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Torsion of Khovanov homology

by

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Abstract. Khovanov homology is a recently introduced invariant of oriented links in $\mathbb{R}^3$. It categorifies the Jones polynomial in the sense that the (graded) Euler characteristic of Khovanov homology is a version of the Jones polynomial for links. In this paper we study torsion of Khovanov homology. Based on our calculations, we formulate several conjectures about the torsion and prove weaker versions of the first two of them. In particular, we prove that all non-split alternating links have their integer Khovanov homology almost determined by the Jones polynomial and signature. The only remaining indeterminacy is that one cannot distinguish between $\mathbb{Z}_{2^k}$ factors in the canonical decomposition of the Khovanov homology groups for different values of k.

1. Introduction. Let L be an oriented link in the Euclidean space $\mathbb{R}^3$ represented by a planar diagram D. In a seminal paper, Mikhail Khovanov [Kh1] assigned to D a family of abelian groups $\mathcal{H}^{i,j}(L)$, whose isomorphism classes depend on the isotopy class of L only. These groups are defined as the homology groups of an appropriate (graded) chain complex $\mathcal{C}(D)$ with integer coefficients. The main property of Khovanov homology is that it *categorifies* the Jones polynomial. More specifically, let $J_L(q)$ be a version of the Jones polynomial of L that satisfies the following identities (called the *Jones skein relation* and *normalization*):

$$-q^{-2}J_{\times_+}(q) + q^2 J_{\times_-}(q) = (q-1/q)J_{)(_0}(q), \tag{1.1}$$
$$J_{\bigcirc}(q) = q + 1/q.$$

The skein relation should be understood as relating the Jones polynomials of three links whose planar diagrams are identical everywhere except in a small disk, where they are different as depicted in (1.1). The normalization

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fixes the value of the Jones polynomial on the trivial knot. $J_L(q)$ is a Laurent polynomial in q for every link L and is completely determined by its skein relation and normalization.

The gist of the categorification is that the (graded) Euler characteristic of Khovanov chain complex equals $J_L(q)$:

$$J_L(q) = \sum_{i,j} (-1)^i q^j h^{i,j}(L), \tag{1.2}$$

where $h^{i,j}(L) = \mathrm{rk}(\mathcal{H}^{i,j}(L))$, the Betti numbers of $\mathcal{H}(L)$. The reader is referred to [BN1, Kh1] for detailed treatment.

There are several numerical conjectures about Khovanov homology. We recall them briefly below.

Given a link L, the ranks $h^{i,j}(L)$ of its Khovanov homology can be arranged into a table with columns and rows numbered by i and j, respectively (see Figure 1). A pair of entries in such a table is said to be a "knight move" pair if these entries have the same (positive) value and their i- and j-positions in the table differ by 1 and 4, respectively. This "knight move" rule is depicted in Figure 1, where a can be an arbitrary positive integer.

	a
a	

$\mathbf{7_7}$	−3	−2	−1	0	1	2	3	4
9								**1**
7							**1**	
5						**2**	**1**	
3					**2**	**1**		
1				**1** + **1**	**2**			
−1			**2**	**1** + **2**				
−3		**1**	**1**					
−5		**2**						
−7	**1**							

Fig. 1. Pattern of the "knight move" rule. Ranks of the Khovanov homology of the knot 7_7 that illustrates Conjecture 1.A.

1.A. Conjecture (Bar-Natan, Garoufalidis, Khovanov [BN1]). *Let L be a knot. Consider the table of the Khovanov ranks $h^{i,j}(L)$ for L. If one subtracts 1 from two adjacent entries in the column $i = 0$, then the remaining entries are arranged in "knight move" pairs.*

1.B. Example. Figure 1 illustrates Conjecture 1.A for the knot 7_7 ([1]). The two 1's to subtract are shown inside circles with a gray background and the rest of the circles joined by lines depict the "knight move" pairs.

Remark. In fact, different "knight move" pairs are allowed to overlap. The common entry in this case is simply the sum of the overlapping entries from both pairs. For example, the knot 13^n_{3663}, whose homology is presented in Section A.4 of the Appendix, has two overlapping pairs $(h^{1,-1}, h^{2,3})$ and $(h^{2,3}, h^{3,7})$. This confusion will be cleared up after we give a more rigorous statement of Conjecture 1.A in Section 2.3.

Remark. Conjecture 1.A was proved by Eun Soo Lee [L1, L2] for the special case of H-thin knots (see below), in particular for all alternating knots.

Let R be a commutative ring with unity. In this paper, we are mainly interested in the cases when $R = \mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{Z}_2$.

1.C. Definitions (cf. Khovanov [Kh2]). A link L is said to be *homologically thin* over a ring R or simply *RH-thin* if its Khovanov homology groups with coefficients in R are supported on two adjacent diagonals $2i-j = \text{const}$. A link L is said to be *homologically slim* or simply *H-slim* if it is $\mathbb{Z}$H-thin and all its homology groups supported on the *upper* diagonal have no torsion. A link L that is not RH-thin is said to be *RH-thick*.

1.D. *If a link is $\mathbb{Z}$H-thin, then it is $\mathbb{Q}$H-thin as well. Conversely, a $\mathbb{Q}$H-thick link is necessarily $\mathbb{Z}$H-thick. If a link is H-slim, then it is $\mathbb{Z}_p$H-thin for every prime p.*

1.E. Examples. The knot 7_7 is $\mathbb{Q}$H-thin since the free part of its homology is supported on the diagonals $2i - j = \pm 1$ (see Figure 1). In fact, it is H-slim as well (see Theorem 1.F below). The first $\mathbb{Q}$H-thick knot is 8_{19} (see Figure 2 later on).

Remark. Most of the $\mathbb{Z}$H-thin knots are H-slim. The first prime $\mathbb{Z}$H-thin knot that is not H-slim is the mirror image of 16^n_{197566}. It is also $\mathbb{Z}_2$H-thick.

1.F. Theorem (Lee [L1, L2]). *Every oriented non-split alternating link L is H-slim and the Khovanov homology of L is supported on the diagonals $2i - j = \sigma(L) \pm 1$, where $\sigma(L)$ is the signature of L.*

([1]) Throughout this paper we use the following notation for knots: knots with 10 crossings or less are numbered according to Rolfsen's table of knots [Ro] and knots with 11 crossings or more are numbered according to the knot table from Knotscape [HTh]. For example, 9_{42} is the knot number 42 with 9 crossings from Rolfsen's table, and 13^n_{3663} is a non-alternating knot number 3663 with 13 crossings from Knotscape's one.

Remark. This theorem was originally conjectured by Bar-Natan, Garoufalidis, and Khovanov [BN1, G] in a somewhat weaker form. Their conjecture stated that every non-split alternating link is $\mathbb{Q}$H-thin and not H-slim. Lee proved a stronger version (see [L1, Corollary 4.3]).

Khovanov homology is very difficult to compute by hand. Conjecture 1.A above was formulated based on extensive computations by Dror Bar-Natan using a Mathematica software package that he developed [BN1]. He computed ranks of Khovanov homology for all prime knots with up to 11 crossings. In 2002, the author developed `KhoHo` [Sh] that used reductions of the Khovanov chain complex to compute its homology faster and over $\mathbb{Z}$. All the conjectures below about torsion of Khovanov homology were formulated based on computations done with `KhoHo`. As of this writing, integer Khovanov homology is known for all prime knots with at most 16 crossings, all prime links with at most 14 crossings, as well as many thousands of other knots and links.

Remark. For several years, `KhoHo` was the fastest program for computing Khovanov homology. It works efficiently for knots and links with up to 17–19 crossings. Only in Summer 2005 was a significantly faster program written by Bar-Natan and Green [BN2, BNG].

As it turns out, torsion of Khovanov homology has very special properties and is at least as interesting and important as the free part of the homology. First of all, every knot and link considered, except the unknot, the Hopf link, their connected sums, and disjoint unions, has torsion of order 2. If proved, this could lead to a (relatively) easy way to detect the unknot.

Conjecture 1. *The Khovanov homology of every non-split link except the trivial knot, the Hopf link, and their connected sums has* 2*-torsion, that is, torsion elements of order* 2*.*

Torsion of orders other than 2 appears very seldom in Khovanov homology. Among all 1,701,936 prime knots with at most 16 crossings, all alternating ones have 2-torsion only. 38 knots with 15 crossings and 129 knots with 16 crossings have 4-torsion. One of the first such knots is the $(4,5)$-torus knot. Its Khovanov homology is presented in Section A.5 of the Appendix.

Remark. The original version of this paper contained a conjecture that no link has torsion of odd order in its Khovanov homology. This turned out to be false. The first known counter-example is the $(5,6)$-torus knot [BN2] with 3-torsion. It has 24 crossings. Nonetheless, a large class of links is proved to have 2-torsion only (see Theorem 1 below).

Let $t^{i,j}_{p^k}(L)$, where p is a prime number and $k \geq 1$, be the p^k-*rank* of $\mathcal{H}^{i,j}(L)$, that is, the multiplicity of $\mathbb{Z}_{p^k}$ in the canonical decomposition of $\mathcal{H}^{i,j}(L)$. Let also $T^{i,j}_p(L) = \sum_{k=1}^{\infty} t^{i,j}_{p^k}(L)$. The complete information about the canonical decomposition of all the groups $\mathcal{H}^{i,j}(L)$ can be combined into a table, where an (i,j)-entry contains the corresponding rank $h^{i,j}(L)$ and a comma-separated list of $t^{i,j}_{p^k}(L)$ for all relevant p and k with the subscript indicating the torsion order. For example, a hypothetical entry of $\mathbf{5}, \mathbf{3_2}, \mathbf{1_4}$ means that the corresponding group is $\mathbb{Z}^5 \oplus \mathbb{Z}_2^3 \oplus \mathbb{Z}_4$ (see Figure 2 as well as the Appendix).

Similarly to the case of ranks, torsion for a majority of knots and links fits into very regular patterns that are explained below (see Figures 2 and 3).

	$\mathbf{a}$
	$\mathbf{a_2}$
$\mathbf{a}$	

$\mathbf{8_{19}}$	0	1	2	3	4	5
17						$\mathbf{1}$
15						$\mathbf{1}$
13				$\mathbf{1}$	$\mathbf{1}$	
11				$\mathbf{1_2}$	$\mathbf{1}$	
9			$\mathbf{1}$			
7	$\mathbf{1}$					
5	$\mathbf{1}$					

$\mathbf{9_{42}}$	-4	-3	-2	-1	0	1	2
7							$\mathbf{1}$
5							$\mathbf{1_2}$
3					$\mathbf{1}$	$\mathbf{1}$	
1				$\mathbf{1}$	$\mathbf{1_2}$, $\mathbf{1}$		
-1				$\mathbf{1_2}$, $\mathbf{1}$	$\mathbf{1}$		
-3		$\mathbf{1}$	$\mathbf{1}$				
-5		$\mathbf{1_2}$					
-7	$\mathbf{1}$						

Fig. 2. Torsion version of the "knight move" rule. Khovanov homology of the knots 8_{19} and 9_{42} that are both $\mathbb{Q}$H- and $\mathbb{Z}$H-thick but are T-fancy and T-simple, respectively.

1.G. Definitions. A link L is said to be *weakly torsion simple* or just *WT-simple* if (1) it satisfies Conjecture 1.A; (2) it has no torsion of odd order; (3) for every "knight move" pair of value a that comprises entries (i,j) and $(i+1,j+4)$, one has $T_2^{i+1,j+2}(L) = a$; and (4) all 2^k-ranks of L fit into such patterns (see Figure 2, where the torsion corresponding to a "knight move" pair is depicted in a gray square). A link L is said to be *torsion simple* or *T-simple* if it is WT-simple and has torsion of order 2 only. It follows that $t_2^{i+1,j+2}(L) = a$ in this case. A link L that is not WT-simple is said to be *T-fancy*.

Remark. Strictly speaking, Conjecture 1.A is stated for knots only. Nonetheless, it was generalized (and proved) by Lee [L2] to the case of $\mathbb{Q}$H-thin links (see Theorem 2.3.C below). Throughout this paper we are going to refer to a link L as being T-simple or T-fancy with the understanding that this notion is assumed to be applicable, that is, L is either a knot or a $\mathbb{Q}$H-thin link.

1.H. Examples. The knot 9_{42} is T-simple (see Figure 2) and the knot 8_{19} is the first T-fancy one. Both of them are $\mathbb{Q}$H- and $\mathbb{Z}$H-thick. Theorems 1.F and 3 (see below) imply that the only T-fancy knots are non-alternating ones. Figure 3 lists the number of prime T-fancy knots with at most 16 crossings.

Remark. The torsion of a T-simple link is completely determined by the ranks of Khovanov homology. In particular, for non-split alternating links it is completely determined by the Jones polynomial and signature [L2].

Conjecture 2. *Every H-slim link is T-simple. In particular, every non-split alternating link is T-simple.*

Most of the T-fancy links have their torsion ranks never greater than the value of the corresponding "knight move" pair.

1.I. Definition. A T-fancy link L is said to be *torsion rich* or just *T-rich* if it satisfies Conjecture 1.A, has no torsion of odd order (that is, all torsion elements have order 2^k for some k) and there is at least one value of (i, j) such that $T_2^{i+1,j+2}(L)$ is *greater* that the value of the corresponding "knight move" pair $(h^{i,j}, h^{i+1,j+4})$.

Number of crossings	8	9	10	11	12	13	14	15	16
Number of prime non-alternating knots	3	8	42	185	888	5110	27436	168030	1008906
Number of T-fancy knots	1	0	6	11	71	322	1736	10889	64341
Number of T-rich knots	0	0	0	0	0	4	14	177	1413

Fig. 3. Number of prime T-fancy and T-rich knots

1.J. Example. The first T-rich knot has 13 crossings. It is the knot 13^n_{3663} mentioned above. This knot is also the first one whose homology is supported on 4 adjacent diagonals. Figure 3 lists the number of prime T-rich knots with at most 16 crossings.

In [Kh2] Khovanov defined a reduced version of his homology. The graded Euler characteristic of this reduced homology is again a version of the Jones polynomial. More specifically, it satisfies the same skein relation from (1.1) but is normalized to be 1, as opposed to $q + 1/q$, on the unknot. Reduced Khovanov homology is always supported on exactly one diagonal less than the standard one. Very few knots have torsion in reduced homology and all known examples of those that have are T-rich.

Conjecture 3. *A knot is T-rich if and only if its reduced Khovanov homology has torsion.*

The following conjecture is a bit optimistic, but its confirmation would be very exciting, since the torsion seems to be easier to read from the diagram than the rest of the homology.

Conjecture 4. *If two knots have the same torsion in their Khovanov homology, then they have the same ranks as well. In other words, the Khovanov homology of a knot is completely determined by its torsion.*

Remark. There are many examples of knots which have the same ranks in Khovanov homology but different torsion. Some of the first ones are 14^n_{9933} and the mirror image of 15^n_{129763}. The former one is T-simple, while the latter one is T-fancy.

In this paper we prove the following results.

Theorem 1. *The Khovanov homology of every H-slim link has no torsion of order p^k for any odd prime p and $k \geq 1$.*

Corollary 2. *The Khovanov homology of every non-split alternating link has no torsion of order p for any p other than a power of* 2.

We prove Theorem 1 by showing that Conjecture 1.A holds true for every H-slim link not only for rational homology, but also for homology over $\mathbb{Z}_p$ for all odd prime p. This is done using a slight modification of Lee's methods from [L2].

Theorem 3 (cf. Conjecture 2). *Every H-slim link is WT-simple.*

Corollary 4. *Every non-split alternating link is WT-simple. In particular, the integer Khovanov homology of non-split alternating links is all but determined by the Jones polynomial and signature except that one cannot distinguish between $\mathbb{Z}_{2^k}$ factors in the canonical decomposition of the Khovanov homology groups for different values of k.*

Remark. Lee proved that the rational Khovanov homology of non-split alternating links is completely determined by the Jones polynomial and signature (see [L2]).

Corollary 5 (cf. Conjecture 1). *Every alternating link except the trivial knot, the Hopf link, their connected sums and disjoint unions has torsion of order* 2.

Remark. Marta Asaeda and Józef Przytycki gave [AP] an independent proof of Corollary 5. Moreover, they proved that an adequate link that satisfies some additional conditions has torsion of order 2 as well. Contrary to our approach, their proof is constructive. They explicitly find a generator in the Khovanov chain complex that gives rise to an appropriate torsion element. More recently, Milena Pabiniak, Józef Przytycki, and Radmila Sazdanović provided similar treatments for semi-adequate links [PPS, PS].

Theorem 3 is a corollary of Theorem 3.2.A that establishes a structure of an exact sequence in Khovanov homology over $\mathbb{Z}_2$. To complete the proof of Conjecture 2, one only has to show that every H-slim link has no torsion elements of order 2^k for $k \geq 2$.

This paper is organized as follows. Section 2 contains the main definitions and facts about Khovanov homology that are going to be used in the paper. Theorems 1 and 3 and Corollary 5 are proved in Sections 4, 3, and 3.3, respectively. The Appendix contains information about the standard and reduced Khovanov homology of knots whose torsion has some remarkable properties.

2. Khovanov chain complex and its properties. In this section we briefly recall the main ingredients of Khovanov homology theory. Our exposition follows the one by Viro, whose paper [V] is recommended for a full treatment.

2.1. Generators and the differential of the Khovanov chain complex. Let D be a planar diagram representing an oriented link L. Assign a number ± 1, called a *sign*, to every crossing of D according to the rule depicted in Figure 4. The sum of such signs over all the crossings is called the *writhe number* of D and is denoted by $w(D)$.

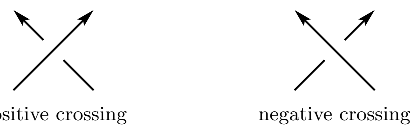


Fig. 4. Positive and negative crossings

At every crossing of D, the diagram locally divides the plane into four quadrants. A choice of a pair of antipodal quadrants at a crossing can be depicted on the diagram with the help of a *marker*, which can be either *positive* or *negative* (see Figure 5). A collection of markers chosen at every crossing of a diagram D is called a (*Kauffman*) *state* of D. There are, clearly, 2^n different states, where n is the number of crossings of D. Denote by $\sigma(s)$ the difference between the numbers of positive and negative markers in a given state s.

Given a state s of a diagram D, one can smooth D at every crossing with respect to the corresponding marker from s (see Figure 5). The result

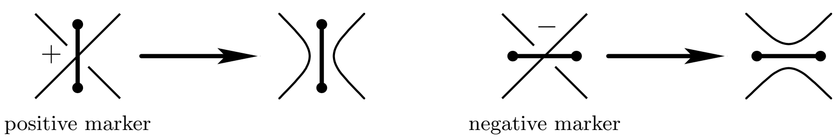


Fig. 5. Positive and negative markers and the corresponding smoothings of a diagram

is a family D_s of disjointly embedded circles. Denote the number of these circles by $|s|$.

Let s be a state of a diagram D. Equip each circle from D_s with either a plus or a minus sign. We call the result an *enhanced* (*Kauffman*) *state* of D that belongs to s. There are exactly $2^{|s|}$ different enhanced states that belong to a given state s. Denote by $\tau(S)$ the difference between the numbers of positively and negatively signed circles in a given enhanced state S.

With every enhanced state S belonging to a state s of a diagram D one can associate two numbers:

$$i(S) = \frac{w(D) - \sigma(s)}{2}, \qquad j(S) = -\frac{\sigma(s) + 2\tau(S) - 3w(D)}{2}.$$

Since both $w(D)$ and $\sigma(s)$ are congruent modulo 2 to the number of crossings, $i(S)$ and $j(S)$ are always integer.

Fix $i, j \in \mathbb{Z}$. It was shown by Viro [V] that the Khovanov chain group $\mathcal{C}^{i,j}(D)$ is generated by all the enhanced states of D with $i(S) = i$ and $j(S) = j$. With the basis of the chain groups chosen, the *Khovanov differential* $d^{i,j} : \mathcal{C}^{i,j}(D) \to \mathcal{C}^{i+1,j}(D)$ can be described by its matrix, called the *incidence matrix* in this context. The elements of the incidence matrix are called *incidence numbers* and are denoted by $(S_1 : S_2)$, where S_1 and S_2 are enhanced states (that is, generators) from $\mathcal{C}^{i,j}(D)$ and $\mathcal{C}^{i+1,j}(D)$, respectively.

The incidence number $(S_1 : S_2)$ is zero unless all of the following three conditions are met:

I. The markers from S_1 and S_2 differ at one crossing of D only, and at this crossing the marker from S_1 is positive, while the marker from S_2 is negative.

Remark. If this condition is satisfied, then D_{S_2} is obtained from D_{S_1} by either joining two circles into one or splitting one circle into two, and hence $|S_2| = |S_1| \pm 1$.

II. The common circles of D_{S_1} and D_{S_2} have the same signs.

III. One of the following four conditions is met:

(1) $|S_2| = |S_1| - 1$, both joining circles from D_{S_1} are negative and the resulting circle from D_{S_2} is negative as well;

(2) $|S_2| = |S_1| - 1$, the joining circles from D_{S_1} have different signs and the resulting circle from D_{S_2} is positive;

(3) $|S_2| = |S_1| + 1$, the splitting circle from D_{S_1} is positive and both the resulting circles from D_{S_2} are positive as well;

(4) $|S_2| = |S_1| + 1$, the splitting circle from D_{S_1} is negative and the resulting circles from D_{S_2} have different signs.

If all the conditions I–III are satisfied, then the incidence number $(S_1 : S_2)$ is defined to be equal to $(-1)^t$, where t is defined as follows. Choose some order on the crossings of D. Let the crossing where one changes the marker to get from S_1 to S_2 have number k in this order. Then t is the number of negative markers in S_1 whose order number is greater than k. As it turns out, the resulting homology does not depend on the choice of the crossing order. More details can be found in [BN1, V].

2.2. Reduced Khovanov homology. Let D be a diagram of a link L. Pick a base point on D that is not a crossing. Let $\widetilde{\mathcal{C}}(D)$ be a subcomplex of $\mathcal{C}(D)$ generated by all the enhanced states of D that have a positive sign on the circle that the base point belongs to. The homology $\widetilde{\mathcal{H}}(L)$ of this subcomplex is called the *reduced Khovanov homology* of L. It can be shown that if L is a knot, then its reduced homology does not depend on the choice of the base point. In general, the reduced homology of a link might depend on the component that the base point is chosen on. See [Kh2] for more details.

2.2.A (Khovanov [Kh2], cf. (1.2)). *The graded Euler characteristic of $\widetilde{\mathcal{C}}(D)$ is a version of the Jones polynomial of L:*

$$\widetilde{J}_L(q) = J_L(q)/(q+1/q) = \sum_{i,j}(-1)^i q^j \widetilde{h}^{i,j}(L), \tag{2.1}$$

where $\widetilde{h}^{i,j}(L)$ are the Betti numbers of $\widetilde{\mathcal{H}}^{i,j}(L)$.

$\widetilde{J}_L(q)$ is completely determined by the following identities (cf. (1.1)):

$$-q^{-2}\widetilde{J}_{\times_+}(q) + q^2\widetilde{J}_{\times_-}(q) = (q-1/q)\widetilde{J}_{)(_0}(q), \qquad \widetilde{J}_{\bigcirc}(q) = 1. \tag{2.2}$$

2.2.B (Khovanov [Kh2]). *For any link L, its reduced Khovanov homology $\widetilde{\mathcal{H}}(L)$ over $\mathbb{Q}$ is supported on exactly one diagonal less than the standard one.*

2.2.C. Corollary. *If L is a $\mathbb{Q}H$-thin link (in particular, a non-split alternating link), then $\widetilde{\mathcal{H}}(L)$ is supported on exactly one diagonal. It follows that $\widetilde{J}_L(q)$ is alternating, that is, its coefficients have alternating signs. More precisely, if $\widetilde{J}_L(q) = \sum_{i\in\mathbb{Z}} c_i q^{2i+\gamma}$, where γ is the number of components of L modulo 2, then $\widetilde{J}_L(q)$ is alternating if and only if $(-1)^{i-j}c_ic_j \geq 0$ for all i and j.*

2.3. Khovanov polynomial and its torsion version. Let L be a link and $\mathrm{Kh}(L)(t,q) = \sum_{i,j} t^i q^j h^{i,j}(L)$ be the Poincaré polynomial in variables t and q of its Khovanov homology. This polynomial is called the *Khovanov polynomial* of L. Now Conjecture 1.A can be reformulated in the following way.

2.3.A (Rigorous statement of Conjecture 1.A). *Let L be a knot. Then there exists a polynomial $\mathrm{Kh}'(L)$ in $t^{\pm1}$ and $q^{\pm1}$ with non-negative coeffi-*

cients only and an even integer $s = s(L)$ such that

$$\mathrm{Kh}(L) = q^{s-1}(1 + q^2 + (1 + tq^4)\,\mathrm{Kh}'(L)). \tag{2.3}$$

In other words, there exist non-negative integers $g^{i,j}(L)$ such that

$$h^{i,j}(L) = g^{i,j}(L) + g^{i-1,j-4}(L) + \varepsilon^{i,j}, \tag{2.4}$$

where $\varepsilon^{0,s\pm1} = 1$ and $\varepsilon^{i,j} = 0$ if $i \neq 0$ or $j \neq s \pm 1$.

Remark. It is clear from the construction that $g^{i,j}(L)$ is the coefficient of the term $t^i q^{j-s+1}$ in $\mathrm{Kh}'(L)$. It must be non-zero for finitely many values of the pair (i, j) only.

2.3.B. *If L is a $\mathbb{Q}$H-thin knot, then the polynomial $\mathrm{Kh}'(L)$ contains powers of tq^2 only. Let $\mathrm{Kh}'(L) = \sum_{i\in Z} a_i t^i q^{2i}$. In this case $g^{i,2i+s-1}(L) = a_i$, and hence*

$$h^{i,2i+s-1}(L) = g^{i,2i+s-1}(L) + g^{i-1,2i+s-5}(L) + \varepsilon^{i,2i+s-1} = a_i + \delta_{i0}, \tag{2.5}$$

$$h^{i,2i+s+1}(L) = g^{i,2i+s+1}(L) + g^{i-1,2i+s-3}(L) + \varepsilon^{i,2i+s+1} = a_{i-1} + \delta_{i0}, \tag{2.6}$$

where δ_{ij} is the Kronecker delta. All other $h^{i,j}(L)$ are zero.

Theorem 1.F implies that $s(L) = -\sigma(L)$ for all alternating knots L, where $\sigma(L)$ is the signature of L.

The following theorem is a counterpart of Conjecture 1.A for the case of $\mathbb{Q}$H-thin links.

2.3.C. Theorem (Lee [L2]). *Let L be an m-component oriented $\mathbb{Q}$H-thin link (for example, a non-split alternating link). Let $\ell_{k,l}$ be the linking number of the kth and lth components of L and let $\sigma(L)$ be the signature of L. Then*

$$Kh(L) = q^{-\sigma(L)-1}\Big[(1+q^2)\Big(\sum_{E\subset\{2,\ldots,m\}} (tq^2)^{2\sum_{k\in E,\, l\notin E} \ell_{k,l}}\Big) + (1+tq^4)\,\mathrm{Kh}'(L)(tq^2)\Big] \tag{2.7}$$

for some polynomial $Kh'(L)$ with non-negative coefficients.

2.3.D. Definition. For a given link L, its *torsion Khovanov polynomial* Kh_T in variables $t^{\pm1}$ and $Q_{p^k}^{\pm1}$ is defined as $Kh_T(L)(t, Q_2, Q_3, Q_4 \ldots) = \sum_{i,j,p,k} t^i Q_{p^k}^j t_{p^k}^{i,j}(L)$, where i and j are arbitrary, $k \geq 1$, and p runs through all prime numbers. Recall that $t_{p^k}^{i,j}(L)$ is the p^k-rank of $\mathcal{H}^{i,j}(L)$ and $T_p^{i,j}(L) = \sum_{k=1}^{\infty} t_{p^k}^{i,j}(L)$.

The following proposition provides a straightforward reformulation of Definitions 1.G and 1.I in terms of Kh_T.

2.3.E. (1) *A link L is T-simple if and only if* $\operatorname{Kh}_T(L)$ *depends on the variables* $t^{\pm1}$ *and* $Q_2^{\pm1}$ *only and* $\operatorname{Kh}_T(L)(t,q) = tq^{s+1}\operatorname{Kh}'(L)$. *The latter equality holds true if and only if* $t_2^{i,j}(L) = g^{i-1,j-2}(L)$ *for all i and j* (*see* 2.3.B).

(2) *A link L is WT-simple if and only if* $\operatorname{Kh}_T(L)$ *depends on the variables* $t^{\pm1}$ *and* $Q_{2^k}^{\pm1}$ *only and* $\operatorname{Kh}_T(L)(t,q,q,\dots) = tq^{s+1}\operatorname{Kh}'(L)$. *The latter equality holds true if and only if* $T_2^{i,j}(L) = g^{i-1,j-2}(L)$ *for all i and j* (*see* 2.3.B).

(3) *A link L is T-rich if and only if* $\operatorname{Kh}_T(L)$ *depends on the variables* $t^{\pm1}$ *and* $Q_{2^k}^{\pm1}$ *only and* $tq^{s+1}\operatorname{Kh}'(L) - \operatorname{Kh}_T(L)(t,q,q,\dots)$ *contains some terms with negative coefficients.*

3. Khovanov homology with $\mathbb{Z}_2$ coefficients. Denote by $\mathcal{H}^{i,j}_{\mathbb{Z}_2}(L)$ the Khovanov homology over $\mathbb{Z}_2$ (instead of $\mathbb{Z}$) of an oriented link L and by $h^{i,j}_{\mathbb{Z}_2}(L)$ the corresponding Betti numbers. In this section we construct an acyclic differential $\overline{\nu}$ of bidegree $(0,2)$ on $\mathcal{H}^{i,j}_{\mathbb{Z}_2}(L)$ and prove Theorem 3.

3.1. Construction of the $(0,2)$-differential. Let D be a planar diagram of a link L, and S be some enhanced state of D. Denote by $\mathcal{N}(S)$ the set of all enhanced states of D that have the same markers and signs on all the circles as S except one circle where '+' is replaced with '−'. It is a straightforward verification that for every enhanced state $S' \in \mathcal{N}(S)$ one has $\sigma(S') = \sigma(S)$ and $\tau(S') = \tau(S) - 2$. Hence, $i(S') = i(S)$ and $j(S') = j(S)+2$.

3.1.A. Definition. A differential $\nu^{i,j} : \mathcal{C}^{i,j}(D;\mathbb{Z}_2) \to \mathcal{C}^{i,j+2}(D;\mathbb{Z}_2)$ of bidegree $(0,2)$ on the Khovanov chain groups with coefficients in $\mathbb{Z}_2$ is defined on the generators of $\mathcal{C}^{i,j}(D;\mathbb{Z}_2)$ as $\nu^{i,j}(S) = \sum_{S' \in \mathcal{N}(S)} S'$ (see Figure 6).

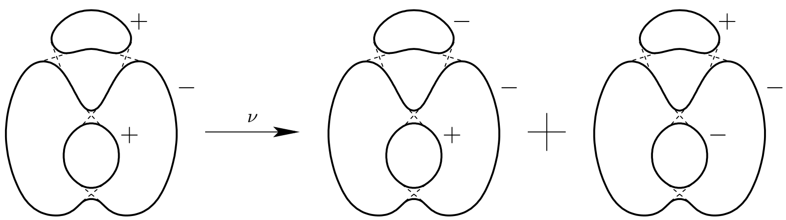


Fig. 6. Action of ν on generators of the chain groups

To show that ν is indeed a differential, i.e. $\nu^2 = 0$, we observe that for any enhanced states S and S'' that have the same markers and signs on all the circles except two circles where S has '+', while S'' has '−', S'' appears exactly twice in $\nu(\nu(S))$.

3.1.B. Lemma. *The map ν is acyclic, i.e. all homology groups with respect to ν are trivial.*

Proof. Let $\mathcal{Z}_n$ be the set of all length n sequences of signs '+' and '−' and let $\mathcal{Z}_n^k \subset \mathcal{Z}_n$ consist of all sequences with the difference between the numbers of minuses and pluses being exactly k (with $k \equiv n \bmod 2$). Then $\mathcal{Z}_n$ and $\mathcal{Z}_n^k$ have 2^n and $b_n^k = \binom{n}{(n+k)/2}$ elements, respectively. Denote by G_n^k the group $\mathbb{Z}_2^{b_n^k}$ whose factors are enumerated by the elements of $\mathcal{Z}_n^k$.

One can construct a differential $\mu_n^k : \mathcal{Z}_n^k \to \mathcal{Z}_n^{k+2}$ similarly to ν: a generator of G_n^k corresponding to a sequence ρ from $\mathcal{Z}_n^k$ is mapped into the sum of the generators of G_n^{k+2} corresponding to all the sequences obtained from ρ by changing exactly one '+' into a '−'.

Let $\mathcal{G}_n$ be the complex

$$0 \to G_n^{-n} \xrightarrow{\mu_n^{-n}} G_n^{-n+2} \xrightarrow{\mu_n^{-n+2}} \cdots \xrightarrow{\mu_n^{n-4}} G_n^{n-2} \xrightarrow{\mu_n^{n-2}} G_n^n \to 0. \tag{3.1}$$

It follows from the definition of the Khovanov chain complex that the complex

$$\cdots \xrightarrow{\nu^{i,j-4}} \mathcal{C}^{i,j-2}(D,\mathbb{Z}_2) \xrightarrow{\nu^{i,j-2}} \mathcal{C}^{i,j}(D,\mathbb{Z}_2) \xrightarrow{\nu^{i,j}} \mathcal{C}^{i,j+2}(D,\mathbb{Z}_2) \xrightarrow{\nu^{i,j+2}} \cdots \tag{3.2}$$

is isomorphic to a direct sum of $\mathcal{G}_{|s|}$ with various shifts, where s runs over all the Kauffman states of D such that $i(s) = i$.

We claim that the complex $\mathcal{G}_n$ is acyclic. Let us prove this by induction on n. The base case of $n = 1$ is trivial. Denote by $\mathcal{G}_n^-$ the subcomplex of $\mathcal{G}_n$ that is obtained by choosing only those sequences that have '−' in the first position. Then $\mathcal{G}_n^-$ is isomorphic to $\mathcal{G}_{n-1}$, and hence is acyclic by the induction hypothesis. It follows that $\mathcal{G}_n$ has the same homology as $\mathcal{G}_n/\mathcal{G}_n^-$. But the latter is again isomorphic to $\mathcal{G}_{n-1}$. Hence, $\mathcal{G}_n$ is acyclic as well. ∎

3.1.C. Lemma. *The map ν commutes with the Khovanov differential d (over $\mathbb{Z}_2$).*

The proof is elementary and is left to the reader as an exercise.

3.2. Patterns in $\mathbb{Z}_2$ homology. Lemma 3.1.C implies that ν can be extended to the $(0,2)$-differential $\overline{\nu}$ in Khovanov homology over $\mathbb{Z}_2$. This differential is also acyclic, although this does not follow from Lemma 3.1.B directly.

3.2.A. Theorem. *The map $\overline{\nu}$ is acyclic. In particular, for every fixed i the following sequence is exact:*

$$\cdots \xrightarrow{\overline{\nu}^{i,j-4}} \mathcal{H}_{\mathbb{Z}_2}^{i,j-2}(L) \xrightarrow{\overline{\nu}^{i,j-2}} \mathcal{H}_{\mathbb{Z}_2}^{i,j}(L) \xrightarrow{\overline{\nu}^{i,j}} \mathcal{H}_{\mathbb{Z}_2}^{i,j+2}(L) \xrightarrow{\overline{\nu}^{i,j+2}} \cdots . \tag{3.3}$$

Consequently, $\sum_{j\in\mathbb{Z}}(-1)^j h_{\mathbb{Z}_2}^{i,2j+\gamma}(L) = 0$ for every i, where γ is the number of components of L modulo 2.

The following proof is due to Khovanov. It replaces the original one that was too technical and unnecessarily complicated.

Proof. Choose a base point b somewhere on the diagram D away from the crossings. In [Kh2] Khovanov introduced another differential $X^{i,j}$: $\mathcal{C}^{i,j}(D) \to \mathcal{C}^{i,j-2}(D)$ of bidegree $(0,-2)$ on the chain groups $\mathcal{C}^{i,j}(D)$ with $\mathbb{Z}$ coefficients. It is defined as follows. Let $S \in \mathcal{C}^{i,j}(D)$ be some enhanced state. If the circle of S that contains b has a positive sign, then $X^{i,j}(S) = 0$. Otherwise $X^{i,j}(S) = S' \in \mathcal{C}^{i,j-2}(D)$, where S' is obtained from S by changing the sign of the circle that contains b from '$-$' to '$+$'. It follows immediately from the definition that $X \circ X = 0$, that is, X is indeed a differential.

It is easy to check that X commutes with the Khovanov differential d and, hence, can be extended to the $(0,-2)$-differential $\overline{X}$ in Khovanov homology. We will abuse the notation slightly and denote reductions of X and $\overline{X}$ modulo 2 by the same symbols.

We first claim that $\nu \circ X + X \circ \nu = \mathrm{id}$. Indeed, let S be some enhanced state of D. If the circle of S that contains b has a positive sign, then $\nu(X(S)) = 0$. Moreover, $\nu(S)$ is a sum of enhanced states such that all but one of them have positive signs on their circles that contain b. Hence, $X(\nu(S)) = S$. On the other hand, if the circle of S containing b has a negative sign, then all the enhanced states from $\nu(S)$ have negative signs on their circles containing b, and $X(\nu(S))$ is the sum of all the states that are obtained from S by changing the sign of the circle that contains b from '$-$' to '$+$' and changing the sign of some other circle from '$+$' to '$-$'. Moreover, $X(S)$ has one more positive sign than S, and $\nu(X(S))$ is the sum of all the same states as $X(\nu(S))$ plus S itself. The claim follows.

Since ν and X both commute with the differential d, one can see that $\overline{\nu} \circ \overline{X} + \overline{X} \circ \overline{\nu} = \mathrm{id}$ at the homology level as well. It follows that $\overline{\nu}$ is acyclic. Indeed, if $\alpha \in \mathcal{H}^{i,j}(D)$ is such that $\overline{\nu}(\alpha) = 0$, then $\overline{\nu}(\overline{X}(\alpha)) = \alpha$, that is, α lies in the image of $\overline{\nu}$. ∎

3.2.B. Corollary. *The map $\overline{X}$ is acyclic on $\mathcal{H}_{\mathbb{Z}_2}(L)$ as well.*

Since the reduced Khovanov homology over $\mathbb{Z}_2$ is isomorphic to the kernel of $\overline{X}$, we obtain

3.2.C. Corollary. $\mathcal{H}^{i,j}_{\mathbb{Z}_2}(L) \simeq \widetilde{\mathcal{H}}^{i,j-1}_{\mathbb{Z}_2}(L) \oplus \widetilde{\mathcal{H}}^{i,j+1}_{\mathbb{Z}_2}(L)$ *for every i, j.*

3.2.D. Corollary. *Let L be an H-slim link. Then it is $\mathbb{Z}_2$H-thin by* 1.D*, that is, its $\mathbb{Z}_2$ homology is supported on the diagonals $2i - j = -s \pm 1$. Theorem* 3.2.A *implies that $h^{i,2i+s-1}_{\mathbb{Z}_2}(L) = h^{i,2i+s+1}_{\mathbb{Z}_2}(L)$ for every i.*

Proof of Theorem 3. Let L be an H-slim knot. It follows from Theorem 1 that L has torsion of order 2^k only. It remains to show that $T^{i,j}_2(L) = g^{i-1,j-2}(L)$ for all i and j (see 2.3.E). Since $T^{i,j}_2(L) = g^{i-1,j-2}(L) = 0$ for $j \neq 2i + s - 1$ and $g^{i-1,2i+s-3}(L) = a_{i-1}$ in the notation of 2.3.B, we only need to prove that $T^{i,2i+s-1}_2(L) = a_{i-1}$.

Observe now that $h_{\mathbb{Z}_2}^{i,j}(L) = h^{i,j}(L) + T_2^{i,j}(L) + T_2^{i+1,j}(L)$. It follows from 2.3.B that $h_{\mathbb{Z}_2}^{i,2i+s-1} = a_i + \delta_{i0} + T_2^{i,2i+s-1}$ and $h_{\mathbb{Z}_2}^{i,2i+s+1} = a_{i-1} + \delta_{i0} + T_2^{i+1,2i+s+1}$. Corollary 3.2.D implies that

$$T_2^{i,2i+s-1} - a_{i-1} = T_2^{i+1,2i+s+1} - a_i \tag{3.4}$$

for all i. Hence, $T_2^{i,2i+s-1} - a_{i-1} = \text{const}$ for some constant independent of i. Since the support of Khovanov homology is finite, there exists i such that $T_2^{i,2i+s-1} = a_{i-1} = 0$. It follows that the constant must be zero.

The case of L being a link can be considered similarly. ∎

3.3. Proof of Corollary 5. Let L be an alternating link with m components. Then its Jones polynomial has the form $\widetilde{J}_L(q) = \sum_i c_i q^{2i+\gamma}$, where $\gamma = m \mod 2$ (cf. Corollary 2.2.C). Define $d(L) = |\widetilde{J}_L(\sqrt{-1})| = \sum_i |c_i|$. In fact, $d(L) = |\det(L)|$, where $\det(L)$ is the determinant of L, hence the notation.

3.3.A. Theorem (Thistlethwaite [Th, Theorem 1]). *Let L be a prime non-split alternating link that admits an irreducible alternating diagram with n crossings, and let $\widetilde{J}_L(q) = \sum_{i=u}^{v} c_i q^{2i+\gamma}$ with $c_u \neq 0$ and $c_v \neq 0$ be its Jones polynomial. Then $v - u = n$ and $c_i c_{i+1} \leq 0$ for every $u \leq i < v$. If, moreover, L is not a $(2,k)$-torus link, then $c_i \neq 0$ for every $u \leq i \leq v$.*

3.3.B. Lemma. *Let L be an alternating link with m components. Then $d(L) \geq 2^{m-1}$. Moreover, if L is not the trivial knot, the Hopf link, their connected sum or disjoint union, then $d(L) > 2^{m-1}$.*

Proof. Assume first that L is non-split and prime. It is easy to deduce from (2.2) (see also [J]) that $\widetilde{J}_L(1) = \sum_i c_i = (-2)^{m-1}$. Hence, $d(L) \geq |\widetilde{J}_L(1)| = 2^{m-1}$. It only remains to show that if L is neither the trivial knot nor the Hopf link, then the polynomial $\widetilde{J}_L(q)$ has both strictly positive and strictly negative coefficients. Theorem 3.3.A implies this for all L except the trivial knot and $(2,k)$-torus links.

Denote a $(2,k)$-torus link by TL_k. It easily follows from (2.2) that $d(\mathrm{TL}_k) = k$. Indeed, if one changes a positive crossing of TL_k into a negative one, one gets TL_{k-2}, and if one smooths such a crossing, one obtains TL_{k-1}. Recall now that a $(2,k)$-torus link has at most two components and that the Hopf link is exactly the $(2,2)$-torus link.

Now the lemma follows from the fact that the Jones polynomial of the connected sum and disjoint union of links L_1 and L_2 is equal to $\widetilde{J}_{L_1}(q)\widetilde{J}_{L_2}(q)$ and $(q+1/q)\widetilde{J}_{L_1}(q)\widetilde{J}_{L_2}(q)$, respectively. ∎

3.3.C. Lemma. *Let L be an alternating link that is not the trivial knot, the Hopf link, their connected sum or disjoint union. Then* $\operatorname{rank}\mathcal{H}(L) > 2^m$, *where m is the number of components of L and* $\operatorname{rank}\mathcal{H}(L) = \sum_{i,j} h^{i,j}(L)$ *is the total rank of the Khovanov homology.*

Proof. Consider first the case when L is non-split. Theorem 2.3.C implies that $\operatorname{rank}\mathcal{H}(L) \geq 2^m$. Assume that this rank is 2^m. In this case $\mathrm{Kh}'(L)$ must be 0 and

$$J_L(q) = \mathrm{Kh}(L)(-1,q) = q^{-\sigma(L)}\Big[(q+1/q)\Big(\sum_{E\subset\{2,\ldots,m\}} (-q^2)^{2\sum_{k\in E,\, l\notin E} \ell_{k,l}}\Big)\Big]. \tag{3.5}$$

Since $\widetilde{J}_L(q) = J_L(q)/(q+1/q)$, one has

$$d(L) = |\widetilde{J}_L(\sqrt{-1})| = \sum_{E\subset\{2,\ldots,m\}} 1^{2\sum_{k\in E,\, l\notin E} \ell_{k,l}} = 2^{m-1}. \tag{3.6}$$

This contradicts Lemma 3.3.B. Hence, $\operatorname{rank}\mathcal{H}(L) > 2^m$.

The general case follows from the fact that $\operatorname{rank}\mathcal{H}(L)$ is multiplicative under disjoint union (see [Kh1, Corollary 12]). ∎

Let us now finish the proof of Corollary 5. Lemma 3.3.C states that $\operatorname{rank}\mathcal{H}(L) > 2^m$, and hence $\mathrm{Kh}'(L) \neq 0$ (in the notation of Theorem 2.3.C). Since L is WT-simple by Theorem 3, it follows from 2.3.E that $\mathrm{Kh}_T(L) \neq 0$ as well. Hence, L has non-trivial torsion. Since L is WT-simple, some torsion elements must be of order 2. ∎

4. Torsion of order p of Khovanov homology.

This section is devoted to proving Theorem 1. We start by showing that the Khovanov homology over $\mathbb{Z}_p$ of an H-slim link satisfies Conjecture 1.A as well.

4.1. Khovanov homology with $\mathbb{Z}_p$ coefficients. Let L be an oriented link and p be an odd prime number. Denote by $\mathcal{H}^{i,j}_{\mathbb{Z}_p}(L)$ the Khovanov homology of L over $\mathbb{Z}_p$ and by $h^{i,j}_{\mathbb{Z}_p}(L)$ its Betti numbers. Let $\mathrm{Kh}_{\mathbb{Z}_p}(L)(t,q) = \sum_{i,j} t^i q^j h^{i,j}_{\mathbb{Z}_p}(L)$ be the corresponding Poincaré polynomial.

4.1.A. Theorem (cf. Theorem 2.3.C and [L2, Theorems 1.2 and 1.4]). *Let L be an m-component oriented H-slim link, for example, a non-split alternating link. Then* $\mathrm{Kh}_{\mathbb{Z}_p}(L)$ *satisfies identity* (2.7) *for the original Khovanov polynomial with some other polynomial* $\mathrm{Kh}'_p(L)$ *instead of* $\mathrm{Kh}'(L)$. *If L is an H-slim knot, then this identity becomes*

$$\mathrm{Kh}_{\mathbb{Z}_p}(L) = q^{-\sigma(L)-1}\big(1+q^2+(1+tq^4)\,\mathrm{Kh}'_p(L)(tq^2)\big), \tag{4.1}$$

where $\sigma(L)$ is the signature of L.

Proof. We will show that the methods used by Lee to prove Theorem 2.3.C for Khovanov homology with $\mathbb{Q}$ coefficients work in our $\mathbb{Z}_p$ case as well if p is an odd prime. Only the main steps are to be outlined and the reader is assumed to be familiar with [L2].

First of all, we define the Lee differential Φ of bidegree $(1,4)$ on the Khovanov chain complex $\mathcal{C}$. The corresponding incidence numbers $(S_1 : S_2)_\Phi$ of two enhanced states $S_1 \in \mathcal{C}^{i,j}_{\mathbb{Z}_p}(D)$ and $S_2 \in \mathcal{C}^{i+1,j+4}_{\mathbb{Z}_p}(D)$ are defined in a similar way to the original ones from page 351 with the only difference being in condition III: The incidence number $(S_1 : S_2)_\Phi$ is zero unless all of the following three conditions are met, in which case $(S_1 : S_2)_\Phi = \pm 1$ with the sign defined as before:

- I_Φ. The markers from S_1 and S_2 differ at one crossing of D only, and at this crossing the marker from S_1 is positive, while the marker from S_2 is negative.
- II_Φ. The common circles of D_{S_1} and D_{S_2} have the same signs.
- III_Φ. One of the following two conditions is met:
  - (1) $|S_2| = |S_1| - 1$, both joining circles from D_{S_1} are positive and the resulting circle from D_{S_2} is negative;
  - (2) $|S_2| = |S_1| + 1$, the splitting circle from D_{S_1} is positive and both the resulting circles from D_{S_2} are negative;

It is easy to see that Φ is indeed a differential and (anti)commutes with the Khovanov differential d. Lee's proofs from [L2] can be applied to our version of Φ to show that it also commutes with the isomorphisms induced on $\mathcal{H}_{\mathbb{Z}_p}(L)$ by the Reidemeister moves. Hence, Φ gives rise to a well defined differential on $\mathcal{H}_{\mathbb{Z}_p}(L)$.

Consider now yet another differential $\Phi + d$ on $\mathcal{C}_{\mathbb{Z}_p}(D)$. It can be best described by changing the labels on the circles comprising enhanced states from '+' and '−' to $\mathbf{a} = (`+') + (`-')$ and $\mathbf{b} = (`+') - (`-')$. In this notation one has a new third condition on the incidence numbers $(S_1 : S_2)_{\Phi d}$:

- III_Φ. One of the following four conditions is met:
  - (1) $|S_2| = |S_1| - 1$, both joining circles from D_{S_1} and the resulting circle from D_{S_2} are marked with $\mathbf{a}$; then $(S_1 : S_2)_{\Phi d} = \pm 2$.
  - (2) $|S_2| = |S_1| - 1$, both joining circles from D_{S_1} and the resulting circle from D_{S_2} are marked with $\mathbf{b}$; then $(S_1 : S_2)_{\Phi d} = \mp 2$.
  - (3) $|S_2| = |S_1| + 1$, the splitting circle from D_{S_1} and both the resulting circles from D_{S_2} are marked with $\mathbf{a}$; then $(S_1 : S_2)_{\Phi d} = \pm 1$.
  - (4) $|S_2| = |S_1| + 1$, the splitting circle from D_{S_1} and both the resulting circles from D_{S_2} are marked with $\mathbf{b}$; then $(S_1 : S_2)_{\Phi d} = \pm 1$.

Denote by $H(D)$ the homology with respect to $\varPhi + d$. It can be shown that $H(D)$ is invariant under the Reidemeister moves, so that we can safely write $H(L)$ instead. The fact that $p \neq 2$ is crucial here, as the proof involves division by 2 (see [L2]).

Theorem 4.4 from [L2] that states

$$H(L) \cong \frac{\operatorname{Ker}(\varPhi : \mathcal{H}_{\mathbb{Z}_p}(L) \to \mathcal{H}_{\mathbb{Z}_p}(L))}{\operatorname{Im}(\varPhi : \mathcal{H}_{\mathbb{Z}_p}(L) \to \mathcal{H}_{\mathbb{Z}_p}(L))} \tag{4.2}$$

still holds true without changes. One needs to use the fact that L is $\mathbb{Z}_p$H-thin, since it is H-slim here.

The only non-trivial generalization is proving an analogue of Theorem 4.2 of [L2] that $\dim_{\mathbb{Z}_p} H(L) = 2^m$. Lee's proof uses Hodge theory arguments which are not applicable to the $\mathbb{Z}_p$ case. Fortunately for us, Hodge theory is only used to provide a *lower* bound on $\dim_{\mathbb{Q}} H(L;\mathbb{Q})$. It follows that $2^m \leq \dim_{\mathbb{Q}} H(L;\mathbb{Q}) \leq \dim_{\mathbb{Z}_p} H(L)$, where the former inequality is provided by Theorem 4.2 from [L2], and the latter by the Universal Coefficient Theorem. In particular, all the enhanced states of D such that at every crossing the two touching circles have different labels, are linearly independent in $H(L)$. Such states are in one-to-one correspondence with all the orientations of L (see [L2]). Lee's proof of the fact that $\dim_{\mathbb{Q}} H(L;\mathbb{Q}) \leq 2^m$ still works without changes for $\mathbb{Z}_p$. Hence $2^m \leq \dim_{\mathbb{Z}_p} H(L) \leq 2^m$ and $\dim_{\mathbb{Z}_p} H(L) = 2^m$.

Filling the remaining technical gaps is left to the reader. ∎

4.2. Proof of Theorem 1. Let L be an H-slim link. Since $h^{i,j}_{\mathbb{Z}_p}(L) = h^{i,j}(L) + T^{i,j}_p(L) + T^{i+1,j}_p(L)$, it follows from Theorems 2.3.C and 4.1.A that $h_{\mathbb{Z}_p}(L) - h(L)$ are arranged in "knight move" pairs everywhere without having to subtract anything (cf., for example, Conjecture 1.A). Hence,

$$T_p^{i,2i-\sigma(L)-1}(L) = T_p^{i+1,2i-\sigma(L)+3}(L) + T_p^{i+2,2i-\sigma(L)+3}(L). \tag{4.3}$$

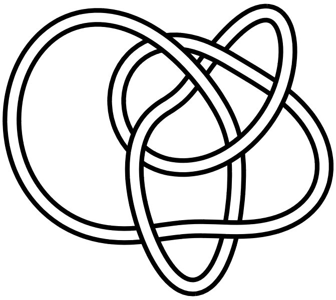
Since the support of Khovanov homology is finite, all $T^{i,j}_p(L)$ must be zero. ∎

Appendix. This section contains information about standard and reduced Khovanov homology of knots whose torsion has some remarkable properties. The knot pictures below were generated using Robert Scharein's program KnotPlot [S].

A.1. How to read the tables. Columns and rows of the tables below are marked with i- and j-grading of Khovanov homology, respectively. For the standard homology, the j-grading is always odd and the corresponding table entries are printed in boldface. The reduced homologies have their j-grading even. They occupy places between the main rows.

Only entries representing non-trivial groups are shown. An entry of the form a, b_2, c_4 means that the corresponding group is $\mathbb{Z}^a \oplus \mathbb{Z}_2^b \oplus \mathbb{Z}_4^c$. If some factors are missing from the group, then the corresponding numbers are absent as well.

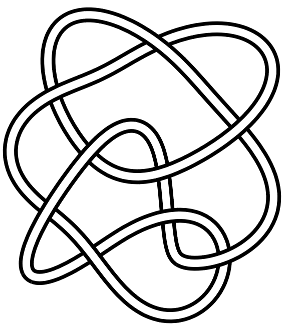
A.2. The knot 8_{19} (see Table 1)**.** This is the first $\mathbb{Q}$H-thick knot. It is T-fancy as well.



	0	1	2	3	4	5
17						**1**
						1
15						**1**
13				**1**	**1**	
				1	1	
11				$\mathbf{1_2}$	**1**	
			1			
9			**1**			
7	**1**					
	1					
5	**1**					

Table 1. The knot 8_{19} and its Khovanov homology (standard and reduced)

A.3. The knot 9_{42} (see Table 2)**.** This knot is $\mathbb{Q}$H-thick but T-simple. Conjecture 2 states that every T-fancy knot should be H-thick as well. This example shows that the converse is not true in general.



	−4	−3	−2	−1	0	1	2
7							**1**
							1
5							$\mathbf{1_2}$
						1	
3					**1**	**1**	
					1		
1				**1**	$\mathbf{1, 1_2}$		
				2	1		
−1				$\mathbf{1, 1_2}$	**1**		
			1				
−3		**1**	**1**				
		1					
−5		$\mathbf{1_2}$					
	1						
−7	**1**						

Table 2. The knot 9_{42} and its Khovanov homology (standard and reduced)

A.4. The knot 13^n_{3663} (see Table 3). This is the first T-rich knot (the groups with excessive torsion are $\mathcal{H}^{-3,-7}$, $\mathcal{H}^{-3,-5}$, $\mathcal{H}^{-2,-5}$, $\mathcal{H}^{-2,-3}$, $\mathcal{H}^{0,-1}$, $\mathcal{H}^{0,1}$, $\mathcal{H}^{1,1}$, and $\mathcal{H}^{1,3}$). This knot has 2-torsion in the reduced homology as well. This supports the claim of Conjecture 3 that a knot is T-rich if and only if its reduced Khovanov homology has torsion. This knot is also the first one whose homology is supported on four diagonals. The only other knots with 13 crossings or less that share the same properties are 13^n_{4587}, 13^n_{4639}, and 13^n_{5016}.

	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7
13														**1**
														1
11														$\mathbf{1_2}$
													1	
9												**1**	**1**	
												1		
7										**1**	**1**	$\mathbf{1_2}$		
										1	2			
5									**1**	$\mathbf{1_2}$	**1**, $\mathbf{1_2}$			
									2	1				
3								$\mathbf{1_2}$	**2**, $\mathbf{1_2}$	**1**				
								1, 1_2	1					
1							**2**, $\mathbf{1_2}$	**1**, $\mathbf{1_2}$	$\mathbf{1_2}$					
							2, 1_2	1						
−1					**1**	**1**	**1**, $\mathbf{2_2}$	**1**						
					1	2								
−3					$\mathbf{2_2}$	**1**, $\mathbf{1_2}$								
				1	1, 1_2									
−5				**1**, $\mathbf{1_2}$	**1**, $\mathbf{1_2}$									
				1_2										
−7		**1**		$\mathbf{1_2}$										
		1												
−9		$\mathbf{1_2}$												
	1													
−11	**1**													

Table 3. Standard and reduced Khovanov homology of the knot 13^n_{3663}

A.5. The $(4,5)$-torus knot (see Table 4). This is one of the first knots whose Khovanov homology has torsion of order 4. Its minimal diagram has 15 crossings. There are no knots with 14 crossings or less that have torsion of order other than 2. This knot is also T-rich and has 2-torsion in reduced homology (cf. Conjecture 3).

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	0	1	2	3	4	5	6	7	8	9	10
29											$\mathbf{1_2}$
											1_2
27										$\mathbf{1}$	$\mathbf{1_2}$
										1	
25								$\mathbf{1}$		$\boxed{\mathbf{1_4}}$	
								1	1		
23						$\mathbf{1}$		$\mathbf{1}$, $\mathbf{1_2}$	$\mathbf{1}$		
						1		1_2			
21						$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1_2}$			
							1				
19				$\mathbf{1}$	$\mathbf{1}$		$\mathbf{1}$				
				1	1						
17				$\mathbf{1_2}$	$\mathbf{1}$						
			1								
15			$\mathbf{1}$								
13	$\mathbf{1}$										
	1										
11	$\mathbf{1}$										

Table 4. Standard and reduced Khovanov homology of the $(4, 5)$-torus knot

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