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Homeomorphism groups of Sierpiński carpets and Erdős space

by

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Abstract. Erdős space $\mathfrak{E}$ is the "rational" Hilbert space, that is, the set of vectors in ℓ^2 with all coordinates rational. Erdős proved that $\mathfrak{E}$ is one-dimensional and homeomorphic to its own square $\mathfrak{E} \times \mathfrak{E}$, which makes it an important example in dimension theory. Dijkstra and van Mill found topological characterizations of $\mathfrak{E}$. Let M_n^{n+1}, $n \in \mathbb{N}$, be the n-dimensional Menger continuum in $\mathbb{R}^{n+1}$, also known as the n-dimensional Sierpiński carpet, and let D be a countable dense subset of M_n^{n+1}. We consider the topological group $\mathcal{H}(M_n^{n+1}, D)$ of all autohomeomorphisms of M_n^{n+1} that map D onto itself, equipped with the compact-open topology. We show that under some conditions on D the space $\mathcal{H}(M_n^{n+1}, D)$ is homeomorphic to $\mathfrak{E}$ for $n \in \mathbb{N} \setminus \{3\}$.

1. Introduction. *All spaces in this paper are assumed to be separable and metrizable.* If X is locally compact then we equip the group $\mathcal{H}(X)$ of homeomorphisms of X with the compact-open topology. If A is a subset of X then $\mathcal{H}(X, A)$ stands for the subgroup $\{h \in \mathcal{H}(X) : h(A) = A\}$ of $\mathcal{H}(X)$.

Let D be a countable dense subset of a locally compact space X. In [5] Dijkstra and van Mill show that if X contains a nonempty open subset homeomorphic to $\mathbb{R}^n$ for $n \geq 2$, to an open subset of the Hilbert cube, or to an open subset of some universal Menger continuum μ^n for $n \in \mathbb{N}$, then $\mathcal{H}(X, D)$ is homeomorphic to $\mathfrak{E}$. In line with these results we consider in this paper the topological group $\mathcal{H}(M_n^{n+1}, D)$ for $n \in \mathbb{N}$. Here M_n^{n+1} is the n-dimensional Menger continuum in $\mathbb{R}^{n+1}$ (see Engelking [6, §1.11]), also known as the n-dimensional Sierpiński carpet, and D is a countable dense subset of M_n^{n+1}. In our main result, Theorem 3.1, we show that under some conditions on D the space $\mathcal{H}(M_n^{n+1}, D)$ is homeomorphic to $\mathfrak{E}$ for $n \in \mathbb{N} \setminus \{3\}$. The proof is based on the proof of [5, Theorem 10.4] where

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Dijkstra and van Mill use their characterization of $\mathfrak{E}$ to deal with the μ^n case. We also heavily rely on Dijkstra [4, §5] where it is shown that there are closed imbeddings of Erdős-type subspaces of ℓ^1 (see Theorem 2.14) in $\mathcal{H}(M_n^{n+1})$ if $n \in \mathbb{N} \setminus \{3\}$. The main complication is that M_n^{n+1}, in contrast to the n-dimensional universal Menger continuum considered in [5, Theorem 10.4], is not homogeneous.

2. Preliminaries. Let $\mathbb{R}^+ = [0, \infty)$. We shall use a number of compactifications of $\mathbb{R}^m$. Let S^m denote the one-point compactification of $\mathbb{R}^m$. We let $\hat{\mathbb{R}}$ denote the compactification $[-\infty, \infty]$ of $\mathbb{R}$. We shall use the convention that $\pm\infty + t = \pm\infty$ when $t \in \mathbb{R}$. This extends the addition operation on $\mathbb{R}^m$ to a continuous function from $\hat{\mathbb{R}}^m \times \mathbb{R}^m$ to $\hat{\mathbb{R}}^m$. An *m-cell* is any space that is homeomorphic to I^m, where $I = [0, 1]$. For a set A in a topological space we let ∂A denote the boundary of A and $\mathrm{Int}(A)$ the interior of A.

Recall that for a compact space X the compact-open topology on $\mathcal{H}(X)$ coincides with the topology of uniform convergence. We denote the identity element of $\mathcal{H}(X)$ by e_X. If O is an open subset of X then we say that $h \in \mathcal{H}(X)$ is *supported on* O if h is equal to the identity on $X \setminus O$, i.e. if $h{\restriction}(X \setminus O) = e_{X \setminus O}$. We write $\mathcal{H}_O(X)$ for the subgroup of $\mathcal{H}(X)$ consisting of all homeomorphisms of X that are supported on O, so $\mathcal{H}_O(X) = \{h \in \mathcal{H}(X) : h{\restriction}(X \setminus O) = e_{X \setminus O}\}$. Furthermore, we let $\mathcal{H}_O(X, A)$ stand for the subgroup $\mathcal{H}_O(X) \cap \mathcal{H}(X, A)$ of $\mathcal{H}(X)$.

We need the following elementary result; see [5, Lemma 10.3].

LEMMA 2.1. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous. If $g \circ f$ is a closed imbedding then so is f.*

We give the definition of an n-dimensional Sierpiński carpet.

DEFINITION 2.2. Let $n \in \mathbb{N}$. A nowhere dense subset X of S^{n+1} is called an *n-dimensional Sierpiński carpet* if the collection $\{U_i : i \in \mathbb{N}\}$ of components of $S^{n+1} \setminus X$ forms a null sequence such that the closures of the U_i's are a pairwise disjoint collection and every $S^{n+1} \setminus U_i$ is an $(n+1)$-cell.

The Menger continuum M_n^{n+1}, constructed according to the "middle third" method (see Engelking [6, §1.11]) is a standard example of an n-dimensional Sierpiński carpet. The following characterization theorem is due to Whyburn [11] (for $n = 1$) and Cannon [2] (for $n \geq 2$).

THEOREM 2.3. *Let X and Y be two n-dimensional Sierpiński carpets for $n \in \mathbb{N} \setminus \{3\}$ and let U and V be components of $S^{n+1} \setminus X$, respectively $S^{n+1} \setminus Y$. If h is a homeomorphism from the boundary of U to the boundary of V, then h can be extended to a homeomorphism from X to Y.*

REMARK 2.4. In Theorem 2.3, let S and T be components of $S^{n+1} \setminus X$, respectively $S^{n+1} \setminus Y$, such that $S \neq U$ and $T \neq V$. The proofs of Lemma 1 and Theorem 1 in [2] together with the Annulus Theorem ([2]), which enables one to control where the boundary of a component of $S^{n+1} \setminus X$ is mapped to, imply that we can extend h to a homeomorphism $\overline{h}\colon X \to Y$ in such a way that $\overline{h}(\partial S) = \partial T$.

Definition 2.5. A point x of an n-dimensional Sierpiński carpet X is called a *boundary point* of X if it lies on a nonseparating copy S of S^n in X, that is, $X \setminus S$ is connected. If x is not a boundary point we call it an *interior point* of X.

Using the notation of Definition 2.2, it follows easily from Brouwer's Invariance of Domain [8, Theorem 3.6.8] and the generalized Jordan Curve Theorem [9, Theorem 36.3] that x is a boundary point of X if and only if $x \in \bigcup_{i=1}^{\infty} \partial U_i$ and that every ∂U_i is homeomorphic to S^n. Note that these definitions of boundary point and interior point of X do not coincide with the usual meaning of these notions since $\operatorname{Int}(X) = \emptyset$. Boundary points and interior points are two topologically different types of points in X, both of which are represented in X. This means that X is not homogeneous. It is well known that these points are topologically the only two different types of points in X if $\dim X \neq 3$ (cf. Theorem 2.3 and Lemma 2.7).

Lemma 2.6. *Let $n \in \mathbb{N} \setminus \{3\}$ and suppose that $x \in \partial U$, where U is a component of $S^{n+1} \setminus M_n^{n+1}$. Then there is a local basis $\mathcal{B}_x$ at x such that for every $B \in \mathcal{B}_x$ and every $y \in B \cap \partial U$ there is a homeomorphism h of M_n^{n+1} with $h(x) = y$ that is supported on B.*

Proof. Note that it follows from Theorem 2.3 and the homogeneity of S^n that all boundary points of M_n^{n+1} are topologically equivalent. Therefore, it is enough to consider the boundary point $x = (0, \ldots, 0) \in \partial(I^{n+1})$, where $\partial(I^{n+1})$ is the boundary of the unbounded component of $\mathbb{R}^{n+1} \setminus M_n^{n+1}$. For $\mathcal{B}_x$ we take the collection $\{B_i : i \in \omega\}$, where $B_i = M_n^{n+1} \cap [0, 3^{-i})^{n+1}$. Now take $i \in \omega$ and a point $y \in B_i \cap \partial I^{n+1}$. If $y = x$ then the identity map obviously satisfies the requirements of the lemma, so we suppose that $y \neq x$. The closure of B_i in M_n^{n+1} is $C_i = M_n^{n+1} \cap [0, 3^{-i}]^{n+1}$, so $C_i = 3^{-i} M_n^{n+1}$, which means that C_i is again an n-dimensional Sierpiński carpet. Note that $D_i = \partial([0, 3^{-i}]^{n+1})$ is the boundary of the unbounded component of $\mathbb{R}^{n+1} \setminus C_i$. Since $B_i \cap D_i$ is open and connected in D_i, and D_i is homeomorphic to S^n, it follows that $B_i \cap D_i$ is path connected and we can use the strong local homogeneity of S^n to see that there is a homeomorphism $g_i\colon D_i \to D_i$, with $g_i(x) = y$, supported on $B_i \cap D_i$. By Theorem 2.3 we can extend g_i to a homeomorphism $\overline{g_i}$ of C_i. If we now define $h_i\colon M_n^{n+1} \to M_n^{n+1}$ by

$$h_i(x) = \begin{cases} \overline{g_i}(x) & \text{if } x \in C_i, \\ x & \text{otherwise,} \end{cases}$$

then h_i is as required. ∎

We want to derive a similar result for the interior points of M_n^{n+1} with $n \in \mathbb{N} \setminus \{3\}$. For this we use the following lemma. Recall that $\partial(I^{n+1})$ is the boundary of the unbounded component of $\mathbb{R}^{n+1} \setminus M_n^{n+1}$.

LEMMA 2.7. *Let $n \in \mathbb{N} \setminus \{3\}$ and let x and y be interior points of M_n^{n+1}. Then there is a homeomorphism $h\colon M_n^{n+1} \to M_n^{n+1}$ with $h(x) = y$ and $h \restriction \partial(I^{n+1}) = e_{\partial(I^{n+1})}$.*

Proof. If $x = y$ we can take $h = e_{M_n^{n+1}}$, so suppose that $x \neq y$. Clearly, we can find quotient mappings $q_x, q_y\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $q_x^{-1}(\{x\}) = I^{n+1}$ and $q_y^{-1}(\{y\}) = I^{n+1}$ such that $q_x\colon \mathbb{R}^{n+1} \setminus I^{n+1} \to \mathbb{R}^{n+1} \setminus \{x\}$ and $q_y\colon \mathbb{R}^{n+1} \setminus I^{n+1} \to \mathbb{R}^{n+1} \setminus \{y\}$ are homeomorphisms. Then $q_x^{-1}(M_n^{n+1}) \setminus \operatorname{Int} I^{n+1}$ and $q_y^{-1}(M_n^{n+1}) \setminus \operatorname{Int} I^{n+1}$ are Sierpiński carpets and we denote them by S_x, respectively S_y.

Let B_x, respectively B_y, be the boundary of the unbounded component of $\mathbb{R}^{n+1} \setminus S_x$, respectively $\mathbb{R}^{n+1} \setminus S_y$. So $B_x = q_x^{-1}(\partial I^{n+1})$ and $B_y = q_y^{-1}(\partial I^{n+1})$. Note that $g = (q_y^{-1} \circ q_x) \restriction B_x$ is a homeomorphism from B_x to B_y such that $q_y \circ g = q_x \restriction B_x$. It follows from Remark 2.4 that we can extend g to a homeomorphism $\overline{g}\colon S_x \to S_y$ such that $\overline{g}(\partial I^{n+1}) = \partial I^{n+1}$.

Now define the function $h\colon M_n^{n+1} \to M_n^{n+1}$ by

$$h(z) = \begin{cases} y & \text{if } z = x, \\ (q_y \circ \overline{g} \circ q_x^{-1})(z) & \text{if } z \neq x. \end{cases}$$

It is easy to see that h is a bijection such that $h \circ q_x = q_y \circ \overline{g}$. Since q_x is a quotient mapping and $q_y \circ \overline{g}$ is continuous, it follows that h is continuous. By compactness of M_n^{n+1} we see that h is a homeomorphism.

Take $z \in \partial(I^{n+1})$. Then $q_x^{-1}(z) \in B_x$ and since $\overline{g}$ is an extension of g we see that

$$h(z) = (q_y \circ \overline{g})(q_x^{-1}(z)) = (q_y \circ g)(q_x^{-1}(z)) = q_x(q_x^{-1}(z)) = z.$$

This shows that $h \restriction \partial(I^{n+1}) = e_{\partial(I^{n+1})}$, so h is as required. ∎

LEMMA 2.8. *Let $n \in \mathbb{N} \setminus \{3\}$ and suppose that x is an interior point of M_n^{n+1}. Then there is a local basis $\mathcal{B}_x$ at x such that for every $B \in \mathcal{B}_x$ and every interior point y of M_n^{n+1} in B there is a homeomorphism h of M_n^{n+1} with $h(x) = y$ that is supported on B.*

Proof. It follows from the construction of M_n^{n+1} that x has arbitrarily small open neighbourhoods B in M_n^{n+1} such that $\overline{B}$, the closure of B in M_n^{n+1} (or in $\mathbb{R}^{n+1}$), is homeomorphic to M_n^{n+1} and the boundary ∂B of B in M_n^{n+1} is the boundary of the unbounded component of $\mathbb{R}^{n+1} \setminus \overline{B}$. Let $\mathcal{B}_x$ be the collection of those neighbourhoods B of x. Clearly, $\mathcal{B}_x$ is a local basis at x. If y is an interior point of M_n^{n+1} such that y is an element of a $B \in \mathcal{B}_x$, then y is an interior point of $\overline{B}$. It follows from Lemma 2.7 that we can find a homeomorphism of $\overline{B}$ that maps x to y and is the identity on the boundary of B in M_n^{n+1}. This homeomorphism can be extended to M_n^{n+1} by taking the identity on $M_n^{n+1} \setminus \overline{B}$. This shows that the local basis $\mathcal{B}_x$ at x is as required. ■

LEMMA 2.9. *Let O be an open subset of M_n^{n+1} for $n \in \mathbb{N} \setminus \{3\}$ and let D_1 and D_2 be countable subsets of O. Suppose that for $j \in \{1, 2\}$ the interior points of M_n^{n+1} contained in D_j are dense in O, and $D_j \cap \partial U_i$ is dense in $\partial U_i \cap O$ for all i. Then there is a homeomorphism h of M_n^{n+1} that is supported on O and satisfies $h(D_1) = D_2$.*

Proof. This proof uses a well known back-and-forth construction; see for instance [1] or [8, Theorem 1.6.9]. Write $D_1 = D_1^{\mathrm{i}} \cup D_1^{\mathrm{b}}$, where D_1^{i} is the set of all points of D_1 that are interior points of M_n^{n+1}, and D_1^{b} is the set of all points of D_1 that are boundary points of M_n^{n+1}. Similarly, write $D_2 = D_2^{\mathrm{i}} \cup D_2^{\mathrm{b}}$. Let $\{a_1, a_2, \ldots\}$ and $\{\tilde{a}_1, \tilde{a}_2, \ldots\}$ be enumerations of D_1^{i}, respectively D_1^{b}, and let $\{b_1, b_2, \ldots\}$ and $\{\tilde{b}_1, \tilde{b}_2, \ldots\}$ be enumerations of D_2^{i}, respectively D_2^{b}. Using the Inductive Convergence Criterion [8, 1.6.2] we construct a sequence $(h_m)_{m\in\mathbb{N}}$ of homeomorphisms of M_n^{n+1} such that $h = \lim_{m\to\infty} h_m \circ \cdots \circ h_1$ exists and is a homeomorphism, and the following conditions are satisfied:

1. (1) h_m is supported on O for all $m \in \mathbb{N}$;
2. (2) $h_m \circ \cdots \circ h_1(a_j) = h_{4j-2} \circ \cdots \circ h_1(a_j) \in D_2^{\mathrm{i}}$ for all j and $m \geq 4j - 2$;
3. (3) $(h_m \circ \cdots \circ h_1)^{-1}(b_j) = (h_{4j-1} \circ \cdots \circ h_1)^{-1}(b_j) \in D_1^{\mathrm{i}}$ for all j and $m \geq 4j - 1$;
4. (4) $h_m \circ \cdots \circ h_1(\tilde{a}_j) = h_{4j} \circ \cdots \circ h_1(\tilde{a}_j) \in D_2^{\mathrm{b}}$ for all j and $m \geq 4j$;
5. (5) $(h_m \circ \cdots \circ h_1)^{-1}(\tilde{b}_j) = (h_{4j+1} \circ \cdots \circ h_1)^{-1}(\tilde{b}_j) \in D_1^{\mathrm{b}}$ for all j and $m \geq 4j + 1$.

These conditions ensure that $h \in \mathcal{H}_O(M_n^{n+1})$, $h(D_1^{\mathrm{i}}) = D_2^{\mathrm{i}}$ and $h(D_1^{\mathrm{b}}) = D_2^{\mathrm{b}}$. Put $h_1 = e_{M_n^{n+1}}$ and assume that $h_1, \ldots, h_{4j-3}$ are defined for certain $j \in \mathbb{N}$.

If $h_{4j-3} \circ \cdots \circ h_1(a_j) \in D_2^{\mathrm{i}}$, take $h_{4j-2} = e_{M_n^{n+1}}$. Otherwise, we use Lemma 2.8 to find a small neighbourhood $V_{4j-2} \subset O$ of $h_{4j-3} \circ \cdots \circ h_1(a_j)$ which is disjoint from the finite set

$$\{b_1, \ldots, b_{j-1}, \tilde{b}_1, \ldots, \tilde{b}_{j-1}\} \cup h_{4j-3} \circ \cdots \circ h_1(\{a_1, \ldots, a_{j-1}, \tilde{a}_1, \ldots \tilde{a}_{j-1}\})$$

and moreover has the property that we can map $h_{4j-3} \circ \cdots \circ h_1(a_j)$ to any other interior point of M_n^{n+1} in V_{4j-2} by a homeomorphism supported on V_{4j-2}. Since D_2^{i} is dense in O we have $D_2^{\mathrm{i}} \cap V_{4j-2} \neq \emptyset$. This means that we can find a homeomorphism f_{4j-2} of M_n^{n+1} supported on V_{4j-2} such that

$$f_{4j-2} \circ h_{4j-3} \circ \cdots \circ h_1(a_j) \in D_2^{\mathrm{i}}.$$

We put $h_{4j-2} = f_{4j-2}$.

If $(h_{4j-2} \circ \cdots \circ h_1)^{-1}(b_j) \in D_1^{\mathrm{i}}$, we take $h_{4j-1} = e_{M_n^{n+1}}$. Otherwise, we use Lemma 2.8 again to find a small neighbourhood $V_{4j-1} \subset O$ of b_j that is disjoint from the finite set

$$\{b_1,\dots,b_{j-1},\tilde b_1,\dots,\tilde b_{j-1}\}\cup h_{4j-2}\circ\dots\circ h_1(\{a_1,\dots,a_j,\tilde a_1,\dots\tilde a_{j-1}\})$$

and has the property that we can map b_j to any other interior point of M_n^{n+1} in V_{4j-1} by a homeomorphism supported on V_{4j-1}. Since $(h_{4j-2}\circ\dots\circ h_1)(D_1^{\mathrm{i}})$ is dense in O, by (2) we know that $(h_{4j-2}\circ\dots\circ h_1)(D_1^{\mathrm{i}})\cap V_{4j-1}\neq\emptyset$. This means that there is a homeomorphism f_{4j-1} of M_n^{n+1} supported on V_{4j-1} such that

$$f_{4j-1}^{-1}(b_j)\in(h_{4j-2}\circ\dots\circ h_1)(D_1^{\mathrm{i}}).$$

We put $h_{4j-1}=f_{4j-1}$.

Using the same argument as above, but now applying Lemma 2.6 instead of Lemma 2.8, we find neighbourhoods $V_{4j},V_{4j+1}\subset O$ of $(h_{4j-1}\circ\dots\circ h_1)(\tilde a_j)$, respectively $\tilde b_j$, and homeomorphisms h_{4j},h_{4j+1} in $\mathcal{H}_{V_{4j}}(M_n^{n+1})$, respectively $\mathcal{H}_{V_{4j+1}}(M_n^{n+1})$, such that

$$h_{4j}\circ h_{4j-1}\circ\dots\circ h_1(\tilde a_j)\in D_2^{\mathrm{b}},\qquad h_{4j+1}^{-1}(\tilde b_j)\in h_{4j}\circ\dots\circ h_1(D_1^{\mathrm{b}}).$$

If the neighbourhoods $V_{4j-2},V_{4j-1},V_{4j},V_{4j+1}$ are chosen small enough, then the conditions of the Inductive Convergence Criterion are satisfied. ■

Remark 2.10. It follows immediately from this lemma that if $D_1\cap\partial U_i$ and $D_2\cap\partial U_i$ are dense in $\partial U_i\cap O$ for every i with $\partial U_i\cap O\neq\emptyset$, and D_1 and D_2 do not contain any interior points of M_n^{n+1}, there is a homeomorphism h of M_n^{n+1} supported on O that maps D_1 onto D_2. Similarly, if D_1 and D_2 both entirely consist of interior points of M_n^{n+1}, there also exists such a homeomorphism.

Now let $p\geq 1$ and consider the Banach space ℓ^p of all sequences $z=(z_0,z_1,\dots)\in\mathbb{R}^\omega$ such that $\sum_{n=0}^\infty|z_n|^p<\infty$. The topology on ℓ^p is generated by the p-norm $\|z\|_p=(\sum_{n=0}^\infty|z_n|^p)^{1/p}$. It is well known that $\|\cdot\|_p$ is a *Kadec norm* with respect to the coordinate projections, that is, the norm topology is the weakest topology that makes all the coordinate projections $z\mapsto z_n$ and the norm function continuous. This fact can also be formulated as follows: the norm topology on ℓ^p is generated by the product topology (inherited from $\mathbb{R}^\omega$) together with the sets $\{z\in\ell^p:\|z\|_p<t\}$ for $t>0$. We extend the p-norm over $\hat{\mathbb{R}}^\omega$ by putting $\|z\|_p=\infty$ for each $z\in\hat{\mathbb{R}}^\omega\setminus\ell^p$.

Definition 2.11. Let X be a space. A function $f\colon X\to\hat{\mathbb{R}}$ is called *lower semicontinuous* (abbreviated *LSC*) if $f^{-1}((t,\infty])$ is open in X for every $t\in\mathbb{R}$.

Note that the norm as a function from $\hat{\mathbb{R}}^\omega$ to $[0,\infty]$ is LSC but not continuous because the norm topology on ℓ^p is much stronger than the topology inherited from $\mathbb{R}^\omega$. It is easily checked that $f\colon X\to\hat{\mathbb{R}}$ is LSC if and only if $f(\lim_{n\to\infty} x_n) \leq \liminf_{n\to\infty} f(x_n)$ for every convergent sequence $(x_n)_{n\in\omega}$ in X.

We define *Erdős space*

$$\mathfrak{E} = \{x \in \ell^2 : x_n \in \mathbb{Q} \text{ for all } n \in \omega\}.$$

Let $\mathcal{T}$ be the zero-dimensional topology that $\mathfrak{E}$ inherits from $\mathbb{Q}^\omega$. We noted that $\mathcal{T}$ is weaker than the norm topology, so clopen sets separate points, that is, $\mathfrak{E}$ is totally disconnected. By the remark above, the graph of the norm function, when seen as a function from $(\mathfrak{E}, \mathcal{T})$ to $\mathbb{R}^+$, is homeomorphic to $\mathfrak{E}$. This means that we can informally think of $\mathfrak{E}$ as a "zero-dimensional space with some LSC function declared continuous".

We point out the following connection between the two topologies on $\mathfrak{E}$. Because the norm is LSC on $\mathbb{R}^\omega$, every closed ε-ball in $\mathfrak{E}$ is also closed in the zero-dimensional space $\mathbb{Q}^\omega$. This means that every point in $\mathfrak{E}$ has arbitrarily small neighbourhoods which are intersections of clopen sets.

Definition 2.12. A subset A of a space X is called a *C-set* in X if A can be written as an intersection of clopen subsets of X. A space is called *almost zero-dimensional* if every point of the space has a neighbourhood basis consisting of C-sets. If Z is a set that contains X then we say that a (separable metric) topology $\mathcal{T}$ on Z *witnesses the almost zero-dimensionality of* X if $\dim(Z, \mathcal{T}) \leq 0, O \cap X$ is open in X for each $O \in \mathcal{T}$, and every point of X has a neighbourhood basis in X consisting of sets that are closed in $(Z, \mathcal{T})$. We will also say that the space $(Z, \mathcal{T})$ is a witness to the almost zero-dimensionality of X.

Thus $\mathfrak{E}$ is almost zero-dimensional. The space $\mathbb{Q}^\omega$ is a witness to the almost zero-dimensionality of Erdős space. More generally, if $\varphi\colon Z \to \mathbb{R}$ is an LSC function with a zero-dimensional domain then it follows easily that Z is a witness to the almost zero-dimensionality of the graph of φ. Clearly, a space X is almost zero-dimensional if and only if there is a topology on X witnessing this fact. Oversteegen and Tymchatyn [10] proved that every almost zero-dimensional space is at most one-dimensional.

Definition 2.13. Let X be a space and let $\mathcal{A}$ be a collection of subsets of X. The space X is called *$\mathcal{A}$-cohesive* if every point of X has a neighbourhood that does not contain nonempty clopen subsets of any element of $\mathcal{A}$. If a space X is $\{X\}$-cohesive then we simply call X *cohesive*.

Again, let $p \geq 1$. As a generalization of the construction of $\mathfrak{E}$, consider a fixed sequence $E_0, E_1, E_2, \ldots$ of subsets of $\mathbb{R}$ and let

$$\mathcal{E} = \{z \in \ell^p : z_n \in E_n \text{ for every } n \in \omega\}.$$

The following two results were proved in Dijkstra [3].

Theorem 2.14. *Assume that $\mathcal{E}$ is not empty and that every E_n is zero-dimensional. The following statements are equivalent:*

(1) *there exists an $x \in \prod_{n=0}^{\infty} E_n$ with $\|x\|_p = \infty$ and $\lim_{n\to\infty} x_n = 0$,*
(2) *every nonempty clopen subset of $\mathcal{E}$ is unbounded,*
(3) *$\mathcal{E}$ is cohesive,*
(4) $\dim \mathcal{E} > 0$.

Recall that if $A_0, A_1, \ldots$ is a sequence of subsets of a space X then $\limsup_{n\to\infty} A_n = \bigcap_{n=0}^{\infty} \overline{\bigcup_{k=n}^{\infty} A_k}$.

Corollary 2.15. *If* 0 *is a cluster point of* $\limsup_{n\to\infty} E_n$ *then every nonempty clopen subset of $\mathcal{E}$ is unbounded* (*and hence* $\dim \mathcal{E} \neq 0$).

We need some new notions. The following definitions are taken from Dijkstra and van Mill [5].

Definition 2.16. If A is a nonempty set then $A^{<\omega}$ denotes the set of all finite strings of elements of A, including the null string λ. If $s = a_0 a_1 \ldots a_{k-1} \in A^{<\omega}$ for some $k \in \omega$, then $|s|$ denotes its *length* k. In this context the set A is called an *alphabet*. Let A^{ω} denote the set of all infinite strings $a_0 a_1 \ldots$ of elements of A. If $s \in A^{<\omega}$ and $\sigma \in A^{<\omega} \cup A^{\omega}$ then we put $s \prec \sigma$ if s is an initial substring of σ, that is, there is a $\tau \in A^{<\omega} \cup A^{\omega}$ with $s^\frown\tau = \sigma$, where $^\frown$ denotes concatenation of strings. If $\sigma = a_0 a_1 \ldots \in A^{<\omega} \cup A^{\omega}$ and $k \in \omega$ with $k \leq |\sigma|$, then $\sigma\restriction k = a_0 a_1 \ldots a_{k-1}$.

Definition 2.17. A *tree* T on an alphabet A is a subset of $A^{<\omega}$ that is closed under initial segments, that is, if $s \in T$ and $t \prec s$ then $t \in T$. An *infinite branch* of T is an element σ of A^{ω} such that $\sigma\restriction k \in T$ for every $k \in \omega$. The *body* of T, written as $[T]$, is the set of all infinite branches of T. If $s, t \in T$ are such that $s \prec t$ and $|t| = |s| + 1$ then we say that t is an *immediate successor* of s, and $\operatorname{succ}(s)$ denotes the set of immediate successors of s in T.

Now we introduce the concept of an *anchor*.

Definition 2.18. Let T be a tree and let $(X_s)_{s\in T}$ be a system of subsets of a space X such that $X_t \subset X_s$ whenever $s \prec t$. A subset A of X is called an *anchor* for $(X_s)_{s\in T}$ in X if for every $\sigma \in [T]$, the sequence $X_{\sigma\restriction 0}, X_{\sigma\restriction 1}, \ldots$ converges to a point in X whenever $X_{\sigma\restriction k} \cap A \neq \emptyset$ for all $k \in \omega$.

Example 2.19. As noted before, $\mathbb{Q}^{\omega}$ is a witness to the almost zero-dimensionality of $\mathfrak{E}$. Let $\mathcal{T}$ be the topology that $\mathfrak{E}$ inherits from $\mathbb{Q}^{\omega}$. Put $T = \mathbb{Q}^{<\omega}$, and for $s = q_0 \ldots q_{k-1} \in T$ with $k \in \omega$, let $\mathbb{Q}^{\omega}_s$ be the closed subset of $\mathbb{Q}^{\omega}$ given by

$$\mathbb{Q}^{\omega}_s = \{x \in \mathbb{Q}^{\omega} : x_i = q_i \text{ for } 0 \leq i \leq k-1\}.$$

Put $\mathfrak{E}_s = \mathbb{Q}^\omega_s \cap \mathfrak{E}$ for $s \in T$ and let B be a bounded subset of $\mathfrak{E}$. We show that B is an anchor for $(\mathfrak{E}_s)_{s\in T}$ in $(\mathfrak{E}, \mathcal{T})$. Let $\sigma = q_0 q_1 \ldots \in [T]$ be such that $\mathfrak{E}_{\sigma\restriction k} \cap B \neq \emptyset$ for all $k \in \omega$. It is clear that $\mathfrak{E}_{\sigma\restriction k}$ converges to the point $\sigma \in \mathbb{Q}^\omega$ in the product topology of $\mathbb{Q}^\omega$, where we identify the string $q_0 q_1 \ldots$ with the sequence $(q_0, q_1, \ldots)$. It suffices to show that $\sigma \in \mathfrak{E}$. Since B is bounded there is an $M \in \mathbb{N}$ such that $B \subset \{x \in \mathbb{Q}^\omega : \|x\| \le M\}$, and because $\mathfrak{E}_{\sigma\restriction k} \cap B \neq \emptyset$ for all $k \in \omega$ this means that $\|(q_0, q_1, \ldots, q_{k-1}, 0, 0, \ldots)\| \le M$ for all $k \ge 0$. We have

$$\|\sigma\| = \lim_{k\to\infty} \|(q_0, q_1, \ldots, q_{k-1}, 0, 0, \ldots)\| \le M,$$

so $\sigma \in \mathfrak{E}$.

Dijkstra and van Mill [5, §8] introduced the following class E' of spaces.

Definition 2.20. E' is the class of all nonempty spaces E such that there exists an $F_{\sigma\delta}$-topology $\mathcal{T}$ on E that witnesses the almost zero-dimensionality of E and there exist a nonempty tree T over a countable set and subspaces E_s of E that are closed with respect to $\mathcal{T}$ for each $s \in T \setminus \{\lambda\}$ such that

- (1′) E_λ is dense in E and $E_s = \bigcup\{E_t : t \in \mathrm{succ}(s)\}$ whenever $s \in T$,
- (2′) each $x \in E$ has a neighbourhood U that is an anchor for $(E_s)_{s\in T}$ in $\big(E, \mathcal{T}\big)$,
- (3′) for each $s \in T \setminus \lambda$ and $t \in \mathrm{succ}(s)$, E_t is nowhere dense in E_s,
- (4′) E is $\{E_s : s \in T\}$-cohesive,
- (5′) E can be written as a countable union of nowhere dense subsets that are closed with respect to $\mathcal{T}$.

In [5, Theorem 8.13] Dijkstra and van Mill prove

Theorem 2.21. *A space E is homeomorphic to $\mathfrak{E}$ if and only if $E \in \mathsf{E}'$.*

As an illustration we show that $\mathfrak{E}$ satisfies the conditions of Definition 2.20. Let $\mathcal{T}$ be the product topology that $\mathfrak{E}$ inherits from $\mathbb{Q}^\omega$, put $T = \mathbb{Q}^{<\omega}$ and let $\mathfrak{E}_s$ for $s \in T$ be as defined in Example 2.19. Since $\mathbb{Q}$ is a σ-compact space, it is easy to see that $\mathbb{Q}^\omega$ is an absolute $F_{\sigma\delta}$-space. Furthermore, $\mathfrak{E}$ is an F_σ subset of $\mathbb{Q}^\omega$, which means that $\mathcal{T}$ is indeed an $F_{\sigma\delta}$-topology on $\mathfrak{E}$ that witnesses the almost zero-dimensionality of $\mathfrak{E}$. It is clear that $\mathfrak{E}_s$ is closed in $(\mathfrak{E}, \mathcal{T})$ for all $s \in T$ and conditions (1′), (3′) and (5′) are easily seen to be satisfied. For (2′) and (4′) note that it follows from Example 2.19 and Corollary 2.15 that every bounded neighbourhood of a point x in $\mathfrak{E}$ is an anchor for $(\mathfrak{E}_s)_{s\in T}$ in $\big(\mathfrak{E}, \mathcal{T}\big)$ that contains no nonempty clopen subsets of any $\mathfrak{E}_s$.

3. Homeomorphism groups of a Sierpiński carpet. We prove the following theorem for n-dimensional Sierpiński carpets as an extension of the results in [5, Chapter 10].

THEOREM 3.1. *Let $n \in \mathbb{N} \setminus \{3\}$, let $\{U_i : i \in \mathbb{N}\}$ be the collection of components of $S^{n+1} \setminus M_n^{n+1}$, and let D be a countable dense subset of M_n^{n+1}. If O is a nonempty open subset of M_n^{n+1} such that either $D \cap \partial U_i = \emptyset$ for every i with $\partial U_i \subset O$, or $D \cap \partial U_i$ is dense in ∂U_i for every i with $\partial U_i \subset O$, then $\mathcal{H}_U(M_n^{n+1}, D)$ is homeomorphic to Erdős space for every open U that contains O.*

As noted before, M_n^{n+1} is not homogeneous, which is why we need the conditions on D here. If we choose for instance a set $D \subset M_n^{n+1}$ such that $|D \cap \partial U_i| = i$ for every i then $\mathcal{H}(M_n^{n+1}, D)$ contains only the identity map.

Note that if $D \cap \partial U_i = \emptyset$ for all $\partial U_i \subset O$, there can still be $j \in \mathbb{N}$ with $D \cap \partial U_j \cap O \neq \emptyset$. Similarly, if $D \cap \partial U_i$ is dense in ∂U_i for all $\partial U_i \subset O$, there can still be $j \in \mathbb{N}$ such that $D \cap \partial U_j \cap O$ is not dense in $\partial U_j \cap O$. The following claim shows that for the proof of Theorem 3.1 we can avoid these situations. Furthermore, it shows that if $D \cap \partial U_i$ is dense in ∂U_i for all $\partial U_i \subset O$, we may assume that the set of interior points of M_n^{n+1} contained in $D \cap O$ is either empty or dense in O. This observation will also be useful in the proof of Theorem 3.1.

CLAIM 3.2. *It suffices to prove Theorem* 3.1 *for the following three cases:*

(i) *$D \cap O$ consists entirely of interior points of M_n^{n+1};*
(ii) *$D \cap \partial U_i \cap O$ is dense in $\partial U_i \cap O$ for every $i \in \mathbb{N}$ and the interior points of M_n^{n+1} contained in $D \cap O$ are dense in O;*
(iii) *$D \cap \partial U_i \cap O$ is dense in $\partial U_i \cap O$ for every $i \in \mathbb{N}$ and $D \cap O$ contains no interior points of M_n^{n+1}.*

Proof. Suppose that we are in the situation of Theorem 3.1. Let D_{i} be the set of all points of D that are interior points of M_n^{n+1}. We define $O' \subset O$ by

$$O' = \begin{cases} O \setminus \overline{D_{\mathrm{i}}} & \text{if } O \setminus \overline{D_{\mathrm{i}}} \neq \emptyset, \\ O & \text{otherwise.} \end{cases}$$

Clearly, O' is a nonempty open subset of M_n^{n+1} such that either $D_{\mathrm{i}} \cap O' = \emptyset$ or $D_{\mathrm{i}} \cap O'$ is dense in O'. Next we define $O'' \subset O'$ by

$$O'' = O' \setminus \bigcup\{\partial U_i : \partial U_i \setminus O' \neq \emptyset\}.$$

Since the interior points of M_n^{n+1} are dense in M_n^{n+1} and the collection $\{U_i : i \in \mathbb{N}\}$ forms a null sequence, it follows that O'' is a nonempty open subset of M_n^{n+1}. Furthermore, if $\partial U_i \cap O'' \neq \emptyset$ then $\partial U_i \subset O'' \subset O$. It is clear that O'' satisfies one of the conditions (i), (ii) or (iii), and if we prove the theorem for O'' then we will have also proved it for O. ∎

We introduce some notation.

Definition 3.3. We define subspaces E_2 and E_4 of ℓ^1 as follows:

$$E_2 = \{z \in \ell^1 : 2^i z_i \in \omega \text{ for all } i \in \omega\},$$
$$E_4 = \{z \in \ell^1 : 4^i z_i \in \omega \text{ for all } i \in \omega\}.$$

We write Z_2 for the space consisting of the *set* E_2 equipped with the zero-dimensional topology inherited from the product space $\mathbb{R}^\omega$, that is, the topology generated by the coordinate projections. Similarly, we write Z_4 for the set E_4 equipped with the zero-dimensional topology inherited from $\mathbb{R}^\omega$. For $i \in \omega$ we let $\xi_i\colon E_4 \to E_4$ denote the projection given by $\xi_i(z) = (z_0, z_1, \ldots, z_i, 0, 0, \ldots)$.

We will use the following proposition in the proof of Theorem 3.1.

Proposition 3.4. *Let $n \in \mathbb{N} \setminus \{3\}$ and let $O \subset M_n^{n+1}$ be open and nonempty. Then there exists a closed imbedding $G\colon E_4 \ni z \mapsto G_z \in \mathcal{H}_O(M_n^{n+1})$, a copy $\hat{\mathbb{R}}_c$ of $\hat{\mathbb{R}}$ in O and a sequence $p_1, p_2, \ldots \in O \setminus \hat{\mathbb{R}}_c$ such that*

(a) $\lim_{i\to\infty} p_i = 0_c \in \mathbb{R}_c$, *where* $\mathbb{R}_c = \hat{\mathbb{R}}_c \setminus \{\pm\infty_c\}$,
(b) *for each $r \in \hat{\mathbb{R}}_c$ and $z \in E_4$ we have $G_z(r) = r + \|z\| \in \hat{\mathbb{R}}_c$,*
(c) *for each $x \in M_n^{n+1} \setminus \mathbb{R}_c$ there is an $i \in \omega$ such that $G_z(x) = G_{\xi_i(z)}(x)$ for every $z \in E_4$,*
(d) *$\beta \circ G\colon Z_4 \to \beta(\mathcal{H}(M_n^{n+1}))$ is a closed imbedding, where $A = \{\infty_c, p_1, p_2, \ldots\}$ and $\beta\colon \mathcal{H}(M_n^{n+1}) \to (M_n^{n+1})^A$ is given by $\beta(h) = h{\restriction}A$ (the restriction of h to A is an element of the infinite product space $(M_n^{n+1})^A$).*

The sets $\mathbb{R}_c$ and A can be chosen such that either both consist of interior points of M_n^{n+1} or both consist of boundary points of M_n^{n+1}. Moreover, for $n = 1$ the sets $\mathbb{R}_c$ and A can be chosen such that $\mathbb{R}_c$ consists of interior points of M_1^2 and A consists of boundary points of M_1^2.

Proof. Dijkstra [4, Remark 3] showed that there exists a closed imbedding $\overline{H}$ of E_2 in $\mathcal{H}(\overline{B})$, where $\overline{B}$ is a topological copy of M_n^{n+1} that contains a copy $\hat{\mathbb{R}}_c$ of $\hat{\mathbb{R}}$ and a sequence $p_1, p_2, \ldots \in \overline{B} \setminus \hat{\mathbb{R}}_c$ such that properties (a)–(d) are satisfied. Note that we can imbed E_4 in E_2 by the map $g\colon E_4 \to E_2$ given by

$$g(z_0, z_1, \ldots) = (z_0, 0, z_1, 0, z_2, 0, \ldots).$$

Now g is even an isometry such that $g(E_4)$ is closed with respect to the weak and (therefore also) strong topology on E_2. This means that we may assume that $\overline{H}$ is a closed imbedding of E_4 in $\mathcal{H}(\overline{B})$ with properties (a)–(d). We prove the proposition for $n = 1$ and $n \geq 2$ separately.

Case I: $n \in \mathbb{N} \setminus \{1, 3\}$. This is the easy case because in the construction of Dijkstra [4, §5] the points of $\hat{\mathbb{R}}_c \cup \{p_1, p_2, \ldots\}$ all lie in the same boundary ∂U of some component U of the complement of $\overline{B}$. From [4, Remark 4] it follows immediately that there is an imbedding G as described in the proposition and such that $\mathbb{R}_c$ and A both consist of boundary points of M_n^{n+1}.

To show that there is also a suitable imbedding G such that both $\mathbb{R}_c$ and A consist of interior points of M_n^{n+1}, we consider two disjoint copies B_1, B_2 of $\overline{B}$ in S^{n+1}. Let ∂U_1, respectively ∂U_2, be the boundary of the component of $S^{n+1} \setminus B_1$, respectively $S^{n+1} \setminus B_2$, that contains the set $\hat{\mathbb{R}}_c \cup \{p_1, p_2, \ldots\}$ in B_1, respectively B_2. Then, using Theorem 2.3, we make a new Sierpiński carpet B from B_1 and B_2 by identifying the points of ∂U_1 with the corresponding points on ∂U_2. This means that the set $\hat{\mathbb{R}}_c \cup \{p_1, p_2, \ldots\} \subset B_1$ now only contains interior points of B. Dijkstra's imbeddings of E_4 in $\mathcal{H}(B_1)$ and $\mathcal{H}(B_2)$ naturally give rise to an imbedding G of E_4 in $\mathcal{H}(B)$ that satisfies the requirements of the proposition and is such that $\mathbb{R}_c$ and A both consist of interior points of B. Applying [4, Remark 4] we see that there exists an imbedding G as in the proposition with $\mathbb{R}_c$ and A both consisting of interior points of M_n^{n+1}.

Case II: $n = 1$. In this case the set $\hat{\mathbb{R}}_c$ consists of boundary points of $\overline{B}$ and the sequence $p_1, p_2, \ldots$ consists, with the exception of one point, of interior points of $\overline{B}$; see [4, §5]. We note that all points of $\hat{\mathbb{R}}_c \cup \{p_1, p_2, \ldots\}$ that are boundary points of $\overline{B}$ lie in the boundary of the same component of $S^2 \setminus \overline{B}$. This means that we can use the same argument as in the case $n \in \mathbb{N} \setminus \{1, 3\}$ to show that we can find an imbedding G as required and such that $\mathbb{R}_c$ and A both consist of interior points of M_1^2.

Now we observe that it follows from the definition of g and the construction of Dijkstra that all the points p_i might as well be chosen as boundary points of $\overline{B}$. By [4, Remark 4] we can then find the desired imbedding G with $\mathbb{R}_c$ and A both consisting of boundary points of M_1^2.

Consider now two disjoint copies B_1, B_2 of $\overline{B}$ in S^2 and assume that all the points p_i in B_1 are boundary points. Let ∂U_1, respectively ∂U_2, be the boundary of the component of $S^2 \setminus B_1$, respectively $S^2 \setminus B_2$, that contains $\hat{\mathbb{R}}_c$. The set $\hat{\mathbb{R}}_c$ in B_1 is an arc in the simple closed curve ∂U_1, and similarly the set $\hat{\mathbb{R}}_c$ in B_2 is an arc in ∂U_2. This means that, using Theorem 2.3, we can form a new Sierpiński carpet B from B_1 and B_2 by simply identifying the points of the set $\hat{\mathbb{R}}_c$ in B_1 with the corresponding points of the set $\hat{\mathbb{R}}_c$ in B_2. Then $\mathbb{R}_c \subset B$ consists of interior points of B and the points $\pm\infty_c$ are boundary points of B. Dijkstra's imbeddings of E_4 in $\mathcal{H}(B_1)$ and $\mathcal{H}(B_2)$ naturally extend to an imbedding G of E_4 in $\mathcal{H}(B)$ that satisfies properties (a)–(d) and is such that $\mathbb{R}_c$ consists of interior points of B, and A consists of boundary points of B. Applying [4, Remark 4] we see that there exists an imbedding G as in the proposition with $\mathbb{R}_c$

consisting of interior points of M_1^2 and A consisting of boundary points of M_1^2. ■

We can now give the proof of Theorem 3.1.

Proof of Theorem 3.1. Take an open subset U of M_n^{n+1} that contains O. Let ρ be a metric on M_n^{n+1} and let $\hat{\rho}$ be the induced metric on $\mathcal{H}(M_n^{n+1})$: $\hat{\rho}(f,g) = \max_{x\in M_n^{n+1}} \rho(f(x), g(x))$ for $f, g \in \mathcal{H}(M_n^{n+1})$. Note that $\hat{\rho}$ is right-invariant: $\hat{\rho}(f \circ h, g \circ h) = \hat{\rho}(f,g)$ for any $h \in \mathcal{H}(M_n^{n+1})$. We prove the theorem by showing that $\mathcal{H}_U(M_n^{n+1}, D)$ satisfies the conditions of Definition 2.20. The result then follows from Theorem 2.21. Without loss of generality we may assume that $D \cap (M_n^{n+1} \setminus U)$ is dense in $M_n^{n+1} \setminus U$. Let $\mathcal{T}$ be the topology that $\mathcal{H}_U(M_n^{n+1}, D)$ inherits from the zero-dimensional product space D^D via the injection $h \mapsto h\restriction D$. It follows from [5, Theorem 10.1] that $\mathcal{T}$ is an $F_{\sigma\delta}$-topology that witnesses the almost zero-dimensionality of $\mathcal{H}_U(M_n^{n+1}, D)$.

Consider the spaces E_4 and Z_4 and the projection map $\xi_i\colon E_4 \to E_4$ for $i \in \omega$ as given in Definition 3.3. We let P be the countable dense subset $\bigcup_{i=0}^{\infty} \xi_i(E_4)$ of E_4. Consider now the Cantor set

$$C' = \{z \in E_4 : z_i \in \{0, 4^{-i}\} \text{ for } i \in \omega\},$$

and note that since $\sum_{i=0}^{\infty} 4^{-i} < \infty$, the norm topology and the product topology coincide on C'. Let $\delta\colon C' \to \mathbb{R}^+$ be the imbedding given by the rule $\delta(z) = \|z\|$. We define $C = \delta(C')$, $\gamma = \delta^{-1}\restriction C$, and $Q = \delta(C' \cap P)$. Thus C is a Cantor set with Q as a countable dense subset and $\|\gamma(r)\| = r$ for each $r \in C$. We define subspaces $\mathcal{E}_{\mathrm{c}}$ and $\mathcal{E}$ of ℓ^1 by

$$\mathcal{E}_{\mathrm{c}} = \{z \in \ell^1 : z_i \in C \text{ for } i \in \omega\}, \qquad \mathcal{E} = \{z \in \ell^1 : z_i \in Q \text{ for } i \in \omega\}.$$

The subscript c refers to the fact that $\mathcal{E}_{\mathrm{c}}$ is a complete space. We let Z_{c} and Z stand for $\mathcal{E}_{\mathrm{c}}$ respectively $\mathcal{E}$ with the witness topologies that these spaces inherit from $\mathbb{R}^\omega$. Let $\nu\colon \omega \times \omega \to \omega$ be a bijection such that $\nu(i,j) \geq j$ for all $i, j \in \omega$. We define an imbedding $\zeta\colon \mathcal{E}_{\mathrm{c}} \to E_4$ by the rule $(\zeta(z))_{\nu(i,j)} = (\gamma(z_i))_j$ for $z \in \mathcal{E}_{\mathrm{c}}$ and $i, j \in \omega$. It is clear from the definition and the fact that the norm and product topology coincide on the compactum C' that $\zeta\colon Z_{\mathrm{c}} \to Z_4$ is a closed imbedding. Note that $\|\zeta(z)\| = \|z\|$ for each $z \in \mathcal{E}_{\mathrm{c}}$, which implies that ζ is also a closed imbedding with respect to the norm topologies (recall that the norm topology is generated by the product topology together with the norm function).

We select a null sequence of nonempty open sets $V_0, V_1, \ldots$ whose closures are disjoint subsets of O. Put $V = \bigcup_{k=0}^{\infty} V_k$. Using Proposition 3.4 we can find for every $k \in \omega$ a closed imbedding $G^k\colon E_4 \to \mathcal{H}_{V_k}(M_n^{n+1})$, a copy $\hat{\mathbb{R}}_k$ of $\hat{\mathbb{R}}$ in V_k and a sequence $p_1^k, p_2^k, \ldots \in V_k \setminus \hat{\mathbb{R}}_k$ such that conditions (a)–(d) of Proposition 3.4, with $\hat{\mathbb{R}}_{\mathrm{c}}$ replaced by $\hat{\mathbb{R}}_k$ and p_i replaced by p_i^k, are satisfied for G^k. If $x \in \hat{\mathbb{R}}$ we write x_k for the representation of x in $\hat{\mathbb{R}}_k$. Let $A_k = \{\infty_k, p_1^k, p_2^k, \ldots\}$ and let $\beta_k \colon \mathcal{H}(M_n^{n+1}) \to (M_n^{n+1})^{A_k}$ be given by $\beta_k(h) = h \restriction A_k$. Then condition (d) of Proposition 3.4 is satisfied for G^k with the set A_k and the map β_k.

We now define $H \colon \mathcal{E}_{\mathrm{c}} \to \mathcal{H}_V(M_n^{n+1})$ by

$$(1) \qquad H_z(x) = \begin{cases} G^0_{\zeta(z)}(x) & \text{if } x \in V_0, \\ G^k_{\gamma(z_{k-1})}(x) & \text{if } x \in V_k \text{ for some } k \in \mathbb{N}, \\ x & \text{if } x \in M_n^{n+1} \setminus V, \end{cases}$$

for $z \in \mathcal{E}_{\mathrm{c}}$. Since the V_k's form a null sequence it is clear that every H_z is a homeomorphism of M_n^{n+1} and that H_z depends continuously on $z \in \mathcal{E}_{\mathrm{c}}$. Let $\varPi \colon \mathcal{H}_V(M_n^{n+1}) \to \mathcal{H}_{V_0}(M_n^{n+1})$ be the continuous map defined by $\varPi(h) = (h \restriction V_0) \cup e_{M_n^{n+1} \setminus V_0}$. Since ζ and G^0 are closed imbeddings and $\varPi \circ H = G^0 \circ \zeta$, Lemma 2.1 implies that $H \colon \mathcal{E}_{\mathrm{c}} \to \mathcal{H}_U(M_n^{n+1})$ is also a closed imbedding. Now we consider the three cases of Claim 3.2 separately.

Case (i). In this case $D \cap O$ consists entirely of interior points of M_n^{n+1}. Choose a $k \in \omega$. By Proposition 3.4 we can choose the imbedding G^k in (1) such that A_k and $\mathbb{R}_k$ consist of interior points of M_n^{n+1}. Note that $\mathbb{R}_k$ is a nowhere dense subset of V_k. This means that we can find a countable dense subset D_k of V_k, consisting of interior points of M_n^{n+1}, with $D_k \cap \mathbb{R}_k = \emptyset$ and $A_k \subset D_k$. Since P is countable and $G_z^k(\mathbb{R}_k) = \mathbb{R}_k$ for all $z \in E_4$ (see property (b) of Proposition 3.4), we may assume that $G_z^k(D_k) = D_k$ for each $z \in P$. Let $\mathbb{Q}_4$ be the additive group $\{i4^j : i, j \in \mathbb{Z}\}$ and note that $C \cap \mathbb{Q}_4 = Q$. Let $\mathbb{Q}_4^k$ be the copy of $\mathbb{Q}_4$ that lies in $\mathbb{R}_k$, so $\mathbb{Q}_4^k$ consists of interior points of M_n^{n+1}. As observed in Remark 2.10, we may assume that the set D has the properties

$$(2) \qquad D \cap V_0 = D_0, \qquad D \cap V_k = D_k \cup \mathbb{Q}_4^k \quad \text{for } k \in \mathbb{N}.$$

We verify that

$$\mathcal{E} = \{z \in \mathcal{E}_{\mathrm{c}} : H_z(D) = D\}$$

and hence that $H \restriction \mathcal{E}$ is a closed imbedding of $\mathcal{E}$ into $\mathcal{H}_U(M_n^{n+1}, D)$ for $n \in \mathbb{N}$. If $H_z \in \mathcal{H}_U(M_n^{n+1}, D)$ and $k \in \mathbb{N}$ then by property (b) of Proposition 3.4 we have $H_z(0_k) = \|\gamma(z_{k-1})\| = z_{k-1} \in \mathbb{Q}_4$. Since $z \in \mathcal{E}_{\mathrm{c}}$ we also have $z_{k-1} \in C$ and hence $z_{k-1} \in Q$. Thus $z \in \mathcal{E}$. Conversely, let $z \in \mathcal{E}$. If $x \in V_k \setminus \mathbb{R}_k$ for some $k \in \omega$ then by property (c) of Proposition 3.4 there is a $z' \in P$ such that $H_z(x) = G^k_{z'}(x)$. Since $G^k_{z'}(D_k) = D_k$ it follows that $x \in D_k = D \cap V_k \setminus \mathbb{R}_k$ if and only if $H_z(x) \in D_k$. Note that $H_z(\mathbb{R}_0) = \mathbb{R}_0$ and that this set is disjoint from D. Consider finally the case that $x \in \mathbb{R}_k$ for $k \in \mathbb{N}$. Then $z_{k-1} \in Q \subset \mathbb{Q}_4$ and $H_z(x) = G^k_{\gamma(z_{k-1})}(x) = x + \|\gamma(z_{k-1})\| = x + z_{k-1}$, which is in $\mathbb{Q}_4$ if and only if $x \in \mathbb{Q}_4$.

Remember that $\mathcal{T}$ is the topology on $\mathcal{H}_U(M_n^{n+1}, D)$ inherited from D^D. Let $\mathcal{T}'$ be the topology that $\mathcal{H}(M_n^{n+1})$ inherits from $(M_n^{n+1})^D$ and note that $\mathcal{T}'$ restricts to $\mathcal{T}$ on $\mathcal{H}_U(M_n^{n+1}, D)$. We first verify that $H\colon Z_{\mathrm{c}} \to (\mathcal{H}(M_n^{n+1}), \mathcal{T}')$ is continuous. Let $x \in D$. If $x \notin V$ or if $x \in V_k$ for some $k \in \mathbb{N}$, then $H_z(x)$ depends on at most a single coordinate of z, so continuity with respect to the product topology is obvious. Let $x \in V_0$ and thus $x \in D_0 \subset V_0 \setminus \mathbb{R}_0$. Then by property (c) of Proposition 3.4, $G^0_{z'}(x)$ depends on only finitely many coordinates of $z' \in E_4$ and hence $H_z(x) = G^0_{\zeta(z)}(x)$ also depends on only finitely many coordinates of $z \in Z_{\mathrm{c}}$. This shows that H is continuous with respect to the product topologies. From property (d) of Proposition 3.4 we find that $\beta_0 \circ H = \beta_0 \circ G^0 \circ \zeta$ is a closed imbedding of Z_{c} into $\beta_0(\mathcal{H}(M_n^{n+1}))$. Since $A_0 \subset D$, the mapping $\beta_0\colon (\mathcal{H}(M_n^{n+1}), \mathcal{T}') \to (M_n^{n+1})^{A_0}$ is continuous. Thus from Lemma 2.1 we conclude that $H\colon Z_{\mathrm{c}} \to (\mathcal{H}(M_n^{n+1}), \mathcal{T}')$ is a closed imbedding. Since $Z = H^{-1}(\mathcal{H}_U(M_n^{n+1}, D))$, also $H\restriction Z$ is a closed imbedding of Z in $(\mathcal{H}_U(M_n^{n+1}, D), \mathcal{T})$.

Consider the point $0_1 \in \mathbb{Q}_4^1 \subset \mathbb{R}_1$. For every $a \in D$ we define $\Gamma_a = \{h \in \mathcal{H}_U(M_n^{n+1}, D) : h(0_1) = a\}$. Note that every Γ_a is closed with respect to $\mathcal{T}$ and that $\bigcup_{a\in D} \Gamma_a = \mathcal{H}_U(M_n^{n+1}, D)$. For $i \in \mathbb{N}$, let $z^i = (4^{-i}, 0, 0, \ldots) \in \mathcal{E}$ and let $h \in \Gamma_a$. Since $\lim_{i\to\infty} z^i = \mathbf{0}$, where $\mathbf{0}$ denotes the zero vector in $\mathbb{R}^\omega$, it follows that $\lim_{i\to\infty} h \circ H_{\mathbf{0}}^{-1} \circ H_{z^i} = h$ in $\mathcal{H}_U(M_n^{n+1}, D)$. However, $h \circ H_{\mathbf{0}}^{-1} \circ H_{z^i} \notin \Gamma_a$. To see this, note that it follows from Proposition 3.4, property (b), that $H_{\mathbf{0}}\restriction\hat{\mathbb{R}}_1 = e_{\hat{\mathbb{R}}_1}$ and $H_{z^i}(0_1) = (4^{-i})_1$. This implies that $h(H_{\mathbf{0}}^{-1}(H_{z^i}(0_1))) = h((4^{-i})_1) \neq h(0_1) = a$. Thus Γ_a is nowhere dense in $\mathcal{H}_U(M_n^{n+1}, D)$ and condition (5′) of Definition 2.20 is satisfied.

We now make an observation which will be the key to satisfying conditions (2′) and (4′) of Definition 2.20.

Claim 3.5. *If A is an unbounded subset of $\mathcal{E}$ then*

$$\operatorname{diam}_{\hat\rho}\{H_z : z \in A\} \geq \rho(-\infty_0, \infty_0).$$

Proof. Let $z \in A$ and let $n \in \mathbb{N}$ be arbitrary. Select a $z^n \in A$ such that $\|z^n\| > \|z\| + 2n$. It follows from (1), condition (b) of Proposition 3.4 and the fact that $\|\zeta(z)\| = \|z\|$ for all $z \in \mathcal{E}_{\mathrm{c}}$ that

$$H_z((-\|z\| - n)_0) = G^0_{\zeta(z)}((-\|z\| - n)_0) = -n_0.$$

Similarly, we see that

$$H_{z^n}((-\|z\| - n)_0) = (\|z^n\| - \|z\| - n)_0.$$

We conclude that

$$\begin{aligned}\operatorname{diam}_{\hat\rho}\{H_z : z \in A\} &\geq \limsup_{n\to\infty} \hat\rho\big(H_z, H_{z^n}\big)\\ &\geq \lim_{n\to\infty} \rho(-n_0, (\|z^n\| - \|z\| - n)_0) = \rho(-\infty_0, \infty_0),\end{aligned}$$

proving Claim 3.5. ■

Let $T = Q^{<\omega}$ and define for $s = q_1 \ldots q_k \in T$ with $k \in \omega$ the subspace $\mathcal{E}_s$ of $\mathcal{E}$ by

$$\mathcal{E}_s = \{z \in \mathcal{E} : z_{i-1} = q_i \text{ for } 1 \le i \le k\}.$$

With the same arguments as given after Theorem 2.21 we see that the spaces $\mathcal{E}_s$ satisfy the conditions of Definition 2.20, every bounded subset of $\mathcal{E}$ is an anchor for $(\mathcal{E}_s)_{s\in T}$ in Z, and every nonempty clopen subset of any $\mathcal{E}_s$ is unbounded. Let $J = \{f_q : q \in Q\}$ be a countable dense subset of $\mathcal{H}_U(M_n^{n+1}, D)$. Since $H\colon Z \to (\mathcal{H}_U(M_n^{n+1}, D), \mathcal{T})$ is a closed map, the set $X_s = \{H_z : z \in \mathcal{E}_s\}$ is closed with respect to $\mathcal{T}$ for each $s \in T$. We define $(E_s)_{s\in T}$ as follows:

$$E_\lambda = X_\lambda \circ J, \qquad E_s = X_{q_1\ldots q_k} \circ f_{q_0} \quad \text{if } s = q_0 \ldots q_k \in T \setminus \{\lambda\}.$$

Note that if $f \in \mathcal{H}_U(M_n^{n+1}, D)$ then the map $h \mapsto h \circ f$ is a homeomorphism of $(\mathcal{H}_U(M_n^{n+1}, D), \mathcal{T})$ as well as of $\mathcal{H}_U(M_n^{n+1}, D)$. So every E_s is closed with respect to $\mathcal{T}$ provided $s \neq \lambda$.

It remains to show that $(E_s)_{s\in T}$ satisfies conditions (1′)–(4′) of Definition 2.20. Since $X_\lambda \neq \emptyset$, the set E_λ, just as J, is dense in $\mathcal{H}_U(M_n^{n+1}, D)$. The other part of condition (1′) follows with the same ease. Since $H\colon \mathcal{E} \to \mathcal{H}_U(M_n^{n+1}, D)$ is an imbedding, condition (3′) is satisfied. Now let W be an arbitrary set in $\mathcal{H}_U(M_n^{n+1}, D)$ such that $\operatorname{diam}(W) < \rho(-\infty_0, \infty_0)$. We show that W works for condition (2′) as well as for (4′). Let $\sigma = q_0 q_1 \ldots \in [T]$ be such that $E_{\sigma\restriction k} \cap W \neq \emptyset$ for each $k \in \omega$. Putting $\tau = q_1 q_2 \ldots \in [T]$ we have $X_{\tau\restriction k} \cap (W \circ f_{q_0}^{-1}) \neq \emptyset$ for each $k \in \omega$. Since $\hat\rho$ is right invariant it follows that

$$\operatorname{diam}_{\hat\rho}(W \circ f_{q_0}^{-1}) < \rho(-\infty_0, \infty_0)$$

and hence $F = \{z \in \mathcal{E} : H_z \in W \circ f_{q_0}^{-1}\}$ is bounded by Claim 3.5. Thus F is an anchor for $(\mathcal{E}_s)_{s\in T}$ in Z and obviously $\mathcal{E}_{\tau\restriction k} \cap F \neq \emptyset$ for each $k \in \omega$. Thus $\mathcal{E}_{\tau\restriction 0}, \mathcal{E}_{\tau\restriction 1}, \ldots$ converges to an element z in Z. Then $X_{\tau\restriction 0}, X_{\tau\restriction 1}, \ldots$ converges to H_z and $E_{\sigma\restriction 0}, E_{\sigma\restriction 1}, \ldots$ converges to $H_z \circ f_{q_0}$, both with respect to $\mathcal{T}$. Thus condition (2′) is satisfied. Now let C be a nonempty clopen subset of some E_s such that $C \subset W$. We may assume that $|s| \ge 1$ and we put $q = s\restriction 1$ and $q^\frown t = s$. So $\operatorname{diam}_{\hat\rho}(C \circ f_q^{-1}) < \rho(-\infty_0, \infty_0)$ and $C \circ f_q^{-1}$ is a nonempty clopen subset of X_t. This means that $\{z \in \mathcal{E} : H_z \in C \circ f_q^{-1}\}$ is a nonempty, clopen, bounded subset of $\mathcal{E}_t$. As mentioned above, this contradicts Corollary 2.15, so we conclude that (4′) is satisfied and $\mathcal{H}_U(M_n^{n+1}, D) \in \mathsf{E}'$. Now apply Theorem 2.21 to see that $\mathcal{H}_U(M_n^{n+1}, D)$ is homeomorphic to $\mathfrak{E}$.

Case (ii). In this case $D \cap \partial U_i \cap O$ is dense in $\partial U_i \cap O$ for every i and the interior points of M_n^{n+1} contained in $D \cap O$ are dense in O. We use the same method as in case (i). Take $k \in \omega$. By Proposition 3.4 we choose the imbedding G^k in (1) again such that the sets A_k and $\mathbb{R}_k$ both consist of interior points of M_n^{n+1}. Noting that $\mathbb{R}_k$ is a nowhere dense subset of M_n^{n+1} we can find a countable dense subset D_k of V_k such that $A_k \subset D_k$, $D_k \cap \mathbb{R}_k = \emptyset$, $D_k \cap \partial U_i$ is dense in $\partial U_i \cap V_k$ for every i with $\partial U_i \cap V_k \neq \emptyset$, and the interior points of M_n^{n+1} in D_k are also dense in V_k. Furthermore, we may assume that $G_z^k(D_k) = D_k$ for each $z \in P$, since P is countable and $G_z^k(\mathbb{R}_k) = \mathbb{R}_k$ for all $z \in E_4$. It follows from Lemma 2.9 that we may assume that D has the properties in (2). We continue in precisely the same way as in case (i) to conclude that $\mathcal{H}_U(M_n^{n+1}, D) \in \mathsf{E}'$ and hence $\mathcal{H}_U(M_n^{n+1}, D)$ is homeomorphic to $\mathfrak{E}$ according to Theorem 2.21.

Case (iii). In this case $D \cap \partial U_i \cap O$ is dense in $\partial U_i \cap O$ for every i and $D \cap O$ contains no interior points of M_n^{n+1}. Again, we want D to have the properties (2) for appropriate sets D_k so that in the same way as in case (i) (and (ii)) we can conclude that $\mathcal{H}_U(M_n^{n+1}, D)$ is homeomorphic to $\mathfrak{E}$. We have to treat the cases $n = 1$ and $n > 1$ separately.

First we consider the case $n = 1$. We want $D \cap V_0 = D_0$, with D_0 a countable dense subset of V_0 with $A_0 \subset D_0$ and $D_0 \cap \mathbb{R}_0 = \emptyset$. Since D only contains boundary points of M_1^2, we want D_0 to consist of boundary points of M_1^2. Furthermore, since we are aiming towards Remark 2.10 again, we also want D_0 to be dense in $\partial U_i \cap V_0$ for every i with $\partial U_i \cap V_0 \neq \emptyset$. This means that $\mathbb{R}_0$ cannot be contained in the boundary of some component U_i of the complement of M_1^2. Therefore, we choose G^0 in (1) such that A_0 consists of boundary points of M_1^2, and $\mathbb{R}_0$ consists of interior points of M_1^2. This is possible according to Proposition 3.4. It is then clear that we can find a set D_0 as required and by Remark 2.10 we may indeed conclude that $D \cap V_0 = D_0$.

Now take $k \in \mathbb{N}$. Just as in (2) we want $D \cap V_k = \mathbb{Q}_4^k \cup D_k$, where D_k is a countable dense subset of V_k with $D_k \cap \mathbb{R}_k = \emptyset$ and $A_k \subset D_k$. Since D consists entirely of boundary points of M_1^2, we choose G^k in (1) such that both A_k and $\mathbb{R}_k$ contain only boundary points of M_1^2. This can be done according to Proposition 3.4. Suppose that $\mathbb{R}_k \subset \partial U_{i_k}$ for some component U_{i_k} of the complement of M_1^2. Noting that $\mathbb{R}_k$ is a nowhere dense subset of M_1^2 we can choose D_k so that it consists of boundary points of M_1^2, it is dense in $\partial U_i \cap V_k$ for every $i \in \omega \setminus \{i_k\}$ with $\partial U_i \cap V_k \neq \emptyset$, and it is dense in $(\partial U_{i_k} \setminus \mathbb{R}_k) \cap V_k$. We see that $D_k \cup \mathbb{Q}_4^k$ is a countable dense subset of V_k, entirely consisting of boundary points of M_1^2, that is dense in $\partial U_i \cap V_k$ for every i with $\partial U_i \cap V_k \neq \emptyset$. It then follows from Remark 2.10 that we may assume that indeed $D \cap V_k = \mathbb{Q}_4^k \cup D_k$.

We conclude that we may assume that D satisfies (2). As before, we may assume that $G_z^k(D_k) = D_k$ for all $k \in \omega$ and $z \in P$, so we can continue in the same way as in case (i) to conclude that $\mathcal{H}_U(M_n^{n+1}, D)$ is homeomorphic to $\mathfrak{E}$.

Now consider the case $n \in \mathbb{N} \setminus \{1,3\}$. This is easier than the one-dimensional case. Take $k \in \omega$. Using Proposition 3.4 we choose the imbedding G^k in (1) such that both the sets A_k and $\mathbb{R}_k$ consist of boundary points of M_n^{n+1}. Note that if $\mathbb{R}_k \subset \partial U_{i_k}$ then $\mathbb{R}_k$ is, in contrast to the case $n = 1$, nowhere dense in ∂U_{i_k}. This means that we can find a countable dense subset D_k of V_k, consisting of boundary points of M_n^{n+1}, such that $A_k \subset D_k$, $D_k \cap \mathbb{R}_k = \emptyset$ and $D_k \cap \partial U_i$ is dense in $\partial U_i \cap V_k$ for all i such that $V_k \cap \partial U_i \neq \emptyset$. From Remark 2.10 it follows that we may assume that $D \cap V_0 = D_0$ if $k = 0$ and $D \cap V_k = \mathbb{Q}_4^k \cup D_k$ if $k \in \mathbb{N}$, so we may assume (2). Again, without loss of generality we have $G_z^k(D_k) = D_k$ for all $k \in \omega$ and $z \in P$, so the same reasoning as in case (i) shows that $\mathcal{H}_U(M_n^{n+1}, D)$ is homeomorphic to $\mathfrak{E}$. ■

In analogy to [5, Theorem 10.4] and [5, Remark 10.7] we can adapt the proof of Theorem 3.1 to produce the following slight generalization.

Theorem 3.6. *Let X be a locally compact space and let D' be a countable dense subset of X. Suppose that X contains an open subset O' that is homeomorphic to an open $O \subset M_n^{n+1}$ for some $n \in \mathbb{N} \setminus \{3\}$, such that $D' \cap O'$ corresponds to a countable dense subset D of O that satisfies the conditions of Theorem* 3.1. *Then $\mathcal{H}_U(X, D')$ is homeomorphic to $\mathfrak{E}$ for every open set U that contains O'.*

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