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The writhes of a virtual knot

by

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Abstract. Kauffman introduced a fundamental invariant of a virtual knot called the odd writhe. There are several generalizations of the odd writhe, such as the index polynomial and the odd writhe polynomial. In this paper, we define the n-writhe for each non-zero integer n, which unifies these invariants, and study various properties of the n-writhe.

1. Introduction. The writhe plays an important roll in classical knot theory; it is the sum of the signs of all the crossings of a knot diagram. Kauffman restricted the definition to the "odd" crossings in a virtual knot diagram and proved that the sum of the signs of all the odd crossings is an invariant of a virtual knot [9]. He called it the *odd writhe* of a virtual knot K and denoted it by $J(K)$. The odd writhe was generalized in two ways; one is the *index polynomial*

$$Q_K(t) = \sum_{n>0} a_n(t^n - 1)$$

due to Henrich [4] and Im–Lee–Lee [6], and the other is the *odd writhe polynomial*

$$f_K(t) = \sum_{n\in\mathbb{Z}} b_{2n}t^{2n}$$

due to Cheng [2]. We remark that $J(K) = -Q_K(0) = f_K(1)$.

The aim of this paper is to define a sequence $\{J_n(K)\}_{n\neq 0}$ of invariants of a virtual knot K and study their properties. We call $J_n(K)$ the *n-writhe* of K.

In Section 2, we prove that the n-writhe is a generalization of the index polynomial and the odd writhe polynomial.

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Proposition 1.1. *Let $\{a_n\}_{n>0}$ and $\{b_{2n}\}_{n\in\mathbb{Z}}$ be the coefficients of the index polynomial and the odd writhe polynomial as above. Then:*

(i) $a_n = J_n(K) + J_{-n}(K)$ *for any* $n > 0$.
(ii) $b_{2n} = J_{1-2n}(K)$ *for any* $n \in \mathbb{Z}$.
(iii) $J(K) = \sum_{n\in\mathbb{Z}} J_{2n+1}(K)$.

We also describe the index polynomial of a virtual "link" in terms of n-writhes and the linking numbers (Corollary 2.5).

In Section 3, we consider the symmetries of a virtual knot. For a virtual knot K, let $-K$ denote the orientation-reversion of K, and K^* and $K^\dagger$ the vertical and horizontal mirror images of K, respectively (cf. [5]).

Proposition 1.2. *For any virtual knot K and $n \neq 0$,*

$$J_n(-K) = J_{-n}(K) \quad \textit{and} \quad J_n(K^*) = J_n(K^\dagger) = -J_{-n}(K).$$

Section 4 is devoted to giving a necessary and sufficient condition for a sequence of integers to be that of the n-writhes of a virtual knot:

Theorem 1.3. *Any virtual knot K satisfies $\sum_{n\neq 0} nJ_n(K) = 0$. Conversely, for any sequence $\{c_n\}_{n\neq 0}$ of integers with $\sum_{n\neq 0} nc_n = 0$, there is a virtual knot K such that $J_n(K) = c_n$ for any $n \neq 0$.*

In Section 5, we consider the classical crossing number, $\mathrm{c}(K)$, which is the minimal number of classical crossings for all the virtual knot diagrams of K. Similarly to a classical knot, we have a lower bound of $\mathrm{c}(K)$ by the span of the Jones polynomial $V_K(t)$ (cf. [11]). We give two kinds of lower bounds in terms of the n-writhes (Proposition 5.2) and prove the following.

Theorem 1.4. *For any $n \geq 3$, there is a virtual knot K with $\mathrm{c}(K) = n$ and $V_K(t) = 1$.*

We also give a lower bound for the virtual crossing number (Proposition 5.3).

In Section 6, we first consider the differences of the n-writhes under the crossing change at a classical crossing. We describe the index polynomial of a "flat" virtual knot due to Im–Lee–Son [7] in terms of the n-writhe (Proposition 6.2). In general, the crossing change is not an unknotting operation for virtual knots.

Theorem 1.5. *Suppose that a virtual knot K can be transformed into the trivial knot by a finite sequence of crossing changes. Then:*

(i) $J_n(K) = J_{-n}(K)$ *for any* $n \neq 0$.
(ii) *The minimal number of such crossing changes,* $\mathrm{u}(K)$, *satisfies*

$$\mathrm{u}(K) \geq \sum_{n>0} |J_n(K)| = \sum_{n<0} |J_n(K)|.$$

Next, we consider a relationship between the n-writhes and Δ-moves. By the property that $J_n(K)$ is invariant under Δ-moves (Lemma 6.4), we have the following.

THEOREM 1.6. *For any $n \geq 1$, there are infinitely many virtual knots K with* $\mathrm{u}(K) = n$ *such that K cannot be transformed into the trivial knot by Δ-moves.*

In Section 7, we define a local move on a virtual knot diagram called a *Ξ-move*. Each n-writhe $J_n(K)$ can be changed by Ξ-moves, but the odd write $J(K) = \sum_{n\neq 0} J_n(K)$ cannot (Lemma 7.1). Furthermore, we have the following.

THEOREM 1.7. *For virtual knots K and K', the following are equivalent:*

(i) *K can be transformed into K' by a finite sequence of Ξ-moves.*
(ii) *$J(K) = J(K')$.*

We remark that Ohyama and Yoshikawa of Tokyo Woman's Christian University obtained the result of Theorem 1.7 independently.

2. Definitions. A *Gauss diagram G* consists of an oriented circle S^1 together with signed, oriented m chords ($m \geq 0$) connecting $2m$ points on S^1. Let $\gamma = \overrightarrow{PQ}$ be a chord in G with sign $\varepsilon(\gamma)$ such that γ is oriented from P to Q. We give signs to the endpoints P and Q, denoted by $\varepsilon(P)$ and $\varepsilon(Q)$, respectively, so that

$$\varepsilon(P) = -\varepsilon(\gamma) \quad \text{and} \quad \varepsilon(Q) = \varepsilon(\gamma).$$

DEFINITION 2.1. (1) For a chord $\gamma = \overrightarrow{PQ}$ in a Gauss diagram G, the *specified arc* of γ is the arc α in S^1 with endpoints P and Q oriented from P to Q with respect to the orientation of S^1. See the left of Figure 1.

(2) The *index* of γ is the sum of the signs of all the points on α except P and Q. We denote it by $\mathrm{Ind}(\gamma)$.

(3) For an integer n, the *n-writhe* of G is the sum of the signs of all the chords with index n and denoted by $J_n(G) = \sum_{\mathrm{Ind}(\gamma)=n} \varepsilon(\gamma)$.

(4) The *parity* of a chord γ is that of $\mathrm{Ind}(\gamma)$; γ is *odd* or *even* if $\mathrm{Ind}(\gamma)$ is odd or even, respectively.

(5) The *odd writhe* of G is the sum of the signs of all the odd chords in G and denoted by $J(G)$.

EXAMPLE 2.2. The Gauss diagram G in Figure 1 (right) has four chords $\gamma_1, \ldots, \gamma_4$ with $\varepsilon(\gamma_1) = 1$ and $\varepsilon(\gamma_2) = \varepsilon(\gamma_3) = \varepsilon(\gamma_4) = -1$. Since the indices are

$$\mathrm{Ind}(\gamma_1) = -1, \quad \mathrm{Ind}(\gamma_2) = 3, \quad \mathrm{Ind}(\gamma_3) = \mathrm{Ind}(\gamma_4) = -2,$$

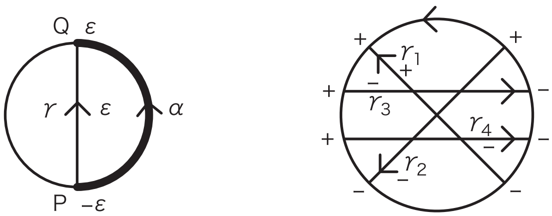


Fig. 1

we have

$$J_3(G) = -1, \quad J_{-1}(G) = 1, \quad J_{-2}(G) = -2, \quad J_n(G) = 0 \ (n \neq 3, -1, -2).$$

Since γ_1 and γ_2 are odd chords, the odd writhe $J(G)$ is 0.

A *virtual knot diagram* is a knot diagram which may have virtual crossings as well as classical crossings. A *virtual knot* is an equivalence class of virtual knot diagrams under the three kinds of classical Reidemeister moves together with the four kinds of virtual Reidemeister moves. Refer to [8] for more details. Throughout this paper, all the virtual knots are oriented.

For a virtual knot diagram D, we take an immersion $S^1 \to \mathbb{R}^2$ whose image is the plane curve obtained from D by ignoring classical/virtual crossing information. If D has m classical crossings, then the preimage consists of $2m$ points on S^1. For each classical crossing c, we connect the preimages $\{P, Q\}$ of c by a chord $\gamma = \overrightarrow{PQ}$, where P is the preimage of the overcrossing and Q is that of the undercrossing. We define the sign $\varepsilon(\gamma)$ of the chord γ to be that of c so that we obtain a Gauss diagram. We denote it by $G(D)$.

Lemma 2.3. *If D and D' are related by a finite sequence of Reidemeister moves, then $J_n(G(D)) = J_n(G(D'))$ for any $n \neq 0$.*

Proof. For a virtual Reidemeister move, we have $G(D) = G(D')$ by definition.

For a Reidemeister move I, the new classical crossing corresponds to a chord with index zero. For a Reidemeister move II, the pair of the new chords have the same index with opposite signs. For a Reidemeister move III, it is sufficient to check the special case with signs and orientations as shown in Figure 2.

In any cases, the signs and the indices of the other chords do not change under the moves. ∎

Definition 2.4. Let n be a non-zero integer. For a virtual knot K and its diagram D, $J_n(G(D))$ is called the *n-writhe* of K and denoted by $J_n(K)$.

Kauffman defines the *odd writhe* $J(K)$ of a virtual knot K by $J(K) = J(G(D))$ for any diagram D of K [9].

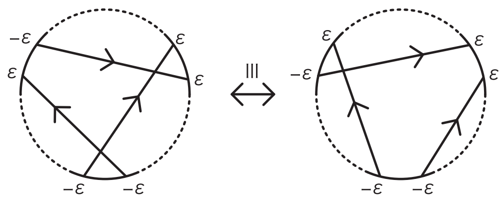


Fig. 2

Proof of Proposition 1.1. (i) The index polynomial of a virtual knot K is given by

$$Q_K(t) = \sum_c \varepsilon(c)(t^{|i(c)|} - 1),$$

where the sum is taken over all the classical crossings of a diagram D and $i(c)$ is the "virtual intersection index" of a classical crossing c [6]. By definition, it holds that $|i(c)| = |\mathrm{Ind}(\gamma_c)|$ for the chord γ_c of $G(D)$ corresponding to c. Therefore,

$$Q_K(t) = \sum_{n>0} \{J_n(K) + J_{-n}(K)\}(t^n - 1).$$

(ii) The odd writhe polynomial of a virtual knot K is given by

$$f_K(t) = \sum_{c \text{ odd}} \varepsilon(c)t^{N(c)},$$

where the sum is taken over all the odd crossings of D; a classical crossing c is *odd* if the corresponding chord γ_c is odd. Refer to [2] for the definition of the integer $N(c)$. Since $N(c) = 1 - \mathrm{Ind}(\gamma_c)$ for any c, we have

$$f_K(t) = \sum_{n\in\mathbb{Z}} J_{1-2n}(K)t^{2n}.$$

(iii) This follows from the definition immediately. ∎

For a virtual link $L = \bigcup_i K_i$, let $\ell(K_i, K_j)$ be the linking number of the ordered pair (K_i, K_j), which is the sum of the signs of all the classical crossings between K_i and K_j such that the overcrossing and the undercrossing belong to K_i and K_j, respectively. In general, we have $\ell(K_i, K_j) \neq \ell(K_j, K_i)$. Then the index polynomial of L is described in terms of the n-writhes and the ordered linking numbers as shown in the following. The proof is easy and we leave it to the reader.

Corollary 2.5. *The index polynomial of a virtual link $L = \bigcup_i K_i$ is given by*

$$Q_L(t) = \sum_i \sum_{n>0} \{J_n(K_i) + J_{-n}(K_i)\}(t^n - 1) + \sum_{i<j} \{\ell(K_i, K_j) + \ell(K_j, K_i)\}(t^{|\ell(K_i,K_j)-\ell(K_j,K_i)|} - 1).$$

3. Orientation-reversion and mirror images. For a Gauss diagram G, we denote by $-G$ the one obtained from G by reversing the orientation of the circle S^1 while keeping the orientation and the sign of each chord. We denote by G^* the diagram obtained by changing both the orientation and the sign of each chord while keeping the orientation of S^1, and by $G^\dagger$ the one obtained by changing the sign of each chord only.

LEMMA 3.1. *For a Gauss diagram G, we have:*

(i) $J_n(-G) = J_{-n}(G)$.
(ii) $J_n(G^*) = -J_{-n}(G)$.
(iii) $J_n(G^\dagger) = -J_{-n}(G)$.

Proof. For a chord γ in G with the specified arc α, let γ' be the chord in $-G$, G^*, or $G^\dagger$ corresponding to γ, and α' the specified arc of γ'.

(i) The arc α' is complementary to α in S^1, and the signs of the chords and the points on S^1 do not change, so that $\operatorname{Ind}(\gamma') = -\operatorname{Ind}(\gamma)$ with $\varepsilon(\gamma') = \varepsilon(\gamma)$. We remark that the sum of the signs of all the points on S^1 is equal to zero.

(ii) The arc α' is complementary to α, the signs of the chords change, and the signs of the points on S^1 do not change, so that $\operatorname{Ind}(\gamma') = -\operatorname{Ind}(\gamma)$ with $\varepsilon(\gamma') = -\varepsilon(\gamma)$.

(iii) The arcs α and α' are the same, and the signs of the chords and the points on S^1 change, so that $\operatorname{Ind}(\gamma') = -\operatorname{Ind}(\gamma)$ with $\varepsilon(\gamma') = -\varepsilon(\gamma)$. ∎

Let D be a diagram of a virtual knot K. We denote by $-D$ the orientation-reversion of D. The *vertical* mirror image D^* is obtained from D by switching over/under-information at all the classical crossings, and the *horizontal* mirror image $D^\dagger$ is obtained by reflecting D across a vertical plane [5]. We denote by $-K$, K^*, and $K^\dagger$ the virtual knots represented by $-D$, D^*, and $D^\dagger$, respectively. In general, the eight virtual knots $\pm K$, $\pm K^*$, $\pm K^\dagger$, and $\pm K^{*\dagger}$ are mutually distinct (cf. [10]).

Proof of Proposition 1.2. Since $G(-D) = -G(D)$, $G(D^*) = G(D)^*$, and $G(D^\dagger) = G(D)^\dagger$, we have the conclusion by Lemma 3.1. ∎

The following is a generalization of [2, Corollary 4.7] and [6, Corollary 4.7].

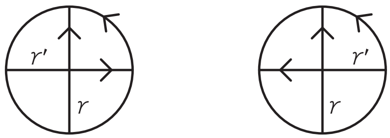
COROLLARY 3.2. *Let K be a virtual knot.*

(i) *If there is an integer $n \neq 0$ with $J_n(K) \neq J_{-n}(K)$, then $K \neq -K$.*
(ii) *If there is an integer $n \neq 0$ with $J_n(K) \neq -J_{-n}(K)$, then $K \neq K^*, K^\dagger$.*

4. Characterization of writhes. Let $\gamma = \overrightarrow{PQ}$ and $\gamma' = \overrightarrow{P'Q'}$ be two chords in a Gauss diagram. We say that the pair of γ and γ' is *intersecting* if the specified arc α of γ contains exactly one of P' and Q', and otherwise *non-intersecting*. We define $\delta(\gamma,\gamma')\in\{-1,0,1\}$ such that

$$\delta(\gamma,\gamma')=\begin{cases}\varepsilon(\gamma)\varepsilon(\gamma') & \text{if } \alpha \text{ contains } Q',\\ -\varepsilon(\gamma)\varepsilon(\gamma') & \text{if } \alpha \text{ contains } P',\end{cases}$$

for any intersecting ordered pair (γ,γ'), and $\delta(\gamma,\gamma')=0$ for any non-intersecting pair. We see that $\delta(\gamma',\gamma)=-\delta(\gamma,\gamma')$ for any ordered pair. See Figure 3.



$\delta(\gamma,\gamma')=\varepsilon(\gamma)\varepsilon(\gamma')$ $\delta(\gamma,\gamma')=-\varepsilon(\gamma)\varepsilon(\gamma')$

Fig. 3

Lemma 4.1. *For any chord γ in a Gauss diagram, we have*

$$\varepsilon(\gamma)\,\mathrm{Ind}(\gamma)=\sum_{\gamma'\neq\gamma}\delta(\gamma,\gamma'),$$

where the sum is taken over all the chords γ' other than γ.

Proof. Let Γ_+ and Γ_- be the sets of chords $\gamma'=\overrightarrow{P'Q'}$ intersecting γ such that the specified arc α of γ contains Q' and P', respectively. By definition,

$$\mathrm{Ind}(\gamma)=\sum_{\gamma'\in\Gamma_+}\varepsilon(Q')+\sum_{\gamma'\in\Gamma_-}\varepsilon(P')=\sum_{\gamma'\in\Gamma_+(\gamma)}\varepsilon(\gamma')-\sum_{\gamma'\in\Gamma_-(\gamma)}\varepsilon(\gamma').$$

Therefore,

$$\varepsilon(\gamma)\,\mathrm{Ind}(\gamma)=\sum_{\gamma'\in\Gamma_+}\varepsilon(\gamma)\varepsilon(\gamma')-\sum_{\gamma'\in\Gamma_-}\varepsilon(\gamma)\varepsilon(\gamma')=\sum_{\gamma'\in\Gamma_+\cup\Gamma_-}\delta(\gamma,\gamma'),$$

which is equal to $\sum_{\gamma'\neq\gamma}\delta(\gamma,\gamma')$. ∎

Proposition 4.2. *For any Gauss diagram G, we have $\sum_{n\neq 0} nJ_n(G)=0$.*

Proof. By Lemma 4.1,

$$\sum_{n\neq 0} nJ_n(G)=\sum_{\gamma}\varepsilon(\gamma)\,\mathrm{Ind}(\gamma)=\sum_{\gamma}\sum_{\gamma'\neq\gamma}\delta(\gamma,\gamma').$$

Since $\delta(\gamma',\gamma)=-\delta(\gamma,\gamma')$, the sum is equal to zero. ∎

For two Gauss diagrams G_1 and G_2 with base points, the connected sum $G_1\#G_2$ is obtained by removing small arcs near the base points from the diagrams and connecting them, keeping the orientations. While $G_1\#G_2$ depends on the positions of base points, the n-writhe of $G_1\#G_2$ does not:

Lemma 4.3. $J_n(G_1 \# G_2) = J_n(G_1) + J_n(G_2)$.

Proof. Since the sum of the signs of all the points on G_i is equal to zero ($i = 1, 2$), the index of any chord in G_i is equal to that in $G_1 \# G_2$. ∎

Lemma 4.4. *For any $n, c \in \mathbb{Z}$ with $n \neq 0, -1$, there is a Gauss diagram $G = G(n, c)$ which satisfies the condition*

$$J_n(G) = c, \quad J_{-1}(G) = nc, \quad \textit{and} \quad J_k(G) = 0 \ (k \neq n, -1).$$

Proof. If $c = 0$, we take the Gauss diagram with no chord. We consider the case $c > 0$. Let $G(n, 1)$ ($n \neq 0, -1$) be the Gauss diagram as shown in Figure 4, and $G(n, c)$ be the connected sum of c copies of $G(n, 1)$. Since

$$J_n(G(n, 1)) = 1 \quad \text{and} \quad J_{-1}(G(n, 1)) = n,$$

$G(n, c)$ satisfies the desired condition by Lemma 4.3. For $c < 0$, the Gauss diagram $G = -G(n, -c)^*$ satisfies the condition by Lemma 3.1. ∎

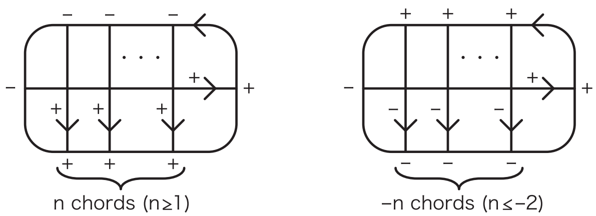


Fig. 4

Proposition 4.5. *Let $\{c_n\}_{n \neq 0}$ be a sequence of integers. If $\sum_{n \neq 0} nc_n = 0$, then there is a Gauss diagram G such that $J_n(G) = c_n$ for any $n \neq 0$.*

Proof. Let $G = \#_{n \neq 0, -1} G(n, c_n)$ be the connected sum of the Gauss diagrams $G(n, c_n)$ for $n \neq 0, -1$. By Lemmas 4.3 and 4.4, we have $J_n(G) = c_n$ ($n \neq 0, -1$). Moreover,

$$J_{-1}(G) = \sum_{n \neq 0, -1} nc_n = c_{-1}$$

by the assumption $\sum_{n \neq 0} c_n = 0$. ∎

Proof of Theorem 1.3. This follows from Propositions 4.2 and 4.5. ∎

We remark that the characterization of the odd writhe polynomial given in [2, Theorem 5.1] follows from Theorem 1.3. Also, we can give a characterization of the index polynomial, which is left to the reader.

Corollary 4.6 (cf. [2]). *The odd writhe $J(K)$ is even.*

Proof. By Proposition 1.1(iii) and Theorem 1.3, we have

$$J(K) \equiv \sum_{n \neq 0} nJ_n(K) = 0 \pmod 2. \ ∎$$

5. Classical crossing number. For a Gauss diagram G, let $c(G)$ denote the number of chords in G.

LEMMA 5.1. *Let G be a Gauss diagram with $c(G) > 0$.*

(i) $c(G) \geq \sum_{n\neq 0} |J_n(G)|$.
(ii) $c(G) \geq \max\{|n| \mid J_n(G) \neq 0\} + 1$.

Proof. (i) Since the number of chords with index n is greater than or equal to $|J_n(G)|$, the total number of chords in G is greater than or equal to $\sum_{n\neq 0} |J_n(G)|$.

(ii) If G has a chord with index n, then there are at least $|n|$ chords intersecting that chord by definition. ■

The number of classical crossings of a virtual knot diagram D is equal to $c(G(D))$. The *classical crossing number* of a virtual knot K is the minimum of $c(G(D))$ over all the diagrams of K, and is denoted by $\mathrm{c}(K)$. There are several lower bounds of $\mathrm{c}(K)$, such as the span of the Jones polynomial [11] and the weighted degree of the Miyazawa polynomial [14].

By Lemma 5.1, we have the following immediately. The inequality (ii) is a generalization of [2, Proposition 4.2].

PROPOSITION 5.2. *Let K be a non-trivial virtual knot.*

(i) $\mathrm{c}(K) \geq \sum_{n\neq 0} |J_n(K)|$.
(ii) $\mathrm{c}(K) \geq \max\{|n| \mid J_n(K) \neq 0\} + 1$.

Let $V_K(t)$ be the Jones polynomial of a virtual knot K. It is known that a local move on a virtual knot diagram D as shown in Figure 5 (left) does not change the Jones polynomial [8]. The local move changes the orientation of a chord in the corresponding Gauss diagram $G(D)$, while keeping the sign of any chord; see Figure 5 (right).

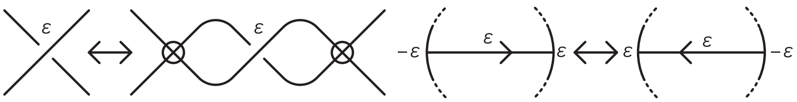


Fig. 5

Proof of Theorem 1.4. For a positive integer n, let K_n be the virtual knot represented by the Gauss diagram in Figure 6. In particular, K_1 and K_2 are the trivial knot. It is easy to see that $J_k(K_n) = 0$ ($k \neq n-1, \pm 1, 0$) and

$$J_{n-1}(K_n) = 1, \quad J_1(K_n) = -\frac{n-1}{2}, \quad J_{-1}(K_n) = \frac{n-1}{2}$$

for odd $n \geq 3$, and

$$J_{n-1}(K_n) = 1, \quad J_1(K_n) = -\frac{n}{2}, \quad J_{-1}(K_n) = \frac{n}{2} - 1$$

for even $n \geq 4$. Therefore, $\mathrm{c}(K_n) = n$ by either Proposition 5.2(i) or (ii).

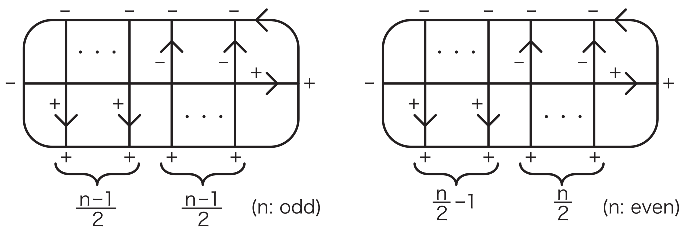


Fig. 6

On the other hand, we change the orientations of some vertical chords so that all the vertical chords are oriented in the same direction. We can apply Reidemeister moves II on the chords to obtain the trivial knot K_1 or K_2. Since the Jones polynomial does not depend on the orientations of chords, we have $V_{K_n}(t) = 1$ for any n. ∎

Let $\mathrm{v}(K)$ denote the *virtual crossing number* of a virtual knot K, which is the minimal number of virtual crossings of all the virtual knot diagrams of K. There are several lower bounds of $\mathrm{v}(K)$ in [1, 3, 6, 12]. The following is a generalization of [6, Corollary 4.14].

PROPOSITION 5.3. *For any virtual knot K, we have*

$$\mathrm{v}(K) \geq \max\{|n| \mid J_n(K) \neq 0\}.$$

Proof. Let D be a virtual knot diagram of K. Assume that the Gauss diagram $G(D)$ has a chord γ with index n. Let α be the specified arc of γ, and α' the complementary arc of α in S^1. We use the same notations to indicate the closed paths in D corresponding to α and α'. The classical crossings between α and α' contribute n to the algebraic intersection number of α and α'. Since the intersection number is equal to zero, the virtual crossings contribute $-n$, which implies that D has at least $|n|$ virtual crossings. ∎

We remark that the virtual knot K_n in the proof of Theorem 1.4 has virtual crossing number $\mathrm{v}(K_n) = n - 1$ by Proposition 5.3. See Figure 7.

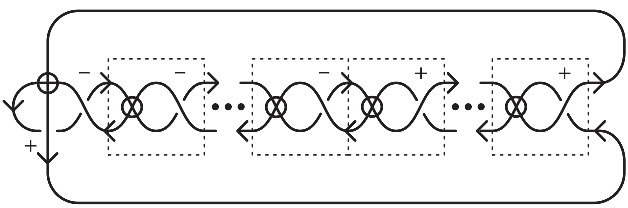


Fig. 7

6. Crossing change and Δ-move. We consider the crossing change at a classical crossing of a virtual knot diagram D. The move changes the orientation and the sign of the chord of $G(D)$ corresponding to the crossing. We remark that the signs of the endpoints of the chord do not change. See Figure 8.

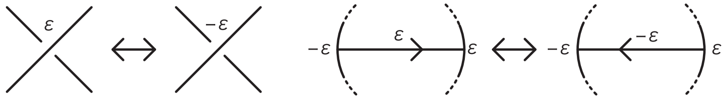


Fig. 8

Lemma 6.1. *Let γ be a chord in a Gauss diagram G with index n and sign ε, and G' the one obtained from G by changing the orientation and the sign of γ.*

(i) *If $n \neq 0$, then*
$$J_k(G') = J_k(G) - \varepsilon \ (k = \pm n) \quad \text{and} \quad J_k(G') = J_k(G) \ (k \neq \pm n).$$
(ii) *If $n = 0$, then*
$$J_0(G') = J_0(G) - 2\varepsilon \quad \text{and} \quad J_k(G') = J_k(G) \ (k \neq 0).$$

Proof. This is similar to the proof of Lemma 3.1(ii). ∎

A *flat virtual knot* [8] is an equivalence class of virtual knots under crossing changes at classical crossings. In [7], the index polynomial of a flat virtual knot K,
$$Q_K(t) = \sum_{n>0} a'_n(t^n - 1),$$
is defined. On the other hand, it follows from Lemma 6.1 that $J_n(K) - J_{-n}(K)$ is also an invariant of a flat virtual knot for any $n > 0$. Then we have the following relationship between these invariants, which is proved by using an "extended" Gauss diagram whose chords correspond to both the classical and virtual crossings. We leave the precise proof to the reader as an exercise.

Proposition 6.2. *Let $\{a'_n\}_{n>0}$ be the coefficients of the index polynomial of a flat virtual knot as above. Then for any $n > 0$, we have*
$$a'_n = 2n\{J_n(K) - J_{-n}(K)\}.$$

Assume that two virtual knots K and K' define the same flat virtual knot; that is, K can be transformed into K' by a finite sequence of crossing changes. Let $\mathrm{d_G}(K, K')$ denote the minimal number of crossing changes which transform K into K'. We call it the *Gordian distance* between K and K'. The following is a generalization of Theorem 1.5.

Theorem 6.3. *Suppose that K and K' are related by crossing changes.*

(i) $J_n(K) - J_n(K') = J_{-n}(K) - J_{-n}(K')$ *for any $n \neq 0$.*

(ii) *The Gordian distance satisfies*

$$\mathrm{d_G}(K,K') \geq \sum_{n>0} |J_n(K) - J_n(K')| = \sum_{n<0} |J_n(K) - J_n(K')|.$$

Proof. (i) By Proposition 6.1, we have $J_n(K) - J_{-n}(K) = J_n(K') - J_{-n}(K')$.

(ii) This follows from Lemma 6.1. ∎

Recall that a Δ-*move* on a virtual knot diagram is the local move as shown in Figure 9 (left). It is known that the Δ-move is an unknotting operation for classical knots [13]. On the other hand, the following lemma implies that the Δ-move is not an unknotting operation for virtual knots in general.

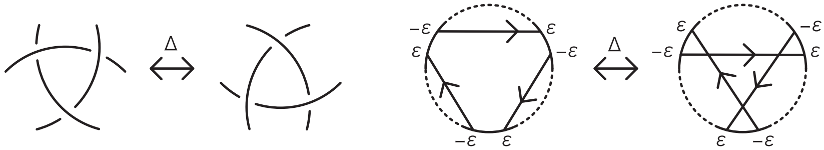


Fig. 9

Lemma 6.4. *For any $n \neq 0$, the n-writhe $J_n(K)$ is invariant under Δ-moves.*

Proof. It is sufficient to consider the change on Gauss diagrams as shown Figure 9 (right). Then the sign and the index of any chord do not change. ∎

Proof of Theorem 1.6. For positive integers n and r, let K_{nr} be the virtual knot represented by the Gauss diagram as shown in Figure 10. It is easy to see that

$$J_r(K_{nr}) = J_{-r}(K_{nr}) = n \quad \text{and} \quad J_k(K_{nr}) = 0 \ (k \neq 0, \pm r).$$

Therefore, K_{nr} cannot be transformed into the trivial knot by Δ-moves, by Lemma 6.4.

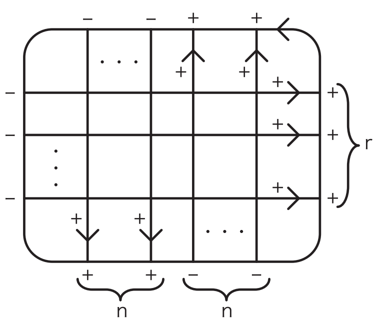


Fig. 10

On the other hand, we change the orientations and the signs of the n vertical chords pointing upward so that we can apply Reidemeister moves II to cancel all the vertical chords. Therefore, $\mathrm{u}(K_{nr}) \leq n$. Since $\mathrm{u}(K_{nr}) \geq |J_r(K_{nr})| = n$ by Theorem 1.5(ii), we have $\mathrm{u}(K_{nr}) = n$. ∎

7. Characterization of odd writhe. Let c_1, c_2, and c_3 be consecutive classical crossings of a virtual knot diagram D. A *Ξ-move* on D changes the positions of c_1 and c_3, while keeping the over/under-information of the crossings. See Figure 11 (left). The corresponding Ξ-move on a Gauss diagram is the transposition of two points P_1 and P_3 next but one to each other on S^1 as shown in Figure 11 (right).

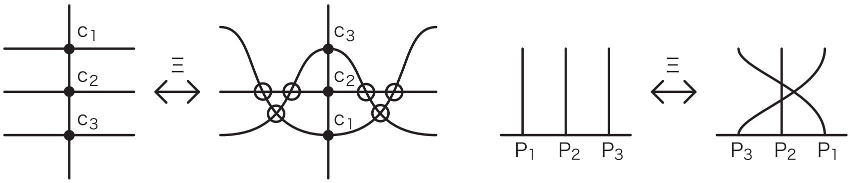


Fig. 11

Recall that the odd writhe $J(G)$ of a Gauss diagram G is the sum of the signs of all the odd chords. Then we have the following.

Lemma 7.1. *If G and G' are related by Ξ-moves, then $J(G) = J(G')$.*

Proof. The parity of three consecutive chords does not change by a Ξ-move. ∎

For an integer n, let $G(n)$ be the Gauss diagram with $2|n|$ chords as shown in Figure 12, where $\varepsilon = +1$ for $n > 0$ and $\varepsilon = -1$ for $n < 0$. In particular, $G(0)$ is the Gauss diagram with no chord.

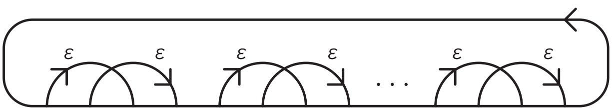


Fig. 12

We use the notation $\gamma = \overline{PQ}$ for a chord with endpoints P and Q when we do not care about the orientation.

Proposition 7.2. *Any Gauss diagram G is related to $G(n)$ for some $n \in \mathbb{Z}$ by a finite sequence of Reidemeister moves and Ξ-moves.*

Proof. Let $\gamma = \overline{PQ}$ be a chord in G. If γ is even, we apply Ξ-moves related to γ so that P is next to Q, and remove it by a Reidemeister move I. Therefore, we may consider the case where all the chords in G are odd.

Since γ is odd, we apply Ξ-moves so that P is next but one to Q. Let $\gamma' = \overline{P'Q'}$ be the chord such that one of the endpoints, say P', lies between P and Q. By applying Ξ-moves for γ', we obtain a Gauss diagram where P, P', Q, and Q' lie on S^1 consecutively in this order. Moreover, we may assume that they are oriented as $\overrightarrow{PQ}$ and $\overrightarrow{P'Q'}$ up to Ξ-moves.

By repeating this process, we obtain a Gauss diagram whose chords are paired such that every pair of chords intersect each other; it is the same as $G(n)$ up to the signs of the chords.

If two chords have opposite signs, then we apply a Reidemeister move II to cancel them. Also, if two consecutive pairs of chords have opposite signs, then we apply Ξ-moves and Reidemeister moves II to eliminate the four chords. See Figure 13. Therefore, G is equivalent to $G(n)$ for some $n \in \mathbb{Z}$. ∎

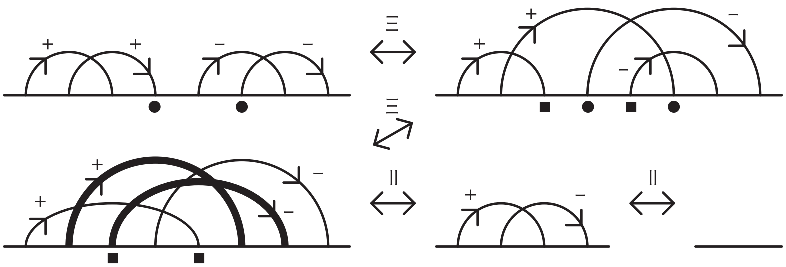


Fig. 13

Proof of Theorem 1.7. Lemma 7.1 implies (i)⇒(ii). Assume that $J(K) = J(K')$. Let D and D' be virtual knot diagrams of K and K', respectively. By Proposition 7.2, there are integers n and n' such that $G(D)$ and $G(D')$ are equivalent to $G(n)$ and $G(n')$ up to Reidemeister moves and Ξ-moves, respectively. Since $J(K) = J(G(n)) = 2n$ and $J(K') = J(G(n')) = 2n'$, we have $n = n'$. This implies (ii)⇒(i). ∎

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