idempotent element of the n-strand Temperley–Lieb algebra TL_n, whose defining property is the annihilation of cap and cup tangles. The coefficients in its expression in terms of Temperley–Lieb tangles are rational (rather than polynomial) functions of q. This suggests that the categorification $\mathbf{P}_n$ of P_n in the universal tangle category TL_n constructed by D. Bar-Natan [BN05] should be presented by a semi-infinite chain complex. In fact, there are two mutually dual categorifications: the complex $\mathbf{P}_n$ which is bound from above and the complex $\mathbf{P}_n^-$ which is bound from below. We will consider only $\mathbf{P}_n$ in detail, since the story of $\mathbf{P}_n^-$ is totally similar.

The construction of $\mathbf{P}_n^-$ by B. Cooper and S. Krushkal [CK12] is based upon the Frenkel–Khovanov formula for P_n and requires the invention of morphisms between constituent TL tangles as well as non-trivial 'thickening' of the complex. An alternative 'representation-theoretic' approach to the categorification of the Jones–Wenzl projector is developed by Igor Frenkel, Catharina Stroppel, and Joshua Sussan [FSS12].

Our approach is rather straightforward: the categorified projector $\mathbf{P}_n$ is a direct limit of appropriately shifted categorification complexes of torus braids (i.e. braid analogs of torus links) with high clockwise twist (the other projector $\mathbf{P}_n^-$ comes from high counterclockwise twists). The limit $\mathbf{P}_n$ can

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be presented as a cone:

$$\mathbf{P}_n \sim \operatorname{Cone}\Big(O^{\mathrm{c}}(2m(n-1)) \to \langle\!\langle\, \overset{m}{\succ\!\vdots\!\prec}\, n \rangle\!\rangle^{\mathrm{s}}\Big), \tag{1.1}$$

where $\overset{m}{\succ\!\vdots\!\prec}\, n$ is a torus braid with m full clockwise rotations of n strands, $\langle\!\langle - \rangle\!\rangle^{\mathrm{s}}$ is the categorification complex with a special grading shift, and $O^{\mathrm{c}}(k)$ denotes a chain complex which ends at the homological degree k. Theorem 2.8 imposes even stronger restrictions on the complex $O^{\mathrm{c}}(2m(n-1))$ in (1.1).

The advantage of our approach is that one can use torus braids with high twist as approximations to $\mathbf{P}_n$ in a computation of Khovanov homology of a spin network which involves Jones–Wenzl projectors: if a spin network ν is constructed by connecting P_n to an (n,n)-tangle τ such that $\langle\!\langle \tau \rangle\!\rangle \sim O^{\mathrm{c}}(k)$, while a spin network ν_m is constructed by replacing P_n in ν with $\overset{m}{\succ\!\vdots\!\prec}\, n$, then the homology of $\langle\!\langle \nu \rangle\!\rangle$ coincides with the shifted homology of $\langle\!\langle \nu_m \rangle\!\rangle$ in all homological degrees i such that $i > k + 2m(n-1)$. Thus one may say that there is a stable limit

$$\langle\!\langle \nu \rangle\!\rangle = \lim_{m\to\infty} \langle\!\langle \nu_m \rangle\!\rangle^{\mathrm{s}}. \tag{1.2}$$

We will define homological limits more precisely in Subsection 2.2.2.

The practical importance of the relation between $\langle\!\langle \nu \rangle\!\rangle$ and $\langle\!\langle \nu_m \rangle\!\rangle$ stems from the fact that ν_m is an ordinary link and its homology can be computed with the help of existing efficient computer programs even for high values of m.

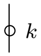
The simplest example of a spin network is the unknot 'colored' by the $(n{+}1)$-dimensional representation of SU(2) with the help of the projector P_n. Its Khovanov homology is approximated by the homology of torus links $\mathrm{T}_{n,-mn}$ which appear as cyclic closures of $\overset{m}{\succ\!\vdots\!\prec}\, n$. The Khovanov homology of torus links has been studied by Marko Stošić [Sto07], who observed that it stabilizes at lower degrees as m grows. This is a particular case of the 'stable limit' (1.2).

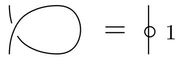
In Section 2 we explain all notations and conventions which are used in the paper. In particular, in Subsection 2.1.4 we define a non-traditional grading of Khovanov homology, which is convenient for our computations. Then we formulate our results.

In Section 3 we review basic facts about homological 'calculus' required to work with limits of sequences of complexes in a homotopy category. In Section 4 we construct a sequence of categorification complexes of torus braids related by special chain morphisms. This sequence yields $\mathbf{P}_n$ as its direct limit. In Section 5 we use the homological calculus of Section 3 in order to prove that $\mathbf{P}_n$ is a categorification of the Jones–Wenzl projector.

2. Notations and results

2.1. Notations

2.1.1. *Tangles and Temperley–Lieb algebra.* All tangles in this paper are framed and we assume the blackboard framing in pictures. We use the symbol  to indicate an addition of k framing twists to a tangle strand:

(2.1)


A tangle is called *planar* if it can be presented by a diagram without crossings. A planar tangle is called *connected* or *Temperley–Lieb* (TL) if it does not contain disjoint circles. Let Tng denote the set of all framed tangles, $\mathrm{Tng}_{m,n}$ the set of (m,n)-tangles and Tng_n the set of (n,n)-tangles. We adopt similar notations for the set TL of TL-tangles.

We use the symbol $\circ$ to denote the composition of tangles, $\tau_1 \circ \tau_2$. The same symbol is used to denote multiplication in a Temperley–Lieb algebra and the composition bifunctor in the category TL.

A Temperley–Lieb algebra TL over the ring of Laurent polynomials $\mathbb{Z}[q, q^{-1}]$ [1] is a quiver ring. The vertices v_n of the quiver are indexed by non-negative integers n, and each pair of vertices v_m, v_n such that $m-n$ is even is connected by an edge e_{mn}. To a vertex v_n we associate a ring $\mathrm{TL}_{n,n}$ (also denoted as TL_n) and to an edge e_{mn} we associate a $\mathrm{TL}_n \otimes \mathrm{TL}_m^{\mathrm{op}}$-module $\mathrm{TL}_{m,n}$. As a module, $\mathrm{TL}_{m,n}$ is generated freely by elements $\langle\lambda\rangle$ corresponding to TL (m,n)-tangles λ, while ring and module structures come from the composition of tangles modulo the relation

(2.2)

$$= -(q+q^{-1}),$$

which is needed to remove disjoint circles that may appear in the composition of Temperley–Lieb tangles.

The map $\mathrm{Tng} \xrightarrow{\langle - \rangle} \mathrm{TL}$ associates an element $\langle\tau\rangle$ to a tangle τ with the help of (2.2) and the Kauffman bracket relation

(2.3)


This relation removes crossings and disjoint circles from the diagram of τ,

(1) It is clear from our normalization of the Kauffman bracket relation (2.3) that we should rather use the ring $\mathbb{Z}[q^{1/2}, q^{-1/2}]$. However, in all expressions in this paper the half-integer power of q appears only as a common factor, so the terms with integer and half-integer powers of q do not mix. Hence we refer to $\mathbb{Z}[q, q^{-1}]$, while keeping in mind that $q^{1/2}$ may appear as a common factor in some expressions.

hence

$$\langle\tau\rangle = \sum_{\lambda\in\mathrm{TL}_n} a_\lambda(\tau)\,\langle\lambda\rangle,\qquad a_\lambda(\tau) = \sum_{i\in\mathbb{Z}} a_{\lambda,i}(\tau)\,q^i \tag{2.4}$$

with only finitely many coefficients $a_{\lambda,i}(\tau)$ being non-zero.

If two tangles differ only by the framing of their strands, then the corresponding algebra elements differ by the q-power factor coming from the following relation associated with the first Reidemeister move:

$$\Big\langle \,\phi\ {\scriptstyle 1}\, \Big\rangle \;=\; -q^{3/2}\;\Big\langle \,\big|\, \Big\rangle \tag{2.5}$$

A $(0,0)$-tangle L is a framed link, so $\langle L\rangle$ is the framing dependent Jones polynomial defined by the Kauffman bracket.

We use the notations QTL and TL^+ for Temperley–Lieb algebras defined over the field $\mathbb{Q}(q)$ of rational functions of q and over the field $\mathbb{Z}[[q,q^{-1}]$ of Laurent power series. A sequence of injective homomorphisms $\mathbb{Z}[q,q^{-1}]\hookrightarrow \mathbb{Q}(q)\hookrightarrow \mathbb{Z}[[q,q^{-1}]$, the latter generated by the expansion in powers of q, produces a sequence of injective homomorphisms of the corresponding Temperley–Lieb algebras.

2.1.2. *The Jones–Wenzl projector.* Let ${}_{n}\overset{i}{\big)} \in \mathrm{TL}_{n-2,n}$ and $\overset{i}{\big(}{}_{n} \in \mathrm{TL}_{n,n-2}$, $1\le i\le n-1$, denote the following TL tangles:

$$ {}_{n}\overset{i}{\big)} \;=\; \overset{1}{\Big|}\ \cdots\ \Big|\ \overset{i\ \ i+1}{\smile}\ \Big|\ \cdots\ \overset{n}{\Big|} \qquad\text{and}\qquad \overset{i}{\big(}{}_{n} \;=\; \underset{1}{\Big|}\ \cdots\ \Big|\ \underset{i\ \ i+1}{\frown}\ \Big|\ \cdots\ \underset{n}{\Big|}$$

Their compositions $U_{n,i} = {}_{n}\overset{i}{\big)} \circ \overset{i}{\big(}{}_{n}$ are standard generators of the Temperley–Lieb algebra TL_n.

The *Jones–Wenzl projector* $P_n\in\mathrm{QTL}_n$ is the unique non-trivial idempotent element satisfying the condition

$$\Big\langle\ \overset{i}{\big(}{}_{n}\ \Big\rangle \circ P_n = 0,\qquad 1\le i\le n-1. \tag{2.6}$$

The Jones–Wenzl projector also satisfies the relation

$$P_n\circ \Big\langle {}_{n}\overset{i}{\big)}\ \Big\rangle = 0,\qquad 1\le i\le n-1. \tag{2.7}$$

We denote the idempotent element of TL^+_n corresponding to P_n as P^+_n.

2.1.3. *Basic notions of homological algebra.* Let $\mathbf{Ch}(\mathsf{A})$ be a category of chain complexes associated with an additive category A. An object of $\mathbf{Ch}(\mathsf{A})$ is a chain complex

$$\mathbf{A} = (\cdots\to A_i \xrightarrow{d_i} A_{i-1}\to\cdots), \tag{2.8}$$

and a morphism between two chain complexes is a chain morphism defined as a multi-map

$$
\begin{tikzcd}
\mathbf{A} \arrow[d, "\mathbf{f}"] & \cdots \arrow[r, "d_{i+1}"] & A_i \arrow[r, "d_i"] \arrow[d, "f_i"] & A_{i-1} \arrow[r, "d_{i-1}"] \arrow[d, "f_{i-1}"] & \cdots \\
\mathbf{B} & \cdots \arrow[r, "d'_{i+1}"] & B_i \arrow[r, "d'_i"] & B_{i-1} \arrow[r, "d'_{i-1}"] & \cdots
\end{tikzcd}
\tag{2.9}
$$

which commutes with the chain differential: $d'_i\, f_i = f_{i-1}\, d_i$ for all i. The *cone* of a chain morphism $\mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B}$ is a complex

$$
\operatorname{Cone}(\mathbf{f}) = \left(
\begin{tikzcd}
\cdots \arrow[r] \arrow[drr] & A_i \arrow[r, "-d_i"] \arrow[drr, "-f_i"] & A_{i-1} \arrow[r] \arrow[drr] & \cdots \\[-1.5em]
 & \oplus & \oplus & \\[-1.5em]
\cdots \arrow[r] & B_{i+1} \arrow[r, "d'_{i+1}"'] & B_i \arrow[r] & \cdots
\end{tikzcd}
\right)
$$

in which the object $A_{i-1} \oplus B_i$ has homological degree i. There are two special chain morphisms $\mathbf{B} \xrightarrow{\iota_{\mathbf{f}}} \operatorname{Cone}(\mathbf{f})$ and $\operatorname{Cone}(\mathbf{f}) \xrightarrow{\delta_{\mathbf{f}}} \mathbf{A}[1]$ associated to the cone:

$$
\begin{tikzcd}
\mathbf{B} \arrow[d, "\iota_{\mathbf{f}}"] & \cdots \arrow[r] & B_{i+1} \arrow[r] \arrow[d, "0\oplus\mathbb{1}"] & B_i \arrow[r] \arrow[d, "0\oplus\mathbb{1}"] & \cdots \\
\operatorname{Cone}(\mathbf{f}) \arrow[d, "\delta_{\mathbf{f}}"] & \cdots \arrow[r] & A_i \oplus B_{i+1} \arrow[r] \arrow[d, "\mathbb{1}\oplus 0"] & A_{i-1} \oplus B_i \arrow[r] \arrow[d, "\mathbb{1}\oplus 0"] & \cdots \\
\mathbf{A}[1] & \cdots \arrow[r] & A_i \arrow[r] & A_{i-1} \arrow[r] & \cdots
\end{tikzcd}
$$

These complexes and chain morphisms form a distinguished triangle:

$$
\mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B} \xrightarrow{\iota_{\mathbf{f}}} \operatorname{Cone}(\mathbf{f}) \xrightarrow{\delta_{\mathbf{f}}} \mathbf{A}[1]. \tag{2.10}
$$

The homotopy category of complexes $\mathbf{Kom}(\mathsf{A})$ has the same objects as $\mathbf{Ch}(\mathsf{A})$ and the morphisms are the morphisms of $\mathbf{Ch}(\mathsf{A})$ modulo homotopies. We denote homotopy equivalence by the sign $\sim$.

In this paper a symbol $\mathbf{A}$ often refers to a chain complex as an object of $\mathbf{Kom}(\mathsf{A})$, that is, up to homotopy. We use a 'sharp' notation $\mathbf{A}_\sharp$ if we have in mind a particular representative of $\mathbf{A}$ in the category $\mathbf{Ch}(\mathsf{A})$, that is, the complex $\mathbf{A}_\sharp$ is defined up to isomorphism rather than up to homotopy.

The notion of a cone extends to $\mathbf{Kom}(\mathsf{A})$ and there are additional relations in that category: $\operatorname{Cone}(\iota_{\mathbf{f}}) \sim \mathbf{A}[1]$ and $\operatorname{Cone}(\delta_{\mathbf{f}}) \sim \mathbf{B}[1]$, so all vertices of a distinguished triangle have equal properties.

2.1.4. *A categorification of the Jones polynomial with modified grading.* In his famous paper [Kho00], M. Khovanov introduced a categorification of the Jones polynomial of links. To a diagram L of a link he associates a complex of graded modules

$$
\langle\!\langle L \rangle\!\rangle = (\cdots \to C_i \to C_{i-1} \to \cdots) \tag{2.11}
$$

so that if two diagrams represent the same link then the corresponding complexes are homotopy equivalent, and the graded Euler characteristic of $\langle\!\langle L \rangle\!\rangle$ is equal to the Jones polynomial of L.

Thus, overall, the complex (2.11) has two gradings: the first one is the grading related to the powers of q and the second one is the homological grading of the complex itself, the corresponding degree being equal to i. In this paper we adopt a slightly different grading convention which is convenient for working with framed links and tangles. It is inspired by matrix factorization categorification [KR08] and its advantage is that it is no longer necessary to assign orientation to link strands in order to obtain the grading of the categorification complex (2.11) which would make it invariant under the second Reidemeister move.

To a framed link diagram L we associate a $\mathbb{Z}\oplus\mathbb{Z}$-graded complex (2.11) with h-degree ($\deg_{\mathrm{h}}$) and q-degree ($\deg_q$). The corresponding shift functors are denoted as h and q, so $\mathsf{h}^l\mathsf{q}^m$ denotes a shift of h-degree by l and a shift of q-degree by m.

Observe that q-degree is homological and its parity determines the sign factors. On the other hand, h-degree is just a $\mathbb{Z}$-grading, it may take both integer and half-integer values and it has no relation to homological sign factors. However, the index i in Khovanov's complex (2.11) reflects h-degree rather than q-degree: $\deg_{\mathrm{h}} C_i = i$, hence we call h-degree 'pseudo-homological'.

The categorification formulas of [Kho00] with new grading conventions take the following form: the module associated with an unknot is still $\mathbb{Z}[x]/(x^2)$ but with a different degree assignment:

$$\left\langle\!\!\left\langle \bigcirc \right\rangle\!\!\right\rangle = \mathsf{q}^{-1}\,\mathbb{Z}[x]/(x^2) \;\cong \mathsf{q}^{-1}\,\mathbb{Z}\oplus\mathsf{q}\,\mathbb{Z}, \tag{2.12}$$

$$\deg_q 1 = 0, \qquad \deg_q x = 2, \quad \deg_{\mathrm{h}} 1 = 0, \tag{2.13}$$

and the categorification complex of a crossing is the same as in [Kho00] but with a different degree shift:

$$\left\langle\!\!\left\langle \diagup\!\!\!\!\diagdown \right\rangle\!\!\right\rangle \;=\; \mathsf{h}^{1/2}\Big(\left\langle\!\!\left\langle)\,(\right\rangle\!\!\right\rangle_1 \xrightarrow{\;d\;} \left\langle\!\!\left\langle \asymp \right\rangle\!\!\right\rangle_0 \Big). \tag{2.14}$$

The differential f is either a multiplication or a comultiplication of the ring $\mathbb{Z}[x]/(x^2)$ depending on how the arcs in the r.h.s. are closed into circles, and in our grading conventions $\deg_{\mathrm{h}} d = -1$, $\deg_q d = 1$. The resulting categorification complex (2.11) is invariant up to homotopy under the second and third Reidemeister moves, but it acquires a degree shift under the first

Reidemeister move:

$$\left\langle\!\!\left\langle \phi\ {}_1 \right\rangle\!\!\right\rangle \;=\; \mathsf{h}^{1/2}\mathsf{q} \left\langle\!\!\left\langle \,\big|\, \right\rangle\!\!\right\rangle. \tag{2.15}$$

The *Khovanov homology* of a framed link L,

$$\mathrm{H}^{\mathrm{Kh}}(L) = \bigoplus_{i,j} \mathrm{H}^{\mathrm{Kh}}_{i,j}(L), \qquad \deg_{\mathrm{h}} \mathrm{H}^{\mathrm{Kh}}_{i,j} = i, \qquad \deg_q \mathrm{H}^{\mathrm{Kh}}_{i,j} = j,$$

is defined as the homology of the complex (2.11), and its graded Euler characteristic or, rather, restricted Poincaré polynomial is related to the Kauffman bracket, i.e. the Jones polynomial, of L,

$$\langle L \rangle = \sum_{i,j} (-1)^j q^{i+j} \dim \mathrm{H}^{\mathrm{Kh}}_{i,j}.$$

Note that both q-degree and h-degree contribute to the power of q.

2.1.5. *A universal categorification of the Temperley–Lieb algebra.* D. Bar-Natan [BN05] described the universal category TL, whose Grothendieck K_0-group is TL considered as a $\mathbb{Z}[q, q^{-1}]$-module. We will use this category with obvious adjustments required by the new grading conventions.

Let $\widetilde{\mathsf{TL}}$ be an additive category whose objects are in one-to-one correspondence with Temperley–Lieb tangles, morphisms being generated by tangle cobordisms (see [BN05] for details). The universal category TL is the homotopy category of bounded complexes associated with $\widetilde{\mathsf{TL}}$. In other words, an object of TL is a complex

$$\mathbf{C} = (\cdots \to C_{i+1} \to C_i \to \cdots), \qquad C_i = \bigoplus_j \bigoplus_{\lambda \in \mathrm{TL}_n} c^{\lambda}_{i,j}\, \mathsf{q}^j \left\langle\!\left\langle \lambda \right\rangle\!\right\rangle, \tag{2.16}$$

where non-negative integers $c^{\lambda}_{i,j}$ are multiplicities; since the complex is bounded, they are non-zero for only finitely many values of i.

A categorification map $\mathrm{Tng} \xrightarrow{\langle\!\langle - \rangle\!\rangle} \mathsf{TL}$ turns a framed tangle diagram τ into a complex $\langle\!\langle \tau \rangle\!\rangle$ according to the rules (2.12) and (2.14), the morphism d in the complex (2.14) being the saddle cobordism. A composition of tangles becomes a composition bi-functor $\mathsf{TL} \times \mathsf{TL} \to \mathsf{TL}$ if we apply the categorified version of the rule (2.2) in order to remove disjoint circles:

$$\left\langle\!\!\left\langle \tau \sqcup \bigcirc \right\rangle\!\!\right\rangle = \mathsf{q}^{-1} \left\langle\!\left\langle \tau \right\rangle\!\right\rangle \oplus \mathsf{q} \left\langle\!\left\langle \tau \right\rangle\!\right\rangle, \tag{2.17}$$

the tangle in the l.h.s. being a disjoint union of a tangle λ and a circle (cf. (2.12)).

A complex $\langle\!\langle \tau \rangle\!\rangle$ associated to a tangle τ is defined only up to homotopy. We write $\langle\!\langle \tau \rangle\!\rangle_\sharp$ for a particular complex with special properties which represents $\langle\!\langle \tau \rangle\!\rangle$.

Overall, we have the following commutative diagram:

$$\begin{tikzcd} & \mathsf{TL} \arrow[d, "\mathrm{K}_0"] \\ \mathrm{Tng} \arrow[ur, "\langle\!\langle - \rangle\!\rangle"] \arrow[r, "\langle - \rangle"'] & \mathrm{TL} \end{tikzcd} \tag{2.18}$$

where the map K_0 turns the complex (2.16) into the sum (2.4):

$$\mathrm{K}_0(\mathbf{C}) = \sum_{\lambda\in\mathrm{TL}_n}\sum_i a_{\lambda,i}\, q^i\langle\lambda\rangle, \qquad a_{\lambda,i} = \sum_{\substack{j,k\\ j+k=i}} (-1)^k\, c^{\lambda}_{j,k}. \tag{2.19}$$

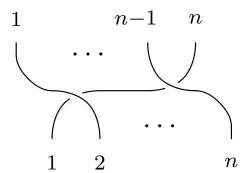
This map is well-defined because the sum in the expression for $a_{\lambda,i}$ is finite for bounded complexes and if complexes $\mathbf{C}$ and $\mathbf{C}'$ are homotopy equivalent, then $\mathrm{K}_0(\mathbf{C}) = \mathrm{K}_0(\mathbf{C}')$ (see [Ros] for the proof).

Since the complex is bounded, the sum in the expression for $a_{\lambda,i}$ is finite.

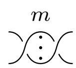
In addition to TL we consider the category TL^+ of complexes bounded from below, that is, the multiplicity coefficients in the sum (2.16) are zero if i is less than certain value. Define the q^+ order of a chain 'module' C_i: $|C_i|_q = \inf\{i+j\colon c^{\lambda}_{i,j} \neq 0\}$. A complex $\mathbf{C}$ in TL^+ is *q^+-bounded* if $\lim_{i\to\infty}|C_i|_q = \infty$. For a q^+-bounded complex, the sum in the expression (2.19) for $a_{\lambda,i}$ is finite, hence the element $\mathrm{K}_0(\mathbf{C})$ is well-defined.

2.2. Results. From now on we assume that $n \geq 2$.

2.2.1. *Infinite torus braid as a Jones–Wenzl projector in a Temperley–Lieb algebra.* A braid with n strands is a particular example of a (n,n)-tangle. A *torus braid* is a braid that can be drawn on a cylinder $\mathbb{S}^1\times[0,1]$ without intersections. In fact, all torus braids have the form $\beta^m_{\mathrm{cyl},n}$, $m\in\mathbb{Z}$, where $\beta_{\mathrm{cyl},n}$ is the elementary clockwise winding torus braid:

(2.20) $\beta_{\mathrm{cyl},n}$ =


We introduce a special notation for the torus braid which corresponds to m full rotations of n strands:



$n = \beta^{mn}_{\mathrm{cyl}}$.

Let $O_+(q^m)$ denote any element of TL^+ of the form

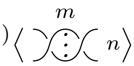
$$\sum_{\lambda\in\mathrm{TL}_n}\sum_{j\geq m} a_{\lambda,j}\, q^j\,\langle\lambda\rangle.$$

We define a *q-order* of an element $\alpha\in\mathrm{TL}^+$ as $|\alpha|_q = \sup\{m\colon \alpha = O_+(q^m)\}$.

Definition 2.1. A sequence of elements $\alpha_1, \alpha_2, \ldots \in \mathrm{TL}^+$ has a *limit* $\lim_{k\to\infty} \alpha_k = \beta$ if $\lim_{i\to\infty} |\beta - \alpha_k|_q = \infty$.

The following theorem may be known, so we do not claim credit for it. It appears here as a by-product and it is an easy corollary of (2.28).

Theorem 2.2. *The TL element corresponding to the infinite torus braid equals the Jones–Wenzl projector:*

(2.21) $\lim\limits_{m\to\infty} q^{\frac{1}{2}mn(n-1)} \big\langle$  $\big\rangle = P_n^+,$

where $P_n^+ \in \mathrm{TL}_n^+$ *corresponds to the Jones–Wenzl projector* $P_n \in \mathrm{QTL}_n$.

In fact, a more general statement is also true:

$$\lim_{m\to\infty} q^{\frac{1}{2}m(n-1)} \langle \beta^m_{\mathrm{cyl},n} \rangle = P_n^+, \tag{2.22}$$

but its proof is more technical and we omit it here.

2.2.2. *A bit of homological calculus.* Consider the chain category $\mathbf{Ch}(\mathsf{A})$ over an additive category A. Define a *chain order* of a complex (2.8) by $|\mathbf{A}|_{\mathrm{c}} = \inf\{m\colon A_m \neq 0\}$. Let $O^{\mathrm{c}}(m)$ be a generic notation of a complex of chain order m. Define the *homotopic order* of a complex $\mathbf{A}$ by $|\mathbf{A}|_{\mathrm{h}} = \sup\{m\colon \mathbf{A} \sim O^{\mathrm{c}}(m)\}$. If $\mathbf{A}$ is contractible, then $|\mathbf{A}|_{\mathrm{h}} = \infty$.

A complex $\mathbf{A}$ is called *chain* (*homotopically*) *small* if it has high chain (homotopic) order. Two complexes connected by a chain morphism $\mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B}$ are considered homotopically if $\mathrm{Cone}(\mathbf{f})$ is homotopically small. In particular, if $\mathbf{f}$ is a homotopy equivalence, then $\mathrm{Cone}(\mathbf{f})$ is contractible, hence it has infinite homological order.

A *direct system* is a sequence of complexes connected by chain morphisms:

$$\mathcal{A} = (\mathbf{A}_0 \xrightarrow{\mathbf{f}_0} \mathbf{A}_1 \xrightarrow{\mathbf{f}_1} \cdots). \tag{2.23}$$

Definition 2.3. A direct system $\mathcal{A}$ is *Cauchy* if $\lim_{i\to\infty} |\mathrm{Cone}(\mathbf{f}_i)|_{\mathrm{h}} = \infty$.

Definition 2.4. A direct system has a *limit* (2) : $\varinjlim \mathcal{A} = \mathbf{A}$, where $\mathbf{A}$ is a chain complex, if there exist chain morphisms $\mathbf{A}_i \xrightarrow{\tilde{\mathbf{f}}_i} \mathbf{A}$ that form commutative triangles

$$\begin{tikzcd} \mathbf{A}_i \arrow[r, "\mathbf{f}_i"'] \arrow[rr, bend left, "\tilde{\mathbf{f}}_i"] & \mathbf{A}_{i+1} \arrow[r, "\tilde{\mathbf{f}}_{i+1}"'] & \mathbf{A} \end{tikzcd}, \qquad \tilde{\mathbf{f}}_i \sim \tilde{\mathbf{f}}_{i+1}\mathbf{f}_i, \tag{2.24}$$

and $\lim_{i\to\infty} |\mathrm{Cone}(\tilde{\mathbf{f}}_i)|_{\mathrm{h}} = \infty$.

(2) This definition differs from the standard categorical definition of a direct limit, but Theorem 3.9 indicates that our definition implies the standard one. We expect that both definitions are equivalent.

In Section 3 we prove the following homology versions of standard theorems about limits (Theorem 3.7, Propositions 3.12 and 3.13):

THEOREM 2.5. *A direct system $\mathcal{A}$ has a limit if and only if it is Cauchy.*

THEOREM 2.6. *The limit of a direct system is unique up to homotopy equivalence.*

Now consider the homotopy category $\mathbf{Kom}(\mathsf{A})$ (we have in mind the particular case of $\mathbf{Kom}(\mathsf{A}) = \mathsf{TL}^+$). The notion of homotopic order of a chain complex, Cauchy property of a direct system and Definition 2.4 of a limit transfer to $\mathbf{Kom}(\mathsf{A})$ for obvious reasons.

2.2.3. *Infinite torus braid as a Jones–Wenzl projector in the universal category.* For a tangle diagram τ let $\langle\!\langle \tau \rangle\!\rangle^{\mathrm{s}}$ denote the categorification complex $\langle\!\langle \tau \rangle\!\rangle$ with h-degree shift proportional to the number $n_\times(\tau)$ of crossings in the diagram τ:

$$\langle\!\langle \tau \rangle\!\rangle^{\mathrm{s}} = \mathsf{h}^{\frac{1}{2} n_\times(\tau)} \langle\!\langle \tau \rangle\!\rangle . \tag{2.25}$$

In Subsection 4.2 we define a direct system of categorification complexes of torus braids connected by special chain morphisms

$$\mathcal{B}_n = \Big(\langle\!\langle \,\vdots\; n \rangle\!\rangle \xrightarrow{\mathbf{f}_0} \langle\!\langle \overset{1}{\cdots}\; n \rangle\!\rangle^{\mathrm{s}} \xrightarrow{\mathbf{f}_1} \cdots \\ \cdots \xrightarrow{\mathbf{f}_{m-1}} \langle\!\langle \overset{m}{\cdots}\; n \rangle\!\rangle^{\mathrm{s}} \xrightarrow{\mathbf{f}_m} \langle\!\langle \overset{m+1}{\cdots}\; n \rangle\!\rangle^{\mathrm{s}} \xrightarrow{\mathbf{f}_{m+1}} \cdots \Big). \tag{2.26}$$

We prove that $|\mathrm{Cone}(\mathbf{f}_m)|_{\mathrm{h}} \geq 2m(n-1)+1$, so $\mathcal{B}_n$ is a Cauchy system and by Theorem 2.5 it has a unique limit: $\varinjlim \mathcal{B}_n = \mathbf{P}_n \in \mathsf{TL}_n^+$.

THEOREM 2.7. *The limiting complex $\mathbf{P}_n$ has the following properties:*

(1) *A composition of $\mathbf{P}_n$ with cap- and cup-tangles is contractible:*

$$\langle\!\langle \overset{i}{(}\, n \rangle\!\rangle \circ \mathbf{P}_n \sim \mathbf{P}_n \circ \langle\!\langle n \overset{i}{)} \rangle\!\rangle \sim 0.$$

(2) *The complex $\mathbf{P}_n$ is idempotent with respect to tangle composition: $\mathbf{P}_n \circ \mathbf{P}_n \sim \mathbf{P}_n$.*

We provide a glimpse into the structure of $\mathbf{P}_n$. A complex $\mathbf{C}$ in TL_n is called 1-*cut* if $\vdots\; n$ never appears in chain 'modules' C_i. A complex $\mathbf{C}$ in TL_n is called *angle-shaped* if the multiplicities $c^\lambda_{i,j}$ of (2.16) satisfy

$$c^\lambda_{i,j} \neq 0 \quad \text{only if } i \geq 0 \text{ and } 0 \leq j \leq i. \tag{2.27}$$

Let $\langle\!\langle \overset{m}{\cdots}\; n \rangle\!\rangle^{\mathrm{s}} \xrightarrow{\tilde{\mathbf{f}}_m} \mathbf{P}_n$ be chain morphisms associated with the limit $\varinjlim \mathcal{B}_n = \mathbf{P}_n$ in accordance with Definition 2.4.

THEOREM 2.8. *There exist* 1*-cut angle-shaped complexes* $\tilde{\mathbf{C}}_{m,n}$ *such that*

$$\mathrm{Cone}(\tilde{\mathbf{f}}_m) \sim \mathsf{h}^{2m(n-1)+1}\,\mathsf{q}^{2m}\,\tilde{\mathbf{C}}_{m,n}.$$

In other words, there exists a distinguished triangle

$$\mathsf{h}^{2m(n-1)}\,\mathsf{q}^{2m}\,\tilde{\mathbf{C}}_{m,n} \xrightarrow{\delta_{\tilde{\mathbf{f}}_m}} \langle\!\langle \overset{m}{\vdots}\; n \rangle\!\rangle^{\mathrm{s}} \xrightarrow{\tilde{\mathbf{f}}_m} \mathbf{P}_n \to \mathsf{h}^{2m(n-1)+1}\,\mathsf{q}^{2m}\,\tilde{\mathbf{C}}_{m,n},$$

in which all morphisms have zero bidegree, and a presentation

$$\mathbf{P}_n \sim \mathrm{Cone}\Big(\mathsf{h}^{2m(n-1)}\,\mathsf{q}^{2m}\,\tilde{\mathbf{C}}_{m,n} \xrightarrow{\delta_{\tilde{\mathbf{f}}_m}} \langle\!\langle \overset{m}{\vdots}\; n \rangle\!\rangle^{\mathrm{s}}\Big), \tag{2.28}$$

where the complex $\tilde{\mathbf{C}}_{m,n}$ is 1-cut and angle-shaped.

At $m = 0$ the formula (2.28) becomes

$$\mathbf{P}_n \sim \mathrm{Cone}\Big(\tilde{\mathbf{C}}_{0,n} \xrightarrow{\delta_{\tilde{\mathbf{f}}_0}} \langle\!\langle \vdots\; n \rangle\!\rangle\Big), \tag{2.29}$$

where the complex $\tilde{\mathbf{C}}_{0,n}$ is 1-cut and angle-shaped.

Since $\tilde{\mathbf{C}}_{0,n}$ is angle-shaped, the complex $\mathrm{Cone}(\delta_{\tilde{\mathbf{f}}_0})$ is also angle-shaped and consequently q^+-bounded. Hence $\mathrm{K}_0(\mathbf{P}_n)$ is well-defined. Also $\mathrm{K}_0(\mathbf{P}_n) \neq 0$, because it contains $\langle\!\langle \vdots\; n \rangle\!\rangle$ with coefficient 1. Theorem 2.7 indicates that $\mathrm{K}_0(\mathbf{P}_n)$ has the defining properties of the Jones–Wenzl projector, hence by uniqueness it is the Jones–Wenzl projector:

COROLLARY 2.9. *The complex* $\mathbf{P}_n$ *categorifies the Jones–Wenzl projector in* TL^+*:*

$$\mathrm{K}_0(\mathbf{P}_n) = P_n. \tag{2.30}$$

3. Elementary homological calculus

3.1. Limits in the category of complexes. Consider a category $\mathbf{Ch}(\mathsf{A})$ of chain complexes associated with an additive category A. The ith *truncation* $\mathbf{t}_{\leq i}\mathbf{A}$ of a chain complex $\mathbf{A}$ is the chain complex $A_i \xrightarrow{d_i} A_{i-1} \to \cdots$. The ith truncation of a chain morphism $\mathbf{f}$ is defined similarly.

Define the *isomorphism order* $|\mathbf{f}|_{\cong}$ of a chain map $\mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B}$ as the largest number i for which the truncated chain morphism $\mathbf{t}_{\leq i}\mathbf{f}$ is an isomorphism of truncated complexes.

REMARK 3.1. Consider a distinguished triangle (2.10). If $\mathbf{A} \cong O^{\mathrm{c}}(m)$, then $|\iota_{\mathbf{f}}|_{\cong} \geq m - 1$.

DEFINITION 3.2. A direct system $\mathcal{A} = (\mathbf{A}_0 \xrightarrow{\mathbf{f}_0} \mathbf{A}_1 \xrightarrow{\mathbf{f}_1} \cdots)$ in $\mathbf{Ch}(\mathsf{A})$ is *stabilizing* if $\lim_{i\to\infty} |\mathbf{f}_i|_{\cong} = \infty$.

Definition 3.3. A direct system $\mathcal{A}$ in $\mathbf{Ch}(\mathsf{A})$ has a *chain limit* $\lim_{\mathbf{Ch}} \mathcal{A} = \mathbf{A}$ if there exist chain morphisms $\mathbf{A}_i \xrightarrow{\tilde{\mathbf{f}}_i} \mathbf{A}$ such that $\tilde{\mathbf{f}}_i = \tilde{\mathbf{f}}_{i+1}\,\mathbf{f}_i$ and $\lim_{i\to\infty} |\tilde{\mathbf{f}}_i|_{\cong} = \infty$.

The following two theorems are easy to prove:

Theorem 3.4. *A direct system has a chain limit if and only if it is stabilizing. If a chain limit exists then it is unique.*

Theorem 3.5. *Suppose that* $\lim_{\mathbf{Ch}} \mathcal{A} = \mathbf{A}$. *Then for a complex* $\mathbf{B}$ *and chain morphisms* $\mathbf{A}_i \xrightarrow{\mathbf{g}_i} \mathbf{B}$ *such that* $\mathbf{g}_i = \mathbf{g}_{i+1}\mathbf{f}_i$, *there exists a unique chain morphism* $\mathbf{A} \xrightarrow{\mathbf{g}} \mathbf{B}$ *such that* $\mathbf{g}_i = \mathbf{g}\tilde{\mathbf{f}}_i$.

Definition 3.6. A sequence of chain morphisms $\mathbf{A} \xrightarrow{\mathbf{f}_0,\mathbf{f}_1,\dots} \mathbf{B}$ has a *chain limit* $\lim_{i\to\infty} \mathbf{f}_i = \mathbf{f}$ if for any N there exists N' such that $\mathbf{t}_{\leq N}\mathbf{f}_i = \mathbf{t}_{\leq N}\mathbf{f}$ for any $i \geq N'$.

3.2. Limits in the homotopy category. A stabilizing direct system is Cauchy, while $\lim_{\mathbf{Ch}} \mathcal{A} = \mathbf{A}$ implies $\varinjlim \mathcal{A} = \mathbf{A}$, hence Definitions 2.3 and 2.4 are expanded versions of stabilization and chain limit which are flexible enough to be transferred to $\mathbf{Kom}(\mathsf{A})$.

Theorem 3.7. *A Cauchy direct system has a limit.*

Proof. Consider a Cauchy system $\mathcal{A} = (\mathbf{A}_0 \xrightarrow{\mathbf{f}_0} \mathbf{A}_1 \xrightarrow{\mathbf{f}_1} \cdots)$ in $\mathbf{Ch}(\mathsf{A})$. We construct another homotopic direct system $\mathcal{A}_\sharp \sim \mathcal{A}$ which is stabilizing. Roughly speaking, we take $\mathbf{A}_{\sharp,0} = \mathbf{A}_0$ and then use the cone construction in $\mathbf{Ch}(\mathsf{A})$ in order to attach to $\mathbf{A}_{\sharp,0}$ chain-small representatives of the cones $\mathrm{Cone}(\mathbf{f}_i)$, one by one. As a result, the sequence $\mathcal{A}_\sharp$ is stabilizing, hence it has a chain limit $\lim_{\mathbf{Ch}} \mathcal{A}_\sharp = \mathbf{A}_\sharp$, and this means that $\varinjlim \mathcal{A} = \mathbf{A}_\sharp$.

Here is a detailed explanation. By Definition 2.3, there exist complexes $\mathbf{C}_i$ such that

$$\mathrm{Cone}(\mathbf{f}_i) \sim \mathbf{C}_i[-1], \qquad |\mathbf{C}_i|_{\mathrm{c}} = m_i, \qquad \lim_{i\to\infty} m_i = \infty. \tag{3.1}$$

The complexes $\mathbf{A}_i$, $\mathbf{A}_{i+1}$ and $\mathbf{C}_i$ form exact triangles:

$$\mathbf{C}_i \xrightarrow{\delta_{\mathbf{f}_i}} \mathbf{A}_i \xrightarrow{\mathbf{f}_i} \mathbf{A}_{i+1} \to \mathbf{C}_i[1]$$

and $\mathbf{A}_{i+1} \sim \mathrm{Cone}(\delta_{\mathbf{f}_i})$. We define recursively a new sequence of complexes $\mathcal{A}_\sharp = (\mathbf{A}_{\sharp,0} \xrightarrow{\iota_{\mathbf{g}_0}} \mathbf{A}_{\sharp,1} \xrightarrow{\iota_{\mathbf{g}_1}} \cdots)$ by the relations $\mathbf{A}_{\sharp,0} = \mathbf{A}_0$, $\mathbf{A}_{\sharp,i} \sim \mathbf{A}_i$ and $\mathbf{A}_{\sharp,i+1} = \mathrm{Cone}(\mathbf{g}_i)$, where the chain morphism $\mathbf{C}_i \xrightarrow{\mathbf{g}_i} \mathbf{A}_{\sharp,i}$ is homotopy equivalent to the chain morphism $\delta_{\mathbf{f}_i}$. In other words,

$$(3.2)\qquad \mathbf{A}_{\sharp,i+1} = \mathrm{Cone}\big(\mathbf{C}_i \xrightarrow{\mathbf{g}_i} \underbrace{\mathrm{Cone}\big(\mathbf{C}_{i-1} \xrightarrow{\mathbf{g}_{i-1}} \cdots \xrightarrow{\mathbf{g}_2} \underbrace{\mathrm{Cone}(\mathbf{C}_1 \xrightarrow{\mathbf{g}_1} \underbrace{\mathrm{Cone}(\mathbf{C}_0 \xrightarrow{\delta_{\mathbf{f}_0}} \mathbf{A}_0)}_{\mathbf{A}_{\sharp,1}})}_{\mathbf{A}_{\sharp,2}}\big)}_{\mathbf{A}_{\sharp,i}}\big)$$

According to Remark 3.1, $|\iota_{\mathbf{g}_i}|_{\cong} \geq m_i$, hence the sequence $\mathcal{A}_\sharp$ is stabilizing, so there exists a chain limit $\lim_{\mathbf{Ch}} \mathcal{A}_\sharp = \mathbf{A}_\sharp$ and consequently $\varinjlim \mathcal{A} = \mathbf{A}_\sharp$. ∎

Simply saying, the complex $\mathbf{A}_\sharp$ is an infinite multi-cone extension of the complex (3.2):

$$(3.3)\qquad \mathbf{A}_\sharp = \cdots \xrightarrow{\mathbf{g}_3} \mathrm{Cone}\big(\mathbf{C}_2 \xrightarrow{\mathbf{g}_2} \mathrm{Cone}(\mathbf{C}_1 \xrightarrow{\mathbf{g}_1} \mathrm{Cone}(\mathbf{C}_0 \xrightarrow{\delta_{\mathbf{f}_0}} \mathbf{A}_0))\big).$$

By Definition 2.4, the limit $\varinjlim \mathcal{A} = \mathbf{A}_\sharp$ implies the existence of morphisms $\tilde{\mathbf{f}}_i$ of equation (2.24). For our applications it is important to express $\mathrm{Cone}(\tilde{\mathbf{f}}_0)$ up to homotopy in terms of the complexes $\mathbf{C}_i$. This can be done by rearranging the infinite multi-cone (3.3) with the help of associativity of cone formation, which exists even within the category $\mathbf{Ch}(\mathsf{A})$:

$$(3.4)\qquad \mathbf{A}_\sharp = \mathrm{Cone}(\tilde{\mathbf{C}} \xrightarrow{\tilde{\mathbf{g}}} \mathbf{A}_0),$$

$$\tilde{\mathbf{C}} = \cdots \xrightarrow{\mathbf{h}_2} \mathrm{Cone}(\mathbf{C}_2[-1] \xrightarrow{\mathbf{h}_1} \mathrm{Cone}(\mathbf{C}_1[-1] \xrightarrow{\mathbf{h}_0} \mathbf{C}_0)),$$

so that $\tilde{\mathbf{f}}_0 \sim \iota_{\tilde{\mathbf{g}}}$, and $\mathrm{Cone}(\tilde{\mathbf{f}}_0) \sim \tilde{\mathbf{C}}[1]$ is expressed in terms of the complexes $\mathbf{C}_i$ arranged into the infinite multi-cone $\tilde{\mathbf{C}}$. Here is a more formal statement.

THEOREM 3.8. *For a Cauchy system $\mathcal{A}$ there exists a stabilizing system $\tilde{\mathcal{C}} = (\mathbf{C}_0 \xrightarrow{\mathbf{h}'_0} \tilde{\mathbf{C}}_1 \xrightarrow{\mathbf{h}'_1} \cdots)$ in $\mathbf{Ch}(\mathsf{A})$ and chain morphisms $\mathbf{C}_i[-1] \xrightarrow{\mathbf{h}_i} \tilde{\mathbf{C}}_i$ such that $\mathrm{Cone}(\mathbf{h}_i) = \tilde{\mathbf{C}}_{i+1}$, $\mathbf{h}'_i = \iota_{\mathbf{h}_i}$ and for the limiting complex $\tilde{\mathbf{C}} = \lim_{\mathbf{Ch}} \tilde{\mathcal{C}}$ there exists a chain morphism $\tilde{\mathbf{C}} \xrightarrow{\tilde{\mathbf{g}}} \mathbf{A}_0$ such that $\mathbf{A}_\sharp = \mathrm{Cone}(\tilde{\mathbf{g}})$, $\tilde{\mathbf{f}}_0 \sim \iota_{\tilde{\mathbf{g}}}$ and consequently $\mathrm{Cone}(\tilde{\mathbf{f}}_0) \sim \tilde{\mathbf{C}}[1]$.*

Proof. Let us recall the associativity of cones in a general setting. For a chain morphism $\mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B}$, a chain morphism $\mathbf{C} \xrightarrow{\mathbf{g}} \mathrm{Cone}(\mathbf{f})$ is a sum: $\mathbf{g} = \mathbf{g}_{\mathbf{A}} \oplus \mathbf{g}_{\mathbf{B}}$,

$$\begin{tikzcd} & \mathbf{A}[1] \arrow[d, "\mathbf{f}"] \\ \mathbf{C} \arrow[ur, "\mathbf{g}_{\mathbf{A}}"] \arrow[r, "\mathbf{g}_{\mathbf{B}}"'] & \mathbf{B} \end{tikzcd}$$

where $\mathbf{C} \xrightarrow{\mathbf{g_A}} \mathbf{A}[1]$ is a chain morphism and $\mathbf{C} \xrightarrow{\mathbf{g_B}} \mathbf{B}$ is a multi-map. Now it is obvious that

$$\text{(3.5)} \qquad \mathrm{Cone}(\mathbf{C} \xrightarrow{\mathbf{g}} \mathrm{Cone}(\mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B})) = \mathrm{Cone}(\mathrm{Cone}(\mathbf{C}[1] \xrightarrow{\mathbf{g_A}} \mathbf{A}) \xrightarrow{\mathbf{g_B}\oplus\mathbf{f}} \mathbf{B}).$$

We apply the associativity relation (3.5) to multi-cones (3.2) consecutively for $i = 1, 2, \ldots$ in order to rearrange them, so $\mathbf{A}_{\sharp,i} = \mathrm{Cone}(\tilde{\mathbf{C}}_i \xrightarrow{\tilde{\mathbf{g}}_i} \mathbf{A}_0)$, while the complexes $\tilde{\mathbf{C}}_i$ and chain morphisms $\tilde{\mathbf{g}}_i$ are defined recursively: $\tilde{\mathbf{C}}_0 = \mathbf{C}_0$, $\tilde{\mathbf{g}}_0 = \delta_{\mathbf{f}_0}$, $\tilde{\mathbf{C}}_{i+1} = \mathrm{Cone}(\mathbf{h}_i)$, and the chain morphisms $\mathbf{C}_i[-1] \xrightarrow{\mathbf{h}_i} \tilde{\mathbf{C}}_i$ and $\tilde{\mathbf{C}}_{i+1} \xrightarrow{\tilde{\mathbf{g}}_{i+1}} \mathbf{A}_0$ are defined by applying the associativity relation (3.5) to the double cone on the second line of the following equation:

$$\text{(3.6)} \qquad \begin{aligned} \mathbf{A}_{\sharp,i+1} &= \mathrm{Cone}(\mathbf{C}_i \xrightarrow{\mathbf{g}_i} \mathbf{A}_{\sharp,i}) \\ &= \mathrm{Cone}(\mathbf{C}_i \xrightarrow{\mathbf{g}_i} \mathrm{Cone}(\tilde{\mathbf{C}}_i \xrightarrow{\tilde{\mathbf{g}}_i} \mathbf{A}_0)) \\ &= \mathrm{Cone}(\mathrm{Cone}(\mathbf{C}_i[-1] \xrightarrow{\mathbf{h}_i} \tilde{\mathbf{C}}_i) \xrightarrow{\tilde{\mathbf{g}}_{i+1}} \mathbf{A}_0) \\ &= \mathrm{Cone}(\tilde{\mathbf{C}}_{i+1} \xrightarrow{\tilde{\mathbf{g}}_{i+1}} \mathbf{A}_0). \end{aligned}$$

The distinguished triangles

$$\mathbf{C}_i[-1] \xrightarrow{\mathbf{h}_i} \tilde{\mathbf{C}}_i \xrightarrow{\iota_{\mathbf{h}_i}} \tilde{\mathbf{C}}_{i+1} \to \mathbf{C}_i$$

determine chain morphisms $\mathbf{h}'_i = \iota_{\mathbf{h}_i}$ of the direct system $\tilde{\mathcal{C}} = (\tilde{\mathbf{C}}_0 \xrightarrow{\mathbf{h}'_0} \tilde{\mathbf{C}}_1 \xrightarrow{\mathbf{h}'_1} \cdots)$. By Remark 3.1 it has a chain limit: $\lim_{\mathbf{Ch}} \tilde{\mathcal{C}} = \tilde{\mathbf{C}}$, which is an infinite multi-cone of (3.4). The chain morphisms $\tilde{\mathbf{C}}_i \xrightarrow{\mathbf{h}'_i} \tilde{\mathbf{C}}_{i+1}$ satisfy $\tilde{\mathbf{g}}_i = \tilde{\mathbf{g}}_{i+1}\,\mathbf{h}'_i$, so by Theorem 3.5 there exists a unique chain morphism $\tilde{\mathbf{C}} \xrightarrow{\tilde{\mathbf{g}}} \mathbf{A}_0$ such that $\tilde{\mathbf{g}}_i = \tilde{\mathbf{g}}\,\tilde{\mathbf{h}}'_i$.

It is easy to show that $\mathbf{A}_\sharp = \mathrm{Cone}(\tilde{\mathbf{C}} \xrightarrow{\tilde{\mathbf{g}}} \mathbf{A}_0)$, and $\tilde{\mathbf{f}}_0 = \iota_{\tilde{\mathbf{g}}}$, hence $\mathrm{Cone}(\tilde{\mathbf{f}}_0) \sim \tilde{\mathbf{C}}[1]$. ∎

It is easy to prove the analog of Theorem 3.5:

Theorem 3.9. *Suppose that a direct system* (2.23) *has a limit* $\varinjlim \mathcal{A} = \mathbf{A}_\sharp$. *For a complex* $\mathbf{B}$ *and chain morphisms* $\mathbf{A}_i \xrightarrow{\mathbf{g}_i} \mathbf{B}$ *such that* $\mathbf{g}_i \sim \mathbf{g}_{i+1}\mathbf{f}_i$, *there exists a unique* (*up to homotopy*) *chain morphism* $\mathbf{A}_\sharp \xrightarrow{\mathbf{g}} \mathbf{B}$ *which forms commutative triangles*

$$\text{(3.7)} \qquad \begin{tikzcd} \mathbf{A}_i \arrow[r, "\tilde{\mathbf{f}}_i"'] \arrow[rr, bend left, "\mathbf{g}_i"] & \mathbf{A}_{i+1} \arrow[r, "\mathbf{g}"'] & \mathbf{B} \end{tikzcd}, \qquad \mathbf{g}_i \sim \mathbf{g}\tilde{\mathbf{f}}_i.$$

In order to complete the proof of Theorems 2.5 and 2.6, we need two simple propositions. The first one establishes a triangle inequality for homological orders of cones.

PROPOSITION 3.10. *If three chain morphisms form a commutative triangle*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\mathbf{f_{AB}}"'] \arrow[rr, bend left, "\mathbf{f_{AC}}"] & \mathbf{B} \arrow[r, "\mathbf{f_{BC}}"'] & \mathbf{C} \end{tikzcd}, \qquad \mathbf{f_{AC}} \sim \mathbf{f_{BC}}\mathbf{f_{AB}}. \tag{3.8}$$

then the homological orders of their cones satisfy the inequalities

$$|\mathrm{Cone}(\mathbf{f_{AB}})|_{\mathrm{h}} \geq \min(|\mathrm{Cone}(\mathbf{f_{AC}})|_{\mathrm{h}}, |\mathrm{Cone}(\mathbf{f_{BC}})|_{\mathrm{h}} - 1), \tag{3.9}$$

$$|\mathrm{Cone}(\mathbf{f_{BC}})|_{\mathrm{h}} \geq \min(|\mathrm{Cone}(\mathbf{f_{AB}})|_{\mathrm{h}} + 1, |\mathrm{Cone}(\mathbf{f_{AC}})|_{\mathrm{h}}). \tag{3.10}$$

Proof. If chain morphisms form a commutative triangle (3.8), then their cones form a distinguished triangle

$$\mathrm{Cone}(\mathbf{f_{AB}}) \xrightarrow{\mathbf{g}_1} \mathrm{Cone}(\mathbf{f_{AC}}) \xrightarrow{\mathbf{g}_2} \mathrm{Cone}(\mathbf{f_{BC}}) \xrightarrow{\mathbf{g}_3} \mathrm{Cone}(\mathbf{f_{AB}})[1],$$

so the first inequality follows from the relation $\mathrm{Cone}(\mathbf{f_{AB}}) \sim \mathrm{Cone}(\mathbf{g}_2)[-1]$ and the second from $\mathrm{Cone}(\mathbf{f_{BC}}) \sim \mathrm{Cone}(\mathbf{g}_1)$. ∎

The second proposition says that if a complex is homologically infinitely small then it is contractible.

PROPOSITION 3.11. *If $|\mathbf{A}|_{\mathrm{h}} = \infty$ then $\mathbf{A}$ is contractible.*

Proof. Since $|\mathbf{A}|_{\mathrm{h}} = \infty$, there exist complexes $\mathbf{A}_i \sim \mathbf{A}$ such that $\mathbf{A}_i = O^{\mathrm{c}}(m_i)$ and $\lim_{i\to\infty} m_i = \infty$. Consider a sequence of chain morphisms establishing homotopy equivalence between the complexes:

$$\mathbf{A} \underset{\mathbf{g}_0}{\overset{\mathbf{f}_0}{\rightleftarrows}} \mathbf{A}_1 \underset{\mathbf{g}_1}{\overset{\mathbf{f}_1}{\rightleftarrows}} \mathbf{A}_2 \rightleftarrows \cdots \rightleftarrows \mathbf{A}_i \underset{\mathbf{g}_i}{\overset{\mathbf{f}_i}{\rightleftarrows}} \mathbf{A}_{i+1} \rightleftarrows \cdots, \qquad \mathbb{1}_{\mathbf{A}_i} - \mathbf{g}_i\mathbf{f}_i = \mathbf{d}_i\mathbf{h}_i + \mathbf{h}_i\mathbf{d}_i,$$

where $\mathbb{1}_{\mathbf{A}_i}$ is the identity chain morphism of $\mathbf{A}_i$, while $\mathbf{A}_i[1] \xrightarrow{\mathbf{h}_i} \mathbf{A}_i$ is a homotopy chain morphism (it does not commute with the chain differential $\mathbf{d}_i$ in the complex $\mathbf{A}_i$).

Consider the compositions $\hat{\mathbf{f}}_i = \mathbf{f}_i \cdots \mathbf{f}_1\mathbf{f}_0$, $\hat{\mathbf{g}}_i = \mathbf{g}_0\mathbf{g}_1 \cdots \mathbf{g}_i$ and $\hat{\mathbf{h}}_i = \hat{\mathbf{g}}_{i-1}\mathbf{h}_i\hat{\mathbf{f}}_{i-1}$. It is easy to see that $\hat{\mathbf{g}}_{i-1}\hat{\mathbf{f}}_{i-1} - \hat{\mathbf{g}}_i\hat{\mathbf{f}}_i = \mathbf{d}\hat{\mathbf{h}}_i + \hat{\mathbf{h}}_i\mathbf{d}$, hence $\mathbb{1}_{\mathbf{A}} - \hat{\mathbf{g}}_i\hat{\mathbf{f}}_i = \mathbf{d}\check{\mathbf{h}}_i + \check{\mathbf{h}}_i\mathbf{d}$, where $\check{\mathbf{h}}_i = \hat{\mathbf{h}}_0 + \hat{\mathbf{h}}_1 + \cdots + \hat{\mathbf{h}}_i$. There is a limit (cf. Definition 3.6) $\lim_{i\to\infty} \check{\mathbf{h}}_i = \check{\mathbf{h}}$, while $\lim_{i\to\infty} \hat{\mathbf{g}}_i\hat{\mathbf{f}}_i = 0$, hence $\mathbb{1}_{\mathbf{A}} = \mathbf{d}\check{\mathbf{h}} + \check{\mathbf{h}}\mathbf{d}$, which means that the complex $\mathbf{A}$ is contractible. ∎

PROPOSITION 3.12. *If a direct system $\mathcal{A}$ has a limit, then it is Cauchy.*

Proof. The inequality (3.9) applied to the commutative triangle (2.24) says that

$$|\mathrm{Cone}(\mathbf{f}_i)|_{\mathrm{h}} \geq \min(|\mathrm{Cone}(\tilde{\mathbf{f}}_i)|_{\mathrm{h}}, |\mathrm{Cone}(\tilde{\mathbf{f}}_{i+1})|_{\mathrm{h}} - 1),$$

hence the limit $\lim_{i\to\infty} |\mathrm{Cone}(\tilde{\mathbf{f}}_i)|_{\mathrm{h}} = \infty$ implies the Cauchy property of $\mathcal{A}$.

PROPOSITION 3.13. *If a direct system $\mathcal{A}$ has a limit, then it is unique.*

Proof. If $\mathcal{A}$ has a limit then by Proposition 3.12 it is Cauchy. Hence it has a special limit $\mathbf{A}_\sharp$ described in the proof of Theorem 3.7. If $\mathcal{A}$ has another limit $\mathbf{A}'$ with chain morphisms $\mathbf{A}_i \xrightarrow{\tilde{\mathbf{f}}'_i} \mathbf{A}'$ then by Theorem 3.9 there is a chain morphism $\mathbf{A}_\sharp \xrightarrow{\mathbf{g}} \mathbf{A}'$ with commutative triangles (3.7). The inequality (3.10) says

$$|\mathrm{Cone}(\mathbf{g})|_{\mathrm{h}} \geq \min\big(|\mathrm{Cone}(\tilde{\mathbf{f}}_i)|_{\mathrm{h}} + 1, |\mathrm{Cone}(\mathbf{g}_i)|_{\mathrm{h}}\big).$$

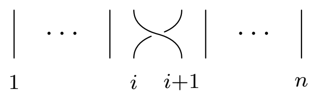
Since both cones in the r.h.s. become homologically infinitely small as $i\to\infty$, the cone $\mathrm{Cone}(\mathbf{g})$ is also homologically infinitely small. Then Proposition 3.11 says that $\mathrm{Cone}(\mathbf{g})$ is contractible; as a result $\mathbf{A}' \sim \mathbf{A}_\sharp$. ∎

We end this section with a theorem which follows easily from Definition 2.4.

THEOREM 3.14. *If a direct system $\mathcal{A}$ satisfies $\lim_{i\to\infty} |\mathbf{A}_i|_{\mathrm{h}} = \infty$ then its limit is contractible: $\varinjlim \mathcal{A} = 0$.*

4. A direct system of categorification complexes of torus braids

4.1. A special categorification complex of a negative braid. Let σ_i denote an elementary negative n-strand braid:

$\sigma_i =$ 

THEOREM 4.1. *If an n-strand braid β can be presented as a product of elementary negative braids: $\beta = \sigma_{i_k} \cdots \sigma_{i_2}\sigma_{i_1}$, then its categorification complex has a special presentation $\langle\!\langle\beta\rangle\!\rangle_\sharp$:*

$$\langle\!\langle\beta\rangle\!\rangle^{\mathrm{s}}_\sharp = \Big(\cdots \to C_2 \to C_1 \to \langle\!\langle\, \overline{\underline{\,\vdots\,}}\; n \rangle\!\rangle\Big) \tag{4.1}$$

such that the complex

$$\mathbf{C} = \mathsf{h}^{-1}(\cdots \to C_2 \to C_1) \tag{4.2}$$

is 1-cut and angle-shaped.

More abstractly, the theorem says that there exists a 1-cut and angle-shaped complex $\mathbf{C}$ and a chain morphism $\mathbf{C} \to \langle\!\langle\, \overline{\underline{\,\vdots\,}}\; n \rangle\!\rangle$ such that $\langle\!\langle\beta\rangle\!\rangle^{\mathrm{s}} \sim \mathrm{Cone}\Big(\mathbf{C} \to \langle\!\langle\, \overline{\underline{\,\vdots\,}}\; n \rangle\!\rangle\Big)$.

REMARK 4.2. Theorem 4.1 implies that the special complex $\langle\!\langle\beta\rangle\!\rangle^{\mathrm{s}}_\sharp$ is angle-shaped.

Proof of Theorem 4.1. First of all, we define a special complex $\langle\!\langle\sigma_i \circ \lambda\rangle\!\rangle^{\mathrm{s}}_\sharp$ for the composition of a TL (n,n)-tangle λ and an elementary negative braid σ_i, $1 \le i \le n-1$. If the composition $\overset{i}{\subset}{}_n \circ \lambda$ does not contain a disjoint circle, then, in accordance with (2.14), we define the special categorification complex of $\sigma_i \circ \lambda$ as

$$\langle\!\langle \sigma_i \circ \lambda \rangle\!\rangle^{\mathsf{s}}_{\sharp} = \left(\langle\!\langle U_{n,i} \circ \lambda \rangle\!\rangle_1 \to \langle\!\langle \lambda \rangle\!\rangle_0 \right). \tag{4.3}$$

If $\overset{i}{\subset}{}_n \circ \lambda$ contains a disjoint circle, then λ must have the form ${}_n\overset{i}{\supset} \circ \lambda'$. Hence $\sigma_i \circ \lambda = \sigma_i \circ {}_n\overset{i}{\supset} \circ \lambda'$. The tangle $\sigma_i \circ {}_n\overset{i}{\supset}$ is the same as ${}_n\overset{i}{\supset}$ with a positive framing twist, so according to (2.15), $\langle\!\langle \sigma_i \circ {}_n\overset{i}{\supset} \rangle\!\rangle \sim \mathsf{h}^{1/2}\mathsf{q} \langle\!\langle {}_n\overset{i}{\supset} \rangle\!\rangle$. Hence in this case we define the special categorification complex of $\sigma_i \circ \lambda$ simply as shifted $\langle\!\langle \lambda \rangle\!\rangle$:

$$\langle\!\langle \sigma_i \circ \lambda \rangle\!\rangle^{\mathsf{s}}_{\sharp} = \mathsf{hq}\, \langle\!\langle \lambda \rangle\!\rangle\, . \tag{4.4}$$

Now we define a recursive algorithm for constructing the complex $\langle\!\langle \beta \rangle\!\rangle^{\mathsf{s}}_{\sharp}$. For $\beta = \,\vdots\, n$ we define $\langle\!\langle \beta \rangle\!\rangle^{\mathsf{s}}_{\sharp} = \langle\!\langle \,\vdots\, n \rangle\!\rangle$. Suppose that we defined the special complex $\langle\!\langle \beta \rangle\!\rangle^{\mathsf{s}}_{\sharp}$ for a braid $\beta = \sigma_{i_k} \circ \cdots \circ \sigma_{i_1}$. We define the special categorification complex of a braid $\beta' = \sigma_{i_{k+1}} \circ \beta$ by applying the rules (4.3) and (4.4) to all constituent tangles λ in the complex $\langle\!\langle \beta \rangle\!\rangle_{\sharp}$ (see (2.16)).

We prove the properties of $\langle\!\langle \beta \rangle\!\rangle^{\mathsf{s}}_{\sharp}$ by induction over k. If $k = 0$ then $\beta = \,\vdots\, n$ and the properties of $\langle\!\langle \beta \rangle\!\rangle^{\mathsf{s}}_{\sharp}$ are obvious.

Suppose that the special categorification complex $\langle\!\langle \beta \rangle\!\rangle^{\mathsf{s}}_{\sharp}$ of a braid $\beta = \sigma_{i_k} \circ \cdots \circ \sigma_{i_1}$ has the form (4.1) and its tail (4.2) is 1-cut and angle-shaped. Consider a longer braid $\beta' = \sigma_{i_{k+1}} \circ \beta$. The object $\langle\!\langle \,\vdots\, n \rangle\!\rangle$ may appear in $\langle\!\langle \beta' \rangle\!\rangle^{\mathsf{s}}_{\sharp}$ if and only if $\lambda = \,\vdots\, n$ and the extra crossing $\sigma_{i_{k+1}}$ is negatively spliced in (4.3), hence $\langle\!\langle \beta' \rangle\!\rangle^{\mathsf{s}}_{\sharp}$ has the form (4.1) and its tail (4.2) is 1-cut.

If the negative crossing $\sigma_{i_{k+1}}$ is composed with the head $\langle\!\langle \,\vdots\, n \rangle\!\rangle$ of the complex (4.1), then the formula (4.3) applies and the tangle U_{n,i_k+1} appearing in the tail of $\langle\!\langle \beta' \rangle\!\rangle^{\mathsf{s}}_{\sharp}$ satisfies (2.27).

If the crossing $\sigma_{i_{k+1}}$ is composed with a TL tangle λ from the ith chain 'module' C_i (see (2.16)) in the tail of the complex $\langle\!\langle \beta \rangle\!\rangle^{\mathsf{s}}_{\sharp}$ with the q-degree shift j satisfying the inequality $0 \le j \le i-1$, then the shifted objects in the r.h.s. of (4.3) and (4.4) also satisfy this inequality. ∎

The picture (2.20) presents a torus braid as a product of negative crossings, hence we obtain

Corollary 4.3. *A torus braid* $\overset{m}{\asymp\!\vdots\!\asymp}\, n$ *has a special angle-shaped categorification complex* $\langle\!\langle\, \overset{m}{\asymp\!\vdots\!\asymp}\, n \rangle\!\rangle^{\mathsf{s}}_{\sharp}$. *In particular, for* $m = 1$,

$$\langle\!\langle\, \overset{1}{\asymp\!\vdots\!\asymp}\, n \rangle\!\rangle^{\mathsf{s}}_{\sharp} = \mathrm{Cone}\Big(\mathbf{C}_{1,n} \to \langle\!\langle\, \overline{\underline{\,\vdots\,}}\; n \rangle\!\rangle\Big), \tag{4.5}$$

where the complex $\mathbf{C}_{1,n}$ *is* 1*-cut and angle-shaped.*

4.2. Special morphisms between torus braid complexes. Relation (4.5) indicates that there is a distinguished triangle

$$\mathbf{C}_{1,n} \to \langle\!\langle\, \overline{\underline{\,\vdots\,}}\; n \rangle\!\rangle \xrightarrow{\mathbf{f}_1} \langle\!\langle\, \overset{1}{\asymp\!\vdots\!\asymp}\, n \rangle\!\rangle^{\mathsf{s}} \to \mathsf{h}\mathbf{C}_{1,n}$$

and

$$\mathrm{Cone}(\mathbf{f}_1) \sim \mathsf{h}\mathbf{C}_{1,n}. \tag{4.6}$$

Composing both sides of the morphism $\mathbf{f}_1$ with the torus braid complex $\langle\!\langle\, \overset{m}{\asymp\!\vdots\!\asymp}\, n \rangle\!\rangle^{\mathsf{s}}$, we get a morphism

$$\langle\!\langle\, \overset{m}{\asymp\!\vdots\!\asymp}\, n \rangle\!\rangle^{\mathsf{s}} \xrightarrow{\mathbf{f}_m} \langle\!\langle\, \overset{m+1}{\asymp\!\vdots\!\asymp}\, n \rangle\!\rangle^{\mathsf{s}}$$

such that

$$\mathrm{Cone}(\mathbf{f}_m) \sim \mathrm{Cone}(\mathbf{f}_1) \circ \langle\!\langle\, \overset{m}{\asymp\!\vdots\!\asymp}\, n \rangle\!\rangle^{\mathsf{s}}. \tag{4.7}$$

Theorem 4.4. *The cone* (4.7) *can be presented by a shifted complex*

$$\mathrm{Cone}(\mathbf{f}_m) \sim \mathsf{h}\,(\mathsf{h}^{n-1}\mathsf{q})^{2m}\, \mathbf{C}_{m,n}$$

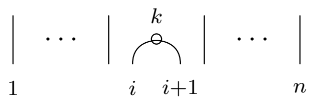
such that $\mathbf{C}_{m,n}$ *is* 1*-cut and angle-shaped.*

The proof is based on a simple geometric lemma:

Lemma 4.5. *For* $n \geq 2$*, the following two compositions of framed tangles are isotopic:*

$$\overset{i}{\subset}\, n \;\circ\; \overset{1}{\asymp\!\vdots\!\asymp}\, n \quad \textit{and} \quad \overset{1}{\asymp\!\vdots\!\asymp}\, n{-}2 \;\circ\; 2\,\overset{i}{\phi\!\!\subset}\, n \tag{4.8}$$

where $2\,\overset{i}{\phi\!\!\subset}\, n$ *is the tangle* $\overset{i}{\subset}\, n$ *with double framing twist on the cap:*

$$k\,\overset{i}{\phi\!\!\subset}\, n =$$ 

Proof. This lemma is geometrically obvious: a cap on a pair of adjacent strands slides down through the torus braid to the bottom. ∎

An immediate corollary of (4.8) and of the framing change formula (2.15) is the following relation:

$$\langle\!\langle\, \overset{i}{\subset}\, n \;\circ \overset{m}{\asymp\!\vdots\!\asymp}\, n \rangle\!\rangle^{\mathsf{s}} \sim \big(\mathsf{h}^{n-1}\mathsf{q}\big)^{2m} \langle\!\langle\, \overset{m}{\asymp\!\vdots\!\asymp}\, n{-}2 \;\circ\; \overset{i}{\subset}\, n \,\rangle\!\rangle^{\mathsf{s}}. \tag{4.9}$$

In order to prove Theorem 4.4, we need three simple propositions. For a positive integer $d \le n/2$, let $\mathbf{I} = (i_1, \dots, i_d)$ be a sequence of positive integer numbers such that $i_k < n - 2k + 2$ for all $k \in \{1, \dots, d\}$. A *cap-tangle* $\overset{\mathbf{I}}{(\!(}\, n$ is an $(n, n-2d)$-tangle which can be presented as a product of d tangles of the form $\overset{i}{(}\, m$:

$$\overset{\mathbf{I}}{(\!(}\, n = \overset{i_d}{(}\; n{-}2d{+}2 \circ \cdots \circ \overset{i_2}{(}\; n{-}2 \circ \overset{i_1}{(}\; n.$$

A *cup-tangle* $n\, \overset{\mathbf{I}}{)\!)}$ is defined similarly:

$$n\, \overset{\mathbf{I}}{)\!)} = n\; \overset{i_1}{)} \circ n{-}2\; \overset{i_2}{)} \circ \cdots \circ n{-}2d{+}2\; \overset{i_d}{)}\, .$$

The first proposition is obvious:

PROPOSITION 4.6. *Every* TL (n,n)*-tangle* λ *has a presentation*

$$\lambda = n\, \overset{\mathbf{I}'}{)\!)} \circ \overset{\mathbf{I}}{(\!(}\, n, \qquad |\mathbf{I}| = |\mathbf{I}'|. \tag{4.10}$$

The number $d_\lambda = |\mathbf{I}| = |\mathbf{I}'|$ is determined by the tangle λ and we call it the *cap-degree* (or *cup-degree*) of λ.

The second proposition is also obvious:

PROPOSITION 4.7. *If at least one of two complexes* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *in* TL_n *is* 1*-cut then their composition* $\mathbf{C}_1 \circ \mathbf{C}_2$ *is* 1*-cut.*

Note that even if both complexes are angle-shaped, then their composition is not necessarily angle-shaped. Indeed, in contrast to the homological degree, the q-degree is not additive with respect to the composition of tangles: if the composition of two TL tangles contains a disjoint circle then the q-degree shifts of the rule (2.17) violate additivity. However, if the upper tangle in the composition has no caps or the lower tangle has no cups then no circles are created and the angle shape is maintained:

PROPOSITION 4.8. *If a complex* $\mathbf{C}$ *in* $\mathsf{TL}_{n-2d_\lambda}$ *is angle-shaped, then the complexes* $\langle\!\langle n\, \overset{\mathbf{I}}{)\!)} \rangle\!\rangle \circ \mathbf{C}$ *and* $\mathbf{C} \circ \langle\!\langle \overset{\mathbf{I}}{(\!(}\, n \rangle\!\rangle$ *are also angle-shaped.*

Proof of Theorem 4.4. In order to construct the 1-cut and angle-shaped complex $\mathbf{C}_{m,n}$, we use the presentation

$$\mathrm{Cone}(\mathbf{f}_m) \sim \mathsf{h}\, \mathbf{C}_{1,n} \circ \langle\!\langle \overset{m}{\rangle\!\vdots\!\langle}\, n \rangle\!\rangle^{\mathrm{s}}, \tag{4.11}$$

which follows from (4.7) and (4.6). We construct $\mathbf{C}_{m,n}$ by simplifying the complexes $\langle\!\langle \lambda \circ \overset{m}{\rangle\!\vdots\!\langle}\, n \rangle\!\rangle^{\mathrm{s}}$ for TL (n,n)-tangles λ appearing in the chain 'modules' of $\mathbf{C}_{1,n}$, with the help of (4.9), thus creating the necessary degree shifts, and then using Corollary 4.3, which says that emerging torus braids have angle-shaped categorification complexes.

Let $\mathsf{h}^i\mathsf{q}^j\,\langle\!\langle\lambda\rangle\!\rangle$ be an object appearing in the ith chain 'module' of $\mathbf{C}_{1,n}$ with a non-zero multiplicity (we made its homological degree explicit by including the shift h^i). We apply (4.9) consequently to every cap $\overset{k}{(}\ n$ appearing in the cap-tangle $\overset{\mathbf{I}}{(\!(}\ n$ in the presentation (4.10) of λ:

$$\langle\!\langle\lambda\rangle\!\rangle\circ\Big\langle\!\Big\langle\ \overset{m}{\rangle\!\vdots\!\langle}\ n\Big\rangle\!\Big\rangle^{\mathrm{s}}\ \sim\ (\mathsf{h}^{n-1}\mathsf{q})^{2m}\,\mathbf{C}_{\lambda,m,n},\tag{4.12}$$

where

$$\mathbf{C}_{\lambda,m,n}=(\mathsf{h}^{n-d_\lambda-1}\mathsf{q}^{d_\lambda-1})^{2m}\Bigg(\Big\langle\!\Big\langle n\ \overset{\mathbf{I}'}{)\!)}\ \Big\rangle\!\Big\rangle\circ\Big\langle\!\Big\langle\ \overset{m}{\rangle\!\vdots\!\langle}\ n-2d_\lambda\Big\rangle\!\Big\rangle^{\mathrm{s}}_{\sharp}\circ\Big\langle\!\Big\langle\ \overset{\mathbf{I}}{(\!(}\ n\Big\rangle\!\Big\rangle\Bigg).\tag{4.13}$$

The object $\langle\!\langle\lambda\rangle\!\rangle$ comes from the 1-cut complex $\mathbf{C}_{1,n}$, hence $d_\lambda>0$ and the complex in big brackets in the r.h.s. of (4.13) is 1-cut in view of Proposition 4.7. Proposition 4.8 implies that the complex

$$\Big\langle\!\Big\langle n\ \overset{\mathbf{I}'}{)\!)}\ \Big\rangle\!\Big\rangle\circ\Big\langle\!\Big\langle\ \overset{m}{\rangle\!\vdots\!\langle}\ n-2d_\lambda\Big\rangle\!\Big\rangle^{\mathrm{s}}_{\sharp}\circ\Big\langle\!\Big\langle\ \overset{\mathbf{I}}{(\!(}\ n\Big\rangle\!\Big\rangle$$

is also angle-shaped. It remains angle-shaped after the shift $(\mathsf{h}^{n-d_\lambda-1}\mathsf{q}^{d_\lambda-1})^{2m}$, as well as after the shift $\mathsf{h}^i\mathsf{q}^j$ which accompanies $\langle\!\langle\lambda\rangle\!\rangle$ in $\mathbf{C}_{1,n}$, because the latter complex is also angle-shaped. The complex $\mathbf{C}_{1,n}\circ\Big\langle\!\Big\langle\ \overset{m}{\rangle\!\vdots\!\langle}\ n\Big\rangle\!\Big\rangle^{\mathrm{s}}$ in the r.h.s. of (4.11) is composed of complexes (4.12) with shifts $\mathsf{h}^i\mathsf{q}^j$, hence Theorem 4.4 is proved. ∎

5. A categorified Jones–Wenzl projector. Consider the direct system (2.26). Theorem 4.4 implies that $|\mathrm{Cone}(\mathbf{f}_m)|_{\mathrm{h}}\geq 2m(n-1)+1$, hence $\mathcal{B}_n$ is Cauchy and it has a unique limit $\varinjlim\mathcal{B}_n=\mathbf{P}_n\in\mathsf{TL}_n^+$.

Now we prove Theorems 2.7 and 2.8, which describe the properties of $\mathbf{P}_n$.

Proof of Theorem 2.8. Consider a tail of the direct system (2.26):

$$\mathcal{B}_{m,n}=\Bigg(\Big\langle\!\Big\langle\ \overset{m}{\rangle\!\vdots\!\langle}\ n\Big\rangle\!\Big\rangle^{\mathrm{s}}\xrightarrow{\mathbf{f}_m}\Big\langle\!\Big\langle\ \overset{m+1}{\rangle\!\vdots\!\langle}\ n\Big\rangle\!\Big\rangle^{\mathrm{s}}\xrightarrow{\mathbf{f}_{m+1}}\cdots\Bigg)\to\mathbf{P}_n.$$

According to Theorem 3.8, the limit $\mathbf{P}_n$ can be presented as a cone (2.28), where $\tilde{\mathbf{C}}_{m,n}$ is an infinite multi-cone:

$$\begin{aligned}\tilde{\mathbf{C}}_{m,n}=\cdots\to\mathrm{Cone}(\mathsf{h}^{-1}(\mathsf{h}^{n-1}\mathsf{q})^{2k}\,\mathbf{C}_{m+k,n}\to\cdots\\ \to\mathrm{Cone}(\mathsf{h}^{2n-3}\mathsf{q}^2\,\mathbf{C}_{m+1,n}\to\mathbf{C}_{m,n})\cdots)\end{aligned}$$

with 1-cut and angle-shaped complexes $\mathbf{C}_{m,n}$ introduced in Theorem 4.4. Hence the complex $\tilde{\mathbf{C}}_{m,n}$ itself is 1-cut and angle-shaped. ∎

Proof of part (1) of Theorem 2.7. The tangle composition with $\langle\!\langle \overset{i}{\subset}\, {}_n \rangle\!\rangle$ is a 'continuous' functor, that is, it can be applied to both sides of $\varinjlim \mathcal{B}_n = \mathbf{P}_n$, hence $\varinjlim \langle\!\langle \overset{i}{\subset}\, {}_n \rangle\!\rangle \circ \mathcal{B}_n = \langle\!\langle \overset{i}{\subset}\, {}_n \rangle\!\rangle \circ \mathbf{P}_n$. According to (4.9),

$$
\begin{aligned}
&\langle\!\langle \overset{i}{\subset}\, {}_n \rangle\!\rangle \circ \mathcal{B}_n \\
&\quad = \Big(\langle\!\langle \overset{i}{\subset}\, {}_n \rangle\!\rangle \circ \langle\!\langle \overline{\underline{\vdots}}\; {}_n \rangle\!\rangle \to \cdots \to \langle\!\langle \overset{i}{\subset}\, {}_n \rangle\!\rangle \circ \langle\!\langle \overset{m}{\asymp\!\vdots\!\asymp}\; {}_n \rangle\!\rangle^{\mathrm{s}} \to \cdots \Big) \\
&\quad \sim \Big(\langle\!\langle \overset{i}{\subset}\, {}_n \rangle\!\rangle \to \cdots \to (\mathsf{h}^{n-1}\mathsf{q})^{2m} \langle\!\langle \overset{m}{\asymp\!\vdots\!\asymp}\; {}_{n-2} \rangle\!\rangle^{\mathrm{s}} \circ \langle\!\langle \overset{i}{\subset}\, {}_n \rangle\!\rangle \to \cdots \Big).
\end{aligned}
$$

Since

$$
\Big| (\mathsf{h}^{n-1}\mathsf{q})^{2m} \langle\!\langle \overset{m}{\asymp\!\vdots\!\asymp}\; {}_{n-2} \rangle\!\rangle^{\mathrm{s}} \circ \langle\!\langle \overset{i}{\subset}\, {}_n \rangle\!\rangle \Big|_{\mathrm{h}} \geq 2m(n-1) \xrightarrow[m\to\infty]{} \infty,
$$

according to Theorem 3.14, $\varinjlim \langle\!\langle \overset{i}{\subset}\, {}_n \rangle\!\rangle \circ \mathcal{B}_n = 0$, hence $\langle\!\langle \overset{i}{\subset}\, {}_n \rangle\!\rangle \circ \mathbf{P}_n$ is contractible. ■

Remark 5.1. The contractibility of $\mathbf{P}_n \circ \langle\!\langle {}_n \overset{i}{\supset} \rangle\!\rangle$ is proved similarly.

Corollary 5.2. *If* $\mathbf{C}$ *is a* 1*-cut complex in* TL_n^+*, then* $\mathbf{C} \circ \mathbf{P}_n$ *is contractible.*

Proof of part (2) of Theorem 2.7. According to (2.29),

$$
\begin{aligned}
\mathbf{P}_n \circ \mathbf{P}_n &\sim \mathrm{Cone}\Big(\tilde{\mathbf{C}}_{0,n} \longrightarrow \langle\!\langle \overline{\underline{\vdots}}\; {}_n \rangle\!\rangle \Big) \circ \mathbf{P}_n \\
&\sim \mathrm{Cone}\Big(\tilde{\mathbf{C}}_{0,n} \circ \mathbf{P}_n \longrightarrow \langle\!\langle \overline{\underline{\vdots}}\; {}_n \rangle\!\rangle \circ \mathbf{P}_n \Big) \sim \mathbf{P}_n,
\end{aligned}
$$

where we used the fact that $\tilde{\mathbf{C}}_{0,n}$ is 1-cut and Corollary 5.2 in order to establish the last homotopy equivalence. ■

Proof of Theorem 2.2. The complexes $\mathbf{P}_n$, $\tilde{\mathbf{C}}_{m,n}$ and $\langle\!\langle \overset{m}{\asymp\!\vdots\!\asymp}\; {}_n \rangle\!\rangle^{\mathrm{s}}$ in (2.28) are angle-shaped, hence they are q^+-bounded and their K_0 images are well-defined. Applying K_0 to this equation and taking into account (2.30) and the definition (2.25), we find

$$
P_n = q^{\frac{1}{2}mn(n-1)} \langle \overset{m}{\asymp\!\vdots\!\asymp}\; {}_n \rangle - q^{2mn+1} \mathrm{K}_0(\tilde{\mathbf{C}}_{m,n}).
$$

The complex $\tilde{\mathbf{C}}_{m,n}$ is angle-shaped, so $|\mathrm{K}_0(\tilde{\mathbf{C}}_{m,n})|_q \geq 0$ and by Definition 2.1 there is a limit (2.21). ■

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